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Bafflement in an Inquiry-based College Mathematics Classroom

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The Oxford Advanced Learner's Dictionary defines bafflement as "a feeling of being completely confused and unable to understand". Indeed, feelings of bafflement could be found throughout the development of mathematics. The Pythagoreans are well known to have been baffled by the fact that the length of a diagonal of a unit square is not a rational number. And when Cantor proved that there is a 1-1 map between the set of all points in a square and those of the side of the square, he is said to have been so unprepared for the result that he exclaimed, "I see it but I don't believe it!" (Dauben, 1983, p. 115).

Bafflement has happened repeatedly in the history of mathematics, and also happens frequently in classrooms (Bunch, 2012). It is often a result of paradoxical or seemingly contradictory perspectives, and typically finds its expression in mathematicians' and students' puzzlement or perplexity, which can spark productive outcomes in advancing knowledge. Accordingly, classroom situations that support bafflement, while epistemologically and didactically challenging, may be potentially constructive for learning processes.

Paradoxes in mathematics are well known. Some concern set theory, like Russell's paradox, but many concern infinity, like Hilbert's hotel, where the manager can accommodate an additional guest, even though the hotel's (countably) infinite number of rooms are all already occupied. Paradoxes are often troubling to students. Mathematics educators have realized their didactic potential early on, and have linked that potential to their propensity for creating cognitive conflict and hence increasing motivation (e.g., Movshovitz-Hadar & Hadass, 1990). A particularly well-known one is Zeno's paradox about Achilles and the tortoise; according to the argument, once Achilles gives the tortoise a head-start, he will never catch it. Zeno's paradox and others deal specifically with infinite iterative processes, which play a central role in calculus and other mathematical topics.

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These paradoxes are intimately linked to the difference between potential and actual infinity. Fischbein (1987) already explained that when dealing with actual infinity we are facing situations which may appear intuitively unacceptable. “Their logic is not our logic, which is rooted in our practical experience” (p. 92). Research in this area (e.g., Mamolo & Zazkis, 2008; Tsamir & Tirosh, 1999; Wijeratne & Zazkis, 2015) has revealed conceptions of limits as unreachable, inconsistencies in students’ thinking about actual infinity, the projection of finite patterns onto the completed state of the iterative process, and an urge to preserve consistency with the physical world.

Learning opportunities that arise from bafflement may potentially be more prevalent and productive for students in inquiry-based classrooms, classrooms in which students are supported in sense-making activities through whole class and small group collaborative exploration and problem solving. Individual students may feel more comfortable expressing non-conventional thoughts in a small group; and different groups in a class may come to opposing conclusions concerning an apparently paradoxical mathematical state of affairs. Laursen and Rasmussen (2019) have made the point that inquiry-based mathematics education (IBME) is becoming more and more common in (US) tertiary mathematics education, and is coherent with student engagement in meaningful mathematics, student collaboration for sensemaking, and instructor inquiry into student thinking. These properties make IBME learning environments fertile for investigating paradoxes so as to maximize the expected in-depth learning.

The aim of this special issue is to approach and illuminate the phenomenon of bafflement in an inquiry-based college mathematics classroom. The papers in the special issue approach this phenomenon with different research questions, different theoretical approaches, different conceptualizations, and different methodological lenses, such as didactical, cultural, cognitive, socio-cognitive, discursive, and affective ones. In this introduction, we highlight the specific insights provided by each of these different approaches, and we identify some commonalities across the papers.

In the special issue, we combine the mathematical and didactical interest in paradoxes with the need for research on IBME at the tertiary level, and we do this with particular attention to the many and varied aspects of IBME. Specifically, we focus on a class of 11 mathematics education MA students in an inquiry-based course on Chaos and Fractals, and within the course we focus on a lesson in which the students investigated the area and perimeter of the Sierpiński triangle, after constructing it by an infinite iterative process. The paradoxical aspect of the situation and the ensuing cognitive conflict stem from the fact that the area decreases and the perimeter increases, both geometrically, along this process. A related situation from calculus is the painter’s paradox investigated by Wijeratne and Zazkis (2015).

The course on Chaos and Fractals took place within a Master’s program in Mathematics Education at a US university. The course had 23 lessons. The instructional approach involved a considerable amount of small group work on tasks followed by whole-class discussions, with sporadic periods of lecture and presentation by the instructor. The small groups were permanent, and each small group had a table-sized white board on which to work collectively.

The papers in this special issue analyze various aspects of a portion of Lesson 9 of the course. In the approximately 45 min of this portion, the class worked alternatively in small groups and in whole class discussions (three times each), and one of the foci of attention is students' interaction about ideas traveling between different settings.

Data were collected in this course in a design-based research project funded by the Israel Science Foundation. Class meetings were recorded with two video cameras. Two focal groups of two to three students were chosen for intensive observation. Group A (Carmen, Jen and Joy) and Group B (Elise, Kevin and Mia) were videotaped during small group work, including their whiteboards and joint work. Group C (Kay, Shani and Soo) and Group D (Curtis and Sam) were not videotaped. During whole-class discussions, the two cameras captured both the instructor and the full class as the discussions unfolded. Transcripts relevant to all papers in this special issue are available as an appendix to this introduction.

The editors and authors of this special issue met for a three-day workshop in January 2020 at Tel Aviv University, Israel. In preparation for the workshop, authors (or author teams) analyzed the relevant classroom video and transcript, each using their own framework and approach. These analyses were completed, discussed, compared, and connections between them established collaboratively at the workshop. As a consequence of this interaction, every author (team) has received considerable input from the other co-authors of the Special Issue. Another journal special issue, which focuses on adaptive instruction in secondary mathematics classrooms, emerged from this same workshop and has been published elsewhere (Swidan & Arzarello, 2022).

The interest of the special issue to mathematics education lies in the different approaches and methodologies by which the same classroom phenomena are analyzed and interpreted, the different points of view afforded by these analyses, and the rich complementary insights provided by them. The specific interest to the readership of this journal stems from the setting of the research and the mathematical problématique in a graduate course at a university, as well as from the fact that most of the theoretical-methodological approaches that will be presented by the authors have so far been used almost exclusively at the elementary or high school level. Hence, we hope that readers of this special issue might be interested in these approaches as they are adapted to the tertiary level.

In her paper, *Taming Fantastic Beasts of Mathematics: Struggling with Incommensurability*, Anna Sfard uses the commognitive perspective to interpret students' explicit bafflement with the area of the Sierpiński triangle as an example of incommensurable discourses. On the one hand, students use the words area and perimeter as part of a plane, which coheres with the finite set discourse. On the other hand, the instructor uses the words area and perimeter as a number in the infinite set discourse. Within each discourse, different meta-level rules are held as true, leading to narratives that are true within each discourse, but not across these two incommensurable discourses. Students' bafflement is attributed to their attempt to reconcile these conflicting narratives, unaware of the fact that such reconciliation is not possible due to the different meta-rules governing the two discourses. Sfard claims that this classroom situation is unique in its explicitness, hence allowing us, as researchers and mathematics educators, a glimpse into the struggle of the students' attempts at reconciliation and into their teacher's moves in response to these attempts. Sfard points out some ways by which the instructor acts in his attempts to help students move to the discourse on infinite numbers and imaginary shapes, such as the Sierpiński triangle. First, he himself talks constantly in the infinite numbers discourse. At no point did the different uses of the words area or perimeter become explicit objects of reflection. Second, while the Sierpiński triangle is an imaginary, discursive idea, the activity is anchored in drawing physical objects on the white board, which might lead students away from the imaginary idea. Again, explicit countering the two might assist students. Finally, resorting to historical accounts of struggles mathematicians have experienced might help, at least, with the discomfort expressed by these students.

Einat Heyd-Metzuyanim and Jason Cooper apply the commognitive perspective taken by Sfard to the same data. Yet, in their paper- *When the Problem Seems Answerable yet the Solution is Unavailable: Affective Reactions Around an Impasse in Mathematical Discourse*, the authors choose to focus on the emotions expressed by the students as they struggle with incommensurable discourses, which they called an impasse. While Sfard refers to the students as a group, Heyd-Metzuyanim and Cooper look closely at individual students' subjectification (affective communication) and positioning while addressing this impasse in whole class discussions. To set the ground for their claim of the impasse students faced, the authors operationalized it as incommensurability between the students' pre-fractal mathematical discourse and the discourse of fractals. Hence, the authors provide an a-priori mathematical analysis of the task in the fractals and in the pre-fractal discourses. This analysis set the ground for making sense of individual students' contributions in the whole class discussion. In their analysis, they show that students who did not express bafflement focused on the process of creating the Sierpiński triangle, while those who explicitly expressed their bafflement considered the process as well as its outcomes. Moreover, the former students were positioned as 'the knowers', while the latter were positioned as 'the followers' who needed help. Heyd-Metzuyanim and Cooper also investigate the instructor's talk moves: he actively repositioned students' contributions to the conversation in order to keep the conflicting narratives on equal ground, to avoid one narrative overpowering other narratives, and to press students towards engaging with the conflicting conclusion, rather than avoiding the impasse.

In their paper, *The Interplay between Individual and Collective Activity: An Analysis of Classroom Discussions about the Sierpiński Triangle*, Geoffrey Saxe and Amelia Farid use a cultural-developmental framework to make sense of the data. The basic assumption of the framework is that cognitive development is constituted through processes that are: Microgenetic—of form-function relations which occur over a short duration of time; Sociogenetic- reproducing and altering form-function relations; Ontogenetic—development over the lifespan of individuals; as well as Phylogenetic. The authors argue that to make sense of the interplay between individual and collective activity, two intertwined analytic strands are needed. The first is developmental analyses of individuals' use of representational forms to serve reasoning and communicative functions as they participate in the collective practices of classroom life. The second is cultural analyses of how individuals' participations often unwittingly reproduce and alter collective practices, including emergent participation structures, social positions, norms, and linguistic registers. The authors conduct micro-ethnographic analyses of face-to-face interactions with a focus on an emerging register. Their analysis of a linguistic register includes word-forms (like area) and action-words-forms (like zooming) that together enable the conceptualizing activity as well as participants' developments of mathematical ideas. Similar to Sfard, they started their analysis by looking at specific words central in the students' discussions, like area and perimeter. However, for Sfard these keywords and the sentences in which they are embedded were an indication for the participating discourses- that of finite or infinite sets. Saxe and Farid investigated the word-form in an attempt to link it to the function of these words in developing registers for the individual and the collective, and the relations between them. The authors also study the participants' positioning. In contrast to Heyd-Metzuyanim and Cooper, they looked at positioning via the relative number of talk turns by each participant. Hence, the instructor was identified as the main participant in the whole class discussion, but he did not participate in the small group discussions. He questioned students to express their ideas, expanding them or clarifying them in 90% of his turns. The analysis shows that students participated considering the inquiry-oriented norms of listening to each other and expressing one's own thinking while referring to others and building on their contributions.

Ways of engagement with others' mathematical ideas and argumentation are the focus of attention by AnnaMarie Conner, Michal Tabach, and Chris Rasmussen. In their paper, *Collectively engaging with others' reasoning: Building Intuition through Argumentation in a Paradoxical Situation*, the authors develop a multimodal methodological-theoretical approach for analyzing different types of teaching moves by which the instructor elicited students' reasoning, as well as the ways by which the students engaged with each other's reasoning. Then, as a proximation of the mathematical progress in the whole class discussions, the authors analyzed the flow of argumentation in the class. Hence, their analysis of the same classroom discussions brings about a different view on these data. Five arguments were found to function-as-if-shared by the students. For two of these five 'accepted mathematical truths', the authors provided detailed analyses in which the contributions of argumentation parts done by the instructor and the students are presented using Toulmin like diagrams, together with the ways of engagement that were identified. Coordinating individual and collective analyses, the authors found that student contributions within the collective argumentation of the class illustrate coordination between engagement and participation in the mathematical progress of the class.

The last paper in this issue, by Tommy Dreyfus, Naneh Apkarian, Chris Rasmussen and Michal Tabach, is *Collective and Individual Mathematical Progress: Layering Explanations in the Case of the Sierpiński Triangle*. Like the paper by Conner et al., this paper presents a Documenting Collective Activity analysis of the argumentations in the class as a way to capture the collective activity. Differently, however, Dreyfus et al. used Abstraction in Context to learn about the knowledge constructed by individual students during the same whole class and group discussions. By applying the two methodologies to the whole data set, the authors gain insight into the complexity of the interplay between Collective and Individual Mathematical Progress (CIMP) in inquiry-oriented classrooms, and demonstrate a methodological approach of Layering Explanations (LE) to the analysis of mathematical progress in such classrooms. The three 'stories' that are detailed in this paper illustrate three different ways by which mathematical progress may take place in class. The *Zooming In* story shows how mathemati-cal progress can relate to an imaginary underlying way of thinking that helps make sense of complex phenomena like infinity. The *Area limit 0* story demonstrates that ideas may be constructed, consolidated, and function-as-if-shared within small groups before they come to function-as-if-shared among the whole class. On the other hand, ideas may sometimes function-as-if-shared in the whole class, even if the associated constructing process has occurred only partially. The *Perimeter of the White = Perimeter of the Black* story shows functioning-as-if-shared in the whole class discussion without any preparatory constructing process of this relationship in the previous small group work. So, the authors demonstrate a multiplicity of ways in which knowledge has developed and mathematical progress has been achieved, by individual contributions to small group work and whole class discussions.

Taken together, the papers in this special issue offer a comprehensive and detailed portrait of the complex learning and teaching processes as undergraduate students engaged with a baffling situation. The various papers also offer novel theoretical and methodological insights and advances.

Appendix: Transcript of Lesson 9

Lesson 9 started with a brief discussion of the term paper [10 min], followed by a video on fractals [18 min], as well as small group work 2 [1 min] and whole class discussion 3 [3 min] discussing the video. The professor then distributed the worksheet in Fig. 1. Although the professor had intended Tasks 1–3 of the worksheet as background for an in-depth discussion of self-similarity in tasks 4 and 5, the class did not progress beyond Task 2 during Lesson 9, and the transcript below refers to Tasks 2c and 2d.

Task 1 gives instructions for recursively constructing the Sierpiński triangle (ST), which took the students quite some time in small group work 4 [20 min] and whole class discussion 5 [6 min]. The professor then asked them to come up with conjectures concerning the area of the ST (small group work 6 [2 min] and whole class discussion 7 [2 min]). During whole class discussion 7 the professor learned that the students were attempting to use computations rather than general considerations to come up with conjectures, mentioned that this was not what he had intended but ended up by encouraging them to go on with their own ways of thinking.

In small group work 8 [8 min], G4 focused on the perimeter; they mentioned factors like 3/2, 9/4 and 27/64. After a brief discussion of the nature of the perimeter when the process is infinite, they returned to computation, agreed that the perimeter keeps increasing forever; but were not completely sure it tends to infinity. G1 kept at first discussing area; they agreed that it is being reduced at every step to three quarters of its current size and ask themselves whether something will be left at the end, or whether it approaches a number. They reasoned that after a few steps, only very little is taken away from the area and that intuitively it would be weird that it should tend

Activity Sierpiński's triangle (Waclaw Sierpiński, 1915)

1. Triangles
   (a) Sketch an equilateral triangle of side $a = 16$ cm (16 has been chosen for convenience).
   (b) Connect the midpoints of the triangle's sides, so as to generate four congruent triangles of side $a/2$.
   (c) "Take away" the triangle in the middle (you may cut it out or simply color it in a dark color).
   (d) You are now left with three equilateral triangles. For each one of them, repeat (b), (c), and (d).
   (e) Discuss the instruction, given in (d), to repeat not only (b) and (c) but also (d). Write your considerations and conclusions, and present them to the class.
2. Stop after carrying out (b) and (c) six times.
   (a) Imagine the shape that results from repeating (b) and (c) "forever". (Remember that the shape under consideration is the part that is not colored.)
   (b) Can you assign an area to this shape? Write your considerations and conclusions, and present them to the class.
   (c) Can you assign a perimeter to this shape? Write your considerations and conclusions, and present them to the class.

Fig. 1 Sierpiński triangle activity – Tasks 1 and 2

to zero. They turn to the perimeter. The instructor, who happens to join, asked the perimeter question in terms of fencing the remaining area. The students noted that for every triangle one removes, one has to add perimeter. They did not compute but the question of infinity did come up. Carmen said that "if we keep zooming in, there's no area, there can be no fence" but they are aware that numerically the perimeter keeps increasing. This is when the transcript starts.

Transcript of Whole Class Discussion 9 [6:15 min].

76	Instructor	Hey, can I ask the class a question?
77	Professor	Yeah, sure. Go ahead
78	Instructor	Guys, so… Let me ask, let me ask the class a question. This group here has been talking about area and perimeter, can you do recount… Wait, first of all, you said area… you did some computations, and you just conceptually thought the area was going to…
79	Joy	Zero
80	Instructor	Zero, right? Okay. And then tell me about… about the perimeter. Tell us about what… Because you guys had different ideas
81	Jen	Yeah, mhm
82	Instructor	So tell us about… Carmen, tell us about your idea, and then Janet, tell us about your idea
83	Joy	Okay

84	Carmen	I was thinking if we keep zooming… Okay, for our area thing, we were going to keep zooming in, keep coloring in, so eventually we're gonna color all in. It's going to be black, so there's no area, so there's nothing to… No area there's nothing to put a fence around it. So, there'd be no perimeter… and then
85	Instructor	So Carmen is thinking that the perimeter then would be zero, because there's no… There's nothing left to put a fence around. And Janet, you were thinking what?
86	Joy	I kind of thought it's toward the opposite end—like, if you zoom in there's more to fence, and if you zoom in there's more to fence, and you just keep putting in more fencing material, because as you zoom in there's more and more to fence. Until… Except that you'd fill in the triangle, to some extent
87	Instructor	Great. So Elise's got a question for you. Go ahead, Elise
88	Elise	So, what I feel, like, what Carmen's saying is when you zoom in… Or she says you color it all in so it's all black, but… What you're coloring in, is perimeter, to some extent. Not totally, because it's also area. But, like, every time you build a little triangle, you have more perimeter in there, right? So, then, all those… I don't know… Does all the black become all the tiny little pieces of all the tiny triangles?
89	Kevin	So what… So the, the perimeter is… also can be considered the perimeter of the black. Part of the perimeter is the perimeter of the black. 'Cause see, when you… When you shade it in, you're adding the perimeter of the black
90	Carmen	Oh, I see what you're saying—so it's actually, like, it's a… the fence is guarding both properties
91	Kevin	Yeah
92	Carmen	Not just yours, but it's doing the other one too. Ok that makes sense haha
93	Instructor	Soo, can you explain with your own words, what this conversation is about, between Kevin and Carmen?
94	Soo	Umm… I think… Umm… Janet is saying, like, you keep zooming in you're going to get more triangles forming, so you have more areas, so you keep adding the numbers, right? And Kevin is saying… I didn't really follow what he said
95	Instructor	Curtis, you were nodding your head when Kevin was talking. Can you say a little about what you interpreted Kevin to say?
96	Curtis	Yes. Kevin… Umm… He was saying, like, a… That, well… Carmen's issue was that the…
97	Soo	I understand Carmen
98	Curtis	There was no area to the… Yeah. But then, umm, Kevin was saying that the perimeter of the… the white is also the same as the perimeter of the… perimeter of the black part. So, since there's area… There is some area of the black… But we didn't talk about that. There could be a…
99	Instructor	Mia, did your group…
100	Mia	I think that another way that might help you visualize it, or the way it's helping me to visualize it, is that we're supposed to take out the shaded triangles, so… Like, if you imagine actually having a piece of paper triangle, shading in the middle triangle, taking it out—you're going to have all these shaded triangles, with perimeters. And that's how I see it. I see the perimeter increasing, and then this… The unshaded area is what's left over, and that's constantly decreasing and going to zero. [***]
101	Instructor	Mia, you're agreeing more with Janet?
102	Mia	I think so, yeah
103	Instructor	You think so?
104	Mia	Yeah
105	Carmen	Wait, are you…

106	Mia	Because I understand, I understand what you are saying, Carmen. But I think if you see us taking out the shaded triangles, and you're going to be left with all these shaded triangles, if you are physically cutting up…
107	Carmen	Right. Is this a cumulative perimeter, or a perimeter at a point in time? Are we saying—is this the perimeter after all these iterations? Or is this the perimeter when we've done this thirty times, and now we're looking at what we've got left over?
108	Mia	I see it the first way.
109	Carmen	Okay, so we're adding up each perimeter.
110	Mia	So it's… Yeah
111	Carmen	So are you doing the same thing with the area? Too when you're thinking about the terms of…
112	Mia	I think the area is… is what from what's left over. From the unshaded part. So if you look it, like the pictures, I think the white part is… just keeps getting smaller and smaller, and the black part keeps getting more… bigger and bigger. The more iterations you do.
113	Instructor	Janet, what're you thinking now? As you've heard these different points of view?
114	Jen	I see the perimeter thing now, the way Janet was saying it.
115	Instructor	Are you sure?
116	Jen	Just the way she said it.
117	Joy	Okay
118	Jen	I'm sorry. Just because when she said, like, if you imagine it being cut out.
119	Instructor	Uh-huh
120	Jen	Then I'm… like oh, okay, well I see it now. Then… It was harder to see here. But I don't know if I'm still like 100%.
121	Instructor	What about your—Sam, how about you? What's your thinking?
122	Sam	I don't know, it seems like… It seems the area go to zero, as N goes to infinity.
123	Instructor	So Sam says area goes to zero, do you people agree with this? Ladies in the back? You think the area goes to zero or not goes to zero?
124	Soo	*** *** ***. Yeah
125	Instructor	I'm sorry, I didn't hear.
126	Soo	Oh, I was just *** ***. It's decreasing, because we tried writing the formulas for the area.
127	Instructor	Uh-huh
128	Soo	And then we have… Uhh… three over four subtracting sum of all those shaded ones, so I'm guessing that eventually it's going to go to zero. So you keep subtracting…
129	Instructor	So, intuitively, you're okay. You want to prove it to yourselves still, I guess, but Sam says that the area goes to zero. So go on, Sam, so what else…
130	Sam	So it goes to zero, but… because we don't reach infinity, umm… So we got to keep adding, umm, more, umm, perimeters. But theoretically probably… we reach infinity in the end, it's going to go to zero. Then we don't have an area. So I'm… I'm not sure.
131	Instructor	Okay, so what Sam is saying—like, you see the perimeter is going off into infinity, like Janet is saying, but then you haven't any area, so then… That was Janet's point originally, that you, that you don't have any fence, you don't have anything to fence, so… take a minute in your groups, and come to some sort of… sort of like tentative conclusion. Don't do any computations, just come to some tentative conclusions about area and perimeter, and then share with the rest of the class. Without computation. Just sort of conceptually what makes sense for you.

Transcript of Small Group Work 10, Group A [1:20 min].

723	Carmen	Do we have to agree?
724	Joy	Oh, I have an example of that
725	Jen	Okay, go ahead, and then I have an idea too
726	Joy	Okay, so if you try… The ink is your perimeter, right?
727	Carmen	Okay
728	Jen	Okay
729	Joy	Just so ink is the perimeter, right? As you, as you draw it in
730	Jen	Oh
731	Carmen	Yeah
732	Joy	And say we didn't shade in the middle, just the drawing. Okay, so we draw it, we draw it, but in all the white spaces we're going to draw more triangles, right?
733	Carmen	Mhm
734	Joy	So we use more and more ink
735	Carmen	Mhm
736	Joy	And as I zoom in I'm going to use more and more in ink. And I'm going to keep zooming in, and I'm going to keep using more ink
737	Carmen	I'm okay with that
738	Jen	So that *** ***
739	Joy	Okay
740	Carmen	What I'm not okay is I feel that we're accumulating the perimeter, but we're not accumulating area
741	Jen	That's maybe…
742	Carmen	That's what makes me uneasy
743	Jen	Maybe this will help you
744	Carmen	Maybe it's not…
745	Jen	Yeah, yeah
746	Joy	Yeah, I know what you mean—like, logically zero should have no
747	Jen	Yeah
748	Carmen	But, like, we're adding up, like we're saying okay—we use this ink here. Like, okay, so… we're looking at perimeters. So, now, okay…. So we use this ink here… [Carmen gestures at the blue triangle between Joy and Jen]
749	Joy	Yeah
750	Carmen	to draw that, so that counts, as a perimeter
751	Joy	Yes
752	Carmen	But when we're calculating the area—this, previously, this big triangle here
753	Joy	Aha
754	Carmen	That previously was an entire area itself, and we counted it
755	Joy	Right
756	Carmen	And we kind of threw it out, and said—okay, so now we're just going to count this triangle here [Carmen gestures at the black triangle close to Joy]
757	Joy	But we still have the perimeter of the original
758	Jen	I think our problem is this drawing, and we can't see it really painted all. If you were to *** ***, like, actually like…

Transcript of Small Group Work 10, Group B [1:20 min].

405	Elise	I'm really trying ***. I thought the perimeter was certainly increasing—because it is. Like, here, if we're counting—tch, tch, tch, tch, tch, tch* [Elise points at the sides of the inner triangles]—it's way more than what we stated with. But… if you're saying you take it out
406	Mia	That's more and more
407	Elise	and then eventually you take out everything, and then there's… nothing?
408	Kevin	Well, if we take, like ***…
409	Mia	There's nothing for the area, but you're still… you're counting the perimeter of what you're taking out
410	Elise	Are you? If you remove it?
411	Mia	And you count the area of what's left over
412	Elise	Why are you counting the perimeter of it? Because it's the cut-outs here?
413	Mia	I see what you're saying
414	Elise	So it… so I agree that if I take this piece out, it still has its perimeter, because it's still connected
415	Mia	Mhm
416	Elise	But if I'm taking… everything out. I mean, if I'm gone so far that there's nothing left
417	Mia	I think it's because… I think specifically we're saying the area is the area of the unshaded part
418	Elise	Right
419	Mia	Right?
420	Kevin	So it sounds like we're still using computation. Because as soon as you say—as N approached infinity, that means you're going to computation. So, I think what we want is something general, like—the area is getting smaller, but the perimeter is getting larger, and just leave it at that general statement [Mia writes on her paper (unclear in the recording)]
421	Mia	You mean perimeter?
422	Elise	Yes

Transcript of Whole Class Discussion 11 [5:02 min].

133	Instructor	Alright, alright. Let's just get some feedback right now. So… Carmen, Janet, Janet, tell us… Tell us what your table… Maybe you don't have a final conjecture but tell us what you're thinking now, as a threesome there
134	Joy	You can go, first
135	Carmen	I don't think we got anywhere
136	Jen	Yeah, we didn't
137	Joy	My opinion is—you're using more and more ink. If your ink is your perimeter, and every time you draw it you use more and more ink. And it's… and you're going to keep using it to get more triangles
138	Carmen	I agree with that, we're using more ink each time. Just… I have a hard time… with the consistency of ok, before we have more area, and then we threw it away, and now we've got the smaller areas. We're saying it's getting smaller, but with the perimeter it's like—okay, we used some fencing, we still going to count that one we previously used, we're going to count that previous things, and keep… That's not… not resonating
139	Instructor	Great. That's a very clear explanation. I understand that there's still something to figure out. Soo, yeah, tell us about your group thinking, please
140	Soo	Well, I think the perimeter is getting smaller

141	Joy	Why?
142	Soo	Because if you're adding the small number each time, so your perimeter in general will get smaller and smaller and smaller and smaller
143	Shani	But you'll still have the…
144	Joy	But it takes more fence to fence smaller sections of an area
145	Soo	You're adding smaller area – smaller number of smaller and ***
146	Instructor	Elise, can you tell us about your group? What did your group talk about?
147	Elise	I think that we think that the area is getting bigger, and the perimeter is getting smaller. Umm… I think that…
148		Wait a second; wait wait wait the area is getting bigger? Hang on… (at least the instructor, Joy, Elise, Mia speaking) [(Students and Instructor talk at once—unclear in the recording)]
149	Mia	We might disagree. We might disagree – that's not
150	Elise	Did I say it right? No, I wasn't. The area is getting smaller and the perimeter is getting bigger
151	Instructor	Okay
152	Elise	I'm sorry
153	Instructor	Okay. You inadvertently flipped them around. Okay
154	Elise	Yes. But I think that… Like Soo is saying—you're adding smaller and smaller pieces, but you're still adding those pieces to what you already have
155	Soo	*** *** *** ***, smaller *** ***
156	Elise	I've got this, but now I'm also adding in this, and now I'm also adding this. So yes, I'm adding smaller pieces, but I'm still adding [Elise points at the inner triangles]
157	Kevin	Oh! Listen… [Kevin raises his hand]
158	Soo	But you're adding small number of small number
159	Elise	But one plus point-oh-one is still bigger than one
160	Soo	No?
161	Kevin	I think what Soo is saying is that individual triangles perimeters' is getting smaller. Is that?…
162	Soo	Yeah
163	Kevin	Yeah, okay
164	Soo	The length of that
165	Kevin	So then she's the entire perimeter…
166	Elise	Definitely
167	Soo	So we're not adding to the… You're not comparing to the original part and adding
168	Elise	So I think that our group thinks… But I shouldn't speak to them… That the perimeter is, like, all of this, combined with all of this, combined with all of this, combined…
169	Soo	Yes
170	Elise	So I think we think that our perimeter is all of the little bars that we have drawn here together. Which means that we're getting more of them, even though they're smaller
171	Joy	I have a question
172	Soo	I don't know
173	Instructor	But what about Sam's point—is that… Eventually he said the same thing, but he said—but then, in the end, there's going to be zero, because you don't have any area
174	Elise	Yeah. I'm still upset about that idea
175	Instructor	Sam, do you want to comment? Or Curtis, you want to comment on…

176	Sam	I and Curtis were talking about, like, uh… it's geometric, so it's going to converge. So the area would be… go to zero… There would be limited amount of areas, so we're going to have a limited number of… perimeters. So we don't have infinite number of perimeters.
177	Joy	So my question is—does it converge to the total area of the triangle, because you can't fit anymore?
178	Instructor	The "it" is the perimeter?
179	Joy	Into the area of the perimeter?
180	Sam	No, the area will converge.
181	Joy	A perimeter… The perimeter converges, but it fills a space that is the area of the original triangle. Can you fit more perimeter into the triangle that you originally started with?
182	Instructor	Oh, the question is—is the perimeter reaching a finite number?
183	Joy	Yes
184	Instructor	Okay
185	Joy	And is that final number close to the area of the original triangle. Because you can't fit anymore in it.
186	Instructor	Right.

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