

# Dynamics of a food chain model with herd behaviour of the prey

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**Abstract** Many prey populations live in herds and so, in spite of all individuals, only the members of outer surface interact with other species. To represent such prey-predator interaction ‘square root functional response’ is more suitable. In this paper, dynamical complexities of a three species food web is investigated where the prey species shows a herd behaviour. Boundedness and stability analysis is carried out. Bifurcation analysis is presented. Our analytical findings are verified through computer simulations.

**Keywords** Food chain model · Herd · Boundedness · Persistence · Stability · Hopf-bifurcation

## Introduction

Ever since the pioneering work of Lotka (1925) and Volterra (1926), most of the researches in biomathematics had been devoted to prey-predator system. Biomathematicians remained silent about three-species systems for a long time.

The reason may be the insufficiency of mathematical tools to handle the increasing number of differential equations. However, urge for incorporating more species had been felt day by day. Prince et al. (1980) remarked that, the study of community behaviour with the help of mathematical models must be based on at least three trophic levels and hence more focus should be made to study the complex behaviour exhibited by deterministic models consists of three and more trophic levels. In reality, different three-species system have become the focus of considerable attention in their own right. For example, in waste treatment process, food chain of waste-bacteria-ciliates has got the attention of researchers (Li and Kuang 2000). The tri-trophic food chain of tea plant-pest-predator (or, pathogen) has emerged for biological control of tea pests (Kabir 2001). Some mathematical models for tri-trophic food chains have been developed and studied in recent past but their is no denying that theoretical studies on such systems are still largely lacking. Some theoretical works on food chain models may be found in the papers of Freedman and Waltman 1977, Gard (1980), Freedman and So (1985), Takeuchi et al. (1992), Freedman and Ruan (1992), Ruan (1993), McCann and Yodzis (1995), Boer et al. (1999), Kuznetsov et al. (2001), Chauvet et al. (2002), Hsu et al. (2001), Maiti and Samanta (2005), Maiti et al. (2006), Pathak et al. (2009).

Among many factors involved in prey-predator models the crucial one is the ‘functional response’. Depending upon the behaviour of populations, more suitable functional response has been developed as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. In this connection, Holling family of functional responses (Holling 1959a, b, 1965) are the most focused, and in particular, the Holling type-II functional response has served as basis

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for a very large literature in prey-predator theory (Maiti and Samanta 2005; Arditi and Ginzburg 1989; Arditi and Saiah 1992; Abrams and Ginzburg 2000; Ruan and Xiao 2001; Murray 2002). The Holling type-II functional response includes the fact that a single individual can feed only until the stomach is not full, and so a saturation function would be better to describe the intake of food. This is similar to the concept of the ‘law of diminishing returns’ borrowed from operations research, via a hyperbola rising up to an asymptotic value. In other words, the functional response would be of the following form

$$F(X) = \frac{\alpha X}{1 + T_h \alpha X},$$

where  $X(T)$  is the prey density at time  $T$ ,  $\alpha$  is the search efficiency of the predator for prey,  $T_h$  is the average handling time for each prey.

### Previous scholarships

In natural ecosystems, most of the prey populations live forming groups, and all members of a group do not interact at a time (Bera et al. 2015, 2016). Major predators of zebra, buffalo, kongoni, toki and Thomsons gazelle are hyena, wild dog, lion, leopard and cheetas and so they form groups for defence against predators. The underlying reasons for group formation more likely depend upon self-defence, group-defence, group alertness within a group and speed, to avoid being killed by a predator (Khan et al. 2004). Models of group formation are used to study environmental and social forces, and individual decision rules that lead to formation of swarms, flocks, schools, herds, and other groups (DeAngelis and Mooij 2005). It is pointed out by Fryxell et al. (2007) that group formation profoundly reduce food intake rates below the levels expected. As a consequence, suitable form of functional response was searched for to describe the social behaviour of such populations. Freedman and Wolkowitz (1986) discussed the characteristics of group defense in this regard. To explore the consequence of forming spatial group of fixed shape by predators, Cosner et al. (1999) made the assumption that the number of predators in a shape is proportional to the area of the group in two dimension, and to its volume in three dimension. As a result, the encounter rate  $E(X, Y) = \alpha XY$  should change its form to  $E(X, Y) = \alpha Y \sqrt{X}$  in two dimension, and in three dimension it should be  $E(X, Y) = \alpha Y X^{\frac{2}{3}}$ . Recently Ajraldi et al. (2011) argued that when prey population exhibit herd behaviour, the functional response should be in terms of square root of prey population. They also claimed that on a large savanna, the herbivorous and their large predators entirely fit their

suggested model. Their work, in which the prey population assumed to form a herd in defense or searching for food purpose, lead to some unexpected behaviours, particularly near origin. Using the form of the Holling type-II functional response, Braza (2012) developed a new functional response, where the prey density term was replaced by its square root.

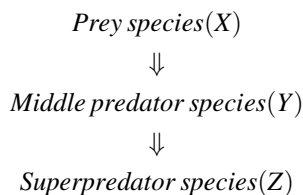
### Objectives

In this work, we have presented the dynamical behavior of a three-species food chain model with a mixed selection of functional response. Our study includes boundedness, stability and bifurcation analysis. The rest of the paper is structured as follows. A brief sketch of the construction of the model and the biological relevance of it have been stated. Next, boundedness of the basic deterministic model is discussed and then the boundary equilibrium points and their stability analysis have been analyzed. The necessary and sufficient condition for the existence of the interior equilibrium point and the conditions for its stability are derived. Criteria for Hopf bifurcation near interior equilibrium has also been studied. Computer simulation of a variety of numerical solutions of the underlying system is presented. At the end general discussions on the results and biological implications of our mathematical findings have been presented. Here our objectives are to enrich the dynamics of a three-species predator-prey system with herd behaviour among the habitat of the prey population which help us to get better understanding of the interaction between prey, middle predator and superpredator in a real ecosystem. It is assumed that the prey is consumed by middle predator and middle predator by superpredator.

### Materials and methods

Before we introduce the mathematical model, let us describe the basic assumptions that we made to formulate it.

1. The biological system we consider is composed of a single prey population whose density is denoted by  $X$ . There is a middle predator and a superpredator (or top predator), whose population density are  $Y$  and  $Z$  respectively. The behavior of the entire community is assumed to arise from the coupling of these three populations, where  $Y$  is the only predator for prey population  $X$ , and superpredator  $Z$  has only food resource  $Y$ .



2. We assume that in the absence of the predators the prey population density grows according to a logistic curve with carrying capacity  $K(K > 0)$  and with an intrinsic growth rate constant  $r(r > 0)$ .
3. In prey-predator relationship, functional response is the key term. Most of the food chain models considered in ecological literature are constructed with same type functional response on predation of prey by middle predator or of middle predator by superpredator. But a mixed selection of functional response would be perhaps more realistic. We assume that the prey population live in herds. To model this herd behaviour we consider the modified square root functional response mentioned earlier (Braza 2012). The middle predator and the superpredator interaction is assumed to be governed by the usual Holling type-II functional response.

The above considerations motivate us to introduce the following tri-trophic food chain model under the framework of the following set of nonlinear ordinary differential equations:

$$\begin{aligned}
 \frac{dX}{dT} &= rX \left(1 - \frac{X}{K}\right) - \frac{\alpha_1 \sqrt{XY}}{1 + T_1 \alpha_1 \sqrt{X}}, \\
 \frac{dY}{dT} &= \frac{c_1 \alpha_1 \sqrt{XY}}{1 + T_1 \alpha_1 \sqrt{X}} - d_1 Y - \frac{\alpha_2 YZ}{1 + T_2 \alpha_2 Y}, \\
 \frac{dZ}{dT} &= \frac{c_2 \alpha_2 YZ}{1 + T_2 \alpha_2 Y} - d_2 Z.
 \end{aligned} \tag{1}$$

Here the parameters  $\alpha_1, \alpha_2$  are the search efficiencies, and  $T_1, T_2$  are the handling times of the middle predator and superpredator, respectively. The parameters  $c_1$  and  $c_2$  denote the biomass conversion factors for prey population to middle predator and for middle predator to superpredator, respectively. Natural death rates for middle predator and superpredator are denoted by  $d_1$  and  $d_2$ , respectively.

The model we have just specified has ten parameters, which makes analysis difficult. To reduce the number of parameters and to simplify the system a little bit, we nondimensionalize the system (1) with the following scaling:

$$x = \frac{X}{K}, \quad y = \frac{Y}{K}, \quad z = \frac{Z}{K} \quad \text{and} \quad t = rT.$$

Then the system (1) takes the form (after some simplification),

$$\begin{aligned}
 \frac{dx}{dt} &= x(1 - x) - \frac{a\sqrt{xy}}{1 + b\sqrt{x}}, \\
 \frac{dy}{dt} &= \frac{c\sqrt{xy}}{1 + b\sqrt{x}} - dy - \frac{pyz}{1 + my}, \\
 \frac{dz}{dt} &= \frac{qyz}{1 + my} - ez,
 \end{aligned} \tag{2}$$

where

$$\begin{aligned}
 a &= \frac{\alpha_1 \sqrt{K}}{r}, \quad b = T_1 \alpha_1 \sqrt{K}, \quad c = \frac{c_1 \alpha_1 \sqrt{K}}{r}, \quad d = \frac{d_1}{r}, \quad e = \frac{d_2}{r}, \\
 p &= \frac{\alpha_2 K}{r}, \quad q = \frac{c_2 \alpha_2 K}{r}, \quad m = T_2 \alpha_2 K.
 \end{aligned}$$

**Boundedness**

Biological validity of a mathematical model is firstly decided by its boundedness. It refers that the solutions of the system will remain in some feasible region forever. The following theorem gives a criterion for boundedness of the system (2).

**Theorem 4.1** *All solutions of the system (2) that start in  $\mathbb{R}^3_+$  are uniformly bounded.*

*Proof* Let  $(x(t), y(t), z(t))$  be any solution of the system (2).

Since

$$\frac{dx}{dt} \leq x(1 - x),$$

we have

$$\limsup_{t \rightarrow \infty} x(t) \leq 1.$$

Suppose

$$W = \frac{c}{a}x + y + \frac{p}{a}z.$$

Then

$$\begin{aligned}
 \frac{dW}{dt} &\leq \frac{c}{a}x - dy - \frac{pe}{q}z \\
 &\leq \frac{2c}{a} - \delta W, \quad \text{where } \delta = \min\{1, d, e\}.
 \end{aligned}$$

Therefore

$$\frac{dW}{dt} + \delta W \leq \frac{2c}{a}.$$

Applying a theorem on differential inequalities (Birkhoff and Rota 1982), we obtain

$$0 \leq W(x, y, z) \leq \frac{2c}{a\delta} + \frac{W(x(0), y(0), z(0))}{e^{\delta t}},$$

and for  $t \rightarrow \infty$ ,

$$0 \leq W \leq \frac{2c}{a\delta}.$$

Thus, all the solutions of (2) enter into the region

$$B = \left\{ (x, y, z) : 0 \leq W \leq \frac{2c}{a\delta} + \epsilon, \text{ for any } \epsilon > 0 \right\}.$$

Hence the theorem.  $\square$

### Boundary equilibria and stability

In the analysis of systems of differential equations it is useful to consider solutions that do not change with time, that is, for which

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \text{ and } \frac{dz}{dt} = 0.$$

Such solutions are called equilibria, steady states or fixed points provided they lie in feasible region. System (2) always has the trivial equilibrium point  $E_0(0, 0, 0)$  and the axial equilibrium point  $E_1(1, 0, 0)$ . The third boundary equilibrium point  $E_2(\hat{x}, \hat{y}, 0)$  exists if and only if  $d < c/(1 + b)$ . When this condition is satisfied,  $\hat{x}, \hat{y}$  are given by

$$\hat{x} = \frac{d^2}{(c - bd)^2}, \hat{y} = \frac{cd((c - bd)^2 - d^2)}{a(c - bd)^4}.$$

The system (2) can not be linearized at  $E_0(0, 0, 0)$  and therefore local stability analysis at  $E_0$  can not be done directly. But rescaling the variable  $x = x_1^2$  the singularity may be overcome (Ajraldi et al. 2011). The eigenvalues of such transformed system being  $\frac{1}{2}, -d$  and  $-e$ , we may conclude that, the trivial equilibrium  $E_0(0, 0, 0)$  has an unstable manifold.

The axial equilibrium point  $E_1(1, 0, 0)$  has the variational matrix

$$V(E_1) = \begin{bmatrix} -1 & -\frac{a}{1+b} & 0 \\ 0 & \frac{c}{1+b} - d & 0 \\ 0 & 0 & -e \end{bmatrix},$$

which has eigenvalues  $-1, (\frac{c}{1+b} - d)$  and  $-e$ . All the eigenvalues be negative iff  $d > c/(1 + b)$ . Hence the prey-only equilibrium is stable when the middle predator death rate  $d_1$  exceeds some threshold value  $d_1^* = \frac{c_1 z_1 \sqrt{K}}{1 + T_1 z_1 \sqrt{K}}$  (in terms of the original parameters of the system (2)).

The variational matrix  $V(E_2)$  at the equilibrium point  $E_2(\hat{x}, \hat{y}, 0)$  is given by

$$V(E_2) = \begin{bmatrix} v_1 & -\frac{d}{c} & 0 \\ v_2 & 0 & -\frac{p\hat{y}}{1+m\hat{y}} \\ 0 & 0 & -e + \frac{q\hat{y}}{1+m\hat{y}} \end{bmatrix},$$

where  $v_1 = 1 - 2\hat{x} - \frac{a\hat{y}}{2\sqrt{\hat{x}(1+b\sqrt{\hat{x}})^2}}$  and  $v_2 = \frac{c\hat{y}}{2\sqrt{\hat{x}(1+b\sqrt{\hat{x}})^2}}$ . The eigenvalues of this variational matrix are  $\lambda_{1,2} = \frac{cv_1 \pm \sqrt{c^2 v_1^2 - 4cdv_2}}{2c}$ , and  $\lambda_3 = -e + \frac{q\hat{y}}{1+m\hat{y}}$ .

Therefore the superpredator-free equilibrium is asymptotically stable if  $\hat{x}$  and  $\hat{y}$  exists along with the following two conditions:

- (i)  $v_1 < 0$ ,
- (ii)  $ae(c - bd)^4 - cd(q - me)((c - bd)^2 - d^2) > 0$ .

### The interior equilibrium point: its existence and stability

**Lemma 4.1** *The first two components of the interior equilibrium point  $E^*(x^*, y^*, z^*)$  of the system (2) exist if*

- (i)  $q > me$ ,
- (ii) *there exist some positive  $x_1$  satisfying the bi-quadratic  $bx_1^4 + x_1^3 - bx_1^2 - x_1 + P = 0$ , where  $P = ae/(q - me)$ .*

When these conditions are satisfied, then

$$x^* = x_1^2$$

and

$$y^* = \frac{m}{q - me}.$$

Furthermore, if  $x^*(1 - x^*) > ade/(q - me)$ , then  $z^*$  exists and is given by

$$z^* = \frac{q}{p} \left( \frac{cx^*(1 - x^*)}{ae} - \frac{d}{q - me} \right).$$

The variational matrix of the system (2) at  $E^*$  is given by

$$V(E^*) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= 1 - 2x^* - \frac{1 - x^*}{2(1 + b\sqrt{x^*})}, \quad a_{12} = -\frac{a\sqrt{x^*}}{(1 + b\sqrt{x^*})}, \\ a_{21} &= \frac{c(1 - x^*)}{2a(1 + b\sqrt{x^*})}, \quad a_{22} = \frac{me}{q} \left( -d + \frac{c\sqrt{x^*}}{1 + b\sqrt{x^*}} \right), \\ a_{23} &= -\frac{pe}{q} \text{ and } a_{32} = \frac{q - me}{p} \left( -d + \frac{c\sqrt{x^*}}{1 + b\sqrt{x^*}} \right). \end{aligned}$$

The characteristic equation of  $V(E^*)$  is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where  $A_1 = -a_{11} - a_{22}, A_2 = a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32}$  and  $A_3 = a_{11}a_{23}a_{32}$ .

Now we have the following theorem guaranteeing the local stability at  $E^*$ .

**Theorem 4.2** *If  $E^*$  exists with*

- (i)  $2(2bx^* + a - b)\sqrt{x^*} + 3x^* - 1 > 0$ ,
- (ii)  $P_1c^2 + P_2c + P_3 > 0$ ,

where

$$P_1 = \frac{mepqa_{12}\sqrt{x^*}(1 - x^*) + 2ax^*(meq(q - me)a_{23} - pm^2e^2a_{11})}{2a(1 + b\sqrt{x^*})^2},$$

$$P_2 = \frac{pqa_{12}(qa_{11} - dme)(1 - x^*) - 2a\sqrt{x^*}(2dmeq(q - me)a_{23} - 2dpm^2e^2a_{11} - mepqa_{11}^2)}{2a(1 + b\sqrt{x^*})},$$

$$P_3 = dmepqa_{11}^2 + d^2(meq(q - me)a_{23} - pm^2e^2a_{11}),$$

then  $E^*$  is locally asymptotically stable.

*Proof* For  $2(2bx^* + a - b)\sqrt{x^*} + 3x^* - 1 > 0$ , it is obvious that  $A_1 > 0$ . It can be shown that  $A_3$  is always positive for all possible values of parameters. And the condition of the theorem gives

$$\Delta = A_1A_2 - A_3 = \frac{P_1c^2 + P_2c + P_3}{pq^2} > 0.$$

Hence the theorem follows from Routh–Herwitz criterion.  $\square$

### Bifurcation analysis

In some models, the stability switch of a system may occur simply by changing a single parameter (when other parameters remain unaltered). Hopf bifurcation is characterised by the stability change of an equilibrium point accompanied by small amplitude limit cycle behaviour around the equilibrium point, as some parameter  $\mu$  (called bifurcation parameter) of the system passes through a critical value  $\mu^*$ . The Hopf bifurcation theorem is stated below (Murray 2002).

**Theorem 4.3** *Let*

$$\frac{dx}{dt} = f(x, \mu)$$

be a real system of ordinary differential equations in  $\mathbb{R}^n$  depending on some parameter  $\mu$  in an open interval  $\mathbf{I}$  of  $\mathbb{R}$ . Assume that the components of function  $f$  are continuously differentiable in all variables. Let  $E(x^*, \mu)$  be an equilibrium point of above system. Assume that the interior equilibrium depends smoothly on some parameter  $\mu \in \mathbf{I}$ . If there exists a  $\mu^* \in \mathbf{I}$  such that

- (i) a pair of complex eigenvalues of the variational matrix  $V(E)$  of the equilibrium point  $E$  exists, say,

$\alpha(\mu) \pm i\beta(\mu)$  with  $\alpha(\mu), \beta(\mu) \in \mathbb{R}$  such that they become purely imaginary at  $\mu = \mu^*$ , whereas the other eigenvalues have negative real parts,

- (ii)  $\left[\frac{d\alpha}{d\mu}\right]_{\mu=\mu^*} \neq 0$ , then at  $\mu^*$  there is a Hopf bifurcation.

Hopf bifurcation theorem has long played a vital role in the dynamics of two-species models. Some three-species

models are also cultured by this theorem but there is no denying that for most three-species models it is really difficult to check those conditions involving eigenvalues of the variational matrix. Liu (1994) derived a criterion of Hopf bifurcation without using the eigenvalues of the variational matrix. We state below the result of Liu for a general three-species system.

**Theorem 4.4** *If the characteristic equation of the equilibrium point is given by  $\lambda^3 + A_1(\mu)\lambda^2 + A_2(\mu)\lambda + A_3(\mu) = 0$ , where  $A_1(\mu), A_3(\mu), \Delta(\mu) = A_1(\mu)A_2(\mu) - A_3(\mu)$  are smooth functions of  $\mu$  in an open interval about  $\mu^* \in \mathbb{R}$  such that*

- (i)  $A_1(\mu^*) > 0, A_3(\mu^*) > 0, \Delta(\mu^*) = 0$ ,
- (ii)  $\left[\frac{d\Delta}{d\mu}\right]_{\mu=\mu^*} \neq 0$ , then a Hopf bifurcation occurs at  $\mu = \mu^*$

In our model (2) we consider the parameter  $c$  as the bifurcation parameter. Using Liu’s criterion we have the following theorem.

**Theorem 4.5** *Let  $E^*$  exists with  $P_1 < 0, P_2 > 0$  and  $P_3 > 0$ . Then a Hopf-bifurcation occurs at the unique value  $c = c^* = \frac{-P_2 - \sqrt{P_2^2 - 4P_1P_3}}{2P_1}$  provided  $2(2bx^* + a - b)\sqrt{x^*} + 3x^* - 1 > 0$ .*

### Results: numerical simulation

Beside analytical studies numerical verification of various results are not less important. It gives a touch of completeness to the analytical findings. Here we have presented computer simulations of the system using MATLAB.

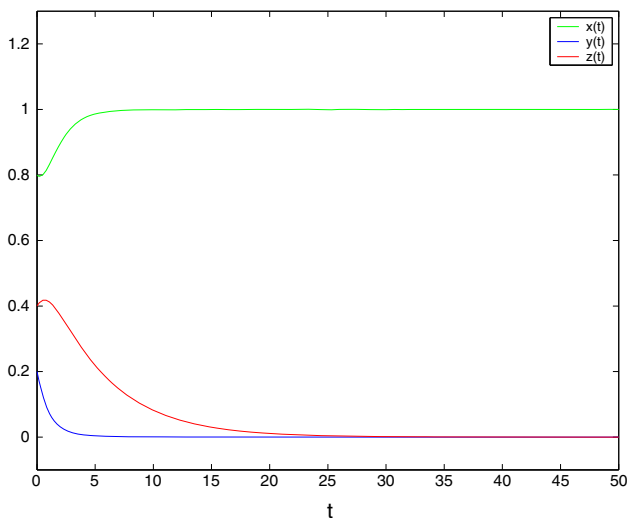
We choose the parameters as  $a = 2.1$ ,  $b = 1$ ,  $c = 2$ ,  $d = 1.1$ ,  $e = 0.2$ ,  $p = 2$ ,  $q = 1.9$ ,  $m = 0.4$ , and  $(x(0), y(0), z(0)) = (0.8, 0.2, 0.4)$ . Since  $d$  exceeds the critical value  $d^* = c/(1 + b) = 1$ , as per discussion in section 4, the prey-only equilibrium is stable. We have verified that both the predator populations dies out and the prey population approaches its equilibrium value in finite time (see Fig. 1).

For  $a = 2.9$ ,  $b = 1.3$ ,  $c = 1.5$ ,  $d = 0.56$ ,  $e = 0.7$ ,  $p = 2.6$ ,  $q = 1.6$ ,  $m = 0.4$  with  $(x(0), y(0), z(0)) = (0.6, 0.4, 0.2)$  we notice that the stability conditions discussed in Sect. 4 are verified as  $v_1 = -0.1743 < 0$  and  $ae(c - bd)^4 - cd(q - me)((c - bd)^2 - d^2) = 0.4079 > 0$ . This is depicted in Fig. 2 which shows that the super-predator-free boundary equilibrium  $E(\hat{x}, \hat{y}, 0)$  is locally asymptotically stable. Clearly  $x \rightarrow \hat{x} = 0.5262$ ,  $y \rightarrow \hat{y} = 0.2303$  and  $z \rightarrow 0$  in finite time.

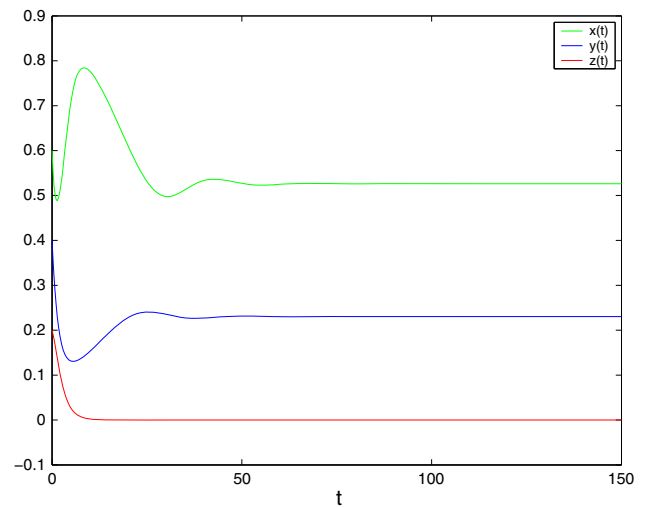
If we take the parameters as  $a = 1.8$ ,  $b = 0.89$ ,  $c = 2.0$ ,  $d = 0.45$ ,  $e = 0.2$ ,  $p = 2.0$ ,  $q = 1.9$ ,  $m = 0.4$ ; the conditions of Lemma 5.1 are satisfied since  $me = 0.08 < 0.4 = m$ , and we have  $x_1 = 0.9411$  satisfying  $bx_1^4 + x_1^3 - bx_1^2 - x_1 + P = 0$ . As the existence criterion is satisfied, the system (2.2) has an interior equilibrium  $E^*(x^*, y^*, z^*) = (0.8856, 0.1099, 0.2998)$ . Now conditions of Theorem 4.2 for stability of  $E^*$  are also satisfied as

- (i)  $2(2bx^* + a - b)\sqrt{x^*} + 3x^* - 1 = 6.3366 > 0$ ,
- (ii)  $P_1c^2 + P_2c + P_3 = 0.0509 > 0$ .

Therefore, for the above choice of parameters,  $E^*$  is locally asymptotically stable. The corresponding phase portrait for



**Fig. 1** Here  $x(0) = 0.8$ ,  $y(0) = 0.2$ ,  $z(0) = 0.4$  and  $a = 2.1$ ,  $b = 1$ ,  $c = 2$ ,  $d = 1.1$ ,  $e = 0.2$ ,  $p = 2$ ,  $q = 1.9$ ,  $m = 0.4$ . It shows that the axial equilibrium is locally asymptotically stable



**Fig. 2** Illustration of the stability of  $E_2$  for  $a = 2.9$ ,  $b = 1.3$ ,  $c = 1.5$ ,  $d = 0.56$ ,  $e = 0.7$ ,  $p = 2.6$ ,  $q = 1.6$ ,  $m = 0.4$  and  $(x(0), y(0), z(0)) = (0.6, 0.4, 0.2)$

different initial choices is presented in Fig. 3a. The paths are clearly stable spirals converging to  $E^*$ . Figure 3b shows that  $x, y, z$  approach their equilibrium values in finite time when  $(x(0), y(0), z(0)) = (0.8, 0.2, 0.4)$ .

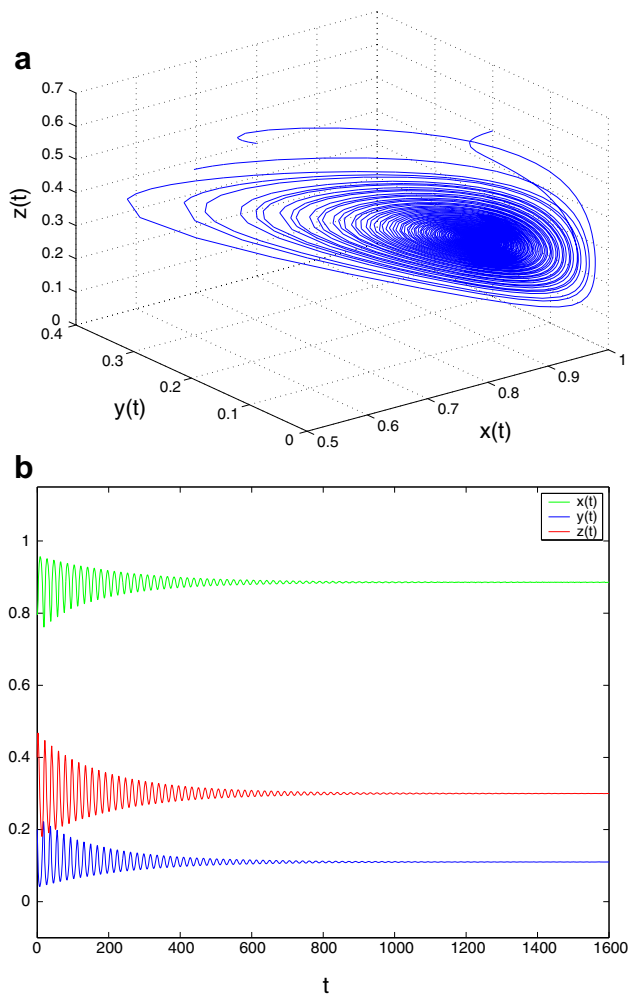
(It may be noticed that there are more than one positive values of  $x_1$  satisfying  $bx_1^4 + x_1^3 - bx_1^2 - x_1 + P = 0$ . But the other values do not satisfy the stability criterion given in Theorem 4.2. For the above choice of parameters, we might obtain  $x_1 = 0.1764$  also but  $P_1c^2 + P_2c + P_3 = -0.5659 < 0$  for this value of  $x_1$ ).

We have considered the parameter  $c$  as the control to break the stability of the system (2.2) near  $E^*$ . If we take values of all the parameters same as Fig. 3. (except that of  $c$ ), the computed critical value of  $c$  becomes  $c^* = 2.8892$ . For  $c = 2.0 < c^*$ , we have already seen that  $E^*$  is locally asymptotically stable (see Fig. 3). For  $c = 3 > c^*$  (with other values unaltered) the stability of system (2.2) breaks and the limit cycles grows out of  $E^*$  (see Fig. 4a). Sustained oscillations of  $x, y, z$  in finite time is shown in Fig. 4b when  $c = 3$ . A bifurcation diagram is also drawn by taking  $c$  as the bifurcation parameter. It shows that  $(x, y, z)$  approaches to a unique value for  $c < c^* = 2.8892$ . Thereafter the stability breaks and the iterated values of  $(x, y, z)$  approaches to different scattered values throughout time (see Fig. 5).

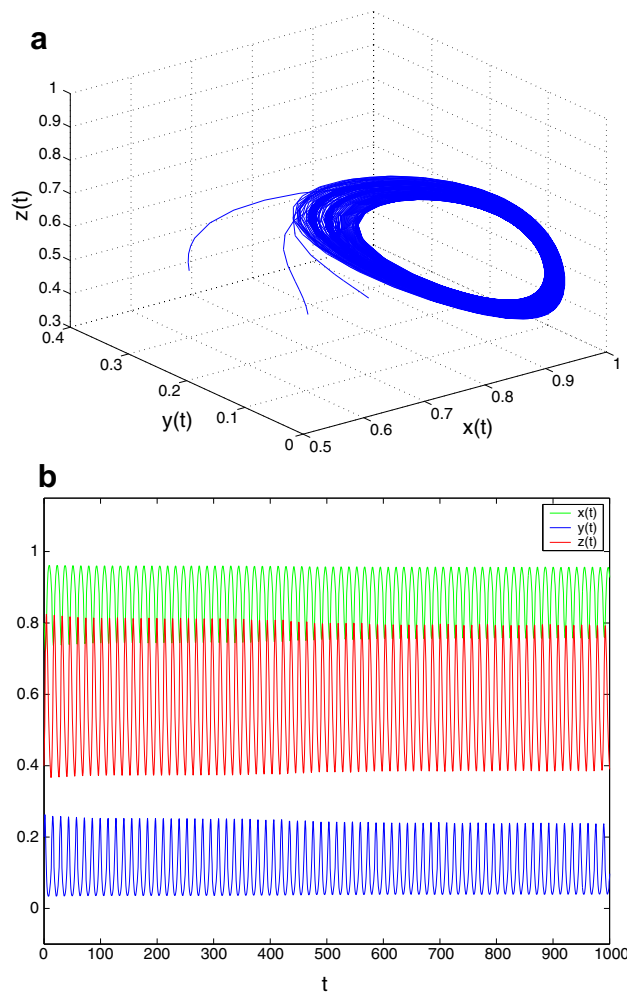
### Discussion and conclusion

The literature abounds with food chains where the prey population exhibits herd behaviour. For example, in marine ecology, the food chain of Pacific sardine (*Sardinops*





**Fig. 3** Here  $a = 1.8, b = 0.89, c = 2.0, d = 0.45, e = 0.2, p = 2.0, q = 1.9, m = 0.4$ . **a** The paths of the system (2.2) for different initial choices showing that  $E^*$  is locally asymptotically stable. **b**  $x, y, z$  approach their equilibrium values in finite time when  $(x(0), y(0), z(0)) = (0.8, 0.2, 0.4)$

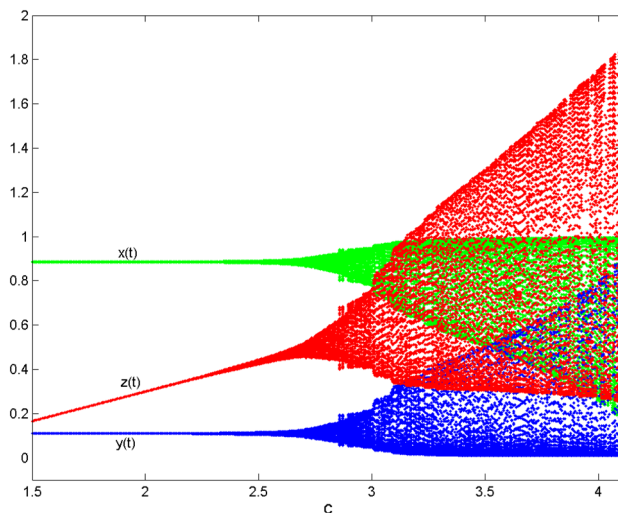


**Fig. 4** Here  $a = 1.8, b = 0.89, c = 3.0 > 2.8892 = c^*, d = 0.45, e = 0.2, p = 2.0, q = 1.9, m = 0.4$ . **a** Phase portrait of the system (2.2) for different initial choices showing periodic orbits near  $E^*$ . **b** Periodic oscillations of  $x, y, z$  with time when  $(x(0), y(0), z(0)) = (0.8, 0.2, 0.4)$

sagax)–Jack mackerel (*Trachurus murphyi*)–Swordfish (*Xiphias gladius*) has got the attention of scientists (Neira et al. 2004; Neira and Arancibia 2004). On the other hand, the herd or schooling behaviour of the sardine is reported by many researchers (Brehmer et al. 2007; Zwolinski and Demer 2012).

In this paper, we have studied the dynamical behaviour of a tri-trophic food chain model where prey population lives in herd. The characteristic of group-living of prey population has been emphasized using square root of prey density in the functional response. The interaction between middle predator and super-predator is assumed to be governed by Holling type-II

functional response. The model we have considered, being bounded, is biologically well behaved. All possible trivial, axial or boundary equilibrium with criteria for their existence and stability has been discussed. In real environment, the coexistence of populations has immense importance for ecological balance in nature. From this viewpoint, study of the stability of the interior equilibrium is emphasized. Bifurcation analysis has also been carried out showing that the conversion rate of prey biomass to that of middle predator may switch the stability of the interior equilibrium. The analysis indicates that if the middle predator is an aggressive consumer, then it might cause ecological imbalance in



**Fig. 5** Bifurcation diagram of the system (2.2) for  $1.5 < c < 4.1$  when  $a = 1.8$ ,  $b = 0.89$ ,  $d = 0.45$ ,  $e = 0.2$ ,  $p = 2.0$ ,  $q = 1.9$ ,  $m = 0.4$

nature. Further studies are required to analyze the dynamics of more realistic but complex systems, such as delayed diffusive predator-prey system with herd behavior.

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