Int. J. Res. Undergrad. Math. Ed.
https://doi.org/10.1007/s40753-023-00233-z

Students' Mathematical Thinking in Movement

Robyn Gandell[1]

Accepted: 20 December 2023

Abstract

Mathematics education research is increasingly focused on how students' movement interacts with their cognition. Although usually characterized as embodiment research, movement research often theorizes the body in diverse ways. Ingold (*Making: Anthropology, archaeology, art and architecture*, 2013) proposes that thinking and knowing emerge from the entwined, dynamic flows of human and non-human materials in a process called making and, following Sheets-Johnstone (*The primacy of movement* (Vol. 82), 2011), contends that humans think in movement. The study that this paper draws on employs Ingold's making to study students' movement during mathematical problem solving. In this paper I also recruit Laban's movement elements (Laban & Ullmann, 1966/2011) as a framework to describe and analyse how the body moves in space and time and to incorporate the often-forgotten dynamic qualities of movement. This paper investigates the movement of a small group of tertiary students as they engage with a mathematical prompt (a task in Abstract Algebra), using thick description, to answer the questions: (1) How do students think mathematically in movement? (2) How do Laban's elements help inform research into students' movement? Through the lens of Laban's movement elements, my analysis demonstrates that students think mathematically in movement. These findings suggest that mathematics educators may be overlooking valuable instances of students' mathematical thinking and knowing: the thinking and knowing in movement which may not be available through verbalizations or artefacts. Although thinking in movement does not fit a traditional conceptualization of undergraduate mathematics, which privileges written communication heavily reliant on notation, to understand students' mathematical cognition more comprehensively, mathematics educators need to reconsider and appreciate students' mathematical thinking in movement.

Keywords Thinking in movement · Mathematics education · Embodiment · Post-humanism · Dynamic movement qualities

✉ Robyn Gandell
robyn.gandell@auckland.ac.nz

[1] University of Auckland, Auckland, New Zealand

Published online: 27 January 2024

> For one hundred years, hordes of finite sequences of signs with no signification have haunted the spaces of the foundations of mathematics and cognition. (Longo, 2005, p.1)

Over the last three decades, theories of embodiment have gained traction in mathematics education with researchers increasingly investigating the role of students' bodies in their thinking and knowing (for example Abrahamson et al., 2020; Alibali & Nathan, 2012; Gerofsky, 2013; Radford, 2014; Roth, 2016; Yoon et al., 2011). Although, embodiment research generally rejects the separation of mind and body in cognition, embodiment research often theorizes the body in different ways. As Roth (2011) explains, some embodiment research considers the body as a support for more abstract thinking or as a tool for learning or communicating. For Ingold (2013), thinking and knowing emerge from the correspondence between the dynamic flows of human and non-human materials in an ongoing process of growth and becoming with the world which he calls *making*. From this perspective, human movement is not a support or tool for thinking in the mind, rather movement, as part of the dynamic flows of materials, is essential for coming to know: humans think in movement.

The aim of the research in this paper is to explore undergraduate students' movement as they engage with a mathematical task through Ingold (2013) lens of *making*. Furthermore, this paper employs Laban's movement elements (Laban & Ullman, 1966/2011) as a framework to describe and analyze the students' movement. This paper addresses two questions: (1) How do students think mathematically in movement? (2) How do Laban's elements help inform research into students' movement?

The paper is divided into four sections: a brief background on movement and mathematics education research; the research design; analysis and findings; discussion and conclusion. The first section positions this study within mathematics education research, particularly undergraduate mathematics education research, offers a description of Ingold (2013) theory of *making* and an overview of movement research with an explanation of Laban's movement elements (Laban & Ullman, 1966/2011). The second section discusses the research design and presents the mathematical task given to the students. The analysis and findings section provides two fragments of the recorded session chosen for this study, delivers a thick descriptive analysis of the fragment using Laban's movement elements and discusses the students' mathematical engagement. The final section offers a discussion of the significance of the findings to undergraduate mathematics education research and teaching.

Background

This section discusses the widening scope of mathematics education movement research, exploring new conceptualizations of thinking and knowing, including Ingold (2013) *making*, and presenting a rationale for adopting Laban's movement elements (Laban & Ullman, 1966/2011) into this study.

Movement in Undergraduate Mathematics Education

For many students moving through compulsory schooling and into tertiary education, there is an increasing expectation to sit and write symbols on paper in the mathematics classroom. Further study of mathematics, for example at undergraduate level, seems to correspond with less movement and more written symbols. By undergraduate level, mathematics students are usually required to sit on chairs behind tables which not only restricts the students' movement but obscures movement from lecturers and other students. Thus, movement generally goes unnoticed and appears irrelevant in undergraduate mathematics. This practice of increasing student immobility at undergraduate level signals to students that movement and bodies are detached from mathematical thinking and knowing. Meanwhile, the increased writing of abstract mathematical symbols, for Longo (2005) above, the "hordes of finite sequences of signs with no signification" (p. 1), signals that mathematical thinking is to be carried out in a reified mind. In an undergraduate mathematics environment, it seems, bodies and movement are considered unnecessary to mathematical cognition.

Nevertheless, recent mathematics education studies are beginning to show that students, from grade school to undergraduate level and mathematicians and mathematics lecturers, all employ movement in their mathematical thinking. Furthermore, these studies link movement and cognition across a range of mathematical concepts including geometry, algebraic thinking, calculus concepts and complex value functions. For example, Nemirovsky et al. (2012) explore how undergraduate students use movement and touch to develop mathematical ideas about the complex plane. Yoon et al. (2011) investigate how secondary mathematics teachers' gestures during differentiation and integration tasks facilitate more complex mathematical understanding. Weinberg et al. (2015), illustrate how lecturers' employ gesture to create both specific and general mathematical meaning for undergraduate students during an abstract algebra lecture. Movement also seems to have a function for mathematicians as they explore mathematical concepts. For instance, Sinclair and Gol Tabaghi (2010) demonstrate how mathematicians recruit gesture and talk to convey temporality and motion into their thinking about the concept of eigenvectors. Furthermore, Oehrtman et al. (2019) demonstrate how mathematicians recruit movement extensively to explain both abstract and concrete (algebraic and geometric) aspects of complex-valued functions. Movement, it seems, is employed for mathematical thinking and knowing by novices and expert mathematicians across a diverse range of mathematical concepts. As mathematics education research into movement grows, researchers are questioning how gesture and movement matter in our understanding of mathematical thinking and knowing at university level with some research suggesting that we may need to re-examine the place of movement in mathematical cognition at undergraduate level.

Mathematics Education Movement Research

Mathematics education embodiment research is supported by a variety of quite different theories. Some mathematics education research, which developed from cognitive science and linguistics, explores how metaphors of the body and movement link physical experiences to language and concepts. Lakoff and Núñez (2000) describe these metaphors as embodied image schemas. Mathematics education researchers employing the ideas of embodied schemas have investigated how gesture and language create conceptual blends in metaphoric spaces (for example, Gerofsky, 2013; Yoon et al., 2011). Theoretical perspectives that have been more recently influenced by movement research include constructivism (Alibali & Nathan, 2012), sociocultural theory for instance, Radford (2014) multimodal sensory-motor activity sensuous cognition, and the biological perspective of enactivism (Maheux & Proulx, 2015). More recently, in movement research, theories have arisen which value non-human and human materials equally, De Freitas and Sinclair (2014) inclusive materialism extends embodiment beyond knowing in bodies to knowing emerging from body-material assemblages, and Roth (2016) recruits Ingold (2013) *making* to consider students emerging mathematical knowing as becoming. While all these theories recognise the entwinement of mind and body, they conceptualize the body in different ways.

Although all embodiment research considers the body as part of mathematical thinking, some of this research continues to position the body as an instrument of a reified, intellectual, knowing mind or as a learning tool to develop more abstract concepts in the mind (Roth, 2011). For instance, research into digital tools, currently popular in mathematics education embodiment research (Abrahamson et al., 2020), seems to embed movement as a learning tool. In general, students using digital tools must use explicit movements to produce the required digital outcomes (often on a computer screen). In their research into digital-user interfaces Swaminathan et al. (2009) note:

> if the user is forced to communicate through specific emblematic actions, context is by definition fixed by the system, as the user must focus on conforming his/ her actions to what he/ she knows the system can understand (p. 2).

As Roth (2011) explains, these activities might be considered embodied, since the body is moving and learning may develop. Nevertheless, thinking and knowing in the body and movement is more than movement awareness or a response to tools.

Over the last ten years, mathematics education research has begun to explore how bodies, movement and mathematical thinking and knowing are enmeshed in a non-hierarchical way (for example, Chorney, 2017; De Freitas & Sinclair, 2014; Ferrara & Ferrari, 2018; Oehrtman et al., 2019; Roth, 2016). Using inclusive materialism, De Freitas and Sinclair (2014) investigate students' diagrams and gestures to show how "the socioculturally immersed body is implicated in the knowing and doing of mathematics" (p. 50). Chorney (2017), and Ferrara and Ferrari (2018) investigate how mathematical knowing emerges from the entwinement of humans and materials through movement. In their research into mathematicians' thinking and movement, Oehrtman et al. (2019) establish not only how movement develops abstract concepts from concrete ideas but also how movement develops concrete concepts from abstract ideas. Furthermore, Roth (2016) uses Ingold (2013) making as growing to explore the emergence of new things as students and tangrams entwine. Thus, movement appears to be more than a means to develop students' abstract thinking, and more than a communication tool; as these studies show, movement is embedded in mathematical thinking and knowing.

Mathematical Thinking in Movement as Making

How then do we account for the process of mathematical thinking and knowing that incorporates movement and bodies? Ingold (2013) proposes a process of *making* in which new things (including mathematical thinking and knowing) emerge. For Ingold (2013), all human and non-human materials are animate, always transforming and evolving. From this perspective, both human and non-human materials are lines of flux in a continual process of becoming. As they flow, materials listen and respond to each other, entwining in an ongoing "dance of animacy" (p. 101) which Ingold calls correspondence. Rather than a reified, disembodied mind building representations of the world in ordered steps, Ingold proposes a process of *making* in which knowing emerges from the dynamic correspondence of material flows (including human flows).

To discuss human flows, Ingold (2013) turns to Sheets-Johnstone (2011) "thinking in movement" (p. 421). For Sheets-Johnstone (2011) and Ingold (2013), humans are primarily tactile-kinaesthetic beings, not reified minds enclosed in body containers. From before birth and before language, humans use their bodies, moving and touching to explore and come to know the world (Sheets-Johnstone, 2011). Learning, then, arises from spontaneous exploratory movements which are then reproduced and repeated to emerge as knowing in the body (Roth, 2011). Consequently, verbalisations, gestures, body movement and body position form a single, inseparable, cognitive whole (Roth, 2011). Instead of abstract thinking in a reified mind somehow isolated from, or contained within a body, animate bodies think and know by moving and responding in their never-ending engagement with the world (Ingold, 2013). Learning does not involve "matching the contents of a mind with objects in the world" (Ingold, 2018, p. 25) but emerges from engagement with the world in the dynamic process of *making* which includes, for humans, thinking in movement.

Ingold (2013) *making*, then, offers a theory enmeshed in movement from which to research students' emerging, entwined movement and cognition. Extending the mathematics education research of Roth (2016) and Ferrara and Ferrari (2018), this paper recruits Ingold (2013) *making* as a theory from which to explore students' emerging thinking in movement as they engage with a mathematical prompt.

Movement Research

Movement analysis has a long history and is well-established in many fields, for example performing arts, sport, cognitive science, neuroscience, anthropology, time-motion studies, computer science and robotics (Bernardet et al., 2019; Challet-Haas, 2016). At least since the fourteenth century, primarily in Europe, over eighty movement classification systems have been devised (Challet-Haas, 2016). Some systems only categorize specified areas of the body (e.g. face or upper body); while other systems include full body movements (such as, kinesics, Benesh movement notation, and Laban's elements) (Challet-Haas, 2016; Moore

& Yamamoto, 2012). Although a growing number of new motion capture tools has supplanted the recording function of movement classification systems, many of these systems continue to be used for analysis in movement research.

At first, mathematics education movement research generally focused on upper body gestural movements, often recruiting McNeill (1998) gestural classification system (e.g., Alibali & Nathan, 2012; Radford, 2014; Yoon et al., 2011). Over the last ten years, perhaps facilitated by the increased use of digital tools or less traditional classroom environments, mathematics education researchers have begun to look beyond gesture to investigate more full-body movements (see for example, Abrahamson et al., 2020; Gerofsky, 2013). Nonetheless, in mathematics education, full-body movement research recruits few of the widely available movement systems. Instead this type of research usually employs descriptive accounts of movement which report which body part/s move where in space over time. A few mathematics education researchers have begun to incorporate movement frameworks into their research. For example, Gerofsky (2013) proposes using the "Elements of Dance" framework derived from Laban and Ullmann (1966/2011) and well-known in dance curricula worldwide (for example, Ministry of Education, 2007; National Coalition for Core Arts Standards, 2015). Ferrari (2020) and Ferrara and Ferrari (2018) incorporate Sheets-Johnstone (2011) dynamic kinesthetic qualities into their research which, as Ferrara and Ferrari explain, enabled them to gain deeper insight into students' mathematical activity.

Both Laban and Ullmann (1966/2011) and Sheets-Johnstone (2011) frameworks account for body, space and time aspects of movement. Furthermore, these frameworks describe qualitative aspects of movement which Laban calls effort actions and Sheets-Johnstone calls dynamic kinaesthetic qualities. These qualitative aspects of movement, which are referred to in this paper as dynamic movement qualities, describe how it feels to perform a movement and how a movement makes an observer feel. For example, pointing towards a page might be performed with sudden and direct qualities (like a jab) or light and free qualities (like a brush). These movements feel different to perform and different to observe: a sudden direct jab might feel and appear as an emphasis or direction, whereas a light, free brush might feel and appear to be a dismissal. Therefore, applying different dynamic qualities to the same body part (an arm), moving along the same trajectory (towards a page), changes both how it feels to perform a movement and how it feels to observe the movement. Although Laban and Sheets-Johnstone agree dynamic qualities are embedded in all movement, these qualities are mostly missing from the descriptive accounts of movement found in mathematics education research. Furthermore, Sheets-Johnstone insists that to fully understand movement researchers should map not only spatial and temporal aspects of movement but also the dynamic qualities of kinesthesia.

Laban's Movement Framework

An ongoing difficulty for movement research is how to present analysis of dynamic three-dimensional performance in static two-dimensional written form. Indeed, as Ferrara and Ferrari (2018) note, it is difficult to "capture movement without reducing it" (p. 425).

Although observers can identify beginning and ending body positions, and positions in between, movement is not a sequence of changes in positions but the process and awareness of the body as it changes position (Moore & Yamamoto, 2012). Laban elements (Laban & Ullmann, 1966/2011) and Sheets-Johnstone (2011) kinaesthetic dynamics, offer quite similar ways to capture how the body moves through space in time and the dynamic qualities of movement. Nonetheless, researchers from a wide variety of research fields favor Laban's elements as these provide a reasonably detailed and not too time-consuming method for classifying movement (Bernardet et al., 2019; Moore & Yamamoto, 2012).

Laban elements (Laban & Ullmann, 1966/2011) are well-established with substantial use and development across a wide variety of research fields. Furthermore, Laban's elements have been used in a variety of non-dance fields, for example industrial time-motion studies, physical rehabilitation, movement research, psychology research and education (Moore & Yamamoto, 2012). More recently, digital-human interaction technologies (such as robotics, computer interaction analysis and motion capture) have employed Laban's movement elements to develop better understanding of human movement and ways to improve digital human interfaces (Bernardet et al., 2019). Consequently, although a few mathematics education researchers have employed Sheets-Johnstone's kinaesthetic dynamics (Ferrara & Ferrari, 2018; Ferrari, 2020) for their research, this paper recruits Laban's elements (Moore & Yamamoto, 2012) as a method to describe and analyse movement.

Over time, a variety of slightly different movement frameworks have developed from Laban and Ullmann (1966/2011) elements. For example, Laban added effort actions to his original framework and Laban Movement Analysis was developed as a more formal system of classification requiring trained practitioners (Moore & Yamamoto, 2012). Regardless of these changes, Laban frameworks generally describe three movement elements (see Table 1): the body (parts and actions); space (reach and direction); and effort (force and timing) (Moore & Yamamoto, 2012). As Laban and Ullmann (1966/2011) and Sheets-Johnstone (2011) note, the performer usually does not, and cannot, easily separate movement into these elements. Moreover, some movement effort actions are difficult to perform separately, although some are also difficult to perform together. For instance, light and free dynamic qualities are usually performed together, whereas executing light movements that are also bound is quite difficult.

The analysis I present in this paper, employs a Laban framework from Moore and Yamamoto (2012) (Table 1). Although the framework in Table 1 has been slightly modified, it describes Laban and Ullmann (1966/2011) three basic elements, body, space, and effort. The effort element is relabeled as dynamic qualities to reflect Sheets-Johnstone (2011) dynamic kinaesthetic qualities and to better express the more qualitative nature of this element of movement. Moreover, the framework reverts to some of the original Laban and Ullmann (1966/2011) descriptors where these are considered clearer than Moore and Yamamoto's descriptors.

For Laban and Ullmann (1966/2011) effort actions/ dynamic qualities lie on a continuum between the descriptor extremes (Table 1). For example, pressure (weight) varies along a continuum between light and strong. Although variations along these continua might be easy to perform, sometimes finding words to describe these subtle variations in performance qualities is difficult. Furthermore, dynamic movement qualities are rarely performed in isolation so that a sudden movement (for

Table 1 Laban's movement elements and sub-elements modified from Moore and Yamamoto (2012)

Basic Element	Sub-elements	Description
Body	Gesture	Action of a single body part
	Posture	Position or action of whole body
	Initiation	Part of the body in which movement begins: • distal (appendages e.g. arms, legs, head) • central (from the central torso) • upper or lower (above or below the waist)
	Sequence	The movement order over time
Space	Location and relationships	General space
		Space between people and things
		Personal space (kinesphere)
	Pathways	In three-dimensional space traces are described: • along lines or curves • through planes and spirals • though volumes
	Dimensions	The three planes of movement: • vertical (up/ down, rising/ falling i.e. against/ with gravity) • horizontal (left/ right, wider/ narrower) • sagittal (forward/ backwards, advance/ retreat)
	Sequence	The movement order through space
Dynamic qualities (Effort)	Focus (how the body attends to environment)	Indirect – Direct
		e.g. multiple – singular, meandering – pinpoint
	Pressure (weight)	Light –Strong
	Time (urgency)	Sustained – Sudden
		e.g. decelerating – accelerating
	Flow (control)	Free – Bound
		e.g. flowing freely – held back

example a jab) is usually also direct and bound. As Laban explains, the continua of dynamic movement qualities mean "the possibilities of combination go on endlessly" (p. 35). As language is post-kinetic (Sheets-Johnstone, 2011), finding appropriate words to describe movement qualities is at times difficult, so that researchers sometimes recruit their own words to explain observed dynamic qualities (Moore & Yamamoto, 2012). In this paper, the dynamic qualities of movement are described using the Table 1 descriptors with additional words enlisted as necessary.

Research Design

This paper is part of a larger study towards a doctoral thesis which explores movement in students' mathematical thinking and knowing and follows on from ideas first addressed in Gandell and Maheux (2019). The aim of the research in this paper is to explore the movement of a group of students as they engage with a mathematical task. Consequently, this research is concerned more with how students' movement and verbalization evolve rather than any learning outcomes from the activity. As Roth (2016) clarifies, in focusing on finished objects, for instance diagrams, written symbols and verbal explanations, researchers may overlook "all those aspects of mathematical making that are not represented by the final product" (Roth, 2016, p. 92).

The research in this paper recruits Ingold (2013) *making* to focus on students' emergent thinking and knowing in movement. Thus, the analysis follows the flows of students forward, as Ingold (2013) suggests, to actively respond to the students flows, to "think through the observations" (p.11) and to focus on the emergent process of students' mathematical thinking in movement. Geertz (1973) thick description offers a way to implement Ingold (2013) advice to follow, observe, think and respond to the students' flows. Thick description is the study of broader and more abstract ideas through an intense and in-depth focus on smaller events (Geertz, 1973). By analyzing small matters in detail, thick description aims "to draw large conclusions from very small but very densely textured facts" (Geertz, 1973, p. 28). Furthermore, observing, recording and analyzing are not distinct activities but entangled in any thick description. Consequently, the research in this paper adopts a thick description approach, repeatedly viewing and reviewing small sections of the audio-visual recordings of the students as they engage with a mathematical prompt. By using a thick description method, this paper follows the students' movements forward critically observing, responding and paying attention to detail while remaining open to different and new readings.

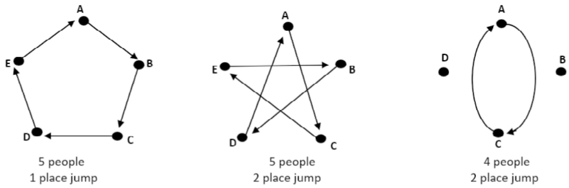
Working together, students offer their actions, including their movements and verbalisations, to each other as indications of their knowing (Roth, 2016). Any actions, made available to the group, can then be used by researchers as a representation of the students' knowing without making assumptions about students' intentions or thoughts (Roth, 2016). Thus, the research in this paper focuses on analysing the verbalizations and movement the students provide for each other as they engage with a mathematical task.

The Setting

In this paper, I draw on a previous study, from Gandell and Maheux (2019) which investigated how students mathematically problematize together over time. Mathematical problematizing is considered from Maheux and Proulx's (2015) perspective, as the students' posing and solving of smaller self-generated problems in response to a mathematical task.

The mathematical task selected for this research, involved a game of throwing a ball to be made into a dance (Fig. 1). In the first five minutes of the hour-long session the researcher led the participants in a movement warm-up and then participants were invited to move around the large open room which contained several vertical whiteboards, with whiteboard markers and magnetic counters, around the edges but no tables or chairs. Consequently, the environment and task were intentionally designed to elicit, but not prescribe, students' verbalization and movement.

Nic watches a game where a ball is being thrown around a group of people in a clockwise direction. The number of people in the group is called the *people number*. Each time the ball is thrown in a game it is thrown in equal size *place-jumps*. Each person throws the ball to the person on their left the same number of *place-jumps* away. When the ball gets back to the first person the game ends.



In some games (like the 5-people 1-place-jump game and the 5-people 2-place-jump game) Nic notices that all the people throw the ball. In other games (like the 4- people 2-place-jump game) only some people throw the ball. Nic wonders whether everyone gets to throw the ball in a 4-people 3-place jump game and a 6-people 3-place-jump game.

Nic wants to make a dance using this game with people moving between each of the positions instead of the ball being thrown. Nic wants to know if everyone gets to move for different size *people number* and *place-jumps*. Create and explain a shortcut that Nic could use for any size of *people number* and *place-jump* size. Present this shortcut in the last 5 minutes of the session.

Fig. 1 The mathematical task

A group of four students, who volunteered to participate in the study, were invited to a session which was not part of their usual course or programme. These participants were non-mathematics major students aged 18 to 22 enrolled in a six-month tertiary undergraduate bridging programme. Bridging programmes offer courses and further credits which students need to enter tertiary degree and diploma programmes. Students in bridging programmes might be returning to study after employment or may be school leavers who do not meet the entry requirements for some programmes (for example, a student needing further algebra knowledge to gain entry to an engineering degree). A small but growing number of undergraduate mathematics education research studies focus on non-mathematics major and bridging students, for example, two special issues in IJRUME (Biza et al., 2022; Pepin et al, 2021) and the Learning and Teaching of Calculus Across Disciplines Conference (Dreyfus et al., 2023). As these research papers and conferences show, students across many non-mathematics major disciplines, such as engineering, physics, economics, chemistry and biology, engage with mathematics specific to their discipline. Nevertheless, there appears to be less undergraduate mathematics education research into more vocation focused programmes such as nursing, architecture, surveying, construction and business. Although the mathematics in many vocational non-mathematics major disciplines may focus on different mathematical concepts to those in mathematics-major and academic disciplines, for example number and proportional thinking in nursing, many students in vocational programmes study mathematics in their undergraduate courses. Therefore, undergraduate mathematics education research may need to further consider the mathematics encountered by undergraduate students in non-mathematics programmes.

Recording and Transcription

Audio-visual recordings include less inherent bias than written records, notes, and after-the-fact recollections which often suffer from a loss of micro-detail (Lemke, 2007). To record movement throughout the open room, I positioned three video cameras around the edges of the room. My transcription of the movement and verbalizations from these video recordings included still images from the video recordings to improve the clarity of the movement descriptions. Although transcriptions may help inform the choice of fragment for thick analysis, as Derry et al. (2010) explain, all transcriptions necessarily embed the transcriber's personal evaluation. To mitigate this personal judgement, I followed Derry et al. (2010) advice, repeatedly viewing and iteratively analysing the recordings. Furthermore, I paid attention to the details of the fragment, as I worked to obtain an increasingly reliable record. As Roth (2007) concedes, although there is some unavoidable bias in analyzing any audio-visual recording these biases generally remain consistent throughout the analysis.

The fragments that I present in the "Analysis and Findings" section of this paper were selected for analysis, because the students in them perform a rich variety of movement sequences including some movement sequences not previously observed in this session. Furthermore, in only six lines of generally repetitive verbalization, the students perform several diverse movement traces, so that the students' movement in this fragment appears more enmeshed in the students' mathematical problematizing than their verbalizations. Moreover, my interest was caught by one student performing a large body movement following his previously small movement traces. For the thick description analysis, the three video recordings of the selected fragments were repeatedly viewed alongside the transcriptions. Following Roth (2016), the analysis here uses the verbalization and movement students make available to each other - and, therefore, to observers - rather than guess at students' motivations.

Transcripts which included all Laban's elements (Table 1) became dense and difficult to read. Thus, the transcripts of the fragments that I present in this paper include students' verbalizations and only the body and space movement elements while the analysis offers more in-depth descriptions using all Laban's elements. Structuring the fragments in this way reduces the density of the transcript and allows a more holistic reading of the fragment. Column 1 of the transcript provides numbers for the sequence of verbalizations and/ or movement traces as they occur, with any simultaneous movement and verbalization described in the same line. Movement sequences that continue beyond a simultaneous verbalization are kept within the line number in which they started.

The description column in Table 1 is used to identify and describe how students perform the sub-elements of their movement. As many of the students' movements in this research belong to the gesture sub-element, the movement transcript generally describes body part along with pathway, and sometimes location of the movement. The sequence for both time and space are provided in the same way as the verbalization transcript, through the sequence of paragraphs and line numbers. Location and relationships between the students and other materials are identified in Fig. 2 with any changes in these sub-elements presented in the transcript and the analysis. As

Fig. 2 Kit, Chas, Ala and Paige at the whiteboard

descriptions of the dimensions and dynamic qualities sub-elements can be lengthy, these sub-elements are most often included in the analysis.

Still images (Figs. 3, 4, 5, and 6) are provided below the transcripts to illustrate the movement descriptions. Any depiction of three-dimensional movement on static, two-dimensional paper, requires removal of one dimension and all the movement. Researchers have developed a variety of methods to illustrate movement on paper including sequences of still images. For this paper, I annotate images from the video recordings with arrows to denote movement. For Ingold (2013), lines drawn by the moving hand contain the dynamic qualities of that movement, so that, although drawn by a moving hand with a cursor, these annotated arrows may help reanimate the still images.

The Mathematical Task

The task (Fig. 1) presents a game in which a ball is thrown clockwise around a group of people until the ball is returned to the first person when the game stops. Both the number of people in the group, and the number of place-jumps (people skipped over) between throws, can be changed. For example, in a four-person one-place-jump game the ball is thrown to each of the four people clockwise in turn. In a four-person two-place-jump game, the third diagram in Fig. 1, the ball is thrown to the next but one person clockwise around the group so only two people throw the ball.

Mathematically, this task can be explored as a modular arithmetic problem. Renaming the positions in the diagram A, B, C, D … as 0, 1, 2, 3 … *n-1*, the number of people in the group is the modulus, *n*, and the place-jump number, *m*, is repeated addition of *m*. In this way, each throw goes to the position in the group which is

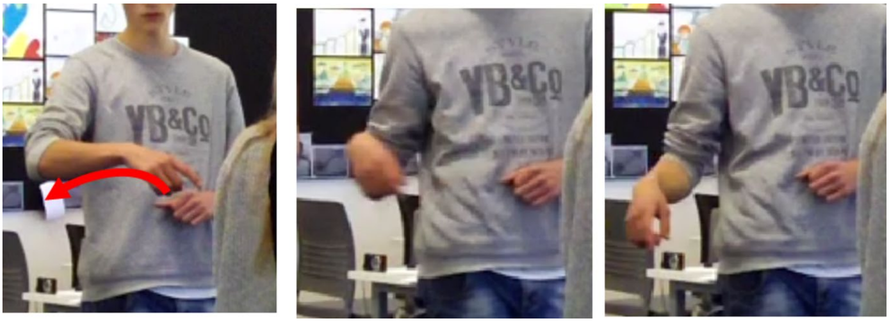


Fig. 3 **a** Kit dabs position one, **b** dabs position two, **c** dabs position three

a multiple of m. For example, in a five-person three-place-jump game, the throws are to multiples of 3 mod 5. Since the first multiples of 3 are 3, 6, 9, 12, 15, the sequence of throws from A goes D, B, E, C, A or the numbers 3, 1, 4, 2, 0. As these are all the numbers in modulus 5, everyone gets to throw the ball (or move/swap places for the dance in the second part of the task). A general rule to find in which games "everyone gets to move", could be stated as: all the games where the people number and place-jump number don't share a common factor.

This task is taken from an undergraduate course in mathematics teaching programme and was used over several years to teach modular arithmetic and group problem solving. Since the participants in this research engaged with the mathematical prompt for fifty minutes, producing a partial final solution, the task seems to provide an appropriate mathematical challenge for the participants. The last two sentences of the task were changed to provide a focus for the participants' solutions, namely Nic who is interested in the outcome, and to encourage the participants to move. Nonetheless, since my research was not part of my participants' usual courses or programme, I was not concerned with learning goals or the students' final solutions.

Trustworthiness and Rigor

For Ingold (2013), researchers, as part of the world, cannot somehow stand outside the world and impartially describe it. Derry et al. (2010) consider that trustworthiness of research is increased when researchers clearly describe in detail what is

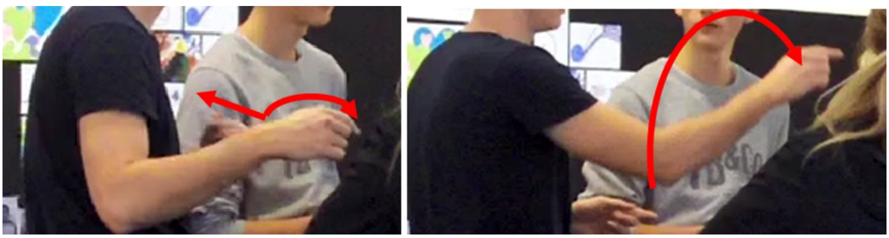


Fig. 4 **a** "instead of swapping" **b** "you go around"

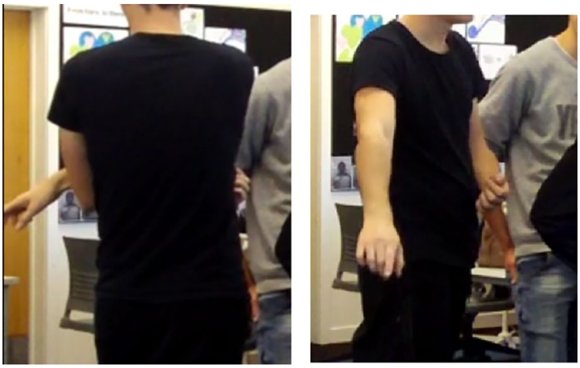
Fig. 5 **a** over and under rolls, **b** spiral around vertically, **c** final held position

recorded and why, and how findings are drawn from the analysis. To establish the trustworthiness and rigor, of my analysis, then, I thoroughly explain the underpinning theory, Ingold's *making*, carefully describe the research elements, and present a detailed analysis using thick description and Laban and Ullmann (1966/2011) movement elements. In addition, the thick description of transcripts occurs after repeated and exhaustive viewing of recordings. Furthermore, Laban's elements are employed to provide detailed movement descriptions, and screenshots are included to further illustrate the students' movement. Moreover, during analysis, the video-recordings of the fragment and discussion of the emerging findings were shared with other mathematics and dance educators to corroborate my analysis and findings.

Any research is also influenced by the researcher's personal history (Denzin & Lincoln, 2017). My history includes experiences as a teacher of mathematics and dance, a dancer, a physiotherapist, a student and a mother. Following Ingold (2013) *making*, these influences correspond with the research as a part of the research apparatus and provide the depth of knowledge in movement and mathematics that is needed for my research design, analysis and discussion.

The first fragment that I present here, starts thirteen and a half minutes after the beginning of the session. It is followed five seconds later by the second fragment. Although the session begins with the researcher leading a movement warm-up, I have no further contact with the students during the session except to remind the students, at ten and five minutes before the end of the session, to present their solution.

Fig. 6 **a** "from there" **b** "to the next person"



Analysis and Findings

In the Beginning

After the warm-up in the centre of the room, the four students walk to, and gaze at, the whiteboard which holds the task sheet. At first, the students initiate a short discussion about what place-jumps means and then use the magnetic counters to attempt a few games on the whiteboard. Next, the students alternate between gazing at the vertical whiteboard and task sheet and acting out two games from the task in the open area of the room. The first enactment the students perform is a four-person three-place-jump game using a counter as a ball. The second enactment the students perform is a six-person three-place-jump game, which they call a "six-three". They perform this six-three as a dance (see Nic's request in the task). Prior to acting out this dance, Kit utters "like we're swapping [places] instead of throwing". The students then return to the whiteboard, with one person uttering "that one doesn't work". The group stand quietly for a few minutes (Fig. 2) gazing at the whiteboard. Kit then utters "so y equals x plus or minus or times whatever …" and, after pausing a few seconds, he performs the following verbalization and movement.

A Mathematical Solution in Movement

1	*Kit*	*A six-two for everyone to move I think*	*Right arm inscribes a circular path in front and to his right side with his right index finger pointing to three discrete positions. He first gazes between Chas and Ala then towards Ala and Paige*	Fig. 3a, b and c
2	*Kit*	*Cos here we've got a …*	*Steps towards the whiteboard and places left index finger, with left arm extended, on to middle diagram on the task sheet*	
3	*Kit*	*oh wait*	*Brings left hand back to his face and moves weight to back leg*	

In Line 1, Kit verbalizes that everyone will move for a six-person two-place-jump game. A six-person two-place-jump game/dance represents 2 mod 6 which has three possible solutions (2,4,0). Since only three people will throw/move in this game, Kit verbalizes an incorrect solution. Simultaneously, Kit performs a movement with his right arm and index finger indicating three places in a roughly horizontal plane (Fig. 3a, b, c).

The three places Kit performs are made clear by changes in pathways and dynamic qualities. Kit moves his right index finger through three successive vertically curving pathways with a light, sustained, partly bound dynamic quality (Fig. 3a, b, c). The end of each vertical curve is performed with a more sudden, direct, bound movement, a combination which Laban and Ullmann (1966/2011) calls a dab. Each dab occurs in a different place but generally on the same horizontal plane. In this way, Kit's movement differentiates three places on a horizontal plane by varying the elements of space (vertical curves) and dynamic qualities (light and sustained curves with sudden and bound ends). Furthermore, the overall movement inheres a sudden, indirect quality with Kit's gaze towards the group rather than his hand. Kit's movement, then, feels more like a sketch of three places than a direction to specific positions, or a labelling of the multiples, in the game.

By performing three places, Kit offers a movement which contradicts his verbalization that everyone moves in a six-two game. Rather, Kit's movement performs a mathematical solution indicating three multiples for 2 mod 6. Kit's movement reveals three unspecified but distinct end points to demonstrate that only three positions, not everyone, moves/throws the ball in a six-person two-place-jump game. This movement answers the question the task asks of each game, whether all six numbers of mod 6 satisfy 2 mod 6. Therefore, Kit's movement provides a mathematical solution for the task, namely only three people, not all six, move in a six-person two-place jump game: the solution for 2 mod 6 is three numbers (0,2,4) not all six numbers of mod 6 (0,1,2,3,4,5).

For Ingold (2013), referencing Sheets-Johnstone (2011), humans think in movement. As Ingold (2013) explains thinking in movement is "not to think by *means* of movement, or to have our thought *transcribed into* movement. Rather the thinking is the movement" (Ingold, 2013, p. 98). Thinking in movement, therefore, does not require students to learn movements to produce answers, or to pre-plan movements to express ideas: thinking in movement is the unplanned, spontaneous and improvised activity of a dynamic thinking body. Kit's three-position movement in Line 1 (Fig. 3a, b, c) seems to be an instance of this thinking in movement. Kit's movement does not correspond to his verbalized solution and the movement does not appear to be a pre-planned or a pre-thought embodiment of that verbalization. Since this is the first appearance of a vertical curve and dab movement in this session, Kit is not reproducing a previously performed and known movement. Thus, in Line 1, a new mathematical movement has emerged from a moving dynamic body as a sketch of an answer to a mathematical prompt. Kit's performance of this mathematical solution in movement demonstrates thinking in movement as Sheets-Johnstone (2011) and Ingold (2013) describe.

Evolving Thinking in Movement

4	*Chas*	*The trouble is instead of swapping*	*Elbows bent index fingers touching, right finger traces horizontal curve forwards, left hand traces straight line backwards and up*	Fig. 4a
5	*Chas*	*you go around*	*Spiral trace with right arm across left, up and forwards*	Fig. 4b
6	*Paige*		*Lifts hands to hips as Chas verbalizes*	
7	*Paige*	*Yeah that's what I was thinking*		

8	*Chas*	*it goes around to the next person*	*Touches left and right index fingers in front midline. Right index finger and arm trace a spiral - left, forward and up across the midline of his body*	
9	*Chas*	*instead of swapping*	*Index fingers touch front midline, right hand traces lines forward, left hand traces line backwards*	
10	*Chas*		*Right hand traces curve backwards under left hand, left hand travels forwards, then right hand traces curves forwards under left hand going back*	Fig. 5a
11	*Chas*	*So that's the swap*	*Right and left hands trace horizontal circles around each other*	Fig. 5b
12	*Chas*		*Holds final position with index fingers pointed upwards for 2 seconds*	Fig. 5c
13	*Chas*	*From there …*	*Rotating his torso and turning his head to the left, right arm with index finger extended points to left side by hip*	Fig. 6a
14	*Chas*	*goes around to …*	*Right arm traces horizontal circle around body rotating torso to point to right side*	Fig. 6b
15	*Chas*	*… the next person*	*Holds arm extended to right and looks back to Kit*	

Thinking in movement arises from a body that resonates with the world (Ingold, 2013; Sheets-Johnstone, 2011). Rather than reified minds creating symbols and representations of the world, new mathematical thinking and knowing, emerge as tactile-kinaesthetic bodies correspond with dynamic material flows. Consequently, thinking in movement is emergent, evolving and adapting to an ever-changing environment, and may take many different forms (Sheets-Johnstone, 2011). In Lines 4 to 15, Chas demonstrates this evolving mathematical thinking in movement as he corresponds with the flows of the mathematical task and other students.

Throughout lines 4 to 15, Chas engages with a problematization which has appeared several times in this session: the relationship between counting around place-jumps and the throwing/ moving positions in the game. Viewing the task games (Fig. 1) as modular arithmetic, counting around place-jumps is a repeated addition of the place-jump number and the throw/ move positions are the multiples of the place-jump number. Counting around as repeated addition, then, offers a way to find the throw positions (multiples) in any game. Although multiplication as repeated addition seems a basic skill in mathematics, modular arithmetic presents the students some conceptual differences. As Schüler-Meyer (2019) describes, students struggle in modular arithmetic to transition from familiar, school-based concepts and processes, such as multiplication and division, into more tertiary discourses of number properties and relations. Some of conceptual difficulties met by students may include: the set of multiples cannot exceed the modular number; each modular number has a finite set of multiples; and the sequence of multiples may create unexpected patterns. For example, 4 mod 5 has only five multiples and these appear in a decreasing sequence 4, 3, 2, 1, 0 instead of the increasing and infinite pattern in multiplication of 4 (4, 8, 12, 16, 20…). Moreover, this group of students have previously enacted a five-person four-place-jump game in which they discussed the disparity between the clockwise direction of counting around place-jumps with the anticlockwise direction of the ball throws (from position 4 to 3 then 2 then 1 then 0). Hence, in Lines 4 to 15 Chas corresponds with the flows of the task and the group's previous problematizations to explore how repeated addition (counting around) relates to finding multiples (throw positions) in modular arithmetic.

In Lines 4 and 5 (Fig. 4a and b), Chas verbalizes and performs two movements which he differentiates both verbally and by using different movement elements. Chas begins in Line 4 with the index fingers of both hands touching in front of and in the midline of his body. He traces a short line towards himself with his left hand and a small curve away from himself with his right hand (Fig. 4a) uttering "instead of swapping back". For the second movement, in Line 5, Chas begins as in Line 4 but performs a larger spiral trace with his right arm (Fig. 4b) uttering "you go around". The small *swap* movement (Fig. 4a) is performed with two hands moving apart, tracing sagittal (forward and backward) and vertical pathways, and with sudden, direct, slightly bound dynamic qualities. In contrast, for the larger spiral *around* movement (Fig. 4b), Chas' right arm traces a path to the left, across his midline, then up and forward, through horizontal, vertical and sagittal planes with a sustained, indirect, light, dynamic quality. In Lines 8 and 9, Chas repeats his *swap* and *around* movements but performs the movement and verbalization from Line 5 in line 8 before the movement and verbalization from Line 4 in Line 9.

Both *swap* and *around* movements appeared to have emerged from previous student activities. The students performed continuous circular traces, like Chas' Line 5 *around* movement, when they enacted games from the task. These circular traces began as pointing to discrete positions in the game to count around place-jumps (repeated addition). This discrete pointing movement transformed into the performance of continuous circular movement traces indicating direction for counting but not positions. Chas' *around* movement, then, appears to have evolved from counting around places (repeated addition). The short direct movement in Line 4, which Chas calls swapping, seems a new movement, although the word swap seems to reference the previous enactment of six-person three-place-jump game as a dance in which the students swap places instead of throwing a ball. The swapping positions, and Chas' *swap* movement, then, are multiples of the place-jump number. In this way, Chas *around* movement appears to reference counting around place-jumps (repeated addition); whereas his *swap* movement references the throw/ move positions (multiples).

From Line 9 Chas carries his movement directly into the Line 10 movement. He begins with his hands slightly apart and, without verbalizing, traces small semi-circles with his hands by rolling his hands over and under each other (Fig. 5a). With sudden, light and more indirect dynamic qualities, Chas' right hand curves down and towards his body under his left hand while his left hand curves up and away from his body over his right hand. This movement is then reversed with the right-hand curving under the left away from his body and the left hand curving up towards his body over his right. Chas then immediately begins the movement and verbalization in Line 11 (Fig. 5b), circling his right hand to the right and away from his body while his left hand circles to the left and towards from his body. Uttering "so that's the swap", the circle movements are reversed (Fig. 5b), with the right hand moving towards his body and left hand moving away. Although Chas changes the orientation of his hands in Line 11, so that the under and over movement (Fig. 5a) becomes a side-to-side movement (Fig. 5b), he continues the light indirect dynamic qualities and semi-circle pathways from Line 10. Chas finally stops and holds the strong, bound position in Fig. 5c (Line 12).

Through Lines 9 to 11, Chas appears to evolve his *swap* movement, while his accompanying verbalizations repeat "swap" or "swapping". This evolving sequence of movements starts, in Line 9, with a repetition of the Line 4 movement. This movement changes in Line 10 with the straight pathways becoming semi-circles, the hands circling around each instead of moving apart and the sudden, direct, slightly bound qualities transforming into sudden, light, indirect qualities. The evolution continues in Line 11, where in Line 10 the hands roll over each other through vertical and sagittal planes (Fig. 5a), in Line 11 the hands circle sideways to each other through horizontal and sagittal planes (Fig. 5b). Although Chas performs three movements that travel along different pathways and through different planes in Lines 9, 10 and 11, all the movements retain similarities: the movements are small, use both hands and inhere sudden dynamic qualities.

With the gradual transformation from straight single dimension pathways to curved multi-dimension pathways and from direct and bound to a less direct and freer movement, the *swap* movement appears to incorporate some elements of the spiral *around* movement (Fig. 4b). The spiral *around* movement, although performed with one arm, travels a more circular pathway, moves through all three planes and inheres free, more indirect and sustained dynamic qualities. Nonetheless, while Chas' movements evolve from Line 9 through to Line 11, he continues to verbalize these movements as a swap. Therefore, without much change in his verbalization, Chas evolves three *swap* movements which bring elements of his *around* (repeated addition) movement into his *swap* (multiples) in movement: Chas, it seems, is thinking in movement to resolve repeated addition and multiplication in modular arithmetic.

After a brief pause in Line 12, Chas traces a large semi-circle around himself. Chas begins in Line 13 by rotating his body left and turning his head to gaze and point with his right arm at a position on his left side (Fig. 6a), uttering "from there". In Line 14, Chas rotates his torso right tracing a pathway around and in front of his body in an almost flat horizontal plane with a light, sustained, direct dynamic quality while verbalizing "goes around to". Chas finishes the circular movement in Line 15, pointing to his left side with his right arm extended, gazing at Kit (Fig. 6b) and uttering "the next person". In this final movement (Lines 13 to 15), Chas seems to complete his combination of elements from his *around* (repeated addition) and *swap* (multiplies) movements. Elements from the *around* movement in Lines 4 and 8 are retained in the large, curved pathway, the light, sustained dynamic quality, and the use of one arm. Whereas the more horizontal, semi-circular pathway and the more direct dynamic qualities inhere elements of the Line 9 to 11 *swap* movements.

Chas' *around* and *swap* movement sequences and verbalizations in Line 4 and 5 seem to represent the two separate problematizations of repeated addition (*around*) and multiples (*swap*) in the modular arithmetic game as a dance. Through Lines 4 to 15, Chas gradually evolves what begin as two distinct movement traces, in Lines 4 and 5, to a single movement in Lines 13 to 15. Through this transformation of movement traces, Chas attempts to resolve the differences between repeated addition and multiples in modular arithmetic. Moreover, this semi-circle trace, which is now both counting around (repeated addition) and swapping places (multiples) is supported by Chas' final verbalization "from there around to the next person" which also incorporates counting around (repeated addition) and position (multiples). Thus, Chas is thinking in movement. He transforms and evolves his movement to perform a resolution the problematization of how repeated addition (counting around) relates to multiples (throw/ swap positions) in modular arithmetic. In this way, Chas performs and verbalises repeated addition (counting around) as the way to find multiples (the next person to throw/move) in this modular arithmetic task.

By adapting and evolving his movements, Chas shows how thinking in movement may evolve and change in an animate body. Rather than thinking in a mind isolated from the body creating "signs with no signification" (Longo, 2005, p. 1), Chas' evolving movement sequences demonstrate how an animate body resonating with the world (here the task, the group and their activities) can resolve mathematical problematizations. Furthermore, Chas shows how verbalizations may lag behind the resonant body in grappling with mathematical tasks.

Discussion and Conclusion

For a long time, mathematics education has often de-privileged movement and the body in mathematical thinking and knowing.). In much mathematics education, particularly as students progress to tertiary study, the body and movement are often considered unnecessary for students' mathematical cognition, with written signs and symbols holding sway as evidence of mathematical thinking. Nevertheless, humans move and know the world through tactile kinaesthetic bodies; they cannot remove themselves from their bodies and think in some disembodied mind (Sheets-Johnstone, 2011). Increasingly, research demonstrates that mathematicians and undergraduates use movement in their mathematical thinking and knowing across a range of mathematical concepts. Movement, it seems, is enmeshed in the mathematical thinking and knowing of undergraduate students.

This paper focuses in depth on fragments from the activity of a group of undergraduate students engaged with a modular arithmetic mathematical task. The data and analysis presented in this paper illustrates not only that movement is integral to mathematical thinking and knowing; but also, that students think mathematically in movement. In the first fragment presented in this paper, Kit performs a movement sketch which indicates 2 mod 6 has three multiples, rather than six as suggested by his verbalization. Kit's movement illustrates how students' moving bodies can perform mathematical thinking: thinking that may not necessarily be articulated, expressed or made available in any other way except in movement. In the second fragment, Chas addresses the problematization of the relationship between repeated addition and multiples in modular arithmetic. By combining the movement elements from two distinct movement traces, Chas evolves the movement traces of repeated addition and multiples into a single movement to resolve how repeated addition finds multiples in modular arithmetic. Therefore, students' mathematical thinking in movement emerges and evolves as students correspond with the world (here the task and other students). Thus, the findings from this paper demonstrate, that rather than moving to support verbalizations or thinking in the mind, students think mathematically in movement.

Use of Laban's elements (Laban & Ullmann, 1966/2011; Moore & Yamamoto, 2012) throughout this analysis provides a way to pay attention to the students' movement and a way to discuss the performance of mathematics. The detailed observation offered by Laban's elements allowed this paper to reveal the sketch-like nature of Kit's three position movement. Furthermore, Laban's elements provided a way to follow Chas' evolving thinking in movement, differentiating Chas' two initial problematizations in movement as they emerged, then offering a way to describe the transformation of these movement traces as elements from both merged into Chas' final movement. As these descriptions and analysis show, Laban's elements offer mathematics education movement researchers a useful analytical tool for examining students' movement in depth.

In this study, students were invited but not required to move. Certainly, the task and environment encouraged movement, but the students could choose, as they sometimes did, to engage with the task by standing at the whiteboard. Furthermore, neither the research nor the task required students to perform any specific movement to succeed: they could move freely and still successfully complete the task. Nonetheless, the students performed a wide variety of movement while engaged with the mathematical task. As this research illustrates given the opportunity, students engaged in a mathematical task may choose to move and think mathematically in movement rather than in written symbols. Furthermore, many undergraduate students, in both mathematics and non-mathematics major programmes, may be expected to use mathematics in careers in which they move. Rather than performing this mathematics by writing symbols and signs on paper, these students may have occupations that demand mathematical calculations as they walk and talk, for example nurses or architects, engineers or builders on a construction site. As teachers and researchers, we should ask how practices which deprivilege movement in mathematics learning and teaching help these students adapt mathematics to their future careers and lives. Instead of denying the enmeshed nature of cognition and movement, undergraduate mathematics education needs to re-privilege students' mathematical thinking in movement both in the classroom and in research.

By ignoring students' spontaneous thinking in movement, mathematics teachers and researchers may be missing valuable instances of students' mathematical thinking and knowing. As Sheets-Johnstone (2011) explains

> thinking in movement is a way of being in the world, of wondering or exploring the world directly, taking it up moment by moment and living it in movement, kinetically. Thinking in movement is clearly not the work of a symbol making body, a body that mediates its way about the world by language, for example, it is the work of an existentially resonant body. (p. 425).

Although mathematics educators approach movement in the classroom in different ways, movement is not simply a learning tool. Students' movement can provide access to elements in their mathematical thinking and knowing which might not be available through their verbalization or artefacts. To understand students' mathematical thinking more comprehensively, mathematics educators need to develop approaches that recognize and support students' thinking in movement, rather than considering movement merely as an adjunct to verbalization or as an expression of concepts held in a mind or as a support for learning. To further understand students' thinking and knowing mathematics, educators need to pay closer attention to students' mathematical thinking in movement.

Acknowledgements An immense thank you to my supervisors Caroline Yoon, Alys Longley and Jean-Francois Maheux. Thank you to the generous scholarships for my doctoral study from the Mathematics Department at the University of Auckland, Unitec Institute of Technology, and a Graduate Women of New Zealand fellowship.

Funding Open Access funding enabled and organized by CAUL and its Member Institutions

Declarations

Conflict of Interest The author declares there is no conflict of interest related to this work.

References

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020, August). The future of embodied design for mathematics teaching and learning. In *Frontiers in Education* (Vol. 5, p. 147). Frontiers. https://doi.org/10.3389/feduc.2020.00147

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286. https://doi.org/10.1080/10508406.2011.611446

Bernardet, U., FdiliAlaoui, S., Studd, K., Bradley, K., Pasquier, P., & Schiphorst, T. (2019). Assessing the reliability of the Laban Movement Analysis system. *PLoS ONE, 14*(6), e0218179. https://doi.org/10.1371/journal.pone.0218179

Biza, I., González-Martín, A. S., & Pinto, A. (Eds.). (2022). Calculus at the intersection of institutions, disciplines and communities [Special issue]. *International Journal of Research in Undergraduate Mathematics Education*, *8*(2), 217–221.

Challet-Haas, J. (2016). The problem of recording human motion. In *Dance Notations and Robot Motion* (pp. 69–89). Springer, Cham. https://doi.org/10.1007/978-3-319-25739-6_4

Chorney, S. (2017). *Re-animating the mathematical concept: A materialist look at students practicing mathematics with digital technology*. https://doi.org/10.14786/flr.v5i1.229

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* Cambridge University Press. https://doi-org.ezproxy.auckland.ac.nz/10.1017/CBO9781139600378

Denzin, N. K., & Lincoln, Y. S. (Eds.). (2017). *The Sage handbook of qualitative research* (5th ed.). Sage.

Derry, S. J., Pea, R. D., Barron, S., Engle, R. A., Erikson, F., Goldman, R., Hall, R., Koschman, T., Lemke, J. L., Sherin, M. G., & Sherin, B. L. (2010). Conducting video research in the learning

sciences: Guidance on selection, analysis, technology, and ethics. *The Journal of the Learning Sciences, 19*(1), 3–53.
Dreyfus, T., González-Martín, A. S., Nardi, E., Monaghan, J., & Thompson, P. W. (Eds.). (2023). *The Learning and Teaching of Calculus across Disciplines – Proceedings of the Second Calculus Conference* (pp.1–188). MatRIC. https://matriccalcconf2.sciencesconf.org/
Ferrara, F., & Ferrari, G. (2018). Thinking in movement and mathematics: A case study. In *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 419–426). Umeå: PME. http://hdl.handle.net/2318/1687322
Ferrari, G. (2020). Moving as a Circle: Folds and Nuances of a Mathematical Concept. *For the Learning of Mathematics, 40*(3), 3–8.
Gandell, R., & Maheux, J. F. (2019). Problematizing: The lived journey of a group of students doing mathematics. *Constructivist Foundations*, *15*(1), 50–60. https://constructivist.info/15/1/050
Geertz, C. (1973). *The interpretation of cultures* (Vol. 5019). Basic Books. https://web.mit.edu/allanmc/www/geertz.pdf
Gerofsky, S. (2013, July). Learning mathematics through dance. In *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture* (pp. 337–344). http://archive.bridgesmathart.org/2013/bridges2013-337.html
Ingold, T. (2013). *Making: Anthropology, archaeology, art and architecture*. Routledge. https://doi.org/10.4324/9780203559055
Ingold, T. (2018). *Anthropology and/as education*. Routledge. https://doi.org/10.4324/9781315227191
Laban, R. V., & Ullmann, L. (2011). *Choreutics*. Alton. Hampshire: Dance Books Ltd. (Original work published 1966).
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from* (Vol. 6). Basic Books.
Lemke, J. (2007). Video epistemology in-and-outside the box: Traversing attentional spaces. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 39–51). Routledge.
Longo, G. (2005). The cognitive foundations of mathematics: Human gestures in proofs and mathematical incompleteness formalisms. In P. Grialou, G. Longo, & M. Okada (Eds.), *Images and reasoning* (pp. 105–134). Keio University.
Maheux, J. F., & Proulx, J. (2015). Doing| mathematics: Analysing data with/in an enactivist-inspired approach. *ZDM, 47*(2), 211–221. https://doi.org/10.1007/s11858-014-0642-7
McNeill, D. (1998). Speech and gesture integration. *New Directions for Child Development, 1998*(79), 11–28.
Ministry of Education. (2007). *The New Zealand curriculum*. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum/The-arts/Achievement-objectives#collapsible1
Moore, C. L., & Yamamoto, K. (2012). *Beyond words: Movement observation and analysis*. Routledge. https://doi.org/10.4324/9780203806074
National Coalition for Core Arts Standards. (2015). *National core arts standards*. https://www.nationalartsstandards.org/
Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences, 21*(2), 287–323. https://doi.org/10.1080/10508406.2011.611445
Oehrtman, M., Soto-Johnson, H., & Hancock, B. (2019). Experts' construction of mathematical meaning for derivatives and integrals of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education, 5*(3), 394–423. https://doi.org/10.1007/s40753-019-00092-7
Pepin, B., Biehler, R., & Gueudet, G. (Eds.). (2021). Mathematics in engineering education: A review of the recent literature with a view towards innovative practices. [Special Issue]. *International Journal of Research in Undergraduate Mathematics Education*, *7*(2), 163–188.
Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM, 46*(3), 349–361. https://doi.org/10.1007/s11858-014-0591-1
Roth, W. M. (2007). Epistemic mediation: Video data as filters for the objectification of teaching by teachers. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 367–382). Routledge.
Roth, W. M. (2011). *Geometry as objective science in elementary school classrooms: Mathematics in the flesh*. Routledge. https://doi.org/10.4324/9780203817872
Roth, W. M. (2016). Growing-making mathematics: A dynamic perspective on people, materials, and movement in classrooms. *Educational Studies in Mathematics, 93*(1), 87–103. https://doi.org/10.1007/s10649-016-9695-6

Schüler-Meyer, A. (2019). How do students revisit school mathematics in modular arithmetic? Conditions and affordances of the transition to tertiary mathematics with a focus on learning processes. *International Journal of Research in Undergraduate Mathematics Education, 5*(2), 163–182.

Sheets-Johnstone, M. (2011). *The primacy of movement* (Vol. 82). John Benjamins Publishing.

Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*(3), 223–240. https://doi.org/10.1007/s10649-010-9235-8

Swaminathan, D., Thornburg, H., Mumford, J., Rajko, S., James, J., Ingalls, T., Campana, E., Qian, G., Sampath, P., & Peng, B. (2009). A dynamic Bayesian approach to computational Laban shape quality analysis. *Advances in Human-Computer Interaction*, *2009*. https://doi.org/10.1155/2009/362651

Weinberg, A., Fukawa-Connelly, T., & Wiesner, E. (2015). Characterizing instructor gestures in a lecture in a proof-based mathematics class. *Educational Studies in Mathematics, 90*(3), 233–258.

Yoon, C., Thomas, M. O., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*(3), 371–393. https://doi.org/10.1007/s10649-011-9329-y

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.