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The Teaching and Learning of Definite Integrals: A *Special Issue Guest Editorial*

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Note from the Editors: Following the Calculus in Upper Secondary and Beginning University Mathematics Conference (Kristiansand, August 6–9, 2019), the Editors received a proposal for a Special Issue on the theme of "The Teaching and Learning of Definite Integrals". The Editors were delighted to accept this proposal. IJRUME co-editor in chief Elena Nardi, also co-chair of the conference with Tommy Dreyfus and John Monaghan, acted as handling editor of the Special Issue. In what follows, Guest Editors Rob Ely and Steven R. Jones introduce its theme and contents.

Introduction to Definite Integrals and to this Special Issue

The definite integral[1] is a central topic in undergraduate mathematics education, as it ranges from introductory calculus through upper-division university mathematics coursework. It is also a crucial concept in science, engineering, economics, and other disciplines, as it is used to model, compute, and define many quantities and systems in those fields. Consequently, we posit that reasoning with definite integrals is a key skill for students to develop in the undergraduate mathematics curriculum. We note that, with the widespread use of computers in the 21st century, techniques for *evaluating* integrals are diminishing in importance for the general calculus student

[1] Integrals come in many varieties, including integrals with fixed-finite bounds, $\int_a^b f(x)\,dx$, integrals with variable bounds, $\int_a^x f(t)\,dt$, integrals with infinite bounds, $\int_a^\infty f(x)\,dx$, and integrals with no bounds, $\int f(x)\,dx$. This special issue on "definite integrals" focuses on those with fixed bounds or variable bounds: $\int_a^b f(x)\,dx$ and $\int_a^x f(t)\,dt$.

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population. Rather, the ability to *interpret* integrals in context and to *model* quantitative structures using them remains crucially important for students to develop when learning calculus. In light of this, calculus education has seen an increased focus on research about learning, understanding, and reasoning with definite integrals. This special issue marks, to our knowledge, the first instance that research about the teaching and learning of definite integrals has been aggregated into one collection. This special issue also provides a rounded set of perspectives across current definite integral research, from snapshots of student understanding following instruction (Kontorovich, this issue; Nilsen & Knutsen, this issue), to a progression of student learning (Stevens & Jones, this issue), to important ways of reasoning with integrals (Jones & Ely, this issue; Oehrtman & Simmons, this issue), and to the application of integrals to science contexts (Bajracharya et al., this issue; Dray & Manogue, this issue).

The idea for this special issue emerged from the *Calculus in upper secondary and beginning university mathematics* conference held in August 2019 in Kristiansand, Norway. In conversations at this conference, it became clear that valuable research was being conducted by a variety of scholars about the teaching and learning of definite integrals, but that these studies were dispersed in various journals and books and were often disconnected from each other. One purpose of this special issue is to bring current research studies into direct conversation with each other. The variety of articles contained in it helps accomplish this goal. This special issue also includes a review of this growing body of research (Jones & Ely, this issue), to help organize, synthesize, and chart out some of the landscape in this field. We hope that this special issue will become a resource for calculus educators and researchers to refer to and to build on, ultimately benefitting the millions of students each year who take calculus courses worldwide.

How these Special Issue Papers Fit into the Broader Literature

While we might speak of "the definite integral" as a singular concept, it has many possible meanings and interpretations (Greefrath et al., 2021; Hall, 2010; Jones, 2013, 2020; Sealey, 2006). One central theme of recent research, and one that is echoed in this special issue, is the role of quantitative reasoning in student thinking about definite integrals. The abilities to recognize relevant quantities that vary in a context, and to assign quantitative meaning to the pieces of integral notation that measure those quantities, have been found to be important for supporting students' usage of integration in context (Chhetri & Oehrtman, 2015; Ely, 2017; González-Martín, 2021; Hu & Rebello, 2013b; Jones, 2015a; Nguyen & Rebello, 2011; Sealey, 2014; Simmons & Oehrtman, 2017). Yet, this kind of quantitative reasoning is unfortunately less common among calculus students (Bressoud, 2009; Jones, 2015b). For instance, a variety of studies in the U.S. have found that students typically leave calculus courses able to interpret definite integral notation $\int_a^b f(x)\,dx$ only as a request to find an anti-derivative for $f(x)$ and evaluate it at b and a, or as a reference to the geometric area contained in a fixed shape bounded by a curve, but rarely as a sum of some kind (Bezuidenhout & Olivier, 2000; Christensen & Thompson, 2010; Grundmeier et al., 2006; Jones, 2015b; Jones et al., 2017; Marrongelle, 2001; Thompson &

Silverman, 2008). These "antiderivative" and "area" interpretations of integrals may be less productive in applying integrals to contextualized problems or quantitative situations (Hu & Rebello, 2013b; Jones, 2015a; Nguyen & Rebello, 2011; Pina & Loverude, 2019; Simmons et al., 2022). Recent years have marked several efforts to develop strong quantitative meanings that can support students' modeling with definite integrals (Ely, 2017, 2020; Samuels, 2022; Sealey & Engelke, 2012; Thompson & Ashbrook, 2019; Thompson et al., 2013; Von Korff & Rebello, 2012).

In this special issue's first article, Jones and Ely discern two main paradigms in the literature for these quantitatively grounded efforts: *adding up pieces* and *accumulation from rate*. The literature review portion of their paper organizes the literature around these two approaches and specifies each approach's meanings, formalizations, foci, and types of reasoning. The theoretical analysis portion of their paper extrapolates from the literature review to compare and contrast what modeling with definite integrals might look like within each paradigm.

Building in this area, the paper by Oehrtman and Simmons provides a detailed example that falls within the *adding up pieces* paradigm. The authors study a set of productive quantitative meanings students draw upon when modeling with integrals. Through a series of teaching experiments and interviews, they identify a sequence of emergent quantitative models students used: basic models, local models, and global models. Informed by these, Oehrtman and Simmons idealize a productive model for reasoning with definite integrals they call Quantitatively-Based Summation (QBS). This includes the specific conceptions that comprise this productive reasoning as well as the processes found to be important for developing these conceptions.

What might it mean to learn integrals in this way? Along with Oehrtman and Simmons's paper, two other papers in this special issue examine this question. In one paper, Stevens and Jones describe a progression of students' learning based on context and quantitative reasoning to develop such meanings of the definite integral. Their study examines learners across an entire teaching unit based on *adding up pieces* that proceeds from the first introductory lesson up through an understanding of integral functions in preparation for the Fundamental Theorem of Calculus[2] (FTC). In another paper, Dray and Manogue extend the topic of learning integrals into a discussion of how one might develop an understanding of line integrals that are found within multivariable calculus. Through an examination of textbooks and research literature, they provide a theoretical discussion on possible lower and upper anchors for learning line integrals, and describe a hypothetical learning trajectory[3]. Their discussion adds an important perspective by discussing line integrals in connection to both mathematics and physics.

There is a strong theme within the existing literature on using differential- and infinitesimal-based understandings and reasonings in connection with definite integrals (Amos & Heckler, 2015; Chhetri & Oehrtman, 2015; Ely, 2020; Hu & Rebello,

[2] The two parts of the Fundamental Theorem of Calculus are often given as: (1) If $f(x)$ is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ on (a,b); and (2) If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

[3] In brief, a hypothetical learning trajectory consists of (1) the goals for student learning, (2) the activities used to promote learning toward the goals, and (3) a hypothesis about how students' learning would progress toward the goal (see Simon, 1995).

2013a; Schermerhorn & Thompson, 2019; Simmons et al., 2022; Thompson & Ashbrook, 2019; Von Korff & Rebello, 2014). That is, differentials and infinitesimals may better support thinking of quantities and their relationships and may prevent the problematic "collapse" metaphor, in which students believe the quantity represented in the differential to have disappeared entirely (Ely, 2012; Oehrtman, 2009). In this vein, Nilsen and Knusten's paper in this issue examines understandings of students who had experienced a strongly limit-based calculus curriculum. The authors report that, despite the curriculum, much of the students' conceptual interpretations of integrals and the FTC were more closely associated with differentials and infinitesimals, in a way compatible with meanings used by Leibniz and his contemporaries (Katz, 2009).

Given the existing literature on definite integrals, perhaps a more fundamental question arises: Is a definite integral an "area under a curve"? Calculus students commonly enough develop such a view of an integral as a unitary area of a Cartesian shape, an understanding that might lead to accurate or inaccurate generalizations alike about properties of definite integrals (Czarnocha et al., 2001; González-Martín, 2005; Jones, 2020; Jones & Dorko, 2015; Kouropatov & Dreyfus, 2013; Rasslan & Tall, 2002; Sealey, 2006, 2014). Students who hold such a view frequently conclude that "the integral is an area, and area is always positive" (Kouropatov & Dreyfus, 2013, p. 643), and that the integral only consistently makes sense when functions are non-negative (Bezuidenhout & Olivier, 2000). Students can also interpret the integral as a total area or an amount rather than net change in an area or amount, in situations where the function is not always positive. Kontorovich's paper in this issue looks at this phenomenon by analyzing a large data set of examination papers and video clips from undergraduate students. About 30% of the students' exams papers employed reasoning consistent with the view that a definite integral measures a total area, not a "net area." The video clips reveal how this type of reasoning can operate in the interplay between a student's model of figures (areas), regions, integral notation, and evaluated integrals (algebraic and numeric).

The paper by Bajracharya, Sealey, and Thompson discerns further difficulties with the common area interpretation of integral, by studying "backward" integrals in which the upper bound is less than the lower bound. The reversal of sign entailed in these integrals made little sense to many students who interpreted the definite integral solely as a monolithic area under a curve. On the other hand, the authors report that students who conceptualized Δx or dx as a difference or change were able to treat it as a signed quantity and could make more sense of a negative "backward" integral. Furthermore, within physical contexts, these students were able to leverage this idea to provide meaning to the sign of the entire definite integral.

The articles in this special issue provide a springboard for further research. In particular, they suggest ways to explore how students work with integrals in higher-level mathematics courses and other STEM domains. The constructs and findings offer directions for studying and supporting student reasoning with the FTC. The articles provide empirically testable guidance for improving curriculum and teaching with definite integrals. Brought together in this special issue, we hope that (unlike a definite integral) the reader will find this collection of papers to be even more than the sum of its parts.

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