

Complex dynamic of plankton–fish interaction with quadratic harvesting and time delay

Amit Sharma¹ · Anuj Kumar Sharma² · Kulbhushan Agnihotri³

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Abstract The present paper deals with a food chain model consisting of three species phytoplankton, zooplankton and fish. We have divided the present paper into two parts. In the first part, we have assumed that the fish population is harvested using a non-linear harvesting function. Considering this rate of harvesting ‘E’ as a control parameter, we have estimated different ranges of harvesting parameter for maintaining the sustainability in the plankton ecosystem. Moreover, the bifurcation analysis of the system is carried out using normal form theorem by taking ‘E’ as bifurcation parameter. In the second part, a digestion delay corresponding to zooPlankton–fish interaction is introduced for more realistic consideration of the real world problem. Taking harvesting parameter in the stability range, the effect of time delay on the given system is investigated. This research demonstrate that for a certain range of delay, system enters into the excited state with the existence of stability switches which seems new findings for the Plankton–fish system. Explicit results are derived for stability and direction of the bifurcating periodic solution by using normal form theory and center manifold arguments. To validate our analytical findings numerical simulations are also executed.

Keywords Plankton · Fish · Quadratic harvesting · Time delay · Stability switches · Normal form · Center manifold theorem

✉ Amit Sharma
amit.daviet@gmail.com

¹ DAV Institute of Engineering and Technology, Jalandhar, Punjab, India

² LRDAV College, Jagraon, Punjab, India

³ SBSSTC, Ferozpur, Punjab, India

Introduction

A major concern in population ecology is to understand how a population of a given species influences the dynamics of populations of other species. The dynamic relationship between predator and their prey has long been and continue to be one of the dominant themes in mathematical ecology due to its universal existence and importance (Berryman and Millstein 1989). The pioneering work of May (1976) has established some mathematical models based on certain ecological principles to explore the complexity of the ecological system. The research of the last two decades have demonstrated that very complex dynamics can arise in three or more species food chain models (Hastings and Powell 1991; Klebanoff and Hastings 1993; Rai and Upadhyay 2004; Gakkhar and Naji 2005; Upadhyay and Chattopadhyay 2005) including quasi-periodic or chaos. A great number of theoretical studies indicates that, indeed, plankton systems are, in principle, capable to generate their own chaos. Prey-predator models (e.g., phytoplankton-zooplankton interaction) in seasonally varying environments have been proved to be chaotic (Inoue and Kamifukumoto 1984; Klebanoff and Hastings 1994; Kuznetsov and Rinaldi 1996). Chaotic behavior in tritrophic food chain interactions have been shown in Gilpin (1979), Hogeweg and Hesper (1978), Scheffer (1991). It is well established that toxin has great impact on phytoplankton-zooplankton interaction and can be used as a mechanism of controlling complexity (bloom) of the plankton ecosystem (Pal et al. 2007; Chakaraborty et al. 2008; Sarkar and Chattopadhyay 2003; Mukhopadhyay and Bhattacharyya 2006; Hansen 1995; Ives 1987; Buskey and Hyatt 1995). Recently, Upadhyay and Chattopadhyay (2005), Upadhyay et al. (2007), Upadhyay and Rao (2009) have studied a series of mathematical models

representing the prey(TPP)-specialist predator(Zooplankton)-generalist predator(Fish) interaction in the context of deterministic chaos in the plankton system. They have studied the role of toxin as a control parameter in stabilizing the plankton system by using different response functions for fish grazing.

Further, the introduction of strictly planktivorous fish to lakes can alter plankton communities via cascading interactions in food webs, where strong fish predation on zooplankton leads to reduction of their grazing pressure on algae. Many limnological studies have focused on this dramatic impact of fish on plankton communities (Carpenter et al. 1987; Leavitt and Findlay 1994; Pace et al. 1999). Furthermore, enhancement of piscivorous fish by stocking and catch restrictions has been attempted in several eutrophic or hypertrophic waters, and has resulted in the elimination of excess algae (Carpenter et al. 1987; Rinaldi and Solidoro 1998; Benndorf et al. 1984; McQueen et al. 1989; Lazzaro 1987). Recently, research carried by authors in Strock et al. (2013) reveals that the introduction of white perch in eutrophic lake reduces the algal pigments concentration and results high label of stability in trophic interaction due to the cascading effects of white perch. Attayde et al. (2010) in their model, also supports the idea that omnivory decreases the amplitude of limit cycles and increases the persistence in plankton dynamics.

Keeping in view the above findings, here, we have considered a tri-trophic food chain model of Plankton–fish interaction. We have assumed that the generalist predator (fish) is harvested using a nonlinear harvesting function (Feng 2014). Our major concern in this research to determine the impact of non-linear harvesting on the Plankton–fish interaction by predicting different ranges of harvesting parameter.

So far many researcher (Upadhyay et al. 2007; Upadhyay and Rao 2009, references there in) have studied that toxicated phytoplankton may be used as controller and can stabilize the Plankton–fish dynamic.

In this paper, we have introduced quadratic harvesting of the fish population and proposed different ranges of harvesting for regulating agencies so that they can utilize it for future harvesting along with the stabilization of the ecosystem.

The organization of our paper is as follows: In Sect. 2, we have developed a mathematical model followed by its properties in Sect. 2.1. The stability and bifurcation analysis of the given model system with rate of harvesting as a bifurcation parameter is presented in Sect. 3. In Sect. 4, we have assumed that there is a gestation delay in fish population and considering this time delay as a bifurcation parameter, we have discussed its possible impact on the Plankton–fish dynamic. The stability and bifurcation

properties are provided in Sect. 5. The justification of our analytical findings by numerical simulation and concluding remarks are given in Sects. 6 and 6.2.

The mathematical model

Let $p(t)$ be population density of the toxin producing phytoplankton (prey) which is predated by individuals of specialist predator zooplankton of population density $z(t)$ and this zooplankton population, in turn, serves as a favorite food for the generalist predator (fish of mollusca) population of size $f(t)$. This tri-trophic interaction is represented by the following system of ordinary differential equation as,

$$\begin{cases} \frac{dp}{dt} = rp - \delta_1 p^2 - \alpha_1 \frac{p}{a+p} z, \\ \frac{dz}{dt} = c_1 \alpha_1 \frac{p}{a+p} z - \delta_2 z - \alpha_2 \frac{z}{b+z} f - \theta \frac{p}{a+p} z, \\ \frac{df}{dt} = -\delta_3 f + c_2 \alpha_2 \frac{z}{d+z} f - E f^2, \end{cases} \quad (1)$$

where r be the growth rate of the TPP species, α_1 be the maximum ingestion rate by zooplankton population, c_1 is the fraction of biomass converted to zooplankton, $\delta_1, \delta_2, \delta_3$ be the mortality rates of phytoplankton, zooplankton and fish population respectively. Let α_2 be the rate at which fish graze on specialist predator (zooplankton) and c_2 is the corresponding biomass conversion by fish predator. a, b and d are the half saturation constants. The effect of harmful phytoplankton species on zooplankton is modeled by Holling-II type response function $\frac{p}{(a+p)}$. It is assumed that the fish population is harvested quadratically at constant rate of harvesting E .

Stability properties of equilibria

The given model system (1) has the following equilibrium points in the closed first octant $R^{3+} = (p, z, f) : p \geq 0, z \geq 0, f \geq 0$

- (i) $R_0 = (0, 0, 0)$, a trivial equilibrium always exist and unstable,
- (ii) $R_1 = (\frac{a\delta_2}{c_1\alpha_1 - \theta - \delta_2}, \frac{1}{\alpha_1}(r - \delta_1 p^*)(a + p^*), 0)$, the boundary equilibrium exists when the conditions $(c_1\alpha_1 - \theta) > \delta_2$ and $p^* < \frac{r}{\delta_1}$ are satisfied and
- (iii) The interior equilibrium point $R^* = (p^*, z^*, f^*)$ of (1) exists if and only if conditions $p^* < \frac{r}{\delta_1}$ and $\frac{z^*}{d+z^*} < \frac{\delta_3}{c_2\alpha_2}$ are satisfied and is given by, $(c_1\alpha_1 - \theta) \frac{p^*}{a+p^*} - \delta_2 = \frac{\alpha_2 f^*}{b+z^*}$, $z^* = \frac{(r - \delta_1 p^*)(a + p^*)}{\alpha_1}$ and $f^* = \frac{1}{E}(-\delta_3 + c_2\alpha_2 \frac{z^*}{d+z^*})$

In this paper, our main interest is to investigate the stability of the interior equilibrium R^* for coexistence of the population.

Using the transformation, $p = x_1 + p^*$, $z = x_2 + z^*$ and $f = x_3 + f^*$, the linearization of given system (1) in the neighborhood of R^* gives the following system,

$$\begin{cases} \frac{dx_1}{dt} = a_{100}x_1(t) + a_{010}x_2(t) + a_{200}x_1^2 + a_{110}x_1x_2 + a_{300}x_1^3 + a_{210}x_1^2x_2 + \dots \\ \frac{dx_2}{dt} = b_{100}x_1(t) + b_{010}x_2(t) + b_{001}x_3 + b_{200}x_1^2 + b_{020}x_2^2 + b_{110}x_1x_2 + b_{011}x_2x_3 + b_{300}x_1^3 \\ \quad + b_{030}x_2^3 + b_{210}x_1^2x_2 + b_{021}x_2^2x_3 + \dots \\ \frac{dx_3}{dt} = c_{010}x_2(t) + c_{001}x_3(t) + c_{020}x_2^2 + c_{002}x_3^2 + c_{011}x_2x_3 + c_{030}x_2^3 + c_{021}x_2^2x_3 + \dots \end{cases} \tag{2}$$

where $a_{100} = r - 2\delta_1 p^* - \alpha_1 \frac{a}{(a+p^*)^2} z$, $a_{010} = -\alpha_1 \frac{p^*}{(a+p^*)}$,
 $a_{200} = -2\delta_1 + \frac{2a\alpha_1 z^*}{(a+p^*)^3}$, $a_{110} = -\frac{a\alpha_1}{(a+p^*)^2}$, $a_{300} = -\frac{6a\alpha_1 z^*}{(a+p^*)^4}$,
 $a_{210} = \frac{2a\alpha_1}{(a+p^*)^3}$, $b_{100} = \frac{c_1\alpha_1 a z^*}{(a+p^*)^2} - \frac{\theta a z^*}{(a+p^*)^2}$, $b_{010} = \frac{(c_1\alpha_1 - \theta)p^*}{(a+p^*)} - \delta_2$
 $-\frac{b\alpha_2 f^*}{(b+z^*)^2}$, $b_{001} = -\frac{\alpha_2 z^*}{(b+z^*)}$, $b_{200} = \frac{-2ac_1\alpha_1 z^*}{(a+p^*)^3} + \frac{2a\theta z^*}{(a+p^*)^3}$, $b_{020} =$
 $\frac{2\alpha_2 b f^*}{(b+z^*)^3}$, $b_{110} = \frac{ac_1\alpha_1}{(a+p^*)^2} - \frac{a\theta}{(a+p^*)^2}$, $b_{011} = -\frac{b\alpha_2}{(b+z^*)^2}$, $b_{300} =$
 $\frac{6a(c_1\alpha_1 - \theta)z}{(a+p^*)^4}$, $b_{030} = -\frac{6b\alpha_2 f^*}{(b+z^*)^4}$, $b_{210} = -\frac{2ac_1\alpha_1}{(a+p^*)^3} + \frac{2a\theta}{(a+p^*)^3}$, $b_{021} =$
 $\frac{2b\alpha_2}{(b+z^*)^3}$, and $c_{010} = \frac{c_2\alpha_2 d f^*}{(d+z^*)^2}$, $c_{001} = -\delta_3 + \frac{c_2\alpha_2 z^*}{d+z^*} - 2E f^*$,
 $c_{020} = \frac{-2c_2\alpha_2 d f^*}{(d+z^*)^3}$, $c_{002} = -2E$, $c_{011} = \frac{dc_2\alpha_2}{(d+z^*)^2}$, $c_{030} = \frac{6c_2\alpha_2 d f^*}{(d+z^*)^4}$,
 $c_{021} = \frac{-2dc_2\alpha_2}{(d+z^*)^3}$.

Rewriting the given system in the form, $\dot{X} = MX + R$, we have

$$M = \begin{pmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{010} & c_{001} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and R is the matrix containing the nonlinear terms of (2).

The characteristic equation corresponding to the matrix M is,

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0. \tag{3}$$

Where, $B_1 = -(a_{100} + b_{010} + c_{001})$, $B_2 = a_{100}(b_{010} + c_{001}) + b_{010}c_{001} - a_{010}b_{100} - b_{001}c_{010}$, $B_3 = a_{010}b_{110}c_{001} - a_{100}(b_{010}c_{001} - b_{001}c_{010})$.

If $B_1 > 0$, $B_3 > 0$ and in addition $B_1B_2 - B_3 > 0$, then using Routh-herwitz criterion, system (1) will be locally asymptotically stable around R^* .

Now, let us choose E , the rate of harvesting as a bifurcation parameter.

Theorem 2.1 Let $B_2 > 0$. Then (3) will have a pair of purely imaginary roots provided $\Theta \equiv B_1B_2 - B_3 = 0$ and system (1) will undergo a hopf-bifurcation if $[\frac{d}{dE}(\Theta)]_{E=E_0} / = 0$ where E_0 is the value of E correspond to $\Theta = 0$.

Proof Proof is obvious and is left for the reader. \square

Analysis of bifurcating solutions

In the last section, we have chosen E as the bifurcation parameter and deduced that the system undergoes a Hopf-bifurcation as E passes through some threshold value E_0 . In this section we perform a detailed analysis about the bifurcating solutions (Hassard et al. 1981).

Let $E_1 = E_0 - E$ and denote the matrix M as $M(E) = F_X(X^*, E)$, where the given system is of the form $\dot{X} = F(X, E)$.

Let $q(E)$ and $\hat{q}(E)$ be the eigenvectors of $M(E)$ and $M'(E)$ respectively corresponding to simple eigenvalues λ and $\bar{\lambda}$. So we have $M(E)q = \lambda(E)q$.

For an imaginary value of $\lambda = i\sigma_0$ and $q = (q_1, q_2, q_3)$, we have

$$\begin{aligned} q_1 &= a_{010}b_{001} = f_1 + if_2, \\ q_2 &= -a_{100}b_{001} + i\omega_0b_{001} = g_1 + ig_2, \\ q_3 &= a_{100}b_{001} - a_{010}b_{100} - \omega_0^2 - i\omega_0(a_{100} + b_{001}) \\ &= h_1 + ih_2, \end{aligned}$$

We normalize \hat{q} relative to q , so that

$$\langle \hat{q}, q \rangle = 1. \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denote the Hermitian product, $\langle u, v \rangle = \sum \bar{u}_i v_i$,

Also

$$\langle \hat{q}, \bar{q} \rangle = 0. \tag{5}$$

$$\text{Let } \hat{q} = \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{pmatrix},$$

From (4) and (5), we get

$$\bar{\hat{q}}_1(f_1 + if_2) + \bar{\hat{q}}_2(g_1 + ig_2) + \bar{\hat{q}}_3(h_1 + ih_2) = 1. \tag{6}$$

$$\hat{q}_1(f_1 - if_2) + \hat{q}_2(g_1 - ig_2) + \hat{q}_3(h_1 - ih_2) = 0. \tag{7}$$

Solving (6) and (7), we have

$$\begin{aligned} \hat{q}_1 &= \frac{k(g_1h_2 - h_1g_2) + \frac{g_2}{2} - \frac{g_1}{2i}}{f_1g_2 - f_2g_1}, \\ \hat{q}_2 &= \frac{k(f_2h_1 - f_1h_2) + \frac{f_1}{2i} - \frac{f_2}{2}}{f_1g_2 - f_2g_1}, \\ \hat{q}_3 &= k, \end{aligned} \tag{8}$$

where k is a constant. So, \hat{q}_1 , \hat{q}_2 and \hat{q}_3 can be determined by taking complex conjugates of (8). Using the notation in Hassard et al. (1981), we substitute

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = uq + \bar{u}\bar{q} + W, \quad \text{where } u = \langle \hat{q}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rangle.$$

At bifurcation parameter $E_1 = 0$, we have,

$$\dot{u} = i\omega_0 u(t) + \bar{q}(0).f(2Re(uq) + W(0)) = i\omega_0 + \bar{q}(0).f(u, \bar{u}),$$

where

$$\begin{aligned} f_0 &= \begin{pmatrix} f_0^1 \\ f_0^2 \\ f_0^3 \end{pmatrix} \quad \text{and} \\ f_0^1 &= u^2(a_{200}q_1^2 + a_{110}q_1q_2) + u\bar{u}(2a_{200}q_1\bar{q}_1 \\ &\quad + a_{110}(q_1\bar{q}_2 + q_1\bar{q}_2)) \\ &\quad + \bar{u}^2(a_{200}\bar{q}_1^2 + a_{110}q_1\bar{q}_2) + u^3\left(\frac{a_{300}}{6}q_1^3 + \frac{a_{210}}{6}q_1^2q_2\right) \\ &\quad + u^2\bar{u}\left(\frac{a_{300}}{2}q_1^2\bar{q}_1 + \frac{a_{210}}{6}q_1^2\bar{q}_2\right) \\ &\quad + u\bar{u}^2\left(\frac{a_{300}}{2}q_1\bar{q}_1^2 + \frac{a_{210}}{6}q_1\bar{q}_1^2q_2\right) \\ &\quad + \bar{u}^3\left(\frac{a_{300}}{6}q_1^3 + \frac{a_{210}}{6}q_1^2q_2\right) + \dots, \tag{9} \\ f_0^2 &= u^2(b_{200}q_1^2 + b_{020}q_2^2 + b_{110}q_1q_2 + b_{011}q_2q_3) \\ &\quad + u\bar{u}(2b_{200}q_1\bar{q}_1 + 2b_{020}q_2\bar{q}_2 \\ &\quad + b_{110}(q_1\bar{q}_2 + \bar{q}_1q_2) + b_{011}q_2\bar{q}_3 + b_{011}\bar{q}_2q_3) \\ &\quad + \bar{u}^2(b_{200}\bar{q}_1^2 + b_{020}\bar{q}_2^2 + b_{110}q_1\bar{q}_2) \\ &\quad + u^3\left(\frac{b_{300}}{6}q_1^3 + \frac{b_{030}}{6}q_2^3 + \frac{b_{210}}{6}q_1^2q_2 + \frac{b_{021}}{6}q_2^2q_3\right) \\ &\quad + u^2\bar{u}\left(\frac{b_{300}}{2}q_1^2\bar{q}_1 + \frac{b_{030}}{2}q_2^2\bar{q}_2 + b_{210}\left(\frac{q_1^2\bar{q}_2}{6} + \frac{q_1\bar{q}_1q_2}{3}\right)\right) \\ &\quad + b_{021}\left(\frac{q_2^3\bar{q}_3}{6} + \frac{q_2q_2\bar{q}_3}{3}\right) \\ &\quad + u\bar{u}^2\left(\frac{b_{300}}{6}q_1\bar{q}_1^2 + b_{030}q_2\bar{q}_2^2 + b_{210}(q_1^2\bar{q}_2 + q_1\bar{q}_1\bar{q}_2)\right) \\ &\quad + b_{021}(q_2^2\bar{q}_3 + q_2\bar{q}_2\bar{q}_3) \\ &\quad + \bar{u}^3(b_{300}\bar{q}_1^3 + b_{030}\bar{q}_2^3 + b_{210}q_1^2\bar{q}_2 + b_{021}\bar{q}_2^2\bar{q}_3) + \dots, \tag{10} \end{aligned}$$

$$\begin{aligned} f_0^3 &= u^2(c_{020}q_2^2 + c_{002}q_3^2 + c_{011}q_2q_3) + u\bar{u}(2c_{020}q_2\bar{q}_2 \\ &\quad + 2c_{002}q_3\bar{q}_3 + c_{011}(q_2\bar{q}_3 + \bar{q}_2q_3)) \\ &\quad + \bar{u}^2(c_{020}\bar{q}_2^2 + c_{002}\bar{q}_3^2 + c_{011}q_2\bar{q}_3) \\ &\quad + u^3\left(\frac{c_{030}}{6}q_2^3 + \frac{c_{021}}{6}q_2q_3\right) \\ &\quad + u^2\bar{u}\left(\frac{c_{030}}{2}q_2q_2\bar{q}_2 + c_{021}\left(\frac{q_2\bar{q}_3}{6} + \frac{q_2q_2\bar{q}_3}{3}\right)\right) \\ &\quad + u\bar{u}^2\left(\frac{c_{030}}{6}q_2\bar{q}_2^2\right) + c_{021}\left(\frac{q_2^2\bar{q}_3}{6} + \frac{q_2q_2\bar{q}_3}{3}\right) \\ &\quad + \bar{u}^3\left(\frac{c_{030}}{6}q_2^3 + \frac{c_{021}}{6}q_2^2\bar{q}_3\right) + \dots \tag{11} \end{aligned}$$

Now taking the dot product of f_0 and $\bar{q}(0)$ and expanding we have,

$$\begin{aligned} \dot{u} &= i\omega_0 u + \bar{q}_1^1 f_0^1 + \bar{q}_2^2 f_0^2 + \bar{q}_3^3 f_0^3, \\ &= i\omega_0 u + \frac{1}{2}g_{20}u^2 + \frac{1}{2}g_{02}\bar{u}^2 + g_{11}u\bar{u} + \frac{1}{6}g_{30}u^3 + \frac{1}{6}g_{03}\bar{u}^3 \\ &\quad + \frac{1}{2}g_{21}u^2\bar{u} + \frac{1}{2}g_{12}u\bar{u}^2 + \dots \tag{12} \end{aligned}$$

From (9), (10), (11) and (12), we obtain

$$\begin{aligned} g_{20} &= 2[\bar{q}_1^1(a_{200}q_1^2 + a_{110}q_1q_2) + \bar{q}_2^2(b_{200}q_1^2 + b_{020}q_2^2 \\ &\quad + b_{110}q_1q_2 + b_{011}q_2q_3) \\ &\quad + \bar{q}_3^3(c_{020}q_2^2 + c_{002}q_3^2 + c_{011}q_2q_3)], \\ g_{02} &= 2[\bar{q}_1^1(a_{200}\bar{q}_1^2 + a_{110}q_1\bar{q}_2) + \bar{q}_2^2(b_{200}\bar{q}_1^2 + b_{020}\bar{q}_2^2 \\ &\quad + b_{110}\bar{q}_1\bar{q}_2 + b_{011}\bar{q}_2\bar{q}_3) \\ &\quad + \bar{q}_3^3(c_{020}\bar{q}_2^2 + c_{002}\bar{q}_3^2 + c_{011}q_2\bar{q}_3)], \\ g_{11} &= \bar{q}_1^1(2a_{200}q_1\bar{q}_1 + a_{110}(q_1\bar{q}_2 + q_1\bar{q}_2)) \\ &\quad + \bar{q}_2^2(2b_{200}q_1\bar{q}_1 + 2b_{020}q_2\bar{q}_2 \\ &\quad + b_{110}(q_1\bar{q}_2 + \bar{q}_1q_2) \\ &\quad + b_{011}q_2\bar{q}_3 + b_{011}\bar{q}_2q_3) + \bar{q}_3^3(2c_{020}q_2\bar{q}_2 \\ &\quad + 2c_{002}q_3\bar{q}_3 + c_{011}(q_2\bar{q}_3 + \bar{q}_2q_3)), \\ g_{21} &= 2\left[\bar{q}_1^1\left(\frac{a_{300}}{2}q_1^2\bar{q}_1 + \frac{a_{210}}{6}q_1^2\bar{q}_2 + \frac{a_{210}}{3}q_1\bar{q}_1q_2\right) \right. \\ &\quad + \bar{q}_2^2\left(\frac{b_{300}}{2}q_1^2\bar{q}_1 + \frac{b_{030}}{2}q_2^2\bar{q}_2 \right. \\ &\quad + b_{210}\left(\frac{q_1^2\bar{q}_2}{6} + \frac{q_1\bar{q}_1q_2}{3}\right) + b_{021}\left(\frac{q_2^3\bar{q}_3}{6} + \frac{q_2q_2\bar{q}_3}{3}\right) \\ &\quad \left. \left. + \bar{q}_3^3\left(\frac{c_{030}}{2}q_2q_2\bar{q}_2 + c_{021}\left(\frac{q_2\bar{q}_3}{6} + \frac{q_2q_2\bar{q}_3}{3}\right)\right)\right]\right]. \tag{13} \end{aligned}$$

Using the notation of Hassard et al. (1981) we write

$$\begin{aligned} c_1(0) &= \frac{i\left\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right\}}{2\omega_0} + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{c_1(0)\}}{\alpha'(0)}, \\ \beta_2 &= 2Re\{c_1(0)\}, \\ T_2 &= -\frac{Im\{c_1(0)\} + \mu_2\omega'(0)}{\omega_0}, \end{aligned}$$

where g_{ij} are given by (13).

Theorem 3.1 *From the above obtained quantities, it can be found that bifurcation is supercritical (stable) if $\mu_2 > 0$ and subcritical (unstable) if $\mu_2 < 0$.*

Coexistence in the presence of digestion delay

In most of ecosystems, both natural and manmade, population of one species does not respond instantaneously to the interactions of other species. Sharma et al. (2014) mentioned that animals must take some times to digest their foods before further activities and responses. To incorporate this idea in the modeling approach, we have considered a gestation delay in the zooPlankton–fish interaction. This delay represents the time lag required for gestation of predator which is based on the assumption that the rate of change of predator depends on the number of prey and of predators present at some previous time (Sarwardi et al. 2012). Further time delays in ecological system has destabilizing effects (Sharma et al. 2014; Cushing 1977; Gopalsamy 1992; Kuang 1993; Macdonald 1989; Beretta and Kuang 2002; Das and Ray 2008; Sharma et al. 2015). So, the main aim of this section is to determined whether the dynamic of given system under investigation in stability region for some parameter values changes in the presence of digestion delay or not.

The model is governed by the following system of the nonlinear delay differential equation,

$$\begin{cases} \frac{dp}{dt} = rp - \delta_1 p^2 - \alpha_1 \frac{p}{a+p} z, \\ \frac{dz}{dt} = c_1 \alpha_1 \frac{p}{a+p} z - \delta_2 z - \alpha_2 \frac{z}{b+z} f - \theta \frac{p}{d+p} z, \\ \frac{df}{dt} = -\delta_3 f + c_2 \alpha_2 \frac{z(t-\tau)}{b+z(t-\tau)} f(t-\tau) - E f^2, \end{cases} \quad (14)$$

The initial condition of the system (14) take the form $p(\theta) = \phi_1(\theta)$, $z(\theta) = \phi_2(\theta)$ and $f(\theta) = \phi_3(\theta)$, $\theta \in [-\tau, 0]$, $\phi_i(0) \geq 0$, $\phi_i(\theta) \in C([-\tau, 0], \mathbb{R}_+^3)$.

Theorem 4.1 *The system (14) remained uniformly bounded in Σ , where $\Sigma = \{(p(t), z(t), f(t-\tau)) : 0 \leq p(t) + z(t) + f(t) \leq \frac{1}{\eta} (\frac{r^2}{\delta_1} - \frac{2(1+c_2\alpha_2)}{c_2\alpha_2 E})\}$ and $\eta = \min(1, r, \delta_2 - c_2\alpha_2)$.*

Proof From (14) for all $t \in [0, \infty)$,

$$\frac{dp}{dt} \leq rp(t) \left(1 - \frac{\delta_1}{r} p(t)\right),$$

or it can be obtain that, $\limsup_{t \rightarrow \infty} p(t) \leq \frac{r}{\delta_1}$.

Let us consider a time dependent function,

$$W(t) = p(t) + z(t) + \frac{1}{c_2\alpha_2} f(t-\tau).$$

Using (14) in above expression, we can obtain

$$\begin{aligned} \frac{dW}{dt} &= rp - \delta_1 p^2 - \alpha_1 \frac{p}{(a+p)} z(t) + (c_1 \alpha_1 - \theta) \frac{p}{a+p} z(t) \\ &\quad - \delta_2 z - \alpha_2 \frac{z}{b+z} f - \frac{\delta_3}{c_2\alpha_2} f(t-\tau) + \frac{z(t-\tau)}{d+z(t-\tau)} \\ &\quad f(t-\tau) - \frac{E}{c_2\alpha_2} f^2(t-\tau) \\ &\leq -rp + \frac{r^2}{\delta_1} - (\delta_2 - c_1 \alpha_1) z - \frac{1}{c_2\alpha_2} f(t-\tau) \\ &\quad + \left(1 + \frac{1}{c_2\alpha_2} - \frac{E}{c_2\alpha_2}\right) f(t-\tau) \\ &\leq -rp + \frac{r^2}{\delta_1} - (\delta_2 - c_1 \alpha_1) z - \frac{1}{c_2\alpha_2} f(t-\tau) - 2 \frac{(1+c_2\alpha_2)}{c_2\alpha_2 E} \end{aligned}$$

or

$$\frac{dW}{dt} + \eta W \leq \left(\frac{r^2}{\delta_1} - \frac{2(1+c_2\alpha_2)}{c_2\alpha_2 E}\right), \quad \text{where } \eta = \min(1, r, \delta_2 - c_2\alpha_2).$$

Now, applying the theorem of differential inequalities (Birkhoff and Rota 1882), we obtain

$$0 < W(t) \leq W(t) e^{-\eta t} + \frac{1}{\eta} \left(\frac{r^2}{\delta_1} - \frac{2(1+c_2\alpha_2)}{c_2\alpha_2 E}\right).$$

As $t \rightarrow \infty$, we have $0 \leq W(t) \leq \frac{1}{\eta} \left(\frac{r^2}{\delta_1} - \frac{2(1+c_2\alpha_2)}{c_2\alpha_2 E}\right)$.

Hence all the solutions of the system (14) are bounded in Σ .

Now, using Taylor’s expansion, (14) can be reduced to the following system of differential equations,

$$\begin{cases} \frac{dx_1}{dt} = a_{100}x_1(t) + a_{010}x_2(t) + a_{200}x_1^2 + a_{110}x_1x_2 + \dots, \\ \frac{dx_2}{dt} = b_{100}x_1(t) + b_{010}x_2(t) + b_{001}x_3 + b_{200}x_1^2 + b_{020}x_2^2 \\ \quad + b_{110}x_1x_2 + \dots, \\ \frac{dx_3}{dt} = c_{001}x_3(t) + c'_{010}x_2(t-\tau) + c'_{001}x_3(t-\tau) + c'_{020}x_2^2(t-\tau) \\ \quad + c_{002}x_3^2 + c'_{011}x_2(t-\tau)x_3(t-\tau) + \dots, \end{cases} \quad (15)$$

where

$$\begin{aligned} c'_{010} &= \frac{c_2\alpha_2 b f^*}{(b+z^*)^2}, & c'_{020} &= \frac{-2c_2\alpha_2 b f^*}{(b+z^*)^3}, \\ c'_{001} &= \frac{c_2\alpha_2 d z^*}{d+z^*}, & c'_{011} &= \frac{c_2\alpha_2 b}{(b+z^*)^2}, & c_{002} &= -2E. \end{aligned}$$

The corresponding characteristic equation is,

$$\Delta(R^*, \tau) = \begin{vmatrix} a_{100} - \lambda & a_{010} & 0 \\ b_{100} & b_{010} - \lambda & b_{001} \\ 0 & c'_{010}e^{-\lambda\tau} & c_{001} + c'_{001}e^{-\lambda\tau} - \lambda \end{vmatrix} = 0.$$

Remark The local asymptotical stability of the non-delayed system around R^* using Routh–hurwitz criterion is already discussed in above section and it follows that the necessary and sufficient conditions for all roots of (3) having negative real parts ($\tau = 0$) are given by $B_2 > 0$ and $B_1B_2 - B_3 > 0$, which means that (14) is remained asymptotically stable around R^* in the absence of gestation delay.

To obtain the effect of digestion delay on the stability of equilibrium point R^* , let us consider the case when $\tau \neq 0$ and the characteristic equation can be written as

$$\Delta(R^*, \tau) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 + e^{-\lambda\tau}(B_4\lambda^2 + B_5\lambda + B_6). \tag{16}$$

Now for $\lambda = 0, \Delta(R^*, \tau) = B_3 - B_6 \neq 0$, it means that $\lambda = 0$ is not a root of the characteristic equation (16) if the interior equilibrium point R^* exists. And the conditions for stability of R^* when τ varies are given by the following theorem.

Theorem 4.2 *If all roots of the Eq. (16) has negative real parts and,*

- (i) *When $H(\sigma, \tau) = 0$ has no positive root, the interior equilibrium point R^* is asymptotically stable for an arbitrary delay τ .*
- (ii) *When $H(\sigma, \tau) = 0$ has at least one positive root say σ_0 , there exists a critical delay $\tau_0 > 0$ such that the interior equilibrium point R^* is asymptotically stable for $\tau \in (0, \tau_0)$.*
- (iii) *When $H(\sigma, \tau) = 0$ has at least two positive roots say σ_{\pm} then there exist positive integer n , such that the equilibrium R^* switches n times from stability to instability to stability and so on such that R^* is locally asymptotically stable whenever $\tau \in [0, \tau_0^+] \cup (\tau_0^-, \tau_1^+) \cup \dots \cup (\tau_{n-1}^-, \tau_n^+)$ and is unstable whenever $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \dots \cup (\tau_{n-1}^+, \tau_{n-1}^-)$.*

The model system (14) undergoes a Hopf-bifurcation around R^* for every $\tau = \tau_k^{\pm}$.

Where

$$\begin{aligned} H(\sigma, \tau) &= f_1^2(\sigma, \tau) + f_2^2(\sigma, \tau) - f_3^2(\sigma, \tau) = 0, \\ f_1(\sigma, \tau) &= (B_3 - B_1\sigma^2)(B_4 - B_6\sigma^2) + \sigma B_5(\sigma^3 - B_2\sigma), \\ f_2(\sigma, \tau) &= (B_4 - B_6\sigma^2)(\sigma^3 - B_2\sigma) - \sigma B_5(B_3 - B_1\sigma^2), \\ f_3(\sigma, \tau) &= -(B_4 - B_6\sigma^2)^2 - B_5^2\sigma^2. \end{aligned} \tag{17}$$

Proof Let $i\sigma_0$ ($\sigma_0 > 0$) is a pair of purely imaginary roots of (16), then we have

$$\begin{aligned} \Delta(R^*, i\sigma) &= -i\sigma^3 - B_1i\sigma^2 + B_2i\sigma + B_3 \\ &\quad - e^{-i\sigma\tau}(-B_4i\sigma^2 + B_5i\sigma + B_6), \end{aligned} \tag{18}$$

Separating the real and imaginary parts, we obtain

$$-B_1\sigma^2 + B_3 = (B_6 - B_4\sigma^2)\cos(\sigma\tau) + B_5\sigma\sin(\sigma\tau), \tag{19}$$

$$B_2\sigma - \sigma^3 = B_5\sigma\cos(\sigma\tau) + (B_6\sigma - B_4)\sin(\sigma\tau), \tag{20}$$

Eliminating τ from above equations, it can be obtained that

$$\begin{aligned} H(\sigma, \tau) &= f_1^2(\sigma, \tau) + f_2^2(\sigma, \tau) - f_3^2(\sigma, \tau) \\ &= \sigma^6 + (B_1^2 - 2B_2 - B_4^2)\sigma^4 \\ &\quad + (B_2^2 - 2B_1B_3 + 2B_4B_6 + B_5^2)\sigma^2 \\ &\quad + (B_3^3 - B_4^2) = 0 \end{aligned} \tag{21}$$

Setting $z = \sigma^2$, (21) can be written as,

$$H(z, \tau) = z^3 + pz^2 + qz + r = 0. \tag{22}$$

where

$$\begin{aligned} p &= B_1^2 - 2B_2 - B_4^2, \quad q = B_2^2 - 2B_1B_3 + 2B_4B_6 + B_5^2, \\ r &= B_3^3 - B_4^2. \end{aligned}$$

Now from the sign of B_1, B_2 and B_3 , it can be obtained that Eq. (22) possesses two positive roots say, $z_{0\pm}$ which implies (21) has two roots $\sigma_{\pm} = \pm\sqrt{z_0}$. Substituting this value in (19) and (20), the corresponding critical value of delay $\tau = \tau_k^{\pm}$ at which R^* stability switches occurs are given by,

$$\begin{aligned} \tau_k^{\pm} &= \frac{1}{\sigma_0^{\pm}} \arctan \frac{(B_4 - B_6\sigma^2)(B_2\sigma^{\pm} - \sigma^3) + B_5\sigma^{\pm}(B_1\sigma^2 - B_3)}{(B_3 - B_1\sigma^2)(B_6\sigma^2 - B_4) + B_5\sigma^{\pm}(B_2\sigma^{\pm} - \sigma^3)} + \frac{2k\pi}{\sigma_0^{\pm}}, \\ &\quad (k = 0, 1, 2, \dots) \end{aligned} \tag{23}$$

Let us define $\theta_{\pm} = \frac{1}{\sigma_0^{\pm}} \frac{(B_4 - B_6\sigma^2)(B_2\sigma^{\pm} - \sigma^3) + B_5\sigma^{\pm}(B_1\sigma^2 - B_3)}{(B_3 - B_1\sigma^2)(B_6\sigma^2 - B_4) + B_5\sigma^{\pm}(B_2\sigma^{\pm} - \sigma^3)}$, we have two sequence of delays $\tau_k^+ = \frac{\theta_+ + 2k\pi}{\sigma_+}$ and $\tau_k^- = \frac{\theta_- + 2k\pi}{\sigma_-}$ for which there are two purely imaginary roots of the (16). We will now study how the real parts of the roots of (16) vary as τ varies in a small neighborhood of τ^+ and τ^- .

Let $\lambda = \xi + i\sigma$ be a root of (16), then substituting $\lambda = \xi + i\sigma$ in (18) and separating real and imaginary parts we have,

$$\begin{aligned} G_1(\xi, \sigma, \tau) &= \xi^3 - 3\xi\sigma^2 + B_1(\xi^2 - \sigma^2) + B_2\xi + B_3 \\ &\quad - e^{-\xi\tau}(B_4 + B_5\xi + B_6(\xi^2 - \sigma^2))\cos\sigma\tau \\ &\quad - e^{-\xi\tau}(B_5\sigma + 2B_6\xi\sigma)\sin\sigma\tau, \\ G_2(\xi, \sigma, \tau) &= -\sigma^3 - 3\xi^2\sigma + 2B_1\xi\sigma + B_2\sigma \\ &\quad + e^{-\xi\tau}(B_4 + B_5\xi + B_6(\xi^2 - \sigma^2))\sin\sigma\tau \\ &\quad - e^{-\xi\tau}(B_5\sigma + 2B_6\xi\sigma)\cos\sigma\tau, \end{aligned}$$

$$\text{and } J = \begin{pmatrix} \frac{\partial G_1}{\partial \xi} & \frac{\partial G_1}{\partial \sigma} \\ \frac{\partial G_2}{\partial \xi} & \frac{\partial G_2}{\partial \sigma} \end{pmatrix}.$$

Thus, we have, $G_1(0, \sigma_{\pm}, \tau^{\pm}) = G_2(0, \sigma_{\pm}, \tau^{\pm}) = 0$ and $|J|_{(0, \sigma_{\pm}, \tau^{\pm})} > 0$. Hence by the implicit function theorem $G_1(\xi, \sigma, \tau) = G_2(\xi, \sigma, \tau) = 0$.

Define ξ, σ as a function of τ in a neighborhood of $(0, \sigma_{\pm}, \tau^{\pm})$ such that $\xi_+(\tau_k^+) = 0, \sigma_+(\tau_k^+) = \sigma_+, \left. \frac{d\xi_{\pm}}{d\tau} \right|_{\tau=\tau_k^+} > 0$ and $\xi_-(\tau_k^-) = 0, \sigma_-(\tau_k^-) = \sigma_-, \left. \frac{d\xi_{\pm}}{d\tau} \right|_{\tau=\tau_k^-} < 0$.

This completes the proof of theorem. □

Stability of bifurcating periodic solution

In this section we are employing the explicit formulae suggested by Hassard et al. Hassard et al. (1981) to determine the direction, stability and the period of the periodic solutions bifurcating from interior equilibrium R_* . Following the procedure of the computation in Hassard et al. (1981), we compute (see Appendix A for details of the computation)

$$g(z, \bar{z}) = g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z\bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{21}(\theta) \frac{z^2\bar{z}}{2} + \dots,$$

where the first four coefficients that we need for determining properties of the Hopf bifurcation are of the form

$$\begin{cases} g_{20} = \frac{2}{d_1} \tau_0 \{ \Delta_{11} + \bar{q}_1^* \Delta_{21} + \bar{q}_2^* \Delta_{31} \}, \\ g_{11} = \frac{1}{d_1} \tau_0 \{ \Delta_{12} + \bar{q}_1^* \Delta_{22} + \bar{q}_2^* \Delta_{32} \}, \\ g_{02} = \frac{2}{d_1} \tau_0 \{ \Delta_{13} + \bar{q}_1^* \Delta_{23} + \bar{q}_2^* \Delta_{33} \}, \\ g_{21} = \frac{2}{d_1} \tau_0 \{ \Delta_{14} + \bar{q}_1^* \Delta_{24} + \bar{q}_2^* \Delta_{34} \}, \end{cases} \tag{24}$$

Where the terms $\bar{d}_1, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{21}, \delta_{22}, \delta_{23}, \delta_{24}, \delta_{31}, \delta_{32}, \delta_{33}$ and δ_{34} are calculated in Appendix A.

Thus, we can compute the following values:

$$c_1(0) = \frac{l}{2\omega_*\tau_k} \left\{ g_{20}g_{11} - 2|g_{11}|^2 - \frac{(|g_{02}|)^2}{3} \right\} + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{Re\{c_1(0)\}}{Re\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}},$$

$$\beta_2 = 2Re\{c_1(0)\},$$

$$T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}}{\omega_*\tau_k}, \quad k = 0, 1, 2, \dots$$

Theorem 5.1 μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau_2 > \tau_{20}$ ($\tau_2 < \tau_{20}$); β_2 determines the

stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

Numerical simulation

In this section, we give the numerical validation of some of our results obtained analytically in above sections. For this consider a numerical example to the model system (1),

$$\begin{cases} \frac{dp}{dt} = 1.75p - 0.05p^2 - 2\frac{p}{10+p}z, \\ \frac{dz}{dt} = -z + 2\frac{p}{10+p}z - 1.45\frac{z}{20+z}f - 0.0126\frac{p}{10+p}z, \\ \frac{df}{dt} = -0.1f + 1.0\frac{z(t-\tau)}{20+z(t-\tau)}f(t-\tau) - Ef^2, \end{cases} \tag{25}$$

Impact of nonlinear harvesting

The primary interest here is to find the impact of quadratic harvesting of the fish population on the stability of the given Plankton–fish system. We obtain numerically that when fish population is harvested at low rate ($E = 0.0001$), given system (1) exhibits chaotic dynamic with initial data $p(0) = 10, z(0) = 5$ and $f(0) = 3$ as shown in Fig. 1. When, we further increase the value of harvesting parameter E from 0.0001 to 0.09, Figs. 2 and 3 and 4 depicts that stability exchange takes place by changing the state of the system from chaos to limit cycle and finally to stable focus. In ecological sense, it means that when the fish population is maintained between $E \in (0.0001, 0.1)$, it has stabilizing effect on the phytoplankton (TPP)-zooPlankton–fish dynamic which is well agreed with the experiment findings of Carpenter et al. (1987), Leavitt and Findlay (1994), Pace et al. (1999), Strock et al. (2013), Attayde et al. (2010). Figure 5 depicts the existence of limit cycles around R^* at $E = 0.15$ and $E = 1.0$ and ecologically it implies the prevalence of bloom like situations in plankton ecosystem. These results indicate that the enhancement in harvesting of fish population induces instability in the plankton system. Moreover, in case of excessive harvesting, the entire fish population will be elimination and phytoplankton-zooplankton population coexist with the existence of limit cycles as depicted in Fig. 6. The numerical results predicts some estimation of the harvesting parameter that will be helpful for regulating agencies to formulate some harvesting policies of fisheries for coexistence of population in ecology. The equilibrium level of all population at different harvesting rates along with nature of stability around R^* is given in Table 1.

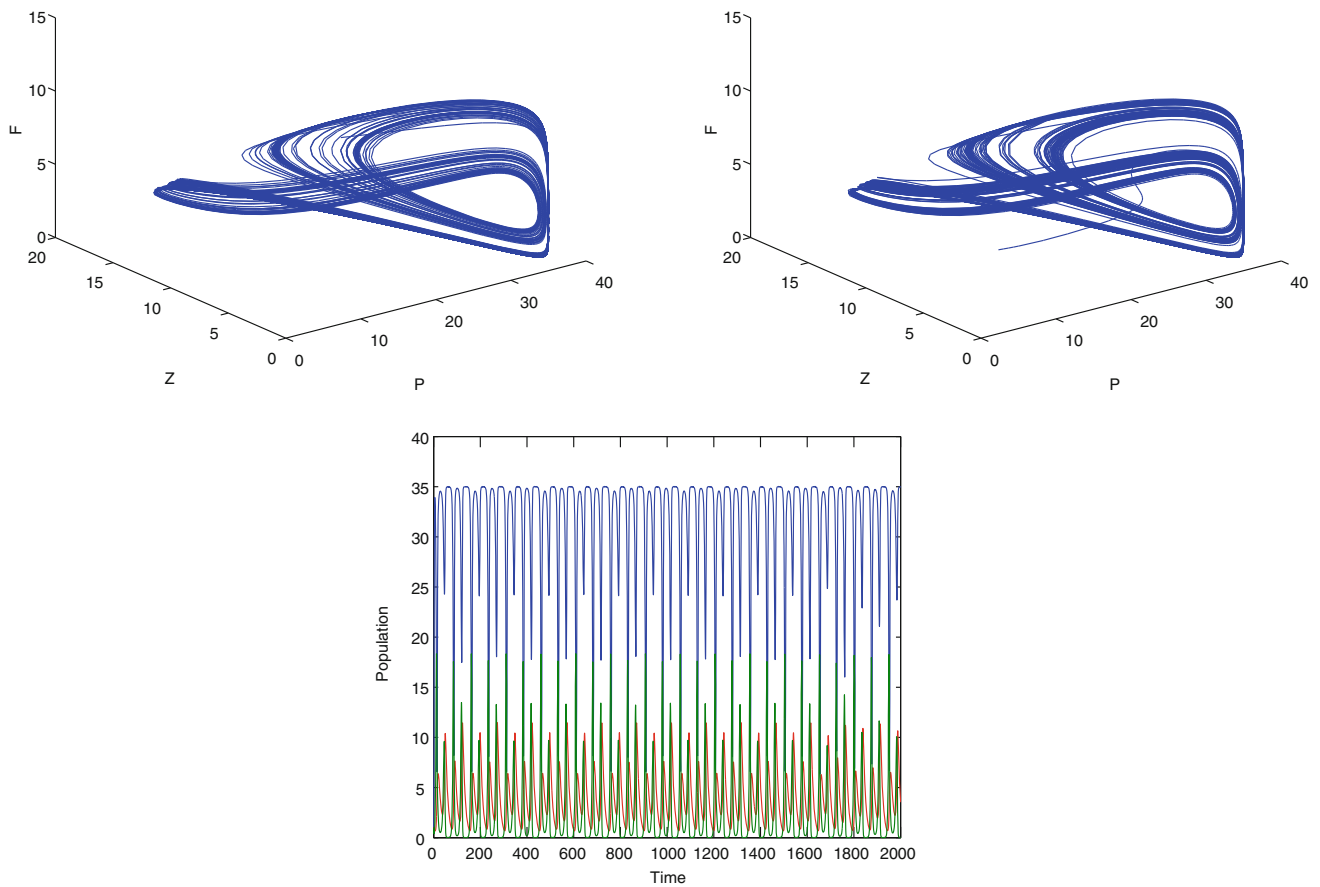


Fig. 1 System exhibits chaotic oscillation around R^* at $E = 0.0001$

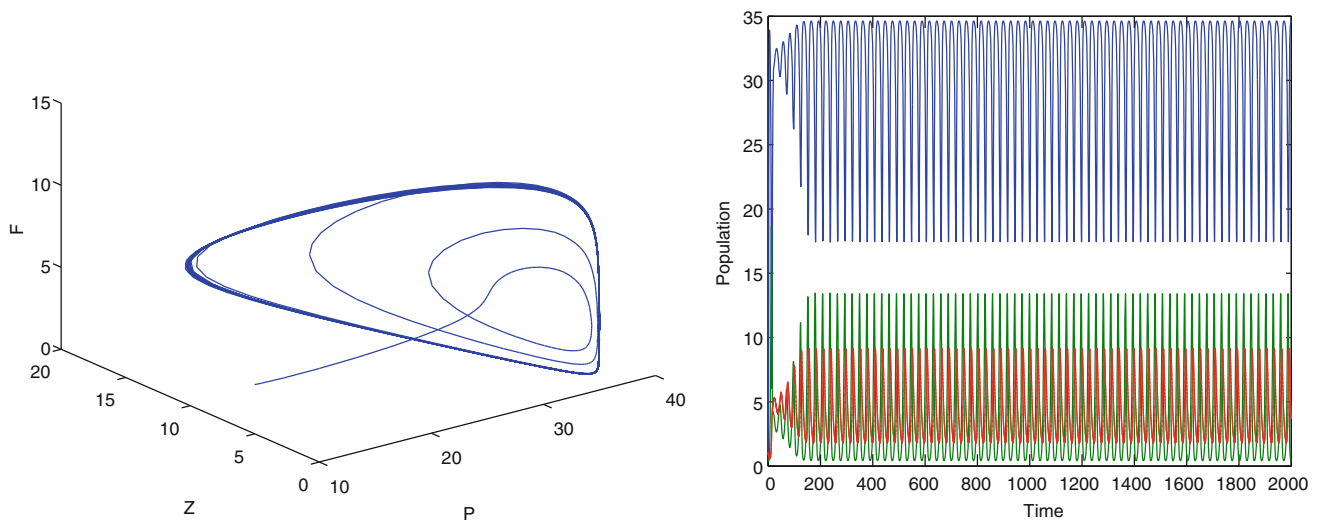


Fig. 2 Change in stability from chaos to limit cycle at $E = 0.01$

It is already well established that toxin has stabilizing effect on the tri-trophic interaction system (Upadhyay and Chattopadhyay 2005; Upadhyay et al. 2007; Upadhyay and

Rao 2009). But, in this research, we have shown that toxin is not the only mechanism of inducing stability in the Plankton–fish system rather harvesting the generalist

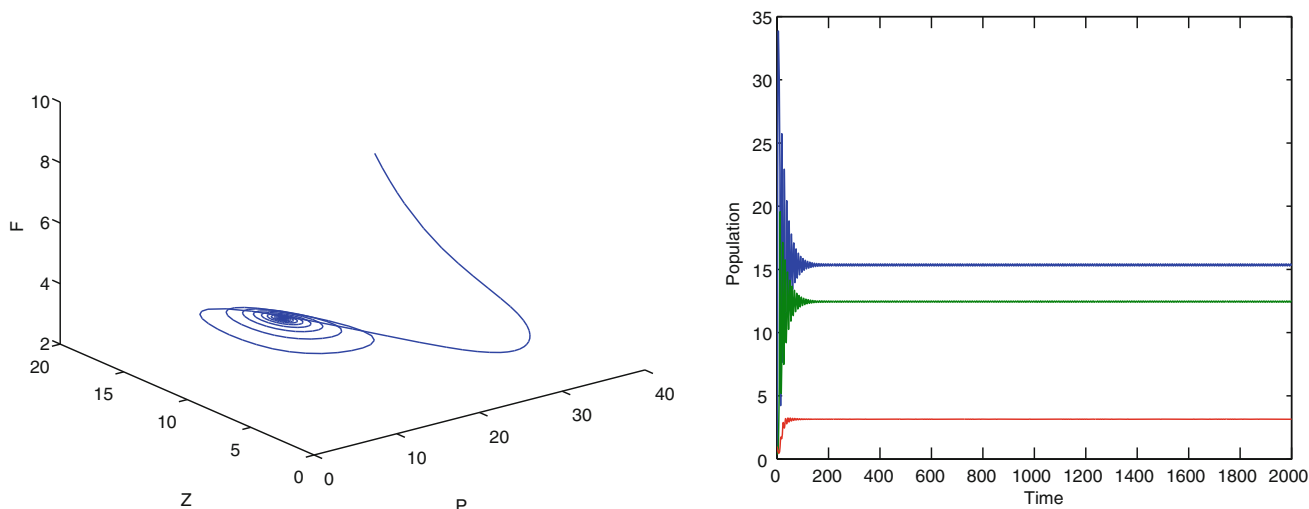


Fig. 3 Change in stability from limit cycle to a stable focus at $E = 0.09$

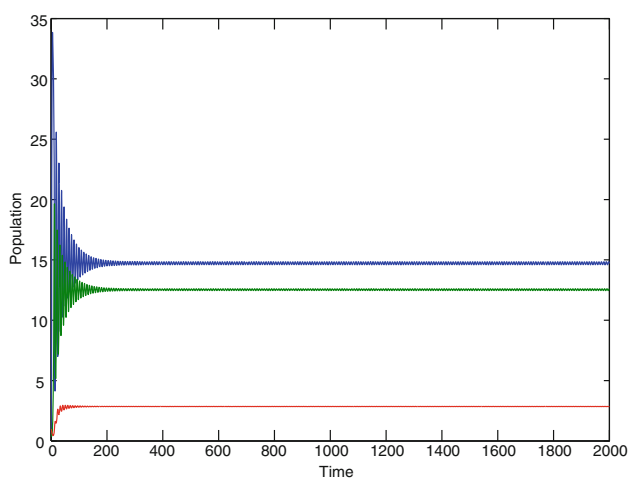


Fig. 4 Time series graph shows the converges of solution trajectories to R^* at $E = 0.1$

predator (fish) at appropriate level of harvesting can induce stability and trigger new mechanism of controlling complexity such as chaos in plankton system. Thus, our findings show that fish may have stabilizing effects if it can be harvested in the proposed harvesting range.

Delay induced instability

In the second part of this paper, we have considered the digestion delay in zooPlankton–fish interaction. We consider the given system in stable state by taking $E = 0.09$ and keeping other parameters unchanged. From the sign of $p = 3.4729$, $q = -3.2571$ and $r = 0.0808$, Eq. (22) has two purely imaginary roots $i\sigma_{0\pm}$ with $\sigma_+ = 0.1612$ and $\sigma_- = 0.0356$. Now from (23) the critical values of parameter delay at which stability exchange take place are given by $\tau_0^+ = 1.013$ and $\tau_0^- = 49.981$ such that R^* remain stable for $\tau \in [0, 1.0)$ (see Fig. 7)

and is unstable when $\tau \in (1, 49.981)$. Thus numerical simulation shows that the interior equilibrium converges in the range $0 < \tau < \tau_0^+$ and it loses stability when τ passes through its critical value τ_0 and a Hopf bifurcation occur which is depicted in Fig. 8. When the parameter τ varies further from critical value τ_0 , it is observed that periodic oscillation appeared around E^* till τ lies in the range $\tau_0^+ < \tau < \tau_0^-$ (see Fig. 9) and regains its stability for $\tau > \tau_0^-$ (see Figs. 10, 11). The equilibrium point R^* remains locally asymptotically stable whenever the delay parameter lies in the range $\tau_0^- < \tau < \tau_1^+$ implies R^* again switches from stability to instability as τ passes through $\tau = \tau_1^+$. Thus it is seen that a finite number of stability switches for the given system occurs when delay varies continuously. Thus it can be seen that there exist intervals of stability and instability as delay parameter varies through its critical value. Ecologically, it signifies the appearance and disappearance of blooms in plankton ecosystem.

Finally the stability determining quantities for Hopf-bifurcating periodic solutions at $\tau = \tau_0$ are given by $c_1(0) = 1.6307e + 009 - 6.7954e + 009i$, $\mu_2 = -1.5409e + 011$, $\beta_2 = 3.2614e + 009$, $T_2 = -3.7051e + 009$.

Hence, we conclude that the Hopf-bifurcation is sub-critical in nature as well as the bifurcating periodic orbits are unstable and decreases as delay increases further through its critical value ' $\tau = \tau_0$ '.

Discussion and conclusion

The consequences of toxin liberation by phytoplankton are of great interest due to its possible detrimental effect on fisheries. The role of top predator on the stability of tri-trophic food chain has been remained an interesting area of research. In this paper the Plankton–fish interaction were

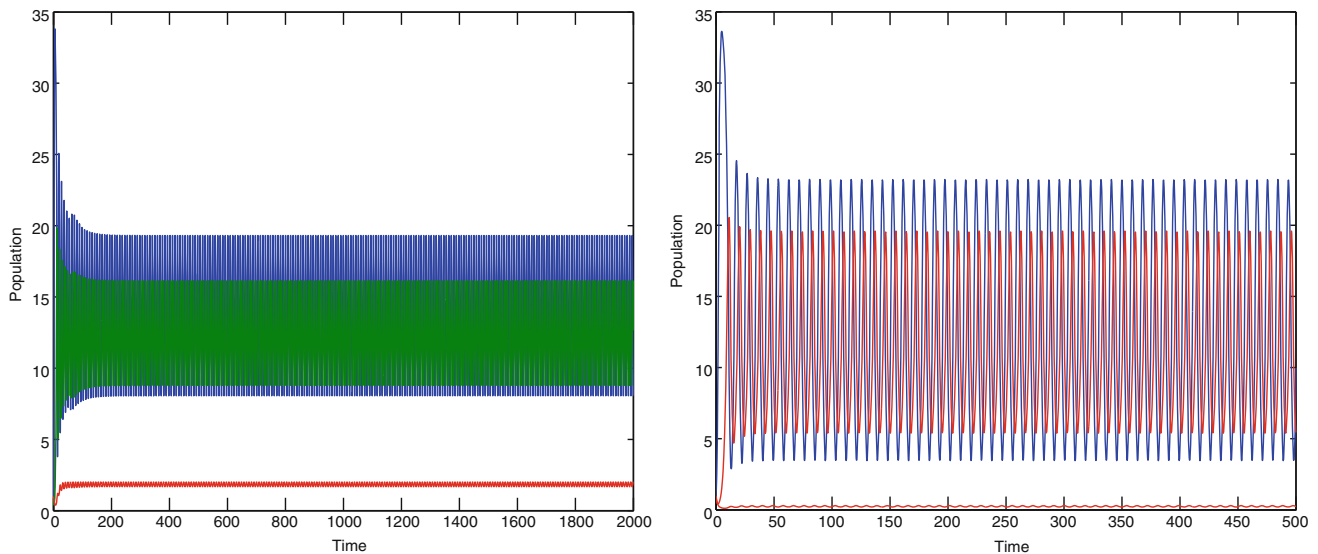


Fig. 5 Limit cycles oscillations of solution curves around R^* at $E = 0.15$ and $E = 1.0$

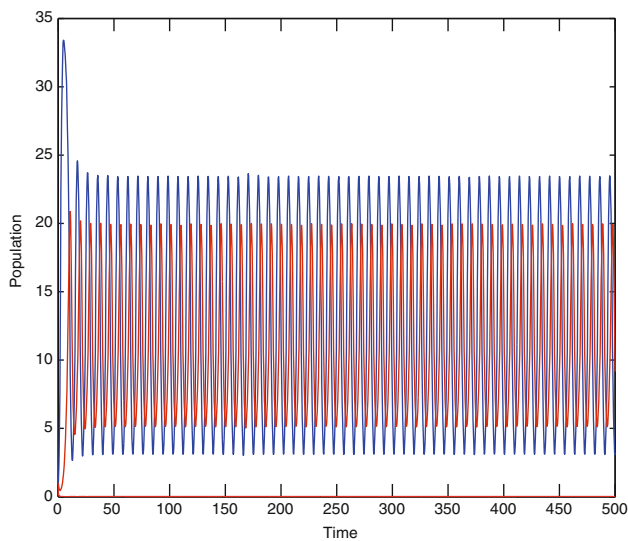


Fig. 6 Time series graph shows that phytoplankton-zooplankton coexist but fish population almost dies out at $E = 30$

studied in the context of obtaining the impact of quadratic harvesting and time delay. We have assumed that the top predator in the tri-trophic food chain is harvested through

non linear (quadratic) harvesting. We have analyzed the given system with various ranges of harvesting parameter keeping other parametric values fixed. It is found numerically that the chaotic oscillation has occurred in the range $0.0001 \leq E < 0.009$ and when values of E is taken in $0.009 \leq E \leq 0.01$, these oscillations disappeared with the existence of limit cycle. Again, for $0.09 \leq E \leq 0.1$, the system converges to a stable focus from limit cycle which shows that the harvesting of fish population induces the stability in the plankton food chain and hence stimulate a possible mechanism for controlling blooms in plankton ecosystem. Our findings has shown that introduction of nonlinear harvesting of fish population in Plankton–fish interaction can be utilized as a controlling agent.

In the second part of the model system we have assumed that the interaction of species in real life are not instantaneous rather it is delayed by time known as time delay. In this paper we have assumed that the zooPlankton–fish interaction are delayed by some time lag known as digestion delay. The main interest of this part is to observe the effect of this delay on the stability of the given system brought by stocking of the fish population. It is found that the given system remained asymptotically stable for delay

Table 1 Impact of Non-linear Harvesting on stability of given system around interior equilibrium R^*

| Rate of harvesting, E | Nature of system | Stability |
|-------------------------|--|----------------|
| 0.0001 | System oscillate chaotically around R^* | Chaotic |
| 0.01 | System oscillate around $R^* = (29, 3, 4)$ | Limit cycle |
| 0.09 | System converge to $R^* = (15.2, 12.4, 3.1)$ | Locally stable |
| 0.1 | System converge to $R^* = (14.7, 2.6, 2.8)$ | Locally stable |
| 0.15 | system oscillate around $R^* = (14.7, 12.6, 2.8)$ | Limit cycle |
| 1.0 | system oscillate around $R^* = (14, 16, 1.6)$ | Limit cycle |
| 10.0 | system oscillate around $R^* = (4, 7, 0.00)$ in pz-plane | Unstable |

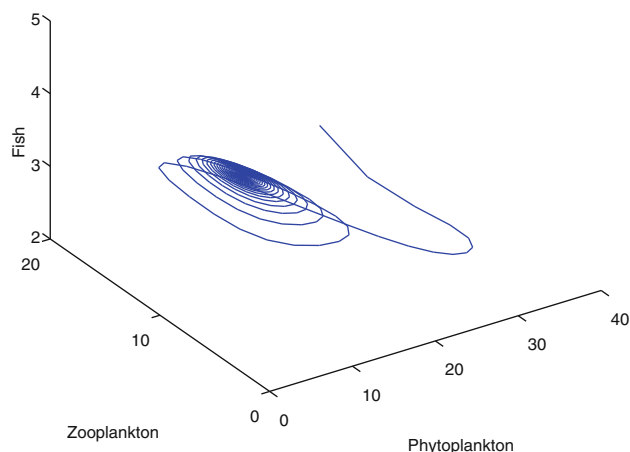
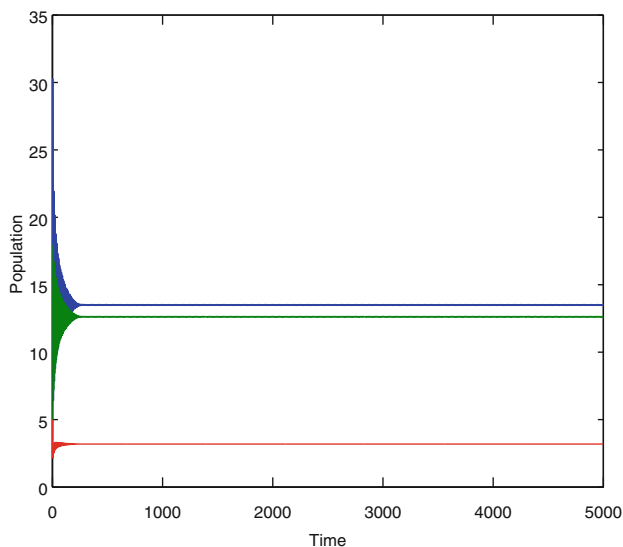


Fig. 7 Stability of interior equilibrium R^* at $\tau = 0.1$

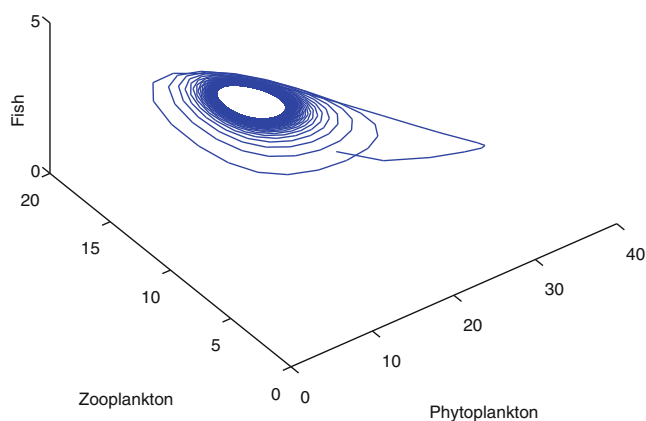
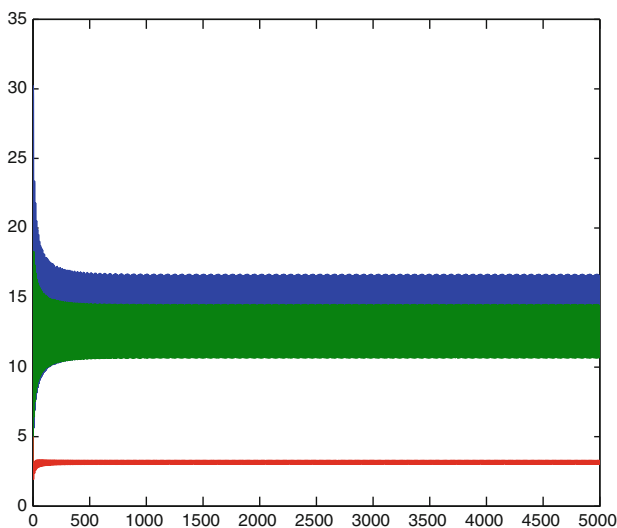


Fig. 8 Hopf bifurcation in the form of limit cycle oscillation at $\tau = 1.0$

in the range $0 \leq \tau \leq \tau_o$ (see Fig. 3). When delay ‘ τ ’ passes through its critical value ‘ τ_0 ’ it is obtained that a phenomenon of switching of stability arises i.e. there exist a sequence of two time delays τ_0^+ ($k = 0, 1, 2, \dots$) and τ_0^- ($k = 0, 1, 2, \dots$) under certain parametric restrictions for which the interior equilibrium point E^* is stable whenever $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \dots \cup (\tau_{m-1}^-, \tau_m^+)$, and unstable when $\tau \in [\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \dots \cup (\tau_{m-1}^+, \tau_{m-1}^-) \cup (\tau_m^+, \infty)$. The above switching behavior of the system is shown in numerical simulation performed in the above section and it seems to be new findings in delayed Plankton–fish interactions. Thus it can be concluded that digestion delay has destabilizing effect on the TPP–zooplankton–fish interaction. Finally it can be predicted

that induction of stability by harvesting of top predator in plankton food chain can be destabilized by digestion delay.

Appendix A

Let $\tau = \tau_k + \mu$, $\bar{u}_i(t) = u_i(\tau t)$ and $u_i(\theta) = u(t + \theta)$ for $\theta \in [-1, 0]$ and dropping the bars for simplification of notations, system (15) becomes a functional differential equation in $C = C([-1, 0], \mathfrak{R}^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{26}$$

where $u(t) = (u_1(t), u_2(t))^T \in \mathfrak{R}^2$ and $L_\mu : C \rightarrow \mathfrak{R}^3$, $f : \mathfrak{R} \times C \rightarrow \mathfrak{R}^2$ are given, respectively, by

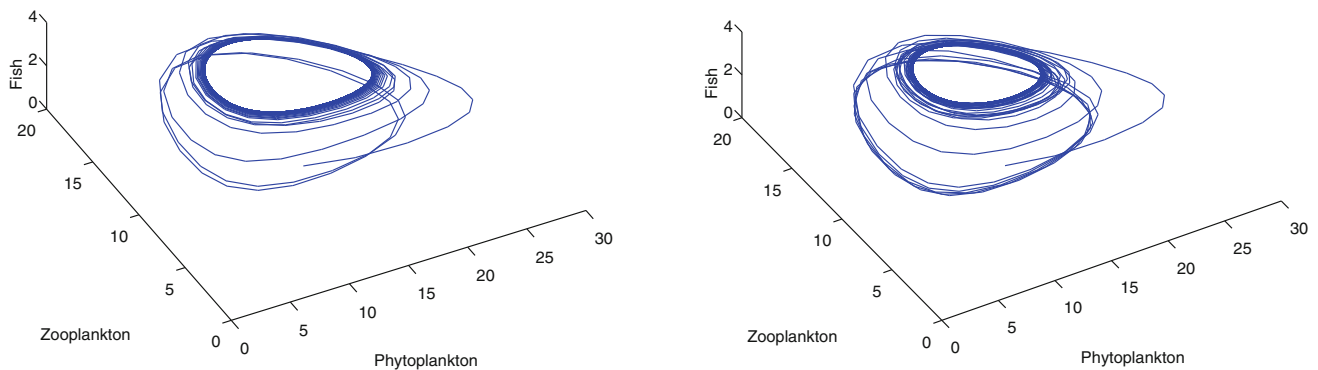


Fig. 9 Limit cycle at $\tau = 40$.

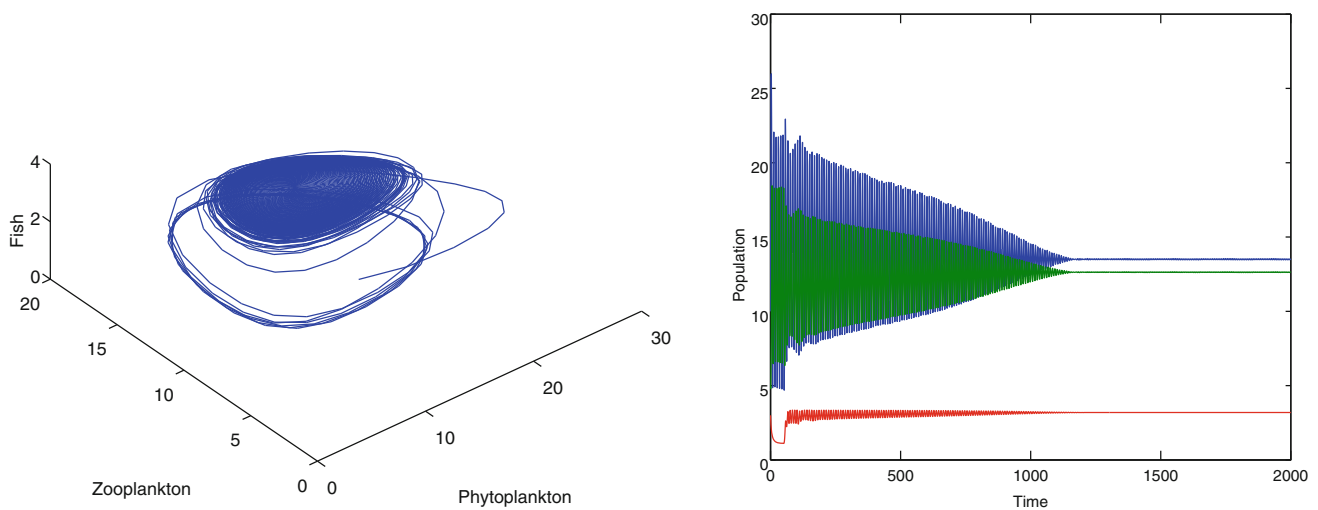


Fig. 10 Switch of stability from limit cycle to stable focus around $\tau = 50$

$$L_\mu(\phi) = (\tau_k + \mu)[A_{11}\phi(0) + A_{22}\phi(-1)],$$

$$A_{11} = \begin{bmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{010} & c_{001} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c'_{010} & c'_{001} \end{bmatrix}, \tag{27}$$

and

$$f(\mu, \phi) = (\tau_k + \mu) \begin{bmatrix} a_{200}\phi_1^2(0) + a_{110}\phi_1(0)\phi_2(0) \\ b_{200}\phi_1^2(0) + b_{020}\phi_1^2(0) + b_{110}\phi_1(0)\phi_2(0) \\ c'_{020}\phi_1^2(-1) + c'_{020}\phi_3^2(0) + c'_{011}\phi_1^2(-1)\phi_3(-1) \end{bmatrix} \tag{28}$$

By Riesz representation theorem, there exists a function $\phi(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C \tag{29}$$

In fact, we can choose

$$\phi(\theta, \mu) = (\tau_k + \mu)[A_{11}\delta(\theta) - A_{22}\delta(\theta + 1)], \tag{30}$$

where δ denote the Dirac delta function.

For $\phi \in C([-1, 0], \mathfrak{R}^3)$, define

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta} & \theta \in [-1, 0), \\ \int_{-1}^0 d\zeta(s, \mu)\phi(s) & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0 & \theta \in [-1, 0), \\ f(\mu, \phi) & \theta = 0 \end{cases}$$

Then system (26) is equivalent to

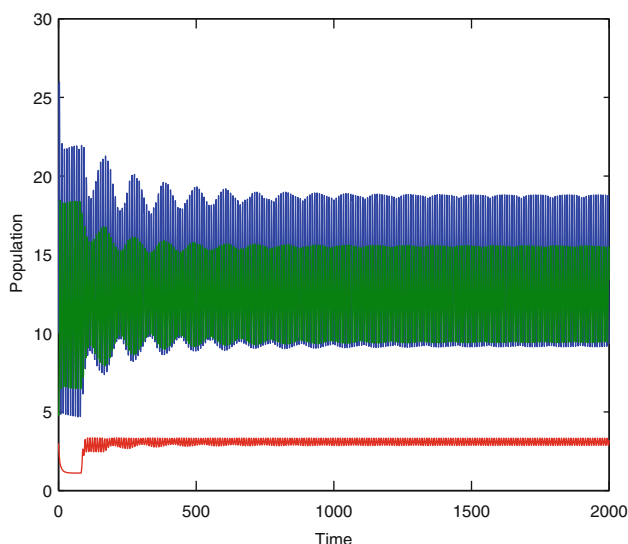


Fig. 11 Another switch of stability from stable focus to limit cycle in the interval $\tau = 60$ to $\tau = 80$

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{31}$$

For $\psi \in C^1([0, 1], (\mathfrak{R})^*)$, define

$$A^*\psi(s) = \begin{cases} \frac{d\psi(s)}{ds} & s \in (0, 1], \\ \int_{-1}^0 d\zeta(t, 0)\psi(-t) & s = 0 \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi. \tag{32}$$

where $\zeta(\theta) = \zeta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. From the results of last section, we know that $\pm i\sigma_0\tau_k$ are the eigenvalues of $A(0)$ and $\mp i\sigma_0\tau_k$ are the eigenvalues of A^* .

Let $q(\theta) = (1, \alpha, \beta)^T e^{i\sigma_0\tau_k\theta}$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_k$ and $q^*(s) = D(1, \alpha', \beta') e^{i\sigma_0\tau_k s}$ be the eigenvector of A^* corresponding to the eigenvalue $-i\sigma_0\tau_k$, then

$$A(0)q(\theta) = i\omega_0\tau_k q(\theta),$$

It follow from the definition of $A(0)$, $L_\mu(\phi)$ and $\eta(\theta, \mu)$ that

$$[(A_{11} + A_{22}e^{-i\sigma_0\tau_0}) - i\sigma_0 I]q(0) = 0,$$

where I is identity matrix of order 2, that is,

$$\begin{pmatrix} a_{100} - i\sigma_0 & a_{010} & 0 \\ b_{100} & b_{010} - i\sigma_0 & b_{001} \\ 0 & c_{010} + c'_{100}e^{-i\sigma_0\tau_k} & c_{001} + c'_{001}e^{-i\sigma_0\tau_k} - i\sigma_0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Solving the above equations, we get

$$q(0) = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\sigma_0 - a_{100}}{a_{010}} \\ \frac{-b_{100} + (b_{010} - i\sigma_0)(a_{100} - i\sigma_0)}{b_{001}a_{010}} \end{pmatrix}.$$

From the definition of A^* , we obtain

$$A^*(\mu)\psi(s) = \int_{-1}^0 d\eta(t, 0)\psi(-t) = A_{11}^T\psi(0) + A_{22}^T\psi(-1).$$

Since $q(\theta)$ is the eigenvector of A^* corresponding to $-i\sigma_0\tau_k$, then we have

$$A^*q^*(0) = -i\sigma_0\tau_k q^*(\theta),$$

or

$$[(A_{11}^T + A_{22}^T e^{i\sigma_0\tau_0}) + i\sigma_0 I](q^*(0))^T = 0,$$

where I is identity matrix of order 2, that is,

$$\begin{pmatrix} a_{100} + i\sigma_0 & b_{100} & 0 \\ a_{010} & b_{010} + i\sigma_0 & c_{010} + c'_{100}e^{-i\sigma_0\tau_k} \\ 0 & b_{001} & c_{001} + c'_{001}e^{-i\sigma_0\tau_k} + i\sigma_0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Solving the above equations, we get

$$q^*(0) = \begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{a_{100} + i\sigma_0}{b_{100}} \\ \frac{-a_{010}b_{100} + (b_{010} + i\sigma_0)(a_{100} + i\sigma_0)}{b_{100}(c_{010} + c'_{010}e^{-i\sigma_0\tau_k})} \end{pmatrix},$$

Now,

$$\begin{aligned} (q^*, \bar{q}) &= \bar{d}_1 [(1, \bar{\alpha}', \bar{\beta}')(1, \alpha, \beta) - \int_{-1}^0 \int_{\xi=0}^{\theta} (1, \bar{\alpha}', \bar{\beta}') e^{-i(\xi-\theta)\sigma_0\tau_k} \\ &\quad d\eta(\theta)(1, \alpha, \beta)^T e^{i\xi\sigma_0\tau_k}] \\ &= \bar{d}_1 [1 + \alpha\bar{\alpha}' + \alpha\bar{\beta}' - \int_{-1}^0 (1, \alpha\bar{\alpha}', \bar{\beta}') \theta e^{i\theta\sigma_0\tau_k} d\eta(\theta)(1, \alpha, \beta)^T] \\ &= \bar{d}_1 [1 + \alpha\bar{\alpha}' + \alpha\bar{\beta}' + \tau_k \bar{\beta}' (\alpha c'_{010} + \beta c'_{001}) e^{-i\sigma_0\tau_k}]. \end{aligned}$$

In order to have $(q^*, q) = 1$, we can choose $d_1 =$

$$\frac{1}{1 + \bar{\alpha}\alpha' + \bar{\beta}\beta' + \beta'\tau_k e^{i\omega_0\tau_k} (\bar{\alpha}c'_{010} + \bar{\beta}c'_{001})},$$

such that $(\psi, A\phi) = (A^*\psi, \phi)$,
 or $-i\sigma_0(q^*, \bar{q}) = (q^*, A\bar{q}) = (A^*q^*, \bar{q}) = (-i\sigma_0q^*, \bar{q}) = i\sigma_0(q^*, \bar{q})$
 i.e. $(q^*, \bar{q}) = 0$.

Next we will compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (31) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{33}$$

On the center manifold C_0 , we have, $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \tag{34}$$

z and \bar{z} are local coordinates for center manifold C_0 in the direction of q and q^* respectively. Here we consider only real solutions as W is real if u_t is real. For the solution $u_t \in C_0$ of (31), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega_0\tau_k z + \bar{q}^*(0)f(0, W(z, \bar{z}, 0) + 2Re\{zq(\theta)\}) \\ &= i\sigma_0\tau_k z + \bar{q}^*(0)f_0(z, \bar{z}), \end{aligned}$$

This equation can be rewritten as

$$\dot{z}(t) = i\sigma_0\tau_k z + g(z, \bar{z}),$$

where

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0)f_0(z, \bar{z}), \\ &= g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{21}(\theta) \frac{z^2\bar{z}}{2} + \dots, \end{aligned} \tag{35}$$

From (33) and (34), we have

$$\begin{aligned} u_t(\theta) &= W(t, \theta) + 2Re\{z(t)q(\theta)\}, \\ &= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + (1, q_1, q_2)^T e^{i\omega_0\tau_k\theta} z \\ &\quad + (1, \bar{q}_1, \bar{q}_2)^T e^{-i\omega_0\tau_k\theta} \bar{z} + \dots \end{aligned} \tag{36}$$

Now from (28) and (35), it follows that

$$\begin{aligned} g(z, \bar{z}) &= \tau_k \bar{d}_1(1, \bar{\alpha}', \bar{\beta}') \\ &\quad \begin{bmatrix} a_{200}u_{1r}^2(0) + a_{110}u_{1r}(0)u_{2r}(0) \\ b_{200}u_{1r}^2(0) + b_{020}u_{2r}^2(0) + b_{110}u_{1r}(0)u_{2r}(0) \\ c'_{020}u_{2r}^2(-1) + c_{002}u_{3r}^2(0) + c'_{011}u_{2r}(-1)u_{3r}(-1) \end{bmatrix}, \\ &= \tau_k \bar{d}_1(1, \bar{\alpha}', \bar{\beta}') \begin{bmatrix} \Delta_{11}z^2 + \Delta_{12}z\bar{z} + \Delta_{13}\bar{z}^2 + \Delta_{14} \frac{z^2}{2}\bar{z} + \dots \\ \Delta_{21}z^2 + \Delta_{22}z\bar{z} + \Delta_{23}\bar{z}^2 + \Delta_{24} \frac{z^2}{2}\bar{z} + \dots \\ \Delta_{31}z^2 + \Delta_{32}z\bar{z} + \Delta_{33}\bar{z}^2 + \Delta_{34} \frac{z^2}{2}\bar{z} + \dots \end{bmatrix}. \end{aligned}$$

Or

$$\begin{aligned} g(z, \bar{z}) &= \tau_k \bar{d}_1 [(\Delta_{11} + \bar{\alpha}'\Delta_{21} + \bar{\beta}'\Delta_{31})z^2 + (\Delta_{12} + \bar{\alpha}'\Delta_{22} + \Delta_{32}\bar{\beta}')z\bar{z} \\ &\quad + (\Delta_{13} + \Delta_{23}\bar{\alpha} + \Delta_{11}\bar{\beta}')\bar{z}^2 + (\Delta_{14} + \bar{\alpha}\Delta_{24} + \Delta_{34}\bar{\beta}) \frac{z^2}{2}\bar{z}]. \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Delta_{11} &= a_{200} + a_{110}\alpha, & \Delta_{12} &= 2a_{200} + 2Re(\alpha)a_{110}, \\ \Delta_{13} &= a_{200} + a_{110}\bar{\alpha}, \\ \Delta_{14} &= (2W_{20}^1(0) + 4W_{11}^1(0))a_{200} + a_{110}(\bar{\alpha}W_{20}^1(0) \\ &\quad + 2\alpha W_{11}^1(0) + W_{20}^2(0)) + 2W_{11}^2(0) \\ \Delta_{21} &= b_{200} + \alpha^2 b_{020} + b_{110}\alpha, \\ \Delta_{22} &= 2b_{200} + 2b_{020}\alpha\bar{\alpha} + 2Re(\alpha)b_{110}, \\ \Delta_{23} &= b_{200} + b_{020}\bar{\alpha}^2 + b_{110}\bar{\alpha}, \\ \Delta_{24} &= (2W_{20}^1(0) + 4W_{11}^1(0))b_{200} + b_{020}(2\bar{\alpha}W_{20}^2 + 4\alpha W_{11}^2) \\ &\quad + b_{110}(\bar{\alpha}W_{20}^1(0) + 2\alpha W_{11}^1(0) + 2W_{11}^2(0)) + W_{20}^3(0), \\ \Delta_{31} &= c'_{200}\alpha^2 e^{2i\sigma_0\tau_k} + c'_{020}\beta^2 + c'_{101}\alpha\beta e^{2i\alpha\tau_k}, \\ \Delta_{32} &= 2c'_{200}\alpha\bar{\alpha} + 2c'_{020}\beta\bar{\beta} + 2Re(\alpha\bar{\beta})c'_{101}, \\ \Delta_{33} &= c'_{200}\bar{\alpha}^2 e^{-2i\sigma_0\tau_k} + c'_{020}\bar{\beta}^2 + c'_{101}\bar{\alpha}\bar{\beta} e^{-2i\sigma_0\tau_k}, \\ \Delta_{34} &= c'_{200}(W_{20}^2(-1)\bar{\alpha}e^{-i\sigma_0\tau_k} + 2W_{11}^2(-1)\alpha e^{i\sigma_0\tau_k} \\ &\quad + 2\alpha W_{11}^2(-1)e^{i\sigma_0\tau_k} + W_{20}^2(-1)\bar{\alpha}e^{-i\sigma_0\tau_k}) \\ &\quad + c'_{020}(2\bar{\beta}W_{20}^3 + 4\beta W_{11}^3) + c'_{101}(W_{20}^2(-1)\bar{\beta}e^{-i\sigma_0\tau_k} \\ &\quad + 2\beta W_{11}^2(-1)e^{i\sigma_0\tau_k} + 2\alpha W_{11}^3(-1)e^{i\sigma_0\tau_k} \\ &\quad + \bar{\alpha}e^{-i\sigma_0\tau_k} + \bar{\alpha}e^{-i\sigma_0\tau_k}W_{20}^3(-1)), \end{aligned}$$

Comparing its coefficients with (35), we find

$$\begin{cases} g_{20} = \frac{2}{d_1} \tau_0 \{ \Delta_{11} + \bar{q}_1^* \Delta_{21} + \bar{q}_2^* \Delta_{31} \}, \\ g_{11} = \frac{1}{d_1} \tau_0 \{ \Delta_{12} + \bar{q}_1^* \Delta_{22} + \bar{q}_2^* \Delta_{32} \}, \\ g_{02} = \frac{2}{d_1} \tau_0 \{ \Delta_{13} + \bar{q}_1^* \Delta_{23} + \bar{q}_2^* \Delta_{33} \}, \\ g_{21} = \frac{2}{d_1} \tau_0 \{ \Delta_{14} + \bar{q}_1^* \Delta_{24} + \bar{q}_2^* \Delta_{34} \}, \end{cases} \tag{38}$$

Since, W_{20} and W_{11} are in g_{21} , we still to compute them.

Now from (31) and (33), we have

$$\begin{aligned} \dot{W} &= \dot{u}(t) - \dot{z}q - \bar{z}\dot{q} \\ &= \begin{cases} A(0)W - 2Re\{\bar{q}^*(0)f_0q(\theta)\} & \theta \in [-1, 0) \\ A(0)W - 2Re\{\bar{q}^*(0)f_0q(0)\} + f_0(z, \bar{z}) & \theta = 0 \end{cases} \triangleq A(0)W + H(z, \bar{z}, \theta) \end{aligned} \tag{39}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \tag{40}$$

Substituting (40) into (39) and comparing the coefficients, we get

$$\begin{aligned} (A(0) - 2i\omega_0\tau_k I)W_{20}(\theta) &= -H_{20}(\theta), \\ A(0)W_{11}(\theta) &= -H_{11}(\theta), \end{aligned} \tag{41}$$

From (39) and for $\theta \in [-1, 0)$

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta), \\ &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta), \end{aligned} \tag{42}$$

Using (35) in (42) and comparing coefficients with (40), we can obtain

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{43}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), \tag{44}$$

From the definition of $A(0)$, (41) and (43), we obtain

$$\dot{W}_{20}(\theta) = 2i\sigma_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta),$$

Solving it and for $q(\theta) = (1, q_1, q_2)^T e^{i\sigma_0\tau_k\theta}$, we have

$$W_{20}(\theta) = \frac{i g_{20}}{\sigma_0\tau_k} q(0)e^{i\sigma_0\tau_k\theta} + \frac{i \bar{g}_{02}}{3\sigma_0\tau_k} \bar{q}(0)e^{-i\sigma_0\tau_k\theta} + E_1 e^{2i\sigma_0\tau_k\theta}, \tag{45}$$

Similarly, from (41) and (44) it follows that,

$$W_{11}(\theta) = -\frac{i g_{11}}{\sigma_0\tau_k} q(0)e^{i\sigma_0\tau_k\theta} + \frac{i \bar{g}_{11}}{\sigma_0\tau_k} \bar{q}(0)e^{-i\sigma_0\tau_k\theta} + E_2, \tag{46}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ and $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ are three dimensional constant vectors, and can be determined by setting $\theta = 0$ in $H(z, \bar{z}, \theta)$.

Again from the definition of $A(0)$ and (41), we have

$$\int_{-1}^0 d\zeta(\theta)W_{20}(\theta) = 2i\sigma_0\tau_k W_{20}(0) - H_{20}(0), \tag{47}$$

and

$$\int_{-1}^0 d\zeta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{48}$$

where $\zeta(\theta) = \zeta(0, \theta)$.

From (39), we know when $\theta = 0$,

$$\begin{aligned} H(z, \bar{z}, 0) &= -2Re(\bar{q}^*(0)f_0q(0)) + f_0(z, \bar{z}) \\ &= -\bar{q}^*(0)f_0q(0) - q^*(0)\bar{f}_0\bar{q}(0) + f_0(z, \bar{z}), \end{aligned}$$

That is,

$$\begin{aligned} H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \\ = -q(0) \left\{ g_{20} \frac{z^2}{2} + g_{11}z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + \dots \right\} \\ - \bar{q}(0) \left\{ \bar{g}_{20} \frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02} \frac{z^2}{2} + \dots \right\} + f_0(z, \bar{z}), \end{aligned} \tag{49}$$

From (33), we have

$$\begin{aligned} u_t(\theta) &= W(t, \theta) + 2Re\{z(t)q(\theta)\} = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(t) \\ &= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \end{aligned}$$

Thus we can obtain,

$$f_0 = 2\tau_k \begin{bmatrix} \Delta_{11} \\ \Delta_{21} \\ \Delta_{31} \end{bmatrix} \frac{z^2}{2} + \tau_k \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \\ \Delta_{32} \end{bmatrix} z\bar{z} + \dots \tag{50}$$

Comparing the coefficients in (49) and using (50), we get

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_k \begin{bmatrix} \Delta_{11} \\ \Delta_{21} \\ \Delta_{31} \end{bmatrix} \tag{51}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_k \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \\ \Delta_{32} \end{bmatrix} \tag{52}$$

Since $i\sigma_0\tau_k$ is the eigenvalue of $A(0)$ corresponding to eigenvector $q(0)$, then

$$\begin{aligned} \{i\sigma_0\tau_k I - \int_{-1}^0 e^{i\sigma_0\tau_k\theta} d\zeta(\theta)\}q(0) &= 0 \quad \text{and} \\ \{-i\sigma_0\tau_k I - \int_{-1}^0 e^{-i\sigma_0\tau_k\theta} d\zeta(\theta)\}\bar{q}(0) &= 0. \end{aligned}$$

Substituting (45) and (51) into (47), we find

$$\{2i\sigma_0\tau_k I - \int_{-1}^0 e^{2i\sigma_0\tau_k\theta} d\zeta(\theta)\}E_1 = 2\tau_k \begin{bmatrix} \Delta_{11} \\ \Delta_{21} \\ \Delta_{31} \end{bmatrix}$$

or

$$\begin{bmatrix} 2i\sigma_0 - a_{100} & -a_{010} & 0 \\ -b_{100} & 2i\sigma_0 - b_{010} & -b_{001} \\ 0 & -c_{010} - c'_{010}e^{-2i\sigma_0\tau_0} & -c_{001} - c'_{001}e^{-2i\sigma_0\tau_0} \end{bmatrix} E_1 = \begin{bmatrix} 2\Delta_{11} \\ 2\Delta_{21} \\ 2\Delta_{31} \end{bmatrix}$$

or

$$E_1 = 2 \begin{bmatrix} \Delta_{11} \\ \Delta_{21} \\ \Delta_{31} \end{bmatrix} \begin{bmatrix} 2i\sigma_0 - a_{100} & -a_{010} & 0 \\ -b_{100} & 2i\sigma_0 - b_{010} & -b_{001} \\ 0 & -c_{010} - c'_{010}e^{-2i\sigma_0\tau_0} & -c_{001} - c'_{001}e^{-2i\sigma_0\tau_0} \end{bmatrix}^{-1}$$

Similarly, substituting (46) and (52) into (48), we obtain

$$E_2 = - \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \\ \Delta_{32} \end{bmatrix} \begin{bmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{010} + c'_{010}e^{-2i\sigma_0\tau_k} & c_{001} + c'_{001}e^{-2i\sigma_0\tau_k} \end{bmatrix}$$

or

$$E_2 = \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \\ \Delta_{32} \end{bmatrix} \begin{bmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{010} + c'_{010}e^{-2i\sigma_0\tau_k} & c_{001} + c'_{001}e^{-2i\sigma_0\tau_k} \end{bmatrix}^{-1}$$

Finally, we can compute the following quantities:

$$c_1(0) = \frac{i\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\}}{2\sigma_0\tau_k} + \frac{g_{21}}{2},$$

$$\mu_2 = \frac{Re\{c_1(0)\}}{Re\{\frac{d\lambda(\tau_k)}{d\tau}\}},$$

$$\beta_2 = 2Re\{c_1(0)\},$$

$$T_2 = -\frac{\Im\{c_1(0)\} + \mu_2\Im\{\frac{d\lambda(\tau_k)}{d\tau}\}}{\sigma_0\tau_k}, \quad k = 0, 1, 2, \dots$$

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