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From Collaborative Construction, Through Whole-Class Presentation, to a Posteriori Reflection: Proof Progression in a Topology Classroom

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Abstract
Coming from a social perspective, we introduce a classroom organizational frame, where students' proofs progress from collaborative construction in small groups, through whole-class presentation at the board by one of the constructors, to a posteriori reflection. This design is informed by a view on proofs as successive social processes in the mathematics community. To illustrate opportunities for mathematics learning of proof progressions, we present a commognitive analysis of a single proof from a small course in topology. The analysis illuminates the processes through which students' proof was restructured, developed previously unarticulated elements, and became more formal and elaborate. Within this progression, the provers developed their mathematical discourses and the course teacher seized valuable teachable moments. The findings are discussed in relation to key themes within the social perspective on proof.

Keywords Commognitive framework · Graduate students · Proof and proving · Topology · University mathematics education

Introduction

In their overview of the mathematics education literature in the area of proof, Stylianides et al. (2017) identify three broad perspectives: the cognitive – proving as generating a logical deduction that links premises with conclusions, the constructivist – proving as convincing, and the social – proving as an activity that is embedded in communities. In the social perspective,

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> […] the emphasis tends to be on *activity* rather than *understanding* […]. In particular, a critical point is that individual proof-related tasks (e.g., constructing proofs, reading proofs) are not viewed in isolation (as is often the case in the other two perspectives) but in the context of a broader mathematical activity. If a student or teacher produces a proof, research in this perspective would frequently place emphasis on the meaning of this artifact and how that individual and members of his or her community could subsequently use it (Stylianides et al., 2017, p. 247, italics in the original).

Stylianides et al. (2017) describe the social perspective as "less developed", "not yet coherent", and lacking "common, widely used concepts" (pp. 247–248). These observations are particularly accurate in the case of university mathematics education research, where the cognitive and constructivist perspectives dominate (for exceptions, see the work of Herbst, Chazan, and colleagues, e.g., Herbst et al., 2011 and Hemmi, 2006 for the university setting). This state of affairs paves the way for socially oriented studies into proof-based university courses.

Within the social perspective, students are expected to prove in a similar manner to that practiced in the mathematics community, and for similar reasons (Stylianides et al., 2017). This expectation has been advocated by many scholars and pursued in several studies (e.g., Legrand, 2001). However, in their comprehensive literature review, Melhuish et al. (2022) concluded that, in proof-based courses, inquiry classrooms have been often positioned as almost a sole alternative to lecturing. Accordingly, research could benefit from expanding the range of ways to engage students with proof in a manner that resembles the mathematics community.

This paper draws on De Millo et al.'s (1993) view of proof as a successive social process in the mathematics community. Our first goal is to elaborate on this view as a preparation for its implementation in proof-based courses. The second goal is to introduce *proof progressions* as a way to organize students' engagement with proof in a classroom. Within this organizational frame, students' proofs progress through three phases: from collaborative construction in small groups, through whole-class presentation at the board by one of the constructors, to a posteriori reflection. As part of a larger project, we study proof progressions in a course in topology – a content area that has been rarely explored in university mathematics education (for exceptions, see Gallagher & Engelke Infante, 2021 and Stewart et al., 2017). Our third goal is to illustrate opportunities for mathematics learning and teaching that proof progressions entail. To do this, we present a single progression and use the commognitive framework to analyze the transformation of the proof and the discursive activity that led to it.

Background

We start with an overview of the main approaches to teaching and learning in proof-based courses. Then, we synthesize selected views of mathematicians on how proofs function in the mathematics community.

Lecturing and Inquiry in Proof-Based Courses

Melhuish et al. (2022) reviewed 104 papers published since 2000 on teaching and learning in proof-based courses. They concluded that "teaching (proof-based) collegiate mathematics is no longer an 'unexplored practice'. As a field, we have learned a lot in the last decade about how these courses are taught and why" (p. 2). And while teaching in these courses may be expected to unfold in myriad ways, the researchers managed to divide their findings into two broad categories: lecturing and inquiry—we structure this section accordingly.

Lecturing is the most researched mode of instruction in proof-based courses (e.g., Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Johnson et al., 2018). In the context of real analysis, Weber (2004) characterized different styles of proof presentation of a single teacher-mathematician[1] (specifically, logico-structural, procedural, and semantic). These styles featured in his classroom monologues, leaving little space for students to take responsibility for any aspect of proof. Other studies show that the gauge of this space may vary, depending on the activities that teacher-mathematicians offer to their students (e.g., Johnson et al., 2018). For instance, Fukawa-Connelly (2012) explored a course in abstract algebra, where the teacher-mathematician consistently raised questions and ceded some responsibility for proof to her students. Nevertheless, the findings showed that students did not play a significant role in the construction of classroom proofs. Other studies (e.g., Dawkins, 2012; Paoletti et al., 2018) also reported on students' limited contribution to the endeavor.

Lecturing is often discussed in relation to teacher-mathematicians' modelling authentic mathematical practice (Melhuish et al., 2022). This approach to lecturing emerges from self-reflective writings (e.g., Krantz, 2015; Pritchard, 2010) and interview studies (e.g., Alcock, 2010; Weber, 2004). For instance, Wood and Weber (2020) found that lecturing allows mathematicians to model such key mathematical processes as proving and defining, establishing notation, and maintaining rigor. Weber (2012) obtained similar findings from the interviews with nine teacher-mathematicians in the context introduction-to-proof courses. Overall, they mostly presented proofs to demonstrate ideas and techniques rather than convince students that a particular mathematical statement is true.

The imperative of lecturing as a means for modelling has not been confined to proof or to advanced mathematics courses (e.g., Artemeva & Fox, 2011; Pritchard, 2010). For example, in the context of functions, Viirman (2021) explored how seven teacher-mathematicians introduced new mathematical objects and explained what counts as a valid mathematical argument. Of particular interest to our study is that Viirman's participants not only enacted the "rules of the mathematical game", but

[1] Research usually uses "mathematicians", "teachers", and "instructors" to refer to mathematics faculty who lead university courses (e.g., Melhuish et al., 2022). While each term is sensible and often consistent with a theoretical lens employed, their singularity unavoidably illuminates one endeavour, leaving the other one in the background. With "teacher-mathematicians" we stress the dual professional identity of research mathematicians for whom teaching is an integral component of their scholarship.

also occasionally articulated them on a more general level (e.g., "In mathematics we need to state exactly what we mean", Viirman, 2021, p. 476).

Encyclopedia of Mathematics Education defines inquiry-based mathematics education as

> […] a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions […], interpret and evaluate their solutions, and communicate and discuss their solutions effectively" (Dorier & Maaß, 2020, p. 384).

Broad approaches of a similar kind can be found in Artigue and Blomhøj (2013) and Laursen and Rasmussen (2019). In practice, inquiry classrooms often include groupwork, student presentations, and active whole-group discussions (e.g., Melhuish et al., 2022). Laursen et al. (2014) comment that inquiry-based learning in college mathematics in the US has grown from a certain method of instruction, where students were assigned mathematical statements to prove, provided with time to do so, and then invited to the board to share their proofs. Originally, the other students "would make sure the proof presented was correct and convincing" (Jones, 1977, p. 275). The followers of this method often modify it make their classrooms more inclusive, collaborative, and dynamic (e.g., Coppin et al., 2009). Ernst et al. (2017) maintain that "proof-based courses are a natural setting for IBL [inquiry-based learning]. In fact, there is a long tradition of using IBL in these courses, where class size and content pressure are typically minimized when compared to other courses we teach" (p. 570).

Several studies have focused on proof in inquiry classrooms, while often drawing on data collected at the end of a course and describing a range of positive developments. For instance, these studies report that students grew "a more robust understanding of the functions of proof than previous studies would suggest" (Cilli-Turner, 2017, p. 2), developed "an understanding of how [to] correctly use definitions and assumptions within the context of their proofs" (Grundmeier et al., 2022, p. 1), and obtained a more humanistic and process-oriented view of proof than their peers in a lecture-based courses (Yoo & Smith, 2007). That said, Melhuish et al. (2022) stress that the studies to date have made modest progress towards an in-depth understanding of the learning and teaching processes that unfold in student-centred proof-based courses. The researchers also note that the studies have revolved around researcher-driven interventions rather than authentic instruction that teacher-mathematicians lead in their classrooms.

To summarize, the readings in the area led us to three conclusions: First, in proof-based courses, teacher-mathematicians rarely prove to confirm the validity of mathematical statements, but mostly to advance students' learning. We highlight this finding to precede its deeper discussion within the social perspective in "Proof Progressions" section. Second, many teacher-mathematicians utilize proof to model various mathematical processes for their students. They view this instructional practice as valuable and useful. Third, research seems to agree that by incorporating a range of activities where students actively engage with proof (e.g., proving in small groups, proving at the board for the whole class to see, and discussing proofs), students are provided with actionable opportunities for mathematics learning. However, little is known about the mechanisms through which classroom organization of proof-related activities shapes students' learning on a micro level. We draw on the first two conclusions in "Proof Progressions" section to justify a particular way proof progressions are implemented in our project. The third conclusion inspired us to investigate student learning within proof progressions in fine grain.

Proofs as Successive Social Processes in the Mathematics Community

De Millo et al. (1993) describe mathematics as a community project, where proofs constitute a "successive social process at work" (p. 272):

> No mathematician grasps a proof, sits back, and sighs happily [...] [They run] out into the hall and [looks] for someone to listen to it. [They burst] into a colleague's office and commandeers the blackboard. [They throw their] scheduled topic and [regale] a seminar with [their] new idea. [They drag their] graduate students away from their dissertations to listen. [They get] onto the phone and [tell their] colleagues in Texas and Toronto. In its first incarnation, a proof is a spoken message, or at most a sketch on a chalkboard or a paper napkin.
> That spoken stage is the first filter for a proof. If it generates no excitement or belief among [their] friends, the wise mathematician reconsiders it. But if they find it tolerably interesting and believable, [the mathematician] writes it up. After it has circulated in draft for a while, if it still seems plausible, [the mathematician] does a polished version and submits it for publication (De Millo et al., 1993, pp. 301–302).

Generalizing De Millo et al.'s examples, we propose that as a successive social process, a proof features in a range of situations that we describe as *socially organized*[2], *proof-transformative*, and *sequential*. The first descriptor accounts for the structuredness of patterned ways in which mathematicians interact with proof and each other, ways that have been broadly practiced and are considered as common in the mathematics community. For another example of such a socially organized situation, consider Thurston's (1994) provocative depiction of formal mathematical presentations:

> Organizers of colloquium talks everywhere exhort speakers to explain things in elementary terms. Nonetheless, most of the audience at an average colloquium talk gets little of value from it. Perhaps they are lost within the first 5 min, yet sit silently through the remaining 55 min. [...] At the end of the talk, the few mathematicians who are close to the field of the speaker ask a question or two to avoid embarrassment (pp. 165–166).

[2] With "socially", we do not imply that mathematicians necessarily interact in these situations "here and now". We use this descriptor as a reverence to Stylianides et al.'s (2017) social perspective on proof and to Sfard's (1998) approach to learning as participation in a historically-established human endeavor.

Regarding the transformational aspect of proof, we note that De Millo et al. (1993), Thurston (1994) and others refer to proof as a discursive artefact that mathematicians produce and reproduce in different media (e.g., “phone”, “blackboard”, “publication”), to different audiences (e.g., “students”, “colleagues” that can come from the same or unrelated fields), and with different purposes (e.g., generate belief or excitement). Given these differences, it seems unreasonable to expect the proof to remain exactly the same in each situation. Thurston (1994) addresses this point explicitly when observing that a proof shared in an hour-long talk to colleagues from the same subfield can turn into a dozen-page long paper in a research journal. Similarly, Raman (2003) distinguishes between private and public aspects of proof to stress that mathematicians’ arguments for self-understanding are usually different but connected to those they generate for a particular mathematics community. A proof may also transform in mathematicians’ interaction, such as a collaboration or a peer-review (e.g., Andersen et al., 2021). To resort to a metaphor, we associate a proof with an elastic object that mathematicians purposefully manipulate to fit the organizational structure of a target social situation.

A proof does not progress arbitrary from one social situation to another; and neither does it proceed in a particular linear direction. With “proof progression”, we are referring to both the chronological shift of a proof from one social situation to another (e.g., see De Millo et al.’s shift from the “spoken stage” to “a polished version […] for publication”); and also to the transformative developments of a proof. These developments are often described in terms of validity, elegance, significance, reduction of gaps, and so on (e.g., Andersen, 2020; De Millo et al., 1993; De Villiers, 1990; Dreyfus & Eisenberg, 1986; Rav, 1999; Thurston, 1994).[3]

De Milo et al.’s perspective offers insights into how proof as a social process fulfils its often-discussed role in mathematics (e.g., Thurston, 1994). Indeed, in the case of each process, proof-constructors validate a particular theorem by means of proof and ground it within the broader area. Proof-focused interactions with colleagues may lead to conceptual innovations, systematization, and dissemination of new results (e.g., De Villiers, 1990). In this way, each proof contributes to mathematics as a discipline. At the same time, different phases of the process provide mathematicians with the opportunity to engage with it in a range of capacities. This engagement is profitable on the individual level: for example, the proof constructors receive feedback on their proof, while others may obtain tools that are relevant to problems at the heart of their current and future research (see Rav, 1999 for proofs as “bearers of mathematical knowledge”, p. 20). In this way, the participation in the social process of proof provides mathematicians with opportunities for mathematical learning. Next, we consider whether and how opportunities of this sort can be created in a university mathematics classroom.

[3] Evidently, proofs differ in the length of their progression sequences. Given the prolificity of the mathematics community, it is somewhat expected that “only a tiny fraction [of proofs] come to be understood and believed by any sizable group of mathematicians” (De Millo et al., 1993, p. 298), a number of proofs are subsequently contradicted or thrown into doubt, when most are simply ignored.

Proof Progressions

Building on De Millo et al.'s (1993) view, we propose that a classroom proving can be organized as a community project, where students participate in a sequence of structured social situations that resemble the successive social processes mathematicians encounter. Specifically, students can construct proofs individually or in small groups first, and then present them at the board, followed by a posteriori reflection. Such proof progressions are at the heart of our larger developmental research that is conducted with a teacher-mathematician, Prof B (for earlier reports from the project see Kontorovich, 2021; Kontorovich et al., 2022).

Jaworski (2004) explains that in developmental research, teachers are not just subjects in didacticians' empirical studies but co-learning partners. In these partnerships, didacticians and teachers work together and "learn something about the world of the other. Of equal importance, however, each may learn something more about [their] own world" (Wagner, 1997, p. 16). The developmental aspect pertains to a multi-layered inquiry into students' mathematics, mathematics teaching, and the contribution of research to teaching development. In this way, research both charts the partners' developmental process and serves as a vehicle for it (Jaworski, 2004). We note that the developmental methodology is consistent with Artigue's (2021) approach to an often-discussed gap between research and practice in the university context. Specifically, she suggests,

> not think in terms of dissemination of research results, but in terms of collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, *problématiques* that make sense for all those involved, and meet their respective interests and needs (p. 14, italics in the original).

Table 1 outlines the structure of the proof progression central to this paper, aligning it with authentic practices of the mathematics community. This progression has been conceived and honed by Prof B, who was initially inspired by what she referred to as the "Moore method" (cf. Coppin et al., 2009)[4] and modified it to suit her students. Prof B's reference to this method is hardly accidental: she is a well-recognized topologist and a regular teacher of a small-size proof-based course in topology that brings together graduate students and mathematics majors in the last year of their studies. Prof B developed this progression to support her course which requires "heaps of proving, which is a common point of weakness for students" (her words). As part of the project, we collaboratively unpacked the connections among the progression, common practices in the mathematics community, and the social perspective on proof.

Clearly, research mathematicians are rarely assigned with statements to prove, and they are seldom expected to prove them publicly after just a short time. Accordingly, from the social viewpoint, the progression hardly leaves room to pursue the first instructional goal of the social perspective: "for students to engage in an authentic

[4] We recognize the pain, harm, and injustices that R. L. Moore's racism, antisemitism, and sexism caused (e.g., Ross, 2007).

way with proving [including] settling debates about the truth of contentious mathematical assertions and for generating and communicating mathematical knowledge" (Stylianides et al., 2017, p. 247) (see Legrand, 2001 for scientific debates as an alternative design to address this goal). Nevertheless, we construe proof progression as a frame that can be implemented in different ways to address a range of goals. Table 1 shows that the particular implementation is consistent with mathematicians' views of proof as a successive social process. It also provides opportunities to pursue the second goal of proving within the social perspective: "providing explanations […], illustrating new methods to solve problems […], and deepening one's understanding of concepts" (Stylianides et al., 2017, p. 247). As we noted beforehand, this goal is enthusiastically endorsed by some mathematics educators (e.g., Alcock, 2010; Hanna, 1990) and consciously pursued by many teacher-mathematicians (e.g., Krantz, 2015; Pritchard, 2010; Weber, 2004, 2012). Furthermore, the implementation constitutes a didactical innovation of Prof B. Thus, by exploring it in its authentic form, we capitalize on Melhuish et al.'s (2022) call "to study the instruction of mathematicians who have been implementing student-centered instruction as a regular part of their practice" (p. 16).

The progression components and their sequencing are not foreign to the mathematics education literature. For example, groupwork, presentations, and whole-classroom discussions are characteristic to the inquiry paradigm (e.g., Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019). Accordingly, we propose that studying Prof B's proof progression provides an opportunity to understand how students can engage in a social activity that is not unique to our project classroom. Specifically, this is an opportunity to address another lacuna that Melhuish et al. (2022) identified in their literature review – explicating the learning and teaching processes in proof-based courses at a fine-grained level. In the next section, we overview the commognitive framework which is our main toolkit to study these processes.

Theoretical Framework

We turn to the commognitive framework (Sfard, 2008) for theoretical foundations and analytical tools. Commognition associates mathematics learning with participation in a collective historically established endeavor, which is consistent with social perspective on proof. Indeed, commognition concerns human activities as they unfold within a certain social context, construes proof as a publicly accessible artefact that interlocutors can use and reflect upon, and it avoids talking about elusive terms, such as understanding (cf. Stylianides et al., 2017). Commognition has been acknowledged for its ability to account for the complexity of university mathematics education (Nardi et al., 2014) and it has been used to investigate teaching and learning in proof-based courses (e.g., Brown, 2018; Karavi et al., 2022; Kontorovich, 2021; Pinto, 2019). This section gives an overview of the framework, with attention to mathematical discourses, learning, proof and proving.

Table 1 Proof progression and its connections to the mathematics community

Progression phase	Description	Resemblance to the mathematics community
Proof construction	Students prove an assigned mathematical statement individually or in small groups.	A “first incarnation” (De Millo et al., 1993, p. 302) or a “‘private argument’ […] which engenders understanding” (Raman, 2003, p. 320).
Proof presentation	A constructor volunteers to re-prove the statement at the board, making it public for the rest of the class and the teacher-mathematician.	The public aspect of proof in the sense of “an argument with sufficient rigor for a particular mathematical community” (Raman, 2003, p. 320). This aspect becomes visible in more formal situations, such as seminars and manuscripts.
A posteriori reflection	A class discusses the presented proof with input from the teacher-mathematician.	Communal feedback on the presented proof (e.g., questions in a seminar, reviews in a journal)

Mathematical Discourses

Commognition maintains that mathematics as a whole and its sub-fields (e.g., topology) can be construed as a *discourse*. Discourses are defined as, "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Operationally speaking, discourses are distinguishable through *keywords* (e.g., "Hausdorff space") and their use, *visual mediators* (e.g., symbols, diagrams) and their use, *narratives* endorsed by the mathematical community (e.g., a definition, a proof), and characteristic *routines* (e.g., proving).

Mathematics is replete with special metadiscursive rules (or *metarules* for short) that govern the actions of the discourse participants when producing and substantiating narratives about mathematical objects. More often than not, metarules are *constraining* rather than deterministic, i.e. they illuminate possible options for actions without dictating a particular one. Sfard (2008) highlights that it is not uncommon for metarules to remain *tacit* in a discourse community. When a metarule is articulated, its status changes from *enacted* to *endorsed*.

Learning

A *personal discourse* refers to patterns in a communication of a discursant when they partake in a mathematical discourse (Sfard, 2008). Lavie et al. (2019) expand the commognitive apparatus to account more fully for the fact that discursive regularities are sensitive to contextual circumstances. The researchers introduce the notion of a *task situation* to refer to "any setting in which a person considers herself bound to act—to do something" (p. 159). Lavie et al. argue that one's capability to act in new circumstances stems from experience. Specifically, when an individual recognizes similarities between a current task situation and precedents that they interpret as sufficiently familiar to the present one, the individual replicates what was done then, while amending their talk and actions to the current circumstances. Eventually, Lavie et al. (2019) define a *task* as an individual's interpretation of the task situation and a *procedure* as a sequence of replicated actions. The procedure-task pair feeds into a definition of a *routine*: "a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure [they] executed to perform the task" (Lavie et al., 2019, p. 161).

Let us acknowledge the challenge of implementing the abovementioned constructs in data analysis. For example, a commognitive analysis of one's task, necessitates accessing their vision of what they wish to achieve under specific circumstances. This vision is clearly sensitive to the task situation that unfolds "here and now", and thus, inquiring into one's vision (e.g., by explicit questioning) changes the original task situation. Indeed, a classic finding in cognitive psychology is that when asked why one took a particular action, a person often provides post hoc rationalizations that are likely to be inconsistent with their in-the-moment thinking (e.g., Nisbett & Wilson, 1977). In the next section, we elaborate on the methods that were employed to cope with these conundra.

Sfard (2008) defines *learning* as a lasting change in one's discourse, i.e. in at least one of the four discourse characteristics. This change can be triggered by *learning opportunities* – "circumstances that call for, and support, *a change* in the learner's participation in a discourse, a transformation that would bring him or her closer to the [target] discourse" (Chan & Sfard, 2020, p. 3, italics in the original). Chan and Sfard (2020) distinguish between opportunities for a change in *the learner's command of a discourse* and for a change in the *discourse* itself. Within the former, the learner becomes more fluent in the target discourse by realizing the opportunity to mathematize according to its metarules. In the latter, the learner enriches their discursive repertoire with new mathematical narratives and routines.

In the above terms, each phase of a proof progression constitutes a different task situation, potentially providing students with opportunities to generate a different proof and change the way they partake in a mathematical discourse. The successive organization of the progression paves the way for the investigation of transformations that students' proofs go through. Proof transformation, students' and teacher-mathematician's routines, and opportunities for mathematics learning are the foci of our analysis.

Proof and Proving

In commognitive terms, proving a mathematical statement is an act of endorsing a narrative. Endorsable narratives are those that can be rendered as valid or not "according to well-defined rules of the given mathematical discourse" (Sfard, 2008, p. 224). Nevertheless, it is worth acknowledging that the mathematics community has been revising these rules throughout history (e.g., Kleiner, 1991), and even today, mathematicians neither agree nor follow the exact same set of proof-related rules (e.g., Inglis et al., 2013; Lew & Mejía-Ramos, 2020).

Notwithstanding this variability, for today's mathematicians, the predominant way of proving consists of manipulation with narratives, and it is thus intradiscursive (Sfard, 2008). This process is expected to unfold as a sequence of utterances (or sub-narratives), each either an "accepted fact", or derived from previous utterances according to some metarules (e.g., deduction) (cf. Kontorovich & Liu, 2023). Coming from the social perspective, we propose that the decision on whether a substantiation constitutes a proof is a matter of social sanctioning. In Manin's (1977) words, "a proof becomes a proof after the social act of 'accepting it as a proof'" (p. 48). Thus, we use "proof" as a discursive label that a particular community can allocate to a narrative that was generated to endorse a mathematical statement (a *proposed proof* or a *proving narrative* hereafter). Consistently, we use "proving" to refer to a discursive activity from which such a proof emanates (cf. Kontorovich, 2021). Note that within this approach, the sanctioning community can consist of the individuals who propose a proof in the first place. Also, the act of sanctioning is time- and context-dependent: to paraphrase a famous saying of Kilpatrick (1985) about mathematical problems, a proof "for you today may not be one for me today or for you tomorrow" (p. 3).

Context and Methodological Underpinnings

Our project is contextualized in a semester-long topology course offered by a Mathematics Department in a large New Zealand university. The department is rather small in terms of the student intake, and this is the only topology course offered. Its syllabus consists of standard topics in point-set and algebraic topologies (e.g., continuity, convergence, homology). The student cohort typically encompasses a mixture of post-graduate students and undergraduates in their final year of a mathematics major; four and two students, correspondingly, in the particular semester. Prof B—an esteem research topologist and experienced university teacher, leads the course. Video-recordings of the course lessons constitute the primary source of our data. In the first phase of the project, the recordings were made with a single video-camera. We recorded the protagonists who led the discursive activity in the whole-class episodes of the lesson; and an arbitrary chosen group when students worked collaboratively.

To illustrate the proof progression in-action we chose a single case. Three main considerations underpinned our selection. First, while no two progressions were identical in terms of the protagonists' interactions and communication, we wanted to share a case that would illustrate patterns that featured in other cases as well. Second, we wanted to illustrate how students' proposed proofs progressed through all of the three phases. Since for each proof we recorded the collaborative work of a single group, the set of cases where a student from the video-recorded group volunteered and was selected to prove at the board was rather small. Third, we wanted to present a case the analysis of which might contribute to the proof literature; in particular, to its social perspective strand. Specifically, we aspired to offer insights regarding the transformation of the proof and the protagonists' proving activity alongside the progression.

Eventually, we selected a progression that took place at the end of the first third of the semester. This was the fourth progression in the classroom, by then, the students were already familiar with the metarules of this activity and the course as a whole. The case illustrates several aspects that are characteristic to many progressions that we recorded and witnessed. First of all, it is concerned with arguably a simple corollary, and its three phases took around ten minutes; all classroom progressions revolved around mathematical statements of this kind and the work on them took less than fifteen minutes. Second, students' communication is terse and deictic, which opens the door to multiple interpretations (more about this shortly). Third, at the board, the prover pays limited attention to the class; this was characteristic to this and a few other students in the course. As one of the reviewers noted, this pattern is not rare among mathematically mature students. Fourth, in spite Prof B's efforts to spur a posteriori discussion, the class remained relatively passive. This occurred almost each time when there were no major issues with the proof at the board. In turn, the activity of Prof B is characteristic of both the kind of feedback that she provided on students' proofs and how she used these proofs in her instruction.

We approached the analysis with two questions: (i) *What were the similarities and differences between the proposed proofs that emerged in each progression* *phase?* Looking ahead, all three proving narratives were endorsed as "proofs". Accordingly, this question can be rephrased as: *how did the students' proof transform along the progression?*; (ii) *What discursive moves and routines underpinned the proposed proofs?*"

We iteratively scrutinized the transcribed data with a focus on the proposed proofs and the protagonists' discursive activity. To address the first question, we attended to the discursive features of the proofs that emerged in each phase, including their structures and formulations of their sub-narratives. To analyze these formulations, we drew on Sfard's (2008) construct of *objectification* as a combination of turning one's talk about mathematical processes into objects (i.e. *reification*) and stripping it from the human agency (i.e. *alienation*). This construct aligns with a broader literature on the characteristics of academic mathematical texts (e.g., for a review, see Morgan, 1998).

To address the second question, we aimed to make sense of the students' procedures and tasks. We divided the transcript into episodes and produced preliminary accounts of what the students were doing and what tasks they could have pursued with their actions. Some actions featured more than once. Building on their similarities, we iteratively amended our analytical accounts to generate descriptions that would fit the patterns. This is how the two main routines, proof-growing and proof-monitoring, came about (see next sections for details).

Some tasks were explicitly articulated by the students. In other cases, we deduced them from the end-points of students' observable actions, suggesting that the set-up tasks were achieved. Yet, figuring out the tasks was especially challenging in some cases. Theoretically, task-identification requires accessing all precedents and past performances that one considers relevant to the current task situation. Lavie et al. (2019) maintain that "precedents for whatever happens in this setting should come from the same discursive, material, institutional, and historical context" (p. 160). Accordingly, we drew on the video-recordings of the previous course lessons, consultations with Prof B, and teaching-and-learning scenarios that were typical to other proof-based courses that the students took earlier.

Within the social perspective, "how and why one proves […] can be viewed as inextricably linked to the social context in which proof occurs" (Stylianides et al., 2017, p. 247). In our classroom, this social context was dynamic. Then, we also accounted for students' awareness of how their proposed proofs could be used in the next progression phase, and of their peers who were in a similar task situation either right now (in the proof construction phase) or a short time ago (in the proof-presentation phase). Speaking more generally, we iteratively considered students' tasks through zooming-in and out of institutional settings in which the current task situation unfolded: from circumstances determined by the protagonists' ecosystem "here and now", through an instance that takes places in a particular lesson, to a situation that unfolds in a cross-level course led by a research mathematician, and so on. Each of these contextual viewpoints gave rise to different interpretations of the task, the conditions for its completion, and the set of relevant precedents. We refer to this interpretative scale as *a broader context* to stress that it employs data beyond that collected by our video-recordings.[5]

The described analytical procedure resulted in a set of candidates for students' tasks. Then, we scrutinized each of them to determine its consistency with the collected data, as these emerged from the consultations with Prof B, the previous lessons, and our familiarity with the relevant institutional setting. In the last stage, we collated viable tasks into single analytical accounts and formulated them so that they would not sound like the "real" task that the students pursued. In this way, an additional outcome of our analysis is an illustration of how considering proof-requiring task situations from a range of social angles can be used to generate plausible hypotheses about students' tasks (see Wetherll & Potter, 1988 for a similar operationalization of the construct of function of language).

One Proof Progression

This progression unfolded in the lesson on Hausdorff spaces. At the beginning of the lesson, Prof B defined Hausdorff spaces as those where open sets separate every two elements (i.e., for each pair of points x and y in $X, x \neq y$, there are open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$). After discussing this definition and related examples, the students self-divided into pairs, and Prof B invited them "to have a go at proving" that if $f : X \to Y$ is a continuous one-to-one function and Y is Hausdorff, then X is also Hausdorff. The protagonists of our case are a doctoral student, Grace, and an undergraduate, Jonah (pseudonyms). At that time, Jonah was in his last semester and considered continuing to a master's in pure mathematics. Grace was not formally enrolled in the course and sat in on it "to fill holes in [her] mathematics education" (her words) as this was the first formal topology course that she participated in. Next, we trace their proof as it progressed over three phases outlined in Table 1.

Collaborative Proof Construction[6]

The dyad embarked on the assigned statement as follows:

1	Jonah:	Okay, so… [sketches two ovals for the sets X and Y in his notebook]
2	Grace:	So we want to show that in X, yeah

[5] In the context of problem posing, Kontorovich et al. (2012) introduced the notion of *considerations of aptness* to refer to one's comprehension of implicit requirements that the request to pose problems entail. We refer to the same ambiguity in relation to proof-requiring task situations in our progression. Overall, our task analysis is consistent with how research operationalizes Brousseau's (1997) *didactical contract* and Herbst and Chazan's (2011) *disciplinary* and *institutional obligations* (2011).

[6] An earlier version of this analysis was presented in Kontorovich and Greenwood (2022)

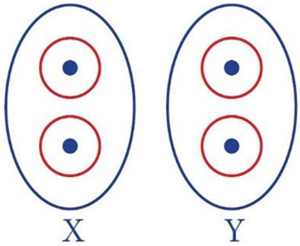


Fig. 1 Reproduction of Jonah's diagram

Jonah completes the diagram reproduced in Fig. 1.

3	Jonah:	That's basically it.
4	Grace:	[a] Yeah, that is kind of it, right? [b] Well, if they weren't disjoint…
5	Jonah:	[a] Oh, that's true. [b] Sounds really simple.
6	Grace:	It's almost too simple. [5-sec pause]
7	Jonah:	I feel like, I fee like something's missing.
8	Grace:	Yeah, I feel like something is missing as well. [15-sec pause]
9	Grace:	So the function is from X to Y. You have two points here [in X], let's call them little x, little y. [notates the points on the sketch]
10	Jonah:	Oh, doesn't it imply that these two actually are in the same… You have an intersection.
11	Grace:	Yeah, that's seems wrong because then it [the statement] is true for all functions. Oh, but the fact that it is a continuous function…
12	Grace:	But why do we need one-to-one? I feel that we got to use that.
13	Grace:	So, so… Let's actually do this super logically. We start with two points.
14	Jonah:	Yeah.
15	Grace:	We want to put an open set around each.
16	Jonah:	Yeah, yeah.
17	Grace:	We go to $f(x)$ and $f(y)$.
18	Jonah:	Yeah, yeah.
19	Grace:	We put an open set here and here, which we can do because it is Hausdorff.
20	Jonah:	Yeah.
21	Grace:	We can pull these back and get two open sets here [pre-images in X].
22	Jonah:	Yeah.
23	Grace:	If there was a point in this intersection but it can't get mapped to two points.
24	Jonah:	Yeah.
25	Grace:	One-to-one means that these two… Oh!!! These two can't get mapped to the same point. Because if they got mapped to the same point, this argument wouldn't work. It has to be two different open sets. That's why [1:1].
26	Jonah:	Oh, wait, what? I still don't see where the one-to-one.
27	Grace:	So our argument would fail. If f wasn't … because of if f wasn't one-to-one, then you could have $f(x)$ equals $f(y)$.
28	Jonah:	Oh, oh, oh [as if realizing this]… Yeah.
29	Grace:	[a] And then you definitely couldn't do this picture. [b] So I think that's where it happened.
30	Jonah:	Okay. Right, right. That's kind of subtle.

Next, we consider students' proving actions and the proposed proof. We start with the latter.

Proposed Proof

In the "Proof and Proving" section, we associated "proof" with a label that some community can attach to a narrative endorsing a mathematical statement. The exchange between Grace and Jonah demonstrates that such a narrative can emerge in fragments and heavily rely on visual mediators (a diagram, in this case). To be able to trace the transformations that the proof went through alongside the progression, we assemble the fragments that the students articulated into what we consider to be the most coherent whole. Specifically, we disregard the order in which the fragments appeared (this is in the focus of the next section), and complete the students' formulations with their apparent intentions. In doing so, we attempt to preserve the original formulations as much as possible to show how they transformed in the second phase.

Our assembled version of the proposed proof goes as follows:

(a) "We start with" two points, "let's call them little x, little y" in X (see [9] and [13]).
(b) They "can't get mapped to the same point" in Y because f is one-to-one (see [25] and [27]).
(c) "We put open sets" around $f(x)$ and $f(y)$, "which we can do because it [Y] is Hausdorff" (see [19]).
(d) "We can pull these [the open sets] back and get two open sets" in X (see [21]).
(e) "If there was a point in this intersection but [it would] get mapped to two points" (see [23] and [4b]).

Two aspects of this proposed proof are noteworthy. First, notice the process-centered and personified formulations of the utterances (a), (c), and (d). Second, the proof is far from elaborate. Indeed, the diagram visualizes that the initial "open sets" around $f(x)$ and $f(y)$ are disjoint, but students do not mention this notion. In (d), the openness of the pre-images in X is declared but not justified. One could reasonably propose that the dyad endorsed this property based on the continuity of f. We concur, but notice that this justification was not articulated. Also, the proof concludes with (e), positing that the pre-images in X have no points of intersection. The existence of such a point is rejected with a rather general argument. And the proof completes without spelling out why the obtainment of two open disjoint sets in X is sufficient to render it a Hausdorff space. We make these points not to criticize students' proof but to prepare the readers for the transformation that it undergoes in the next progression phase.

Proving Process

In regard to the proving process the pair went through, we divide their interaction into three rounds. The utterances [1–8] constitute the first round. It revolves around a diagram, to which both students referred as "basically it" and "kind of it". In [2], Grace commences the construction of the proving narrative, but the construction is relinquished once Jonah completes the visual. The brevity of students' utterances prevents us from proposing a univocal candidate for the task that that they pursued. Yet, their decisive tones, affirming intonations and gestures suggest that the obtainment of the diagram brings the task to a closure. Hence, we conclude that generating a verbal version of the proof was not necessary for the students at that point. In [3–8], the dyad implements what we term as *proof-monitoring*: a process of "looking back" at the previous discursive activity (i.e. a procedure) to assess whether it can be sanctioned as a proof of the assigned statement (i.e. a task). In this round, Grace and Jonah look at the diagram for a time, and only the outcomes of this monitoring are articulated: both do not identify issues with their proof.

Notwithstanding the affirming proof-monitoring, both agree that "it" (the proof or its construction) was "too simple". Notice the tension between the students not identifying a mathematical issue with their proposed proof and still not being satisfied with it. We account for this tension by drawing on the broader context: the proof-requiring task situation was set up by a research mathematician in a cross-level course. From this viewpoint, it can appear unlikely that the dyad could come up with a sanctionable proof just in seconds. This hypothesis is consistent with students' decision to continue their proving further, while locating their previous work (i.e. the diagram) at the center of their discursive activity.

In [7–8], the holistic "it" turns into a focused "something's missing", and the identification of a potentially problematic element turns into a task for the second round. Pursuing this task is impossible without generating a proving narrative, which happens in [9–11]. We refer to these students' utterances and actions as *proof-growing*: a procedure through which a proposed proof is not constructed "from scratch" but it becomes more extensive, elaborate, and detailed based on previously conducted work. In this case, Grace uses Jonah's diagram to name the sets and points, and, in [10], Jonah rephrases Grace's utterance from [4b].

This round ends in [11–12] with Grace identifying that their proposed proof does not capitalize on f being one-to-one. This proof-monitoring is not unlike the one in the first round: on the one hand, it shows that Grace does not detect an issue in their proposed proof. On the other hand, she expands the scope of her monitoring to include the function's injectivity – a part of the assigned statement that does not feature in their proof narrative yet. In our experience, it is typical for mathematically mature students to interpret a gap between given conditions and those utilized in their work, as a univocal marker of an issue in *their* mathematics (e.g., Kontorovich, 2018). That said, both students agree that their proof endorses the assigned statement "for all functions", and injective functions are a subset of "all" – so why wouldn't they see their proving mission as accomplished? As before, we propose that this tension can be explained by considering the students' task situation within a broader, this time historical, context: not including redundant conditions in a statement and proving theorems with a broad scope of applicability is characteristic to many, if not most, pure mathematics courses. Drawing on these precedents, the students could sense that the validity of the statement "for all functions" would be just too strong considering the statement's conditions.[7] In this way, we account for students' decision to initiate another proving round by the mismatch between the proposed proof and their previous proving experiences.

Consistently, delineating the role of injectivity becomes the task for the third round. The utterances [13–30] capture proof-growing and proof-monitoring, but the students' interaction changes: now Grace leads the implementation of both: she narrates one proof element at a time, while Jonah only endorses her statements. In [25], this interaction bears fruit: Grace realizes that their diagram had highlighted the function's injectivity at the start, by depicting $f(x)$ and $f(y)$ as distinct points. Accordingly, in this round, the students' proving narrative expanded with a new component about a visual element of the diagram that had been taken for granted until then.

Proving at the Board[8]

Almost as Grace and Jonah's conversation concluded, Prof B asked "who is ready to present?" and Jonah volunteered. He approached the board holding the classroom notes with definitions and propositions but the notebook with the much-discussed diagram remained on the desk.

Table 2 presents the transcript of Jonah's discursive activity at the board. The columns demarcate between the oral and written components of his communication. In some cases, Jonah spoke as he wrote (more or less), while there was an evident time gap between his articulated and written narratives in other instances. We use "…" to point to cases where Jonah did not continue his oral sentences, "↓" to mark him glimpsing at his notebook, and square brackets for our commentary. Throughout the whole episode, Jonah stood facing the board and with his back to the class. Figure 2 presents a snapshot of Jonah's board on the completion of his work.

Proposed Proof

To be fully consistent with our approach to proof, we should consider Jonah's discursive activity at the board in its totality, i.e. as the one that unfolds in different media, including the board (e.g., text, diagrams, annotations), oral verbatim (e.g., utterances, intonations), gestures, movements, facial expressions, gazes, and so on (Kontorovich, 2023; Kontorovich et al., 2022). But given that Jonah's work on the board overlapped and subsumed his communication in other media (see the next section for details), it is sufficient to confine the discussion to his written narrative.

[7] This would also entail that all topological spaces are Hausdorff as they can be mapped to a single point.

[8] In Kontorovich et al. (2022) we used the data presented in this section to illustrate the potency of the commognitive framework to study students' mathematizing at the board as part of inquiry. Here, the focus is on board proving as the second progression phase.

Table 2 Transcript of Jonah's proof at the board

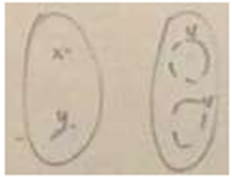
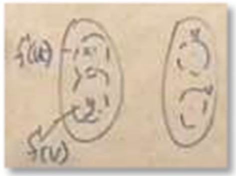
Move	What Jonah said at the board	What Jonah wrote on the board
1	So, let x not equal to y and suppose they're in X	let $x \neq y \in X$.
2	Then f of… f of x is not equal to y because f is one to one	then $f(x) \neq f(y)$ because f is 1–1
3	And… mmmh… Y is Hausdorff so there exist UV open in Y such that	Y is Hausdorff, so there exist U, V open in Y such that
4		$f(x) \in U$, $f(y) \in V$, $U \cap V = \varnothing$.
5	And… mhh… suppose [glimpses at the statement to be proved and at what he wrote on the whiteboard; looks hesitant]	
6		[sketches on the lower left part of the board] 
7	[Giggles and smiles as if embarrassed, then, goes back to his seat and returns with the notes that he made when working on the statement earlier.]	
8	[↓] Oh yeah! Because f is continuous so… the pre-image of U and the pre-image of V [points at the symbols of $f^{-1}(U)$ and $f^{-1}(V)$ that he just wrote while standing with his back to the classroom] are open in X	f is continuous, so $f^{-1}(U)$, $f^{-1}(V)$ are open in X.
9	[↓] Suppose for a contradiction [says in a fading voice]	Suppose for a contradiction
10	[as he sketches the dotted circle around y] So this is the pre-image of V	[continues the sketch from 6] 

Table 2 (continued)

Move	What Jonah said at the board	What Jonah wrote on the board
11		, $\exists z \in f^{-1}(U) \cap f^{-1}(V)$ then $f(z) \in f\left(f^{-1}(U)\right) = U$ and $f(z) \in f^{-1}(f(V)) = V$[a] *□

[a]We assume that Jonah intended to write $f(z) \in f\left(f^{-1}(V)\right) = V$

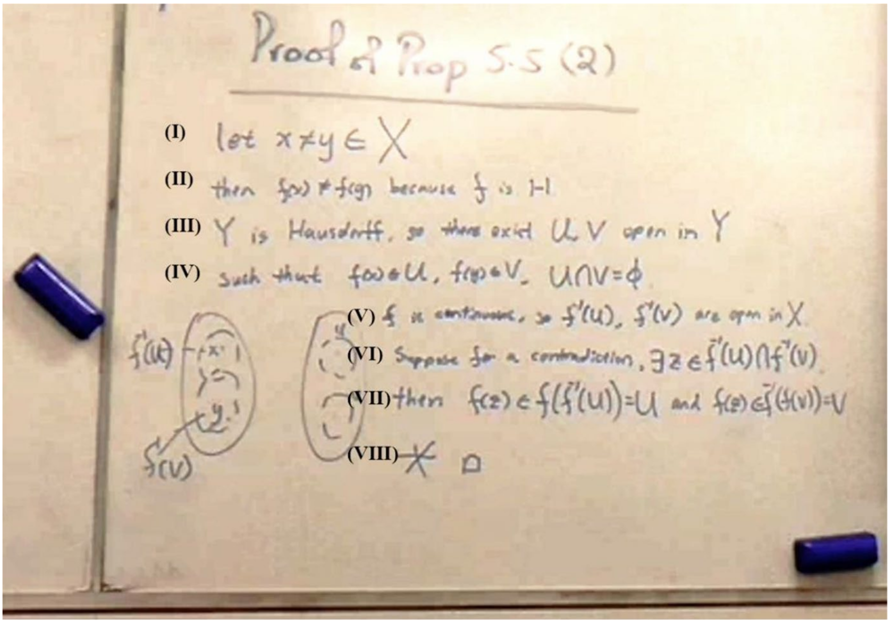


Fig. 2 Snapshot of Jonah's board (numbering added)

On the face of it, Jonah's proof maps rather neatly to the one presented in the previous progression phase, with most components of the two proofs appearing isomorphic to each other (i.e. (I)↔(a), (II)↔(b), (III)↔(c), (V)↔(d), (VI-VII)↔(e)). Yet, substantial differences can be discerned in regard to *restructuring*, *formalization*, and *growth* of the proof from the previous phase:

- Recall that in the previous phase, we were the ones to assemble the students' oral fragments spread over three proving rounds into a coherent whole. In this phase, Jonah generated a linearly structured proof. For instance, note how effortlessly he builds on the injectivity of f in (II), considering that it took the dyad a while to realize the role of this condition.
- The formalization aspect is evident in the sub-narratives of the previously proposed proof transforming from being process-centered and grounded in human actions to becoming fully objectified (e.g., compare "which we can do because it is Hausdorff" in [19] and "Y is Hausdorff, so…" in (III)). This change is inseparable from the board-narrative being written and containing multiple symbols, compared to its oral predecessor. Together with formal names of mathematical objects (e.g., "continuity"), the symbols replace the deictic "it", "they", and "here" that the students used beforehand.
- Perhaps the most visible instance of proof growth is evident in the transition from "if there was a point in this interaction […] but it can't" in [23] to the sub-narrative in (VI)-(VII). These lines declare the sub-proof by contradiction and

spell out its steps. The board-diagram also grows new visual elements and annotations compared to its predecessor in Fig. 1.

That said, Jonah's proof preserves some of the gaps that featured in the previous phase. For instance, both proofs are silent regarding how the fact that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint yields the Hausdorff-ness of X.

Proving Process

Jonah's proving unfolds smoothly in [1–5], where he unpacks the notions of "Y is Hausdorff", "f is one-to-one", and "f is continuous" into detailed statements about open sets and points. However, the central problem of the assigned statement (cf. Selden & Selden, 1995) is in showing that the pre-images of U and V are disjoint. A diagram played a key role in Jonah's collaboration with Grace, and thus, it is not surprising that he generates one at the board as well (see [5]). But this diagram does not help, and Jonah resorts to his notebook. We do not find it arbitrary that he stumbled where he did since this is the element of the proof that the Jonah and Grace did not substantiate in the previous phase. A short glance at his notebook affords Jonah to complete the proof rather effortlessly (see [8–11]). He concludes the writing with "*□" and instantly goes back to his seat while cracking a smile to our video-camera, which suggests that he was content with the work. Thus, we propose that Jonah proved the statement, in his eyes at least.

Several aspects of Jonah's proving are noteworthy. While proving publically, he made almost no effort to acknowledge the rest of the class. We also notice his reluctance to use the notes that he made when collaborating with Grace. We account for these aspects by proposing that Jonah's task at the board was to autonomously generate a self-contained written narrative that the class would endorse as a proof. This task is consistent with Jonah standing with a back to the class, often blocking the text that he put on the board with his body; initially dupicating his writing orally without additional elaborations; and gradually "turning off" the oral component. This task is consistent with the broader view on students' board-proving in our project: the proof-presentation phase followed on from proof-construction that the whole class engaged in just a short time ago. Thus, as Prof B put it, the proofs at the board were expected to not only "present the main ideas", something thas is legitimate in proof-construction, but "to stand alone". In our case, this expectation translated to the generation of written narratives that the class would sanction as a proof in the following phase without using the prover's commentaries. We also note that, in the previous phase, Grace did most of the "heavy lifting" in turning Jonah's initial diagram into a proving narrative. Thus, Jonah's volunteering to the board and the half-hearted usage of the notes may also relate to his wish to generate a proof of the same mathematical statement on his own.

A Posteriori Discussion

The following exchange unfolded once Jonah returned to his seat:

1	Prof B:	Very nice, beautifully laid out. But I do have two quibbles. Anybody? [3-sec pause] And this is just for me. [10-sec pause]
2	Rick:	You want to specify the quantifiers?
3	Prof B:	Yep, but I'll let you off. The other thing, I do think people write this but I'm not keen on this either cause it sort of says x doesn't equal y, and y is not in X, but doesn't really say anything about where x comes from.
4	Jonah:	I wanted to say that.
5	Prof B:	I've seen people write that, but yeah yeah, I'm not [keen on this]. So, I would recommend [adds on the board, see Fig. 3] and they're different, yeah?

We consider this last progression phase from the perspective of proof transformation and opportunities for mathematics learning. The first thing Prof B does is sanctions Jonah's proposed proof and even dubs it as a "beautifully laid out" one. Yet, she does not move on before drawing the classroom attention to two "quibbles". As a result, Jonah's proof transforms through Prof B reformulating its first sub-narrative. This transformation instantiates a proof-growing routine in a way that is not very different from how Grace and Jonah implemented it in the first progression phase: Prof B did not introduce a new element to a proof but built on and amended Jonah's previous work. Evidence of proof-monitoring feature in Prof B's move as well: she suggests that while Jonah's symbolic statement "$x \neq y \in X$" would be accepted by some, it will raise an issue for others due to the unspecified residence of x. And while there is no doubt regarding the notational camp to which Prof B belongs, she does not position herself as the ultimate authority on the matter. In the post-lesson reflection, Prof B argued that the same correction is likely to be made if Jonah submitted his proof to a topology journal. Accordingly, we suggest that she monitored Jonah's proof not

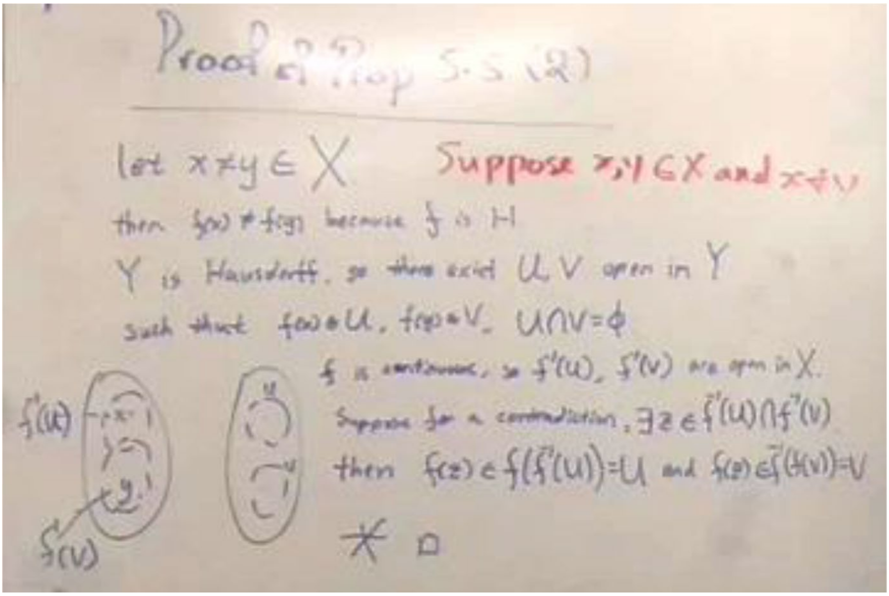


Fig. 3 The proof with Prof B's addendum

only through the lens of a teaching-and-learning situation that unfolded in the classroom, but also from a broader perspective of a professional topological discourse. This perspective is appropriate to an advanced mathematics course that is taken by research students and undergraduates in the last stages of their degrees.

Regarding opportunities for mathematics learning, Prof B's "quibbles" revolve around notational metarules. In the post-lesson reflection, she explained that the first "quibble" pertains to a common issue of "unspecified quantifiers", i.e. situations where mathematicians do not spell out whether their statement refers to all, some, or particular cases. She discussed this issue several times with the class, which is evident in Rick guessing this "quibble" right away. In the second "quibble", Prof B draws students' attention to two competing metarules to denote the act of taking two distinct elements from the same set (i.e. "$x \neq y \in X$" and "$x, y \in X$ and $x \neq y$"), critiques one of these metarules, and positions the other as a preferable alternative. We see this instructional move as her way to model how students can select between competing metarules (cf. Viirman, 2021).

Lastly, Prof B draws classroom's attention to the particular metarules due to Jonah's proof deviating from them. However, note that their applicability goes beyond the particular mathematical statement under discussion. Furthermore, notice that Prof B "let off" the first metarule and addressed the second one more suggestively than prescriptively. Accordingly, we conclude that drawing class attention to these metarules was not less important for her than making the specific proof abide by them.

Discussion

In this section, we discuss our main arguments and findings with attention to four themes that Stylianides et al. (2017) render as critical to research within the social perspective:

> (1) understanding mathematical practice with respect to proof, especially mathematicians' reasons for engaging in proving; (2) identifying what proving is for students and teachers (rather than assuming a priori that proving is convincing; (3) designing classroom environments where proof can be seen as a tool for generating and communicating mathematical knowledge; and (4) creating social norms with respect to proof that invite students to prove and provide learning opportunities for students when engaged in proving activities. (p. 247–248)

In regards to (1), mathematics education research often builds on conceptualizations of proof in mathematics (e.g., Davis, 1986; De Villiers, 1990; Rav, 1999) and investigates how individual mathematicians engage with proof (e.g., Inglis et al., 2013; Lew & Mejía-Ramos, 2020; Weber et al., 2022). Less attention is paid to the social mechanisms through which proof functions in the mathematics community and how mathematicians interact with proof and with each other within broader communal activities (e.g., Andersen, 2020; Andersen et al., 2021). In this paper, we elaborated on De Millo et al.'s (1993) view on proofs as successive social processes and argued that these processes unfold in situations that are socially organized, proof-transformative, and sequential. Then, we proposed that mathematicians participate in such proof-centered processes not only to contribute to the discipline but also for self-learning purposes. We believe that further conceptualization and study of disciplinary mechanisms of this sort is paramount to understanding proof as a collective endeavor of the mathematics community. From the social perspective, this understanding is essential to designing learning environments where students engage with proof as mathematicians do.

In regards to (3), this paper stems from a larger developmental project that is centered around proof progressions – an organizational frame that is consistent with De Millo et al.'s view on proofs. The key idea underpinning this frame is that a proof progresses through a sequence of task situations where students engage with it in different ways and capacities. By taking part in each progression phase, the students agree to play according to the rules of the "game of proving", rules that appear similar but not identical in each phase. From some theoretical perspectives, the progression may appear as revolving around "the same" proof, or at least "the same key ideas" (cf. Raman, 2003). However, on a discursive level, our findings show that the progression requires students to generate different proving narratives, when sanctioning each as a "proof" needs to be negotiated each time with a relevant classroom sub-community.

In our project, students' proofs progress from collaborative construction in small groups, through a whole-class presentation at the classroom board by one of the constructors, to a posteriori reflection. In the illustrated case, the progression from the first to the second phase resulted in restructuring of the proof to one more formal, elaborate, and involved the growth of previously non-articulated elements. These aspects are similar to Zazkis et al.'s (2016) findings in real analysis. In that study, individual students translated graphical arguments to written proofs, when this process entailed elaborating, that is, adding more details and spelling out warrants that were implicit; and syntactifying, that is, converting a graphical argument into a verbal-symbolic one. Developments of this sort are common to many progressions in our data corpus, and we believe that they are inextricable from the proofs' shift from an oral to a written communication medium. Indeed, academic mathematical texts are renowned for being dense with terminology and symbols, modest in their use of "grammatical words", having impersonal and authoritative formulations, et cetera (e.g., Morgan, 1998). Then, it is somewhat expected that such experienced students as our participants would write in this way in the presence of a mathematically mature audience. Overall, the presented case illustrates how the focal organizational frame afforded students to engage in task situations with different requirements for proof generation and communication.

Regarding (2), let us start with the teacher-mathematician's angle. In the last progression phase, we showed that Prof B sanctioned Jonah's proof promptly and shifted the discussion to two metarules. She mentioned one of them briefly, while the other was endorsed and yielded another proof transformation. These moves are similar to the previous findings on teacher-mathematicians leveraging classroom proofs to model certain practices for their students (e.g., Fukawa-Connelly, 2012; Viirman, 2021; Weber, 2004). Yet, there is a noteworthy difference: Prof B re-acted to a student's proof that deviated from metarules that she perceived as conventional.

Havighurst (1952) popularized the notion of teachable moments, which refer to a special timing that affords learning and instruction. In this case, the metarules were highly relevant to the proof, and they appeared easy for Prof B to generate. This ease cannot be taken for granted: for experienced participants of a mathematical discourse, its metarules often become the "second skin" to the extent that they are invisible. Furthermore, note that both metarules pertained neither to Hausdorff spaces (the key mathematical object in the proof) nor to routines of proof-generation per se. In this way, the presented case provides evidence that students' public proving can occasion teachable moments with a broad scope of applicability. To put this finding in terms of Stylianides et al. (2017), a student's proof served as a trigger for the teacher-mathematician to generate and communicate relevant mathematical knowledge that the class could miss otherwise.

We considered "what proving is for students" in terms of the tasks that they self-imposed in the proof progression. In the proof construction phase, we saw Grace and Jonah referring to a wordless diagram as "basically it" and "kind of it". The students knew that in their course and beyond, a request for proof is tantamount to the generation of a written narrative. We argued that the generation of such a narrative was not part of the students' task in the first progression phase. Later, the students gradually introduced narrative layers to uncover blind spots in their proposed proof (this can be seen as knowledge generation through the proving activity, point (3)). Similarly, in the second phase, we offered numerous insights in regard to the task that Jonah pursued with his proving at the board.

Let us generalize the previous paragraph with a nod to Stylianides et al.'s (2017) remark, on research within the social perspective currently lacking "common, widely used [theoretical] constructs" (p. 248). The nuances presented in this progression emerged from the analysis of the tasks that Grace, Jonah, and Prof B pursued in proof-requiring task situations. Our attention to students' self-imposed tasks was informed by commognitive research showing time and again that newcomers to a mathematical discourse often do not share tasks with its oldtimers (e.g., Sfard, 2008). Drawing on research with smaller children and school mathematics, Lavie et al. (2019) explain this phenomenon by "a person's interpretation of a given task situation" (p. 161). The presented case elaborates that a task one pursues can be a matter of a deliberate choice. Indeed, Jonah's work at the board shows that his initial satisfaction with a diagram cannot be explained as an idiosyncratic interpretation nor an incapability to write down a formal proof. In a similar fashion, Raman (2003) offers the construct of private and public aspects of a proof, and Lew and Mejía-Ramos's (2020) consider the contextuality of mathematicians' expectations from proofs. Accordingly, we specify Stylianides et al.'s call for the use of theoretical context-sensitive constructs that afford identifying what proving is for provers without making a priori assumptions.

Now we arrive to point (4): did the opportunities provided by the proof progression result in Grace and Jonah learning? The presented analysis captures short-term developments in the students' participation in mathematical discourse, rather than longitudinal changes as per the commognitive definition (Sfard, 2008). In the case of Jonah, these developments concern his limited contribution to the generation of a proving narrative in the first phase compared to a fully-fledged proof he created in the second phase. Some may challenge us on this point, noting that Jonah was the one to generate the original diagram and declare “that’s basically it”. He delivered on this declaration at the board, something that may be explained by him “holding the proof in his mind” all along. We remind the sceptics that commognition operates with communication that rests in publicly accessible spaces and it recognizes the effort that is often needed to switch from inner dialogue with oneself (i.e., thinking) to conversing with others (e.g., recall how Jonah “stumbled” at the board). By volunteering to mathematize publicly, Jonah realized an opportunity to change his command of an academic topological discourse. He missed this opportunity when collaborating with Grace, who did the “heavy lifting” of growing proof narrative. Indeed, in the second progression phase, Jonah not only proved through using conventional keywords, symbols, narratives, and routines, but he did so at the board, which is characteristic to research mathematics (e.g., Kontorovich et al., 2022). A somewhat similar argument can be made regarding Grace’s discursive development. She led the proof growth throughout the interaction to broaden proof components and expand it to new elements. Specifically, Grace’s discourse was enriched by a narrative about the role of the function’s injectivity, a condition that initially appeared redundant. In this way, we argue that the presented case illustrates how students can actualize learning opportunities to change their mathematical discourses that the proof progression affords.

Overall, the proof transformations and discursive developments that we witnessed throughout our project appear sufficiently promising for us to pursue studying proof progressions in additional courses and student cohorts. We notice that this organizational frame is student-centered to a significant extent, and it is consistent with the goals of many teacher-mathematicians in proof-based courses (cf. Melhuish et al., 2022). Its design is flexible enough to pursue a range of didactical goals and it can be embedded in a classroom without necessarily revising the structure of the whole course. Accordingly, we invite mathematics educators and teacher-mathematics to join us in further explorations of how students’ proofs can progress in university classrooms.

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. Thompson (Eds.), *Research in Collegiate Mathematics Education VII* (pp. 63–92). American Mathematical Society.

Andersen, L. E. (2020). Acceptable gaps in mathematical proofs. *Synthese*, 197, 233–247. https://doi.org/10.1007/s11229-018-1778-8

Andersen, L. E., Johansen, M. W., & Sørensen, H. K. (2021). Mathematicians writing for mathematicians. *Synthese, 198*(26), 6233–6250. https://doi.org/10.1007/s11229-019-02145-5

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication, 28*(4), 345–379. https://doi.org/10.1177/0741088311419630

Artigue, M. (2021). Mathematics education research at university level: Achievements and challenges. In V. Durand-Guerrier, R. Hochmuth, E. Nardi, & C. Winsløw (Eds.), *Research and development in University Mathematics Education* (pp. 3–21). Routledge.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education, 45*, 797–810. https://doi.org/10.1007/s11858-013-0506-6

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer Academic Publishers.

Brown, S. (2018). Difficult dialogs about generative cases: A proof script study. *Journal of Mathematical Behavior, 52*, 61–76. https://doi.org/10.1016/j.jmathb.2018.02.002

Chan, M. C. E., & Sfard, A. (2020). On learning that could have happened: The same tale in two cities. *The Journal of Mathematical Behavior, 60*, 100815. https://doi.org/10.1016/j.jmathb.2020.100815

Cilli-Turner, E. (2017). Impacts of inquiry pedagogy on undergraduate students conceptions of the function of proof. *The Journal of Mathematical Behavior, 48*, 14–21. https://doi.org/10.1016/j.jmathb.2017.07.001

Coppin, C. A., Mahavier, W. T., May, E. L., & Parker, E. (2009). *The Moore method: A pathway to learner-centered instruction*. MAA.

Davis, P. J. (1986). The nature of proof. In M. Carss (Ed.), *Proceedings of the 5th International Congress on Mathematical Education* (pp. 352–358). Springer Science + Business Media, LLC.

Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *The Journal of Mathematical Behavior, 31*(3), 331–343. https://doi.org/10.1016/j.jmathb.2012.02.002

De Millo, R. A., Lipton, R. J., & Perlis, A. J. (1993). Social processes and proofs of theorems and programs. In T. R. Colburn, J. H. Fetzer, & T. L. Rankin (Eds.), *Program verification* (pp. 297–319). Kluwer Academic Publishers.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

Dorier, J. L., & Maaß, K. (2020). Inquiry-based mathematics education. *Encyclopedia of mathematics education*, 384–388. https://doi.org/10.1007/978-3-030-15789-0

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics, 6*(1), 1–10.

Ernst, D. C., Hodge, A., & Yoshinobu, S. (2017). What is inquiry-based learning? *Notices of the American Mathematical Society, 64*(6), 570–574. https://doi.org/10.1090/noti1536

Fukawa-Connelly, T. (2012). A case study of one instructor's lecture-based teaching of proof in abstract algebra: Making sense of her pedagogical moves. *Educational Studies in Mathematics, 81*, 325–345. https://doi.org/10.1007/s10649-012-9407-9

Fukawa-Connelly, T., Weber, K., & Mejía-Ramos, J. P. (2017). Informal content and student note-taking in advanced mathematics classes. *Journal for Research in Mathematics Education, 48*(5), 567–579. https://doi.org/10.5951/jresematheduc.48.5.0567

Gallagher, K., & Engelke Infante, N. (2021). A case study of undergraduates' proving behaviors and uses of visual representations in identification of key ideas in topology. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-021-00149-6

Grundmeier, T. A., Retsek, D., Berg, A., Mann, S., & Prieto, H. (2022). Assumptions and definition use in an inquiry-based introduction to proof course. *Primus, 32*(1), 1–13. https://doi.org/10.1080/10511970.2020.1827321

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.

Havighurst, R. J. (1952). *Human development and education*. Longmans, Green.

Hemmi, K. (2006). *Approaching proof in a community of mathematical practice* (Doctoral dissertation). Stockholm University.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405–462. https://doi.org/10.54870/1551-3440.1225

Herbst, P., Nachlieli, T., & Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry. *Cognition and Instruction*, *29*(2), 218–255. https://doi.org/10.1080/07370008.2011.556833

Inglis, M., Mejia-Ramos, J. P., Weber, K., & Alcock, L. (2013). On mathematicians' different standards when evaluating elementary proofs. *Topics in Cognitive Science, 5*(2), 270–282. https://doi.org/10.1111/tops.12019

Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17–36). PME.

Johnson, E., Keller, R., & Fukawa-Connelly, T. (2018). Results from a survey of abstract algebra instructors across the United States: Understanding the choice to (not) lecture. *International Journal of Research in Undergraduate Mathematics Education, 4*, 254–285. https://doi.org/10.1007/s40753-017-0058-1

Jones, F. B. (1977). The Moore method. *The American Mathematical Monthly, 84*(4), 273–278.

Karavi, T., Mali, A., & Avraamidou, L. (2022). Commognition as an approach to studying proof teaching in university mathematics lectures. *EURASIA Journal of Mathematics Science and Technology Education, 18*(7), 1–10. https://doi.org/10.29333/ejmste/12173

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–15). Lawrence Erlbaum.

Kleiner, I. (1991). Rigor and proof in mathematics: A historical perspective. *Mathematics Magazine, 64*(5), 291–314.

Kontorovich, I. (2018). Unacceptable discrepancy: The case of the root concept. *For the Learning of Mathematics*, *38*(1), 17–19.

Kontorovich, I. (2021). Minding mathematicians' discourses in investigations of their feedback on students' proofs: A case study. *Educational Studies in Mathematics, 107*(2), 213–234. https://doi.org/10.1007/s10649-021-10035-2

Kontorovich, I. (2023). When learning stumbles upon identity and affect: A loaded collaboration in Linear Algebra. *International Journal of Mathematics Education in Science and Technology*. https://doi.org/10.1080/0020739X.2023.2173102

Kontorovich, I., & Greenwood, S. (2022). Mathematics learning through a progressive transformation of a proof: A case from a topology classroom. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 2437–2445). ERME/Free University of Bozen-Bolzano.

Kontorovich, I., & Liu, N. (2023). Students leverage their struggles with proof to script fictional dialogues about the rules of proving. *In Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. RUME.

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior*, *31*(1), 149–161. https://doi.org/10.1016/j.jmathb.2011.11.002

Kontorovich, I., L'Italien-Bruneau, R., & Greenwood, S. (2022). From "presenting inquiry results" to "mathematizing at the board as part of inquiry": A commognitive look at the familiar practice. In R. Biehler, G. Gueudet, M. Liebendörfer, C. Rasmussen, & C. Winsløw (Eds.), *Practice-oriented research in tertiary mathematics education: New directions* (pp. 491–512). Springer.

Krantz, S. (2015). *How to teach mathematics* (3rd ed.). American Mathematical Society.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129–146. https://doi.org/10.1007/s40753-019-00085-6

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406–418. https://doi.org/10.5951/jresematheduc.45.4.0406

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*, 153–176. https://doi.org/10.1007/s10649-018-9817-4

Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 127–135). Kluwer Academic Publishers.

Lew, K., & Mejía-Ramos, J. P. (2020). Linguistic conventions of mathematical proof writing across pedagogical contexts. *Educational Studies in Mathematics, 103*, 43–62. https://doi.org/10.1007/s10649-019-09915-5

Manin, Y. I. (1977). *A course in mathematical logic*. Sage.

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we know now and what we have yet to learn. *Journal of Mathematical Behavior, 67*, 100986. https://doi.org/10.1016/j.jmathb.2022.100986

Morgan, C. (1998). *Writing mathematically: The discourse of 'investigation'*. Routledge.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16*(2), 182–198. https://doi.org/10.1080/14794802.2014.918338

Nisbett, R. E., & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review, 84*(3), 231–259. https://doi.org/10.1037/0033-295X.84.3.231

Paoletti, T., Krupnik, V., Papadopoulos, D., Olsen, J., Fukawa-Connelly, T., & Weber, K. (2018). Teacher questioning and invitations to participate in advanced mathematics lectures. *Educational Studies in Mathematics, 98*, 1–17. https://doi.org/10.1007/s10649-018-9807-6

Pinto, A. (2019). Variability in the formal and informal content instructors convey in lectures. *The Journal of Mathematical Behavior, 54*, 100680. https://doi.org/10.1016/j.jmathb.2018.11.001

Pritchard, D. (2010). Where learning starts? A framework for thinking about lectures in university mathematics. *International Journal of Mathematical Education in Science and Technology, 41*(5), 609–623. https://doi.org/10.1080/00207391003605254

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*(3), 319–325. https://doi.org/10.1023/A:1024360204239

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica, 3*(7), 5–41. https://doi.org/10.1093/philmat/7.1.5

Ross, P. (2007). R. L. Moore: Mathematician & teacher. *The Mathematical Intelligencer, 29*, 75–79. https://doi.org/10.1007/BF02986178

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*(2), 123–151. https://doi.org/10.1007/BF01274210

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4–13. https://doi.org/10.2307/1176193

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Stewart, S., Thompson, C., & Brady, N. (2017). Navigating through the mathematical world: uncovering a geometer's thought processes through his handouts and teaching journals. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education* (pp. 2258–2265). DCU Institute of Education and ERME.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 237–266). National Council of Teachers of Mathematics.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30*(2), 161–177. https://doi.org/10.1007/0-387-29831-2_3

Viirman, O. (2021). University mathematics lecturing as modelling mathematical discourse. *International Journal of Research in Undergraduate Mathematics Education, 7*, 466–489. https://doi.org/10.1007/s40753-021-00137-w

Wagner, J. (1997). The unavoidable intervention of education research: A framework for reconsidering researcher-practitioner cooperation. *Educational Researcher, 26*(7), 13–22. https://doi.org/10.3102/0013189X026007013

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior, 23*, 115–133. https://doi.org/10.1016/j.jmathb.2004.03.001

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematics Education in Science and Technology, 43*, 463–482. https://doi.org/10.1080/0020739X.2011.622803

Weber, K., Mejía-Ramos, J. P., & Volpe, T. (2022). The relationship between proof and certainty in mathematical practice. *Journal for Research in Mathematics Education, 53*(1), 65–84. https://doi.org/10.5951/jresematheduc-2020-0034

Wetherll, M., & Potter, J. (1988). Discourse analysis and the identification of interpretative repertoires. In C. Antaki (Ed.), *Analysing everyday explanation: A casebook of methods*. Sage Publications.

Wood, C., & Weber, K. (2020). The relationship between mathematicians' pedagogical goals, orientations, and common teaching practices in advanced mathematics. *The Journal of Mathematical Behavior, 59*, 100792. https://doi.org/10.1016/j.jmathb.2020.100792

Yoo, S., & Smith, C. (2007). J. Differences between mathematics majors' view of mathematical proof after lecture-based and problem-based instruction. In T. Lamberg and L. R. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 84–86). University of Nevada.

Zazkis, D., Weber, K., & Mejía-Ramos, J. P. (2016). Bridging the gap between graphical arguments and verbal-symbolic proofs in a real analysis context. Educational Studies in Mathematics, 93,155–173. https://doi.org/10.1007/s10649-016-9698-3