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Transitions Between Domains of Activity as "Domestications of the Eye" for the Learning of Mathematics with GGBot

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Abstract

We present a theoretical study that allows us to attempt framing in an embodied perspective the effectiveness of the drawing robot GGBot in the learning of geometry. The aim of the article is to set the intertwining of activity, semiotics, perception, and knowledge at the crossover of Radford's theory of objectification (TO) and Borba and Villareal's notion of humans-with-media. Such a crossover is articulated in four building blocks: (1) processes of objectification and the role of semiotic means of objectification, where we state that digital artifacts such as the GGBot change the topology of the semiotic means of objectification; (2) cognition is sensuous and learning is a process of domestication of the eye, where perception is theoretically shaped by the interaction with GGBot; (3) GGBot and humans-with-media, where we outline new thinking collectives and their modes of activity; (4) domestication of the eye triggered by transitions between domains of activity. Each building block of our theoretical discussion is empirically anchored to four episodes involving primary school students' learning geometrical figures using the GGBot. To conclude, we focus on two basic concepts of geometrical thinking that unfold in the shift between domains.

Keywords GGBot · Geometry · Theory of objectification · Humans-with-media · Sensuous cognition · Drawing · Domestication of the eye · Angle · System of Reference

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Digital technologies open new possibilities in the teaching and learning of mathematics. They are molding educational scenarios where digital artifacts play a crucial role in the interaction between individuals and the modes of mathematical activity, thinking, and learning. Exploiting the opportunities provided by technology for teaching and learning requires a rethinking of educational paradigms and strategies (Clark-Wilson, 2010; Sacristán et al., 2010). In the context of mathematics education, mathematical knowledge has the potential to be expanded to powerful digital technologies (DTs), which challenge traditional teaching and learning paradigms and practices. Drijvers et al. (2016) have pointed out three main features of mathematics education affected by DTs: evidence for effect; digital assessment; communication and collaboration.

For the scope of this article, we focus on the effectiveness of DTs in the learning of mathematics, that is, benefits of the integration of DTs in mathematics education paradigms for the improvement of student learning. Our aim is to scrutinize which exact effect one form of DT has on mathematics understanding and learning. There is a body of research on the impact of DTs in learning and teaching mathematics. Heid and Blume (2008) provide cases that describe the development of technology-intensive curricula and tools. The authors describe and analyze the roles that research played in their development work and ways in which research, curriculum development, and tool development can inform each other.

On that note, Clements et al. (2013) make it very clear that, these days, the world of mathematics education is changing very rapidly, and that technology is a major factor influencing the directions of change. They emphasize that recent technological developments are challenging traditional views on curriculum, teaching, learning, and assessment. They look for teaching, learning, and assessment that are the most appropriate given the rapid technological developments. Heid and Blume (2008) identify four categories that evolve across research about technology in teaching and learning mathematics: interaction of teachers and students with technology; changes in curriculum; mathematical activity in technological settings; consequent changes in mathematical thinking as a result of such an activity.

Quantitative research review studies (Cheung & Slavin, 2013; Li & Ma, 2010; Rakes et al., 2010) show significant and positive effects of the implementation of DTs in mathematics education, which nevertheless do not appear to be sufficiently solid in operationalizing the impact of technology in teaching–learning. In fact, the nature of quantitative studies does not allow us to delve into the teaching–learning processes involving DTs. They disregard the type of DT, the educational level, the theoretical frameworks in which DT is inserted, the teaching design, and the mathematical activity the students carry out resorting to the semiotic potential deployed by DT.

Qualitative research meets the need to understand how and why DT is effective in mathematics education by embedding it in suitable theoretical perspectives. Fey (1984) highlights that DT has the potential to alter both what mathematics students learn and how they learn it. Drijvers' (2004) qualitative research shows that, although DT can make the learning of particular mathematical content more easily accessible to students, it can also make that learning problematic. He provides the example of straightforward procedures to solve equations with computer algebra that turned out to produce both technical and conceptual obstacles to students, and the two types of obstacles were shown to be related. This research issue is framed within Rabardel's (2003) instrumental genesis, focusing on the relationship between the student and the tool in computer algebra system environments.

Hadas et al. (2000) designed and conducted a qualitative study in a Dynamic Geometry Software (DGS) to account for the explanations that students gave, within instrumental genesis, beyond student explanations researchers had expected them to offer. The researchers identified a previously unexpected genre of explanations (visual/variational) that either were based on the dynamic displays or stemmed from students' (probably DGS-based) imagery. A possible new norm for mathematical explanations was discovered. To account for student explanations, Leung (2015) highlights that, "the variation in different aspects of a phenomenon unveils the invariant structure of the whole phenomenon" (p. 452). In fact, DGS allows conceptualization by recognizing the operational invariants emerging from the action of dragging geometrical objects in a digital and visual environment. Recognizing operational invariants can lead a learner to transform activity into perceiving conceptual and theoretical aspects of Euclidean geometry.

Qualitative studies (Anabousy & Tabach, 2016; Jacinto & Carreira, 2017; Maschietto, 2018) also address the impact of activity and semiotic representation springing from DT in students' problem-solving skills. The common denominator to the technological approach toward problem solving is the intertwining of technological knowledge and mathematical knowledge that molds students' cognition and solving strategies. The key element in the encounter between mathematical thinking and DT is the access to new forms of action and representations that occur in the relationship between the individual, or groups of individuals, and the technological artifact.

Roberts et al. (2013) and Maschietto and Trouche (2010) remark that the complexity of the interplay among technologies (in general), mathematics, and education is related not only to DT but also to artifacts human beings have used in the cultural and historical development of mathematics. Of course, DTs feature new forms of action and semiotic potentials not even considered possible in the past. The research issue is to investigate the relation between digital and non-digital mediated forms of mathematical activity. Despite advances in DT, there is still strong value in using a combination of physical tools and DT in mathematics education.

In line with the standpoint that recognizes the common thread that links digital and non-digital technology, Borba and Villareal (2005) offer an important contribution to the understanding of the interplay between technology and mathematics education. They characterize the intertwining among individuals, technology, and mathematical knowledge as the emergence of a new social and cultural identity termed *humans-with-media.* Such a notion encourages a merged vision of humans and technology (whether physical artifacts, software, etc.), overcoming the classical and more dichotomous conceptions. More precisely, the authors refer to Lévy's (1993) expression of *thinking collectives* to highlight that knowledge is a product of collectives composed by human and non-human actors.

The present study is positioned in this line of research that looks at the effectiveness of DT in the teaching and learning of mathematics. We carry out our analysis within the notion of *humans-with-media* that frames the interplay between digital and non-digital technologies. We bring the aforementioned studies on this theme a step forward, in order to operationalize the mathematical thinking and learning of *humans-with-media* within a socio-cultural semiotic perspective. Our objective is to continue along the line traced by Jacinto and Carreira (2017), who propose a model for problem solving conceived as a techno-mathematical fluency. The researchers highlight the symbiosis between humans and technology that shapes mathematical thinking into the interplay of mathematical and technological knowledge. The core feature of techno-mathematical fluency is the activity individuals carry out with DT according to their perception of the representational affordances of the digital tool.

The aim of our article is to set out the intertwining of activity, semiotics, perception, and knowledge at the crossover of Radford's (2021) theory of objectification (TO) and Borba and Villareal's (2005) notion of human collectives. We conceive DT as every feasible combination of hardware and software (Del Zozzo & Santi, 2020) and the DT we deal with in this contribution is the robot GGBot (complete name GREATGeometryBot), a 3D-printed drawing robot along with SNAP!, which is the visual programming language that allows humans to communicate with it (more details in the section "Drawing with GGBot") .

In our line of thought, articulated against the backdrop of Radford's TO, mathematical learning is rooted in the student's mathematical activity interwoven with the use of signs and artifacts (e.g., material objects, gestures, icons, drawings, indexical use of language, natural language, symbolic language). We are interested in signs not only for what they *represent*, but rather for what they allow individuals *to do*. Thus, we include as semiotic resources not only structured systems of signs typical of mathematics, but also material objects, gestures, indexical use of language, kinesthetic activity, and rhythm that do not have a specific representational role. The TO conceives learning as a *process of objectification*, that is, the movement toward knowledge on the part of the student to notice and become aware of cultural–historical mathematical knowledge, in terms of understanding and argumentation, even if it does not fit classical and ideal approaches to rationality (Asenova, 2022).

This process allows the student to transform the mathematical object into an object of consciousness. Learning has a strong phenomenological nature where noticing occurs in an enlarged notion of mind and consciousness, termed by the TO *sensuous cognition* (Radford, 2014), that includes not only ideal and mental features, but also embodied ones such as perception, feelings, and kinesthetic activity. The outcome of objectification is the *domestication of the eye* (Radford, 2010), a long process that allows students—in cultural–historical activity intertwined with the use of signs and artifacts—to *transform the eye (and other senses)* into *sophisticated theoretical organs* able to become aware and make sense of mathematical knowledge. Perception is transformed beyond its biological function into a *theoretical perception.*

Borba and Villareal's (2005) idea of *humans-with-media* overcomes the dichotomic conception of humans and technology (in general), instead prompting a blended vision. More precisely, they refer to Lévy's (1993) expression of *thinking collectives* to highlight that knowledge is a product of collectives composed by human and non-human actors:

> We believe that humans-with-media, humans-media or humans-with-technologies, are metaphors that can lead to insight regarding how the production of knowledge itself takes place, in the same way that human being is also a metaphor for the epistemological subject that is so deeply rooted that is assumed, by many, to be natural. [...] In our opinion, this metaphor synthesizes a view of cognition and of the history of technology that makes it possible to analyse the participation of new information technology 'actors' in these thinking collectives in a way that we do not judge whether that is 'improvement' or not but rather identify transformations in practice. In other words, this notion is appropriate for showing how thinking is reorganized with the presence of information technologies, and what types of problems are generated by collectives that include humans and media such as paper-and-pencil, or various information technologies. (Borba & Villareal, 2005, p. 23)

As the authors state, to such deeply entangled relation between the thinking collectives and the knowledge production follows that:

> a new technology of intelligence results in a new collective that produces new knowledge, which is qualitatively different from the knowledge produced by other collectives. Once writing was introduced, orality was also transformed. As memory was extended to paper it was possible for theories to be born. In the case of mathematics, the opportunity emerged for long demonstrations to be developed and stored. It is relevant to note that writing did not abolish orality. On the contrary, it created what Lévy (1993) labeled secondary orality, which would be orality related to reading what has been written. (p. 24)

If *humans-with-media* focuses on the interaction between individuals and technology, there is no characterization on how such an interaction occurs, or why it is effective in mathematics education in a cultural–historical semiotic approach. On the other hand, if the TO is a robust and profound theory that encompasses the complexity of mathematical thinking and learning, taking into account perception, agency, and signs and artifacts, there is no specific reflection on the role of DT in the processes of objectification. DT has been studied within the TO (Prieto & Arredondo, 2021; Salinas & Miranda, 2018), but as one of the possible signs and artifacts in teaching–learning activities that allow students to become acquainted with mathematical knowledge. The focus is not on the technological artifact per se, but rather on forging perception to think mathematically.

The research issue we tackle in this contribution is about the role of DT in teaching and learning processes. More precisely, we investigate how theoretical perception is transformed when digital technologies come to supplement initial learning with non-digital technology. The TO broadens and operationalizes the notion of *humans-with-media* by conceiving it as the intertwining among activity, signs and artifacts, and mathematical knowledge. The point we want to investigate is if the introduction of a DT changes the topology of the signs and artifacts that mediate students' mathematical activity, or if DT is another artifact we add to the traditional non-digital ones. If the first option holds true, the notion of *humans-with-media* unfolds in two possible distinct although dialoguing subjectivities, *humans-with-non-digital-technology* and *humans-with-digital-technology*, where non-digital technology refers to artifacts in general according to Roberts et al. (2013) mentioned above.

Our theoretical view is in favor of the first option, and we ask ourselves how embodied cognition is affected by digital and non-digital technology. We remark that we are not establishing a dichotomy between digital and non-digital resources. Our objective is to characterize mathematical activity once a digital technology is included in non-digital environments. We also focus on the intertwining between non-digital contexts and contexts that combine digital and non-digital technologies. Such a distinction allows us to frame transitions between domains when digital technologies are involved and their impact on the learning of mathematics in an embodied cognition stance.

We propose a theoretical framework developed at the crossover of the TO and *humans-with-media*, consisting of four building blocks: processes of objectification; cognition is sensuous; GGBot and *humans-with-media*; domestication of the eye as transitions between domains of activity. The theoretical framework characterizes *humans-with-media* as a domestication of the eye in different domains shaped by a semiotic topology that includes digital and non-digital technologies. Learning with DT (GGBot) is understood as processes of objectification that occur as transitions between different domains, with or without the presence of DT.

In the following section, we describe the GGBot, after which we present four episodes with primary school students exposed to geometry activities with the robot, introduce the four building blocks, and finally discuss and draw conclusions.

Drawing with GGBot

The DT we deal with is the drawing robot GGBot, which is a descendant of the original Papert's drawing turtle (Baccaglini-Frank et al., 2020). GGBot's hardware component assembles the 3D-printed electronic structure, which contains an Arduino controller to activate two motors connected to its wheels, and a wireless communicator, which communicates over a dedicated channel. The GGBot presents two marker-holders, one at each end, where one can insert one or two markers to let it draw (Fig. 1a).

As highlighted in Baccaglini-Frank et al. (2020), GGBot is an artifact explicitly designed for the teaching and learning of geometry: the physical properties of the GGBot's design allow us to control the geometrical properties of the resulting drawing. Rubber rings around its wheels touch the ground in a small area (evoking a point); the drawing marker is inserted into a prism with a square base and the drawing tip of the marker comes out exactly at the square's center, touching the ground at the mid-point between the points of contact between the wheels and the ground, which is also the pivot point around which the GGBot rotates in the basic "turn" commands, evoking the vertex of an angle (Fig. 1c).

These features are designed so that the human interaction with the GGBot can have a multimodal nature, that is, it involves more than one sensory channel linked

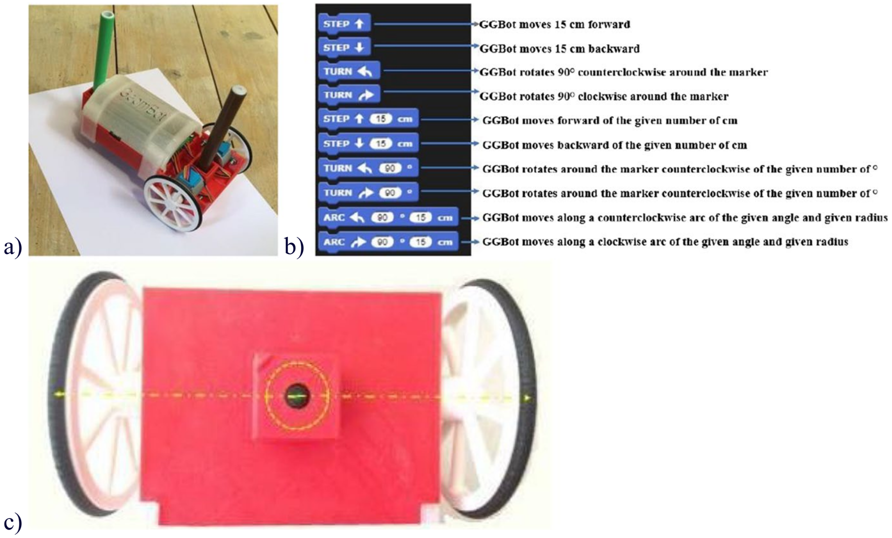


Fig. 1 **a** View of the GGBot; **b** SNAP! commands list, each with the explanation of the correspondent GGBot movement; **c** detail of the GGBot underside with the marker tip halfway between the wheels

to the kinesthetic activity. The drawing robot allows students to be exposed to the geometrical meanings of point, distance, and angle. The software component can be implemented through the visual programming language SNAP!, and an internet browser that lets it run. More precisely, SNAP! makes available specific commands that run on the GGBot, as for example the 10 listed in Fig. 1b. Each combination of SNAP! commands corresponds to a sequence of GGBot movements that, if the marker is threaded, will generate a drawing.

So far, we have provided a naïve description of the GGBot drawing robot as a digital artifact, but we need to elaborate more in-depth on the drawing act of a geometrical figure. In fact, such an act has different specifics whether it is performed by the movement of a human being's hand or by the movement of a GGBot (both possibly equipped by a marker).

Drawing Act Performed by Human Being Hand

For a human being, to draw a certain geometrical entity involves the following steps: at first, evoking an imaginative perception of the geometrical entity, building on that person's previous experiences; then, it requires taking the marker with the hand; finally, that person has to reproduce on the paper the imaginative perception through a sequence of traits that may or may not be linked to one another (when a human is drawing something, she can easily detach the marker from the paper).

The previous procedure lets the marker (guided by the will of the human being) leave a trace on the paper. Moreover, performing a drawing of a geometrical figure by the human hand does not necessarily require linguistic control. For example, we can draw a square on a sheet of paper without wording the process with, “draw a segment, draw a second segment congruent and perpendicular to the first one, draw a third segment such that …” This type of drawing act contributes to shaping the students’ perception of geometrical figures in the “traditional” paper-and-pencil geometry. We can call the result *human drawing*.

Drawing Act Performed by GGBot

It sounds obvious to say this, but GGBot has no will and no autonomous previous experience in geometry. Moreover, it needs a human being to work (and to interact with a human being also needs an internet connection and SNAP!). This means that the drawing act starting step is, again, the human’s evoking of an imaginative perception of the geometrical entity, building on her previous experiences. Nevertheless, the human needs straightaway to modify their perception of the geometrical figure, since they have to insert a list of SNAP! commands, taking into account that such commands will be executed by the GGBot as a concatenated sequence of movements. More precisely, starting from the overall image of the geometrical figure, they have to imagine the first step to start the drawing, which then has to be expressed as a SNAP! command. After the first step, they need to imagine what the next step needs to be, and then express it into a command, and so on and so forth until the (imagined) drawing has been accomplished.[1] Finally, possibly after inserting the marker in the GGBot, the SNAP! code can be run. Each movement of the GGBot allows the marker to leave a trace on the paper. Once the movement ends (i.e., when the GGBot stops), the marker will have left a series of concatenated traces on the paper. We can call the result the *GGBot drawing*.

The creation of the *GGBot drawing* requires a constant back-and-forth movement between the students’ perception of geometrical figures forged by the *human drawing* experience and the activity intertwined with the GGBot and its features as digital technology. Differently from the *human drawing* case, performing a *GGBot drawing* of a geometrical figure requires linguistic control of the action of drawing. For instance, to draw one side of a square, one SNAP! command “step forward” can be used. To draw another side, perpendicular to the first one, a pair of commands is needed: a 90° rotation (for the first one, it is irrelevant whether the rotation is clockwise or counter-clockwise) and another step forward. To draw a third side, another pair of commands is needed: a 90° rotation (clockwise or counter-clockwise depending on the previous one) and another step forward, and so on.

So far, we have provided a naïve description of the GGBot as a DT and some first reflections regarding the drawing act, whether it is performed by the movement of a human being’s hand or by the movement of the GGBot. A

[1] The authors would like to thank one of the anonymous referees for sharing this insightful reflection.

complexity already emerges, which, by means of the empirical examples we describe in the next section and by the following theoretical elaboration, we aim to disentangle.

Empirical Examples

In this section, we describe four episodes involving four primary school students, which we consider emblematic in anchoring our theoretical discussion. The episodes are drawn from one session of a sequence of three that were conducted by Anna Baccaglini-Frank (Baccaglini-Frank & Mariotti, 2022). Each session lasted a fixed time-frame of one school period (about 45 min) and involved one class of grade 3 students (8 years old), the classroom teacher, and the researcher. In this contribution, we focus on the first of the three sessions, which allows us to detect the genesis of the new geometrical perception enabled by GGBot. We consider some video-recorded sequences where students, the teacher, and the researcher were seated in a circle around a large sheet of paper and worked in groups by intervening in turn.

In the transcription that follows, we use pseudonyms for students, R for the researcher, and the abbreviation SoR stands for System of Reference. Transcripts include not only utterances, but also non-verbal information (such as gestures, signs, and drawings). In order to take into account the semiotic complexity and to be faithful to the synchronicity between signs, we present the data in tables with three columns: one for the gestures and non-verbal signs; the second for the utterances; the third for the marker trace analysis (in Table 1) and for the drawings in the other cases (in Tables 2 and 3, the drawings are just for the reader's convenience since students traced the figure with a finger on the paper). We also numbered the lines, using the same number to indicate simultaneity.

During the session we are considering in this article, students were asked to carry out various types of tasks. First, activities, guided by questions, to explore the movement of the GGBot are conducted: such a type of task is the one involved in Episode 1 that follows.

Episode 1: Serena During the Exploration of GGBot

The GGBot executed a series of commands (two steps forward, a 90° clockwise turn and a further step forward), and both its movement and the trace it left on the paper by the marker are observed and then analyzed by the students. The subsequent extract regards Serena, and it refers to the following trace (Fig. 2):

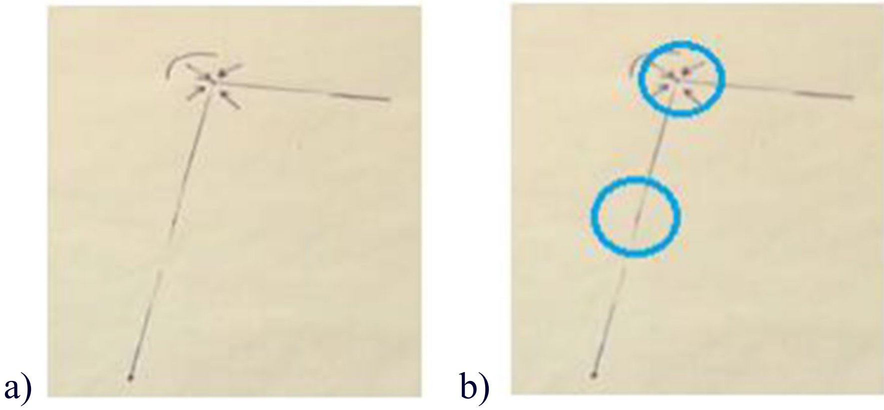
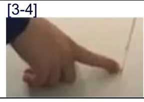
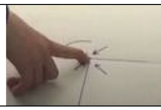


Fig. 2 **a** Trace on the paper analyzed by Serena; **b** for the reader's convenience, we highlight with blue circles the position of two particular points in the trace: the first one (from the bottom) is due to the separation between the first two steps forward (it will be circled by the researcher during the dialogue with the student at line 6 in Table 1); the second one is due to the GGBot's rotation and it had been highlighted by the researcher with four arrows and an arc before the dialogue with the student

Table 1 Serena during the exploration of GGBot, transcripts

Gestures and non-verbal signs	Utterances	Marker trace analysis
[1] Serena indicates the GGBot, lying next to the student on the floor, with her finger. [1] R nods.	[1] Serena: Instead, in my opinion, the little robot when you put in it this (she means the marker)	
[2] Serena slides her finger along the trace of the marker on the paper.	[2] Serena: it has done the two steps	
[3] Serena points her finger on the middle point that separates the two steps. [4] Serena punctuates the words by rhythmically tapping her finger three times in the same spot.	[3] Serena: but here, also here that you can see here, it stopped [4] Serena: and the marker was always on the same point	[3-4] 
[5] Serena moves her finger away from the paper.		
[6] R takes the marker and draws a circle around the point that separates the two steps with an arrow that points to it.	[6] R: so, you are saying…explain well why do you say that also in this point you know that the robot was still. What do you see it from?	
[7] Serena points again her finger on the middle point that separates the two steps (which now has also been circled by R).	[7] Serena: (I see it) thanks to the most pressed point because the marker was still	[7] 
[8] Serena moves her finger away from the paper.		
[9] Serena points her finger also in the center of the 90° rotation that the robot has made.	[9] Serena: and also here, the marker was still	

After the GGBot technical exploration, students continued working in pairs and they were asked to use SNAP! to code the movement of the GGBot that would lead it to draw with the marker a certain given figure, first a square, then a rectangle, and then the letter “V” (figure-to-code tasks). Each SNAP! code was then projected on the wall, asking the class whether it was correct or not, in order to let the GGBot draw the requested figure. Episode 2 that follows refers to code validation in the case of a square.

Episode 2: Daniele in a Figure-to-Code Task

The SNAP! code proposed by one pair was projected on the wall, and students were asked to determine whether the projected code allows the GGBot to actually draw a square. The code under analysis is like the following (Fig. 3). For the reader’s convenience, we recall in Fig. 3 the commands explanation (the explanation was not projected nor shared with students during the session).

To perform the code validation, Daniele traced the figure he thought the GGBot would draw, by drawing with his finger on the paper.

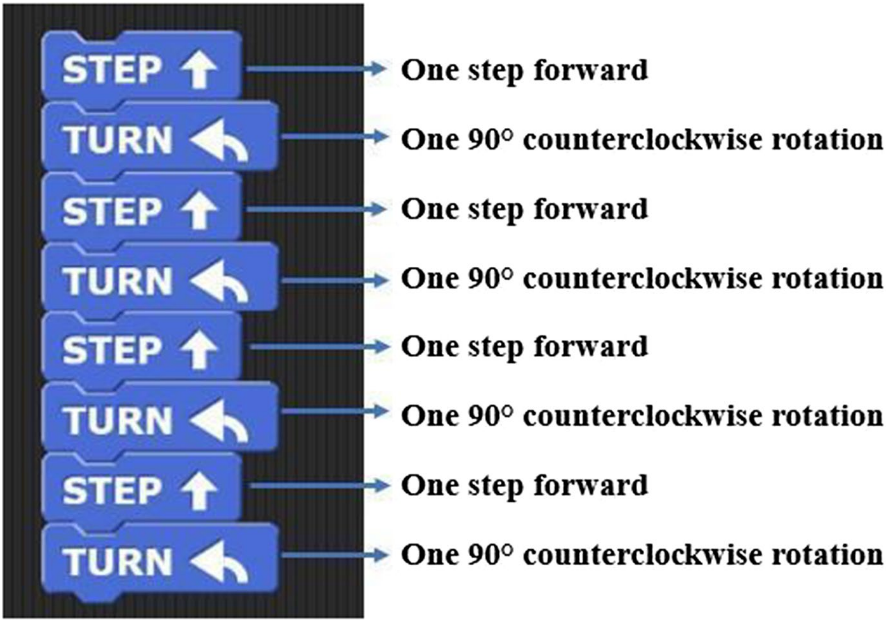
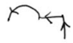


Fig. 3 List of commands under analysis (with their explanation) in Episode 2

Table 2 Daniele in a figure-to-code task, transcripts

Gestures and non-verbal signs	Utterances	Daniele's drawings (with his finger)
	[1] Daniele: In my opinion it is not correct because …	
[2] Daniele approaches the center, points his finger at the paper, and traces a little line in the vertical direction (according to his egocentric SoR) moving his finger away from himself.	[2] Daniele: … it takes a step	[2] 
[3] Daniele traces on the paper a line in the horizontal direction (according to his egocentric SoR) towards his left.	[3] Daniele: it goes on the left	[3] 
[4] Daniele gazes at the projected code, staring it for 2 seconds.	[4] Daniele: then …	
[5] Daniele gazes at his hand on the paper, which remains still.	[5] Daniele: and then … then then it does … it goes forward	
[6] Daniele rotates his index in a counter-clockwise direction using the thumb as a pivot .	[6] Daniele: and it does like that	[6] 
[7] Daniele gazes again at the projected code.		
	[8] Daniele: Wait, wait	
[9] Daniele points his finger on the paper and traces a line in the vertical direction (according to his egocentric SoR) moving away from himself.	[9] Daniele: it takes a step	[9-10] 
[10] Daniele keeps his hand still.	[10] Daniele: it goes on the left	
[11] Daniele rotates his hand in the counter-clockwise direction using his index as a pivot , and then he gazes again at the projected code.	[11] Daniele: it turns on the left [11] R: yes	[11] 
[12] Daniele gazes on his hand on the paper and with his index traces a line in the horizontal direction (according to his egocentric SoR) towards his left.	[12] Daniele: it takes a step [12] R: yes	[12] 
[13] Daniele rotates his hand in the counter-clockwise direction iterating the same movement with his hand as before, using his index as a pivot.	[13] Daniele: it turns on the left [13] R: yes	[13] 
[14] Daniele traces a line in the vertical direction (according to his egocentric SoR) towards himself.	[14] Daniele: it takes a step [14] R: yes	[14] 
[15] Daniele rotates his hand in the counter-clockwise direction iterating the same movement with his hand as before, using his index as a pivot.	[15] Daniele: it turns on the left [15] R: yes	[15] 
[16] Daniele traces a line in the horizontal direction (according to his egocentric SoR) towards his right.	[16] Daniele: it takes a step [16] R: yes	[16] 

In the last part of the session, the students were involved in another type of task, a collective one where they could answer in turn. Starting with a given SNAP! code, they were asked to predict the GGBot's movement and, consequently, foresee the trace that the marker would have left on the paper (code-to-figure tasks). This type of task is the one involved in Episodes 3 and 4, and we also analyzed them in Del Zozzo and Santi (2022). In Fig. 4a, we show the SNAP! code projected on the wall. For the reader's convenience, in Fig. 4b, we recall the commands explanation and we show the expected figure drawn by the GGBot according to the code alongside (what is shown in Fig. 4b was not projected nor shared with students during the session). In the transcriptions on both Tables 3 and 4, in order to overcome the ambiguity between the transcription numbers and the SNAP! code numbers, we indicate the i-th transcription line number with TLi and the i-th SNAP! code line number with SLi (with respect to Fig. 4b).

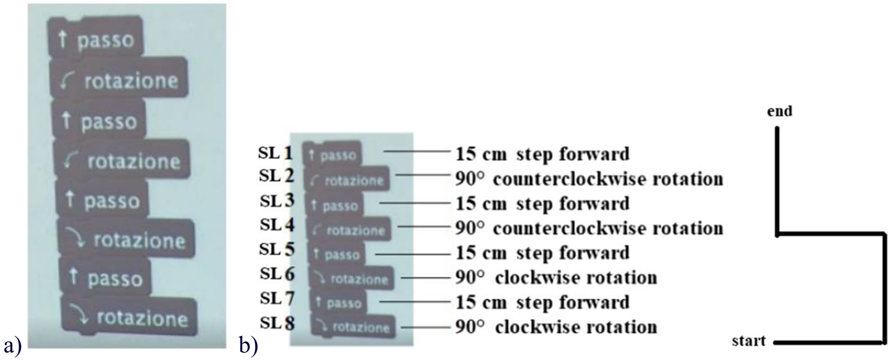
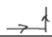
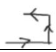
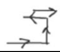
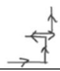


Fig. 4 **a** SNAP! commands list (projected); **b** commands explanations and the expected figure

Episode 3: Angela in a Code-to-Figure Task

To perform the requested prediction, Angela traced the figure she thought the GGBot would draw, by drawing with her finger on the paper.

Table 3 Angela in a code-to-figure task, transcripts

Gestures and non-verbal signs	Utterances	Angela's drawings (with her finger)
[TL1] Angela points her index finger on a white area of the paper and then she traces a first segment in the horizontal direction (according to her egocentric SoR) towards her right. .	[TL1] R: like that	[1]
[TL2] Angela goes on with a second segment in the vertical direction (according to her egocentric SoR) moving away from herself and articulating the movement in two steps.		[2] 
	[TL3] Angela: ehm	
[TL4] Angela traces a third segment in the horizontal direction (according to her egocentric SoR) towards her left. This segment is shorter than the first two. [TL5] Angela traces a fourth segment in the horizontal direction (according to her egocentric SoR) towards her right. This segment is shorter than the first two as well. [TL6] Angela traces a fifth segment in the vertical direction (according to her egocentric SoR) moving away from herself. Then, she stops moving and gazes at R. [TL7] Angela gazes at her finger on the paper.		[4]  [5]  [6] 
[TL8] Angela moves horizontally her finger right and left various times.	[TL8] Angela: and like …	[8-9] 
	[TL9] Angela: and like so and so …	
[TL10] R points her finger towards Angela.	[TL10] R: So, it is an excellent idea. It is a sort of stair, did you see?	

During the same activity, some minutes after Angela, Vanessa took the floor. To perform the requested prediction, she traced the figure she thought the GGBot would draw, by drawing with the marker on the paper.

Episode 4: Vanessa in a Code-to-Figure Task

Table 4 Vanessa in a code-to-figure task, transcripts

Gestures and non-verbal signs	**Utterances**	**Vanessa's drawings**
	[TL1] Vanessa: So, before she (Vanessa is referring to a classmate's answer in a previous figure-to-code task) did a square, ok? [TL2] R: Ok	
[TL3] Vanessa draws on the paper a line in the vertical direction (according to her egocentric SoR) moving away from herself.	[TL3] Vanessa: So, he took a step forward, no?	[TL3] 
[TL4] Vanessa draws on the paper a line in the horizontal direction (according to her egocentric SoR) towards her right.	[TL4] Vanessa: a rotation	[TL4] 
[TL5] Vanessa separates the marker from the paper and starts to oscillate over the second segment.	[TL5] Vanessa: then another, a ... another step forward, so the rotation …	
[TL6] Vanessa continues to oscillate repeatedly over the second segment.	[TL6] Vanessa: yes well, the, the step forward	
[TL7] Vanessa draws a third short segment in the vertical direction (according to her egocentric SoR) towards herself. For the reader's convenience, in the image, we have circled the short segment in blue.	[TL7] Vanessa: a rotation	[TL7] 
[TL8] Vanessa draws out the line towards herself.	[TL8] Vanessa: and then after the rotation again a step forward	[TL8] 
[TL9] Vanessa draws on the paper a line in the horizontal direction (according to her egocentric SoR) towards her right.	[TL9] Vanessa: then he put the rotation the opposite way, and so like this	[TL9] 
[TL10] Vanessa draws quickly another two lines.	[TL10] Vanessa: and like this	[TL10] 

So far, we have described four episodes involving four primary school students during their first day of interaction with GGBot. In the next section, we present some theoretical building blocks that will allow us to frame and understand the impact of GGBot in transforming theoretical perception in geometry.

Theoretical Building Blocks for Learning in a *Humans-with-Media* Environment

In order to characterize the mathematical learning carried out with the GGBot and the ensuing *humans-with-media*, as mentioned in the introduction, we articulate a theoretical discussion to frame the complexity emerging from the dialogue between individuals and the GGBot.

Building Block 1: Process of Objectification—Activity, Signs and Artifacts, Culture

In the theory of objectification (TO), the issue of *learning* is rooted in the dialectics between the individual and their culture. Learning is a movement pushed by the intrinsic differential between individual and cultural knowledge. In fact, in attending to knowledge, the student has to cope with something that in the beginning is different from her, an alterity that challenges, resists, and opposes her. Learning is the process of objectification that fades such a difference away to make sense of cultural knowledge and transform it into something familiar that allows new forms of action, thinking, imagination, and feeling. In order to reduce the distance between the individual and cultural knowledge, activity as a specific human endeavor is required on the part of the student.

We remark that signs and artifacts are constitutive of processes of objectification, referred to as *semiotic means of objectification*:

> These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (Radford, 2003, p. 41; *italics in original*)

Radford (2008) refers to the configuration of semiotic means of objectification as the *territory of artifactual thought.* The semiotic means of objectification are constitutive of mathematical activity and enable specific *modes of activity*. Mathematical agency is embedded in *domains of activity* inscribed by social, cultural, and historical elements connected to the features of the semiotic means of objectification. Even though GGBot as a whole is a digital artifact, it cannot be considered a semiotic means of objectification in and of itself. In fact, GGBot is a complicated object, which brings together ideal and material features such as the signs of SNAP!, the drawings of the marker, and the physical movements made up of steps and rotations. In its interaction with other semiotic means of objectification involved in geometrical activity, the

GGBot implies a reconfiguration of the topological structure of signs and artifacts (i.e., the territory of artifactual thought) students use in the learning of geometry. In fact, GGBot is an artifact, which has been designed a priori with an already embedded mathematical culture that establishes a set of constraints and potentials in the modes of activity and the modes of thinking. Nevertheless, in line with the TO, when a student is involved in activities with the GGBot, it becomes an engine that mobilizes a rich set of semiotic means of objectification, dynamically entangling one another and entangled with the activity itself.

On the one hand, the GGBot is designed as a multimodal artifact that activates the interplay of different sensorial channels connected with sensuous cognition and the semiotic means of objectification specific of the artifact (SNAP! command, drawings, and its material features). On the other, it triggers the use of "familiar" signs (such as drawings, gestures, indexical use of language, natural language) that students have to interpret in light of its features. We stress that the introduction of the GGBot changes the topology of the territory of artifactual thought and forges forms of action and thinking that would not be available without this DT resorting to "familiar" signs and artifacts.

Episode 1 (transcription in Table 1) occurs during the first moments of the interaction between the students and the GGBot, and it illustrates this idea. The activity is strictly related to the use of the GGBot, but the student is only dealing with the static drawing that the marker has left on the paper. Serena was trying to give sense to signs present on the paper in terms of the movements of the GGBot. She was coordinating two ways of attending to the geometric figure, facing a struggle between the two interpretations of the signs that she was trying to reconcile. In terms of the activity deployed by the GGBot, the geometric figure drawn on paper (Fig. 2a) is the result of two steps forward, a clockwise turn, and a further step forward. Instead, if what is shown in Fig. 2a was a human drawing, the figure would have been made by drawing two perpendicular segments, the first segment twice as long as the second.

While analyzing the trace on the paper, in line 2, Serena slides her finger along the trace of the marker on the paper, uttering "it has done the two steps." In lines 3–4, Serena pointed her finger on the middle point that separated the two steps and punctuated the words by rhythmically tapping her finger three times in the same spot, saying, "but here, also here that you can see here, it stopped" and, in line 7, she explained, "(I see it) thanks to the most pressed point because the marker was still." The rhythmical gesturing and the wording "most pressed point" allowed Serena to grasp the figure with a broader meaning that connects the *GGBot drawing* and the *human drawing*.

The same line of reasoning holds for the angle, where in line 9 where Serena said, "and also here, the marker was still," pointing her finger also in the center of the 90° rotation that the robot made. We note that the two points where the marker is still have a different nature from a geometrical point of view. In the case of the segment, the marker is still because the robot stopped. In the case of the angle, the marker is still because it stands on the fixed point of the robot's rotation. In her analysis of the *GGBot drawing*, Serena got that, when it came to the angle, "also here the marker was still." Serena's interpretation of the drawing produced by the GGBot testified a specific form of action and thinking interwoven with the new topology of territory of artifactual thought originating from the GGBot.

Building Block 2: cognition is sensuous and learning is conceived as "domestication of the eye"

As mentioned above, the outcome of learning as a process of objectification is the encounter with mathematical cultural objects and their transformation into objects of consciousness. Recalling the enlarged notion of mind and consciousness mentioned in the introduction, the encounter with knowledge and its transformation into an object of consciousness occurs within the features of *sensuous cognition*. In light of the dialectic–materialist approach underpinning the TO, the basic tenet behind the notion of sensuous cognition is that the body, the senses, and the objects of sensation are not a priori entities, but are mutually transformed by cultural–historical activity *entangled with the use of signs and artifacts.*

Within sensuous cognition, the issue of learning as the process triggered by the conflict between the student's personal knowledge and the cultural one leads us to the following question: how do students change their perception from "spontaneous" forms of attending to objects to a mathematical and theoretical one? As mentioned above, the change of perception occurs as a *domestication of the eye*. The domestication of the eye has a multimodal nature, both in the various sensorial channels and the richness of signs and artifacts interwoven with cultural–historical activity involved in the transformation of perception (Radford, 2021). Radford highlights that mathematics is something that appears when students and a teacher engage in classroom activities. Thus, as something produced by human activity, school mathematics can be seen as a sensible phenomenon: it is an entity that is at the same time ideal, sensible, and material, and it appears as the activity unfolds.

The GGBot restructures the topology of the territory of artifactual thought, and it is the pivotal artifact that gives rise to modes of activity that *domesticate the eye* for the objectification of geometrical knowledge. For example, the shift from the drawing act performed by a human hand and the drawing act performed by the GGBot entails a transformation of geometrical perception from a visual and unitary perception of a geometrical figure to a more sensory-motor and discretized one. Such a transformation is prompted by the conflict between:

- An already theoretically domesticated eye—which encounters a geometric figure as unitary geometrical, in terms of segments and angles perceived as portions of a plane between two half-lines
- The new theoretical eye introduced by the presence of the GGBot—which encounters a geometric figure as something constructed in terms of steps and rotations

The student overcomes the conflicting ways of attending to the figure giving rise to modes of activity that brings to the fore a co-ordination of gestures and natural language that contributes to the ongoing process of domesticating the eye. In the episodes presented in the section "Empirical examples," we can pinpoint precise moments where the students' modes of activity change to solve the conflict, and we can delineate their attempt of domesticating the eye. For instance, Episode 2 (transcription in Table 2) occurs during the middle part of the session. In the figure-to-code task, Daniele needed to predict whether the SNAP! code under scrutiny allowed the GGBot to move so as to draw a square. In this case, there is no trace of the marker on the paper. In line 3, after drawing the segment corresponding to the GGBot first step, Daniele drew a segment uttering, "it goes on the left."

Daniele here was interpreting the angle as a portion of plane (with two consecutive segments you obtain an angle conceived as a portion of plane). The conflict started immediately after when he should perform a step forward following the rotation (lines 4–5): he first gazed at the projected code, staring at it for 2 seconds, then at his hand on the paper, which remained still, and puzzled, uttering, "and then … then then it does … it goes forward." In line 6, he changes the interpretation of the angle trying to merge its new objectification as rotation (three concatenated sequences of movements—step forward, rotation, step forward—disregarding portions of the plane) with the previous one. Indeed, in line 6, he rotated his index in a counter-clockwise direction using the thumb as a pivot, uttering, "and it does like that."

The movement performed by the student's finger looks very similar to the icon of the "turn" SNAP! command, while still, the GGBot's actual movement does not seem to be intuitively graspable from the icon on the SNAP! turn button.[2] Thus, Daniele goes through another conflict given by the prior observation of the behavior of the GGBot and the visual sign for this behavior available on the rotating SNAP! command. Then, Daniele decides to start over again (line 8), and in lines 9–11, the new modes of activity and the ensuing domestication of the eye become evident: he points his finger on the paper tracing a line and uttering "it takes a step"; then, from line 10 to line 11, he made a change in his language from "it goes on the left," uttered keeping his hand still, to "it turns on the left," uttered rotating his hands using his index as a pivot. From that moment on, he used only the word "turn," linking it to a rotation of his hand and reconciled two ways of attending to the angle as objectified with marker and paper, without and with the GGBot. Daniele overcame the conflict with the transformation of the perceived cultural object into an object of consciousness. We remark that Daniele's use of gestures and language is linked to the structure of semiotic means of objectification established by the GGBot.

A similar conflict, but with a diverse evolution, is Vanessa's in Episode 4 (transcription in Table 4, lines numbered as TL1, …, TL10). She was facing a code-to-figure task: she needed to predict what figure will result from the SNAP! code under scrutiny (Fig. 4a, where code lines , in Fig. 4b, are numbered SL1, …, SL8). To perform the requested prediction, she drew with a marker on the paper. Vanessa started to draw a line, uttering, "he took a step forward" (TL3, interpreting SL1),

2 The authors would like to thank one of the anonymous referees for sharing this insightful reflection.

followed by the drawing of the second perpendicular segment, explicitly linked to the noun "rotation" (TL4, interpreting SL2). As for Daniele, also for Vanessa, the conflict started when she should go a step forward (SL3) with the marker: due to the previous interpretation of the rotation, she was puzzled about where to go.

Vanessa's confusion is highlighted by her oscillating the marker over the second segment, and her uttering "then another, a … another step forward, so the rotation …" and "yes, well, the, the step forward" (TL 5–6). Vanessa was confused because she lived in a conflict between the two ways in which the notion of angle co-emerged with her sensuous act: the angle as the part of the plane between two half-lines and the angle as the rotation of the GGBot. The gesturing with the marker and the utterances described above testify such a conflict; she was able to draw the consecutive segments, but, when trying to relate them with the SNAP! commands, she was at odds with what she was doing.

In TL 7, Vanessa was managing the second rotation of the SL4: unlike Daniele, she avoided the conflict linking the word "rotation" with a little portion of a perpendicular segment (TL 7, in Table 4 the short segment is circled in blue) that she extended synchronously with the words, "then after the rotation again a step forward" (TL 8). Vanessa's use of natural language is always assertive and explicit: she used words to scan the imagined movement and the resulting trace. Nevertheless, her co-ordinated use of natural language and the drawing with the marker (TL 7–8) shows in an evident and interesting way her domestication of the eye related to the angle and her struggle to erase the differential between her previous form of noticing and the new (GGBot's) one that is challenging, resisting, and opposing her. We observe Vanessa's difficulty in fully accomplishing the domestication of her sensuous cognition to "see" angles with the "eyes" of the GGBot.

Notwithstanding the difficulties in managing the angle, Vanessa arrived at a human drawing consistent with the drawing that the GGBot with the marker would have left (Fig. 4b). However, her mode of perception avoided the objectification of the angle as a rotation required by the predictive task. In this episode, despite the interesting mobilization of new forms of activity entangled with semiotic means of objectification beyond the realm of the GGBot, Vanessa's domestication of the eye did not result in a fully-fledged theoretical mode of perceiving the geometrical figure consistent with the constraints and potentials. The two episodes show that the role of the GGBot changes as the activity unfolds and the domestication is accomplished. It acquires new meanings and supports the creation of new mathematical meanings.

So far, the discussion delineates a peculiar complexity in the interaction between students and GGBot: it is as if there are two entangled eyes being domesticated. It seems that a new "collective" eye, complex to domesticate, is emerging. Thus, to cope with the need to properly face such hybridization, we now add a third theoretical building block, enriching our discussion with the construct of *humans-with-media*.

Building Block 3: GGBot and Humans-with-Media

The constructs *human-with-media* and its actualization as techno-mathematical fluency (Jacinto & Carreira, 2017) allow us to analyze the participation and transformation of students into new information technology "actors" (Borba & Villareal, 2005), according to a view of perception, agency, and cognition in mathematics forged by digital and non-digital technology.

We embrace Borba and Villareal's idea of *humans-with-technologies* to analyze the learning of geometry as a *domestication of the eye* when we consider, in our context, two collectives: *humans-with-paper-and-marker*[3] and *humans-with-paper-and-marker-and-GGBot* belonging to two different domains that stem from the topology of their territory of artifactual thought. *Humans-with-paper-and-marker* is the starting point: it is the collective that produces the knowledge that, with respect to the experience with the GGBot, represents the "spontaneous" forms of attending to geometrical figures. *Humans-with-paper-and-marker-and-GGBot* is the new collective, the one that triggers new modes of thinking and perception that domesticate the eye for the objectification of geometry in terms of the cultural knowledge condensed in the GGBot.

Following the above description of the GGBot, we shared some first reflections upon the drawing act, distinguishing whether it is performed by the movement of a human being's hand or by the movement of the GGBot. We called the former trace *human drawing* and the latter trace *GGBot drawing*. The notion of *humans-with-media* allows us to frame better such a crucial issue: the *human drawing* trace corresponds to the drawing act of the collective *humans-with-paper-and-marker*, and the *GGBot drawing* trace corresponds to the drawing act of the collective *humans-with-paper-and-marker-and-GGBot.* Nevertheless, recalling Borba and Villareal's citation in the introduction, as for writing and orality, also in our context, there is a before and an after the introduction of GGBot. The process that domesticates the eye (and other senses) into theoretical organs pertains to thinking collectives, intertwined with one another. Once the new thinking collective *humans-with-paper-and-marker-and-GGBot* is introduced, the transformation of perception and knowledge enlarges the modes of activity of *humans-with-paper-and-marker* collective.

The previous analysis of Daniele and Vanessa shows that the new collective *humans-with-paper-and-marker-and-GGBot* allowed the two students to encounter a new perception of angle as a rotation. Let us now consider the case of Angela in Episode 3 (transcription in Table 3, lines numbered as TL1, …, TL10). Angela traced correctly with her finger the first three segments of the figure (TL 1–4 that, with respect to Fig. 4b, correspond to Angela's interpretation of the SLs from 1 to 5). When reaching SL6 (90° clockwise rotation), she was not able correctly to handle the change of direction in the rotation (the previous two in SL2 and SL4 are counter-clockwise). This student lived the same struggle of Daniele and Vanessa between the new objectification of the angle as rotation.

Furthermore, the student experienced a conflict regarding the harmonization of different Systems of Reference. The egocentric System of Reference related to the production of *human drawings* and the allocentric one related to the production of *GGBot drawings* (forced by the need of linguistic SNAP! expressions). She interpreted the change in the direction of the rotation as a reverse direction along the same segment (TL 5). At the 7SL, Angela traced with her finger the vertical

[3] In this article, we identify the case of drawing with a finger on paper with the drawing with the marker on paper.

segment (TL 6). Then, the last command puzzled her even more and she went back and forth with her finger (TL 8–9). In her activity, Angela resorted only to gestures and the SNAP! commands. She was able to notice in her imaginative perception the corresponding movement of the GGBot related to the first three segments of the figure. Nevertheless, when it comes to the third rotation, her perception was not able to grasp the angle of the figure as it would be conveyed by the movement of the GGBot.

The co-ordination of gestures and the SNAP! icons require attending to the angle of the figure in terms of step and rotation of the GGBot and not as the portion of the plane delimited by the two half-lines (i.e., the two sides of the figure). Furthermore, it requires a transformation of imaginative perception able to consider also the connection between egocentric (Angela's) and allocentric (GGBot's) System of Reference that does not emerge in the use of pencil-and-paper. Her perception did not encompass the change of direction in the rotation, due both to the new way of encountering the angle of the figure as a rotation of the GGBot and the conflict between the egocentric and allocentric Systems of Reference.

The back-and-forth gesturing along the side of the figure (TL 8–9) and the global absence of structured natural language are tokens of her blurred perception and her struggle in domesticating the eye to transform her perception of the angle with respect to the one she realized before. Nevertheless, Angela's episode shows us that the new modes of activity triggered by the new collective *humans-with-paper-and-marker-and-GGBot* and the ensuing domestication of the eye into novel and broader forms of theoretical perception to attend to geometrical figures regard not only the angle of a figure, but also managing different Systems of References. The episode of Angela stresses the interplay—generative and conflicting—between the two collectives, reified in two entangled domains of activity and two modes of domesticating the eye, when the digital technology appears in the landscape of mathematical learning.

Building Block 4: Domestication of the Eye as Transitions Between Domains of Activity

The introduction of the drawing robot defines three main domains of activity. The first domain is the *drawing act for humans-with-paper-and-marker* before the introduction of the GGBot, where the outcome of the activity is the *human drawing* to objectify geometrical objects. This domain refers to students' objectification of geometrical figures before the introduction of GGBot. The second domain is the *drawing act for humans-with-paper-and-marker-and-GGBot*, where the outcome of the activity is a *GGBot drawing* to objectify geometrical objects. In other words, this domain refers to students' activities with the GGBot where the drawing on the paper is left directly by the GGBot itself.

The third domain is again the *drawing act for humans-with-paper-and-marker*, but *after* the activity with GGBot where the outcome of the activity is again a *human drawing*, but one as a predictive task of a *GGBot drawing* prompted by the previous experience with the robot. In other words, this domain refers to activities of *human* *drawing* realized *as if* it were the *GGBot drawing*. We remark that, in the third domain, the notion of *human drawing* has to be enlarged also to consider gestures and/or a marker as semiotic means of objectification. In the third domain, the role of the GGBot is transformed compared with the previous two, because the technology affects the activity without being physically present. This situation confirms (and is confirmed by) L. Radford (personal communication, 1 September 2022) when he claims that "As for the activity itself, its [the digital technology's] role changes as the activity unfolds; it acquires new meanings and supports the creation of new mathematical meanings (much as the language we use when learning something)."

We conceive such domains as socio-cultural spaces where objectification processes are carried out according to their specific topologies of semiotic means of objectification. We can think of the following:

- The first domain as related to *humans-with-paper-and-marker* collective, where activity and the ensuing domestication of the eye are intertwined with "familiar" semiotic means of objectification such as gestures, drawings, and natural language
- The second domain as related to *humans-with-paper-and-marker-and-GGBot* collective, where activity and the ensuing domestication of the eye are intertwined with the constraints and potentials of the GGBot in mobilizing a network of semiotic means of objectification
- The third domain as related to *humans-with-paper-and-marker-after-the-GGBot*, which is a third collective where the activity and the ensuing domestication of the eye are intertwined again with "familiar" semiotic means of objectification, such as gestures, drawings, and natural language that recollect the constraints and potentials of the GGBot

The domains introduced above, the collectives, and the ensuing domesticated eye are co-constitutive, and in what follows, we will refer to one of the three terms according to the context of the discourse.

Given these domains, we introduce the notion of *transition between* consistently with our theoretical perspective and outline its features, and how it occurs. In our understanding, a *transition between* is a transformation of sensuous cognition in terms of *domestication of the eye* that occurs by shifting between domains. In *transitions between*, students are thus exposed to two different domains among the three mentioned above—*humans-with-paper-and-marker*, *humans-with-paper-and-marker-and-GGBot* and *humans-with-paper-and-marker-after-the-GGBot*—each one marked by its mode of activity molded by a specific configuration of semiotic means of objectification that identifies such a domain. The transformation of sensuous cognition occurs via a further *domestication of the geometrical eye* as the student engages in the activity of the new domain.

The different configuration of semiotic means of objectification of each domain molds perception in ways of attending to geometrical objects unknown to the student. The dialectics between the different forms of activity prompted by the shift from one domain to the other pushes further the *domestication of the eye* to objectify geometrical figures in a more encompassing manner. The introduction of the

GGBot for the learning of geometry as a process of objectification is truly effective when the student's eye is domesticated to see geometrically as they shift from one domain to the other. The domestication of the eye is fully accomplished when the *humans-with-paper-and-marker-after-the-GGBot* domain is successfully involved in the transitions among the three domains. In fact, the student is able to notice and become aware of geometry with the "eye of the GGBot," even when the GGBot is no longer physically present. They thus reach a deeper and more potent understanding of geometry, otherwise inaccessible in the modes deployed by the GGBot.

Managing transitions between the three domains is a challenging objective to accomplish on the part of the student. All the students in the episodes we present faced transitions between domains of activity, including *humans-with-paper-and-marker-after-the-GGBot* one. In domesticating the eye, they show discomfort in acknowledging the transformation of perception as they transition between the three domains, especially to the third one. Of the four students, only Daniele fully accomplished the domestication of the eye as he transitioned between the first and the third domain. He started interpreting the code in terms of the first collective, until he experienced two subsequent conflicts. The first (lines 4–5, Table 2) occurred when he drew the second side of the square disregarding the rotation and living a conflict with the SNAP! code. The second (lines 6–8, Table 2) occurred when he interpreted the rotation by drawing as an arc after the first side of the rectangle. He overcame the conflict after domesticating the eye with the support of gestures and natural language (lines 10–11, Table 2). From line 11, his drawing and utterances were consistent with the domesticated eye of *humans-with-paper-and-marker-after-the-GGBot* domain.

Wrap-up: Objectification of Geometrical Figures Before and After the Activity with GGBot

In the present article, we follow a line of research about the effectiveness of DT in the learning of mathematics, focusing on the exact effect the GGBot has on the understanding and learning of geometry. Our aim here is to frame how theoretical perception is transformed when the GGBot comes to supplement and interacts with initial learning with non-digital technology. The analysis is carried out according to a theoretical framework constructed at the crossover of the TO and *humans-with-media* and structured in four building blocks: processes of objectification; cognition is sensuous; GGBot and *humans-with-media*; domestication of the eye as transitions between domains of activity.

The connection of the four building blocks builds a two-fold theoretical framework that features what the transformation of perception by a GGBot is and how it occurs. In regard to the first aspect, the TO claims that the transformation of perception is at the core of mathematical thinking and learning, and allows the student to encounter mathematical knowledge as an object of consciousness. In sensuous cognition, the transformation of perception molds the individual's senses into cultural and historical artifacts able to recognize mathematical concepts as objects of consciousness. The basic tenet behind the notion of sensuous cognition is that the body and the senses are not biological a priori entities, but instead mutually transformed by cultural–historical activity *entangled with the use of signs and artifacts.*

In the context of the present study, we consider the interplay between configurations of signs with and without the GGBot and the ensuing *human-with-media* collectives. In regard to the second aspect, building blocks 3 and 4 tell us how the transformation occurs and how eventually the students understand and learn geometry. The process of domestication of the eye is prompted by the dialogue between the two *humans-with-media* collectives we have identified, specific to the GGBot: *humans-with-paper-and-marker* and *humans-with-paper-and-marker-and-GGBot*. They are two human collectives forged by their territory of artifactual thoughts and their ensuing mode of activity, i.e., manners of drawing geometrical figures.

In order to accomplish effective domestications of the eye with the GGBot, our framework singles out the notion of *transitions between* domains of activity that derive from the two *humans-with-media* collectives mentioned above. In fact, the introduction of the GGBot in the learning path creates a discontinuity with previous activities and sets off an articulated form of activity that involves three domains: *humans-with-paper-and-marker*, *humans-with-paper-and-marker-and-GGBot* and *humans-with-paper-and-marker-after-the-GGBot*. The *transitions between* the three domains of activities allow a learning path with the GGBot that could accomplish the domestication of the eye.

Our contention is that the domestication of the eye is not the outcome just of the intertwining of more signs and artifacts including the GGBot, but also of the intertwining of the aforementioned three domains via *transitions between.* The *transitions between* forge the theoretical transformation of perception to the point that the student acquires techno-mathematical fluency (Jacinto & Carreira, 2017), and is transformed into a new information technology "actor" (Borba & Villareal, 2005) able to perceive, act, and think like the GGBot, even if it is not physically present. In the context of the GGBot, we have outlined three domains of activity that trigger *transitions between*, but it is not necessarily a general outcome for other DTs. The episodes we have analyzed show how our theoretical machinery frames the domestication of the eye with the GGBot to perceive theoretically angles of geometrical figures and Systems of Reference as mathematical cultural–historical knowledge.

We discuss the geometrical knowledge objectified in the four episodes above according to the forms of action carried out in transitions between *humans-with-paper-and-marker*, *humans-with-paper-and-marker-and-GGBot*, and *humans-with-paper-and-marker-after-GGBot.* We focus on two fundamentals of geometrical knowledge that unfold as different modes of theoretical perception in the shift between domains: the Systems of Reference and the notion of angle.

Regarding the System of Reference, as Baccaglini-Frank et al. (2014) pointed out, there is a large consensus in distinguishing between egocentric and allocentric Systems of Reference. Indeed, in expressing the location of an object in the context of the GGBot, we can refer to the observer's perspective (egocentric System of Reference) or not (allocentric System of Reference regarding the GGBot). As an example, the meaning of deixis in space (i.e., terms as forward, backward, left, right, etc.) depends on the perspective we consider. The introduction of the GGBot externalizes with respect to the student the point of view of the observer in the act of drawing. Moreover, to perform a *GGBot drawing*, a linguistic expression of space is forced by the employment of SNAP! commands. Thus, the students' perception of space needs to be domesticated, in order to handle geometrical relations interpreted from the two systems of reference, that of the student and of the GGBot.

The shifts and interplay between *drawing for humans-with-paper-and-marker* and *drawing for humans-with-paper-and-marker-and-GGBot* open the way to a conception of space without a priori privileged Systems of Reference or directions that students will objectify at higher school levels. We remark that there is a symmetry between egocentric and allocentric Systems of Reference in that SNAP! commands have to be interpreted as egocentric for the GGBot and allocentric for the students. The reverse relation holds when we refer to the student's System of Reference.

Regarding the notion of angle, in *drawing for humans-with-paper-and-marker*, an angle is perceived as the portion of the plane between two subsequent segments/half-lines. In terms of activity, the angle emerges from two movements with a marker to draw the segments/half-lines and perceive the angle as the ensuing portion of the plane. The same activity can be enriched with a ruler and/or a protractor to make drawing precise and of a precise measure for example in degrees.

In *drawing for humans-with-paper-and-marker-and-GGBot*, an angle emerges both in a static and a dynamic way. Strictly speaking, the angle is perceived as the rotation of the robot around the mid-point of the (imaginative) segment that connects the points of contact of the wheels on the ground (see the earlier section *Drawing with GGBot*). However, the GGBot recollects the angle bound to *human drawing* via three movements: a step, a rotation, and another step. Indeed, the result of such three movements is a static trace on paper that recalls the notion of angle as the part of the plane identified by two consecutive segments/half-lines. In the shift and dialectics between activity with *drawing for humans-with-paper-and-marker* and activity with *drawing for humans-with-paper-and-marker-and-GGBot*, the student's perception is domesticated to encompass both a static objectification of angle as portion of the plane and a dynamic objectification as a rotation of the GGBot as described above.

Both for the System of Reference and for the angle, the domestication of the eye according to the modes of action and thinking of the GGBot is fully accomplished in transitions that include the three domains of activity.

Concluding Remarks

This article is positioned in the line of research about the effectiveness of DT on mathematics understanding and learning, and aims at investigating how theoretical perception is transformed when the drawing robot GGBot comes to supplement initial learning with non-digital technology.

We propose a theoretical framework constructed at the crossover of the TO and the *humans-with-media* made of four building blocks: processes of objectification; cognition is sensuous; GGBot and *humans-with-media*; domestication of the eye as transitions between domains of activity. Our theoretical approach frames the unfolding of geometrical perception of primary school students in activities involving the drawing robot GGBot. We discuss four illustrative examples that anchor our theoretical discussion and show the struggle students have to face to domesticate their eye via the *transitions between* domains of activity as they engage in new forms of agency that directly or indirectly entail the use of the GGBot.

We share some crucial points that our theoretical study can offer to mathematics education with the GGBot. The first element is more general and regards the encounter of a socio-cultural semiotic (TO) approach with Borba's and Villareal's *humans-with-media* perspective whose subject matter is the reorganization of thinking in a merged vision of human and digital and non-digital technologies. On the one hand, Radford's semiotic approach allows us to operationalize with specific variables such a merging. On the other hand, the *human-with-media* approach sheds light on the nature of semiotic means of objectification and the social-cultural activity they enable when the GGBot comes into play.

The basic tenet of our theoretical study is that the introduction of the GGBot transforms the topology of the territory of artifactual thought. In particular, we show in our episodes how the GGBot determines the geometrical activity with its constraints and potential thereby shaping the forms of thinking of the students that we framed as objectification. The topological change of the territory of artifactual thought is testified by the mobilization of semiotic means of objectification strongly bound to the features of the GGBot. For example, Daniele domesticated his eye recurring to a configuration of gestures and natural language coherent with the forms of activity embedded in the robot. On the contrary, Angela's gesturing and utterances were incoherent with the activity embedded in the robot and she failed in domesticating her eye and carrying out the code-to-figure task correctly.

Another important element that characterizes our theoretical approach is the introduction of three domains of activity related to the GGBot:

- Drawing for *humans-with-paper-and-marker* (before the introduction of the GGBot), related to the *humans-with-paper-and-marker* collective. The outcome of activity here is a human drawing.
- Drawing for *humans-with-paper-and-marker-and-GGBot*, related to the *humans-with-paper-and-marker-and-GGBot* collective. The outcome of activity here is a GGBot drawing.
- Drawing for *humans-with-paper-and-marker-after-GGBot*, related to the *humans-with-paper-and-marker-after-GGBot* collective. The outcome of activity here is a human drawing, but realized as the prediction of a GGBot drawing.

They provide a broader definition of *humans-with-media*, actualizing the symbiosis between humans and GGBot in the interconnection between activity, semiotic means of objectification, and knowledge. *Humans-with-media* are identified with social individuals who, in the symbiosis with the digital technology, develop new forms of cultural–theoretical perception (domestication of the eye).

The notion of *transitions between* the three domains of activity offers a new theoretical tool to identify the effectiveness of the GGBot in the learning of mathematics—as domestication of the eye—and the possible origin and nature of the difficulties that students have to face. Transitions between domains of activity featured by the GGBot allow us to shed new light on the learning of geometry as a process of objectification.

Our study confirms, in the context of *transitions between domains of activity*, Miragliotta's (2019) findings that show the power of predicting tasks in providing insights about the learning of geometry. In particular, in our case, geometrical predictions were revealed to be windows onto the students' imaginative perception of the geometrical entity that internalizes the forms of activity carried out with the GGBot, and the semiotic means of objectification it mobilizes, according to knowledge condensed in it. In this respect, as we mentioned above, the transitions to the *drawing for humans-with-paper-and-marker-after-GGBot* play a prominent role in recognizing via objectification the general geometrical meaning emerging as operational invariants of the activity with the GGBot independently of its physical presence.

In regard to Jacinto and Carreira's (2017) techno-mathematical fluency, we observe that mathematical and technological knowledge are both present in the topology of the territory of artifactual thought and the forms of activity it enables. In fact, the GGBot and the topology of territory of artifactual thought it establishes condense mathematical and technological knowledge that are mobilized in activity. In this perspective, the perception of the representational affordances of technological tool—a keystone in Jacinto and Carreira's perspective—can be operationalized in terms of domestication of the eye, as highlighted by Daniele's episode.

We believe that the theoretical reflection we propose in this contribution allows us properly to frame the complexity of the *transitions between domains of activities* triggered by the introduction of the GGBot in the learning of geometry. Nevertheless, in this article, we focus on the combination of four basic SNAP! commands (the first four in Fig. 1b): further studies are necessary to fathom cases where other commands are involved. For instance, the parametric SNAP! commands for steps and rotations (the second group of four in Fig. 1b) allow us not only to unfold an enlarged (and more general) set of geometrical figures, but also to offer the opportunity to encounter mathematical cultural objects related to measurements and metrics.

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Data availability The data analyzed in this study have been collected by Anna Baccaglini-Frank who authorized us to use them in our study. The data sets generated and analyzed in the current study are not available due the participants' privacy declared in the consent forms.

Declarations

Conflict of Interest The authors declare no competing interests.

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