Evolution of a Task Domain

John Mason¹

Published online: 9 January 2019

© The Author(s) 2019

Abstract

My interest here is in the way in which a single task can develop, gaining both in mathematical richness and in pedagogic purpose. This leads me to consider pedagogical affordances which can turn tasks from having rich potential into tasks used richly. The result is a domain of variations and extensions, together with possible pedagogical actions which could come to mind for a teacher while their students explore. As these tasks evolved, I experienced moments of struggle myself, trying to keep track of my (often rather clumsy) notation. This gave me confidence that the tasks might be useful to teachers and would-be teachers for the opportunity afforded to experience struggles parallel to those experienced by students working either on similar tasks or on the initial task itself. This, in turn, can sensitise teachers to something of what their learners may experience, inspiring them to pose tasks in a way which avoids the *didactic transposition* (Chevallard 1985).

Keywords Structured variation grids · Didactic transposition · Rich tasks · Tasks used richly · Exploration · Investigation · Tunja sequences

Getting Started

I used the following task in Open University Summer Schools for many years and I have used it in many different workshops all over the world, for reasons which I will give once I have stated the task and once you have had a chance to think about it. I am going to resist giving indications of solutions for as long as possible, so as not to spoil the possibility for you to experience the thrill of (minor) discoveries.

One More Than

What numbers can be expressed as, or arise as, one more than the product of four consecutive integers?

✉ John Mason

¹ Oxford, UK

Comment

If you have not tried this yourself, you will miss an experience which is hard to appreciate simply through description. Please do not read on until you have had a go.

I have found the task fruitful because people who are confident with algebra often jump straight into symbols. They end up with a quartic equation, but are usually none the wiser for this. They sometimes remain stuck for a time, but eventually most turn to constructing some examples, and, if their calculations are correct and if they are familiar with a certain sequence of numbers, they soon realise something about what numbers arise or can be achieved. The tendency at first is to conjecture too broad a class of numbers. Some people start their examples with $1 \times 2 \times 3 \times 4 + 1$, while others think to use $0 \times 1 \times 2 \times 3 + 1$; some recognise 361, or are led to conjecture and test a property of 361, while others do not. Those who look further realise that they can predict the final number from the four consecutive numbers being multiplied, and that they can do this in more than one way.

I hope that you did try the task yourself, because in my experience it really helps to work on tasks you are going to offer students. Furthermore, in my experience, working on tasks that challenge you along similar lines to challenges faced by learners can help to sensitise you to what learners might experience. This is the basis of how I have worked throughout my time in mathematics education, which is summarised in Mason (2002a) but which is present in almost all of my writing (see, for example, Mason et al. 2010).

In this case, I naturally wanted to extend the result, but at the time the only thing I could find was the product of four consecutive numbers in an arithmetic progression, with the added 1 replaced by a certain square number (I promised not to give everything away!). In what follows, the heading *Disruptive Remark* signals an indication that what follows may interfere with your own investigation.

The starting task now suggests the presence of a domain of tasks. The term *domain* indicates a space of tasks which include variations that have related but possibly different pedagogical affordances. These include mathematical and pedagogical extensions. For example, I could offer learners the following variation on the original task.

Something More Than

What constant needs to be added to the product of any four consecutive numbers from an arithmetic progression so as to produce a perfect square? Can it always be done?

Interlude: Informative Distinctions

Having at one time responded favourably to the notion of *rich tasks*, I subsequently became convinced that *it is not the task that is rich, but whether the task is used richly*. In other words, what matters when considering a task is not the task itself, but rather pedagogical actions that become available to me to make the experience pedagogically rich for participants. I found, for example, that I could invoke various pedagogical actions to transform a routine set of exercises from a text into a productive experience

for learners (Watson and Mason 2006). That is why, in addition to charting the emergence of a domain of tasks arising from *One More Than*, I choose to add various pedagogical comments which may be of assistance in alerting readers to possible pedagogical actions, not simply with these tasks, but with others as well.

My pedagogical comments are based upon a variety of distinctions made and actions described in the literature, including:

Scaffolding and fading (Brown et al. 1989): the notion stems from Wood and Bruner's insights from their reading of Vygotsky (1978), in which scaffolding is only effective if it is gradually faded, so that learners begin to take it on for themselves (the original notion of *zone of proximal development*, according to van der Veer and Valsiner 1991). A similar approach uses the trio *directed–prompted–spontaneous* as a reminder to the teacher (Love and Mason 1995).

Pausing: an action intended to provide participants with an opportunity to rehearse their own *personal narrative* or *self-explanation* (Chi and Bassok 1989), in order to locate uncertainties and to be prepared for the next step.

Didactic transposition (Chevallard 1985): the tendency for expert awareness (derived or arising from personal exploration) to be transposed into *instruction in behaviour* (learners being told what to do). This arises from four 'onlys': only awareness is educable (derived from Gattegno 1970); only behaviour is trainable; only emotion is harnessable; only attention is directable, derived from an image in the Upanishads (see Mason 1992). This article offers an antidote to didactic transposition by offering alternatives to framing tasks as instructions to learners as to what to do.

The Principle of Variation (Marton and Booth 1997; Marton 2015) and the notion of *Example Spaces* (Watson and Mason 2002): something is available to be learned (about a concept or procedure) only when something has been varied. The space of examples available to a learner when encountering some stimulus is the domain in which they are comfortable and familiar. Their appreciation and comprehension (their understanding) of a topic is based on the complexity of their associated accessible example spaces, part of their concept images (Tall and Vinner 1981).

Natural Powers: as many authors have noted, children arriving at school have already displayed wonderful powers, such as being able to imagine and to express; to specialise and to generalise; to conjecture and to convince. Learning to think mathematically can be seen as refining these powers in mathematical ways. The question is whether learners' powers are being evoked, invoked and developed, or whether the instruction usurps those powers (Mason 2002b).

Shifts of Attention (Mason 1998): what learners are attending to is important; so too is how they are attending. Distinguishing between *holding wholes* (gazing); *discerning details*; *recognising relationships in a situation*; *perceiving properties as being instantiated*; *reasoning on the basis of agreed properties* are distinguishable states and unless teacher and learners are attending to the same things in the same way, communication between them is likely to be incomplete. *Say What You See* is a useful pedagogic action to invoke, in order to stimulate the focusing of attention on details, recognising relationships and expressing them to others, thereby preparing to perceive properties being instantiated (generalising).

Asking *What Is the Same and What Different* (Brown and Coles 2000) is also effective.

Going with (and across) the grain (Watson 2000): generating facts by following a very familiar number pattern, such as that of the natural numbers, and then examining relationships between the calculation and the result in order to perceive a property or to detect a generality.

Return to the Flow

Comment on One More Than

I look especially for tasks which highlight an interplay between familiarity with the use of symbols (working with generality), and the need to specialise, in order to get a sense of structural relationships which might then be turned into properties (generalisations) to be justified, because this is the essence of mathematical thinking for me. Furthermore, every opportunity to experience this interplay adds to the richness of experience and the complexity of one's personal example space of instances of specialising and generalising. Put another way, it contributes to increased sophistication and complexity of the use of the power to specialise and to generalise.

Some people using an algebraic approach sensibly use $(n-1)n(n+1)(n+2)$ to try to simplify the bracket expansion. It turns out not to make a great deal of difference in recognising relationships or perceiving properties.

Comment on Something More Than

Note that this reformulation of *One More Than* might arise quite naturally among a group of learners for themselves. It leaves invariant salient features of the structure of the previous task, varying only the domain of *consecutive* (which is one manifestation of the principle of variation), by extending it to consecutive terms in an arithmetic progression. It does, however, make the property of 'being a square number' explicit rather than leaving it for learners to discover for themselves. A suitable pedagogic choice depends on sensitivity to learners and their current state.

Such a task could be appropriate for students gaining facility in algebraic manipulation and who are confidently constructing their own special cases (*specialising* is the term used by Pólya 1962).

Interlude: Formats for Presenting Generalisation Tasks

A less-demanding version of *One More Than* would be to provide learners with some examples and to invite generalisation. I call such sequences Tunja sequences, because two teachers in Tunja, Colombia – whose students were supposed be learning to factor

quadratic equations without having a firm grip on multiplication by negative numbers – asked me for advice. My solution was to exploit the structure of a sequence of arithmetic algebraic statements which students can extend and generalise, and then to use the same structure with polynomials while gaining practice in algebra (Mason 1999, 2001). Here is an instance.

One More Than (Tunja Style)

Check, extend and generalise the following sequence of arithmetic relationships:

$$\begin{aligned}1 \times 2 \times 3 \times 4 + 1 &= 25 \\2 \times 3 \times 4 \times 5 + 1 &= 121 \\3 \times 4 \times 5 \times 6 + 1 &= 361\end{aligned}$$

Comment

I like to present each line in silence, pausing after writing the product, then after the addition, and the equals sign, and then acting out mentally calculating before writing the answer. The idea is to provoke learners into anticipating in the moment rather than constantly trying to catch up. Also, I use the pausing to indicate where my attention is, and to invoke their natural *going with the grain* tendency stimulated by natural numbers.

Here, the examples provided are ‘consecutive’. It is important sometimes to offer non-sequential examples so that learner attention is drawn to functional relationships as well as to inductive relationships.

The opportunity to specialise has been deferred until students have detected and exploited apparent relationships, using them as properties to extend and then check conjectures. The right-hand sides could also be presented as squares if that was deemed appropriate for particular learners. It would be pedagogically valuable when reflecting on the activity with learners to draw attention to the way in which systematic variation revealed underlying relationships.

Something More Than can also be presented in Tunja style as a sequence of polynomials, which takes abstraction to a new level because the objects are now polynomials. The pedagogic choice would again depend on learners’ mathematical maturity and experience with expressing generality.

Something More Than (Tunja style)

Check, extend and generalise the following sequence of polynomial relationships:

$$\begin{aligned}n(n+1)(n+2)(n+3) + 1 &= (n^2 + 3n + 1)^2 \\n(n+2)(n+4)(n+6) &= 16 = (n^2 + 6n + 4)^2 \\n(n+3)(n+6)(n+9) + 81 &= (n^2 + 9n + 9)^2 \\n(n+4)(n+8)(n+12) + 256 &= (n^2 + 12n + 16)^2\end{aligned}$$

Pedagogical Comment

As with any Tunja sequence, I would present each term in each line slowly, pausing to give learners time to conjecture what is coming next. To anticipate what is coming next, they need to discern appropriate details, recognise relationships amongst those details and begin to perceive possible properties that might be being instantiated. This, in turn, requires shifting both the focus of attention and the way of attending: seeking details, recognising relationships, perhaps even perceiving properties as being instantiated. Engaging learners in these sorts of tasks on multiple occasions enables a gradual fading of scaffolding prompts to generalise, to conjecture and to convince.

Learners also need time to self-explain, to formulate their own personal narrative. Recalling something you have been told very often involves re-formulating the sense and expressing that in your own words. This takes time.

The constant terms could be displayed as 4th powers if deemed necessary for some learners.

An astute questioner might wonder whether any of the quadratics on the right-hand side factor further. Detecting and expressing relationships can lead to a generalisation from which it is clear that these quadratics can never factor further.

This version of the task is completely opposite to the first one. Whereas the first task stimulates students to try examples, preferably systematically, until they detect structural relationships which they can then express in general, here students cannot produce more ‘examples’ until they have sought out potential structural relationships among the numbers that appear in the examples which have already been provided. *Going with the grain* makes it possible to predict and extend, but it is vital to *go across the grain* in order to justify any conjectures. Along the way, learners will be seeking relationships between different discerned details (the added constants in the brackets on the left-hand side; the added constant on the left-hand side; the coefficients on the right-hand side; the middle term of the quadratic and how it might relate to the bracketed terms on the left).

Learners with access to a computer algebra system can use it to test conjectures about factorings. They can also use it to generate sequences of examples, or to search for possible examples, saving themselves the effort of manipulating polynomials by hand, and greatly increasing the range of examples to consider.

When I began to make use of the notion of Tunja sequences in different settings many years ago now, I realised that it could be helpful to develop a grid structure so that two different parameters could be varied at once. This led to the notion of a *Structured Variation Grid* (SVGrid: see www.pmtheta.com/structured-variation-grids.html). An SVGrid is a window on a potentially infinite, two-dimensional array of cells, each usually divided into an upper part containing a calculation and a lower part containing the result of that calculation. The window can be moved over the underlying grid using the left–right and up–down buttons. Clicking on a cell reveals or hides it, so that participants can be exposed to a sequence of calculations (upper cells), a sequence of results (lower cells) or a sequence of both, providing different opportunities to generalise. A cell reference system can be used or hidden using the Labels button (Fig. 1).

The window notion makes it possible to invite ‘far generalisations’ by moving around on the effectively infinite grid. The invitation is to predict the contents of the upper and lower cells in row r and column c . The labels help keep track of where you are on the grid.

The use of two adjacent cells (upper and lower) is designed to emphasise the difference between following and using a recognisable pattern (*going with the grain*), and articulating a relationship between two expressions (*going across the grain*). The grain image exploits the fact that a cut across a tree reveals the structure of the rings, while *going with the grain* is the process of splitting logs with an axe.

Any one row or column, or indeed any diagonal, constitutes a Tunja sequence using the upper, the lower or both cells. Individual parts call upon *going with the grain* to reach an expression of generality, revealed term by term by clicking, and pausing to predict. Both upper and lower cells together invoke *going across the grain* to justify the equality of the two expressions. This is, of course, the crux and purpose of these grids.

For example, an SVGGrid containing $c \times r$ in the r th row and c th column in the upper cell, and the answer in the lower cell, turns out to be a useful way to invite learners to discover that $(-1)(-1)$ is most naturally assigned the value $+1$, because of continuing arithmetic relationships of a multiplication table as the window is moved to the left and down.

Figures 2 and 3 shows part of the Four-fold Products Grid: the first three terms of a Tunja sequence made up of the blue cells of the first row, followed, below that, by the corresponding yellow cells in that row.

Having filled in some cells in a line, either uppers, lowers or both, attention can be directed to continuing to fill in sequentially, *going with the grain*, hoping to evoke a property being instantiated, and, hence, expressing a generality. At some point, attention can shift to a second line.

If participants do more than watch and wait, they may begin to get a taste for the difference between following a sequence of instructions and making use of their own

Fig. 1 Empty grid showing buttons

natural powers to generate and so make sense of some situation. This includes extending and varying for themselves. They then become immersed in the mathematical theme of *invariance in the midst of change* (the variation principle).

Even presenting the full (visible) part of the grid all at once (like a poster) can be used productively to engage learners' powers if effective pedagogic actions are employed, as in the next task (Fig. 4).

Something More Than (SVGrid as poster style)

Say What You See happening in the SVGrid below. How are adjacent cells related? Predict entries in the upper and lower cells in row r and column c .

Comment

The grid can be used in poster mode as suggested here, where everything is displayed from the beginning, and participants are invited to detect and express relationships on the way to generalising. Initially rather over-whelming, poster style can be useful to provide experience of how, when something seems over-whelming, discerning details and working with them can provide entry into making sense of the whole. The grid can, of course, be used as a normal SVGrid, gradually revealing various cells and inviting participants to attend to relationships that might be instances of properties, and so expressing those as generalisations. The main diagonal gives the original Tunja sequence.

Notice how the presence of the grid changes the tenor of the task. It no longer stimulates specialising in order to locate some structural relationships that might generalise. It is now a collection of examples, laid out in a structured format, but making it much easier to locate structural relationships. This might best suit learners working on detecting, expressing and justifying generalities. Of course, the window can be shifted, prompting (re-) construction of further instances, perhaps as part of checking a conjecture. An even more explicit version indicates the form of the additive constant and the result, by clicking the 'factor' button so that the constants in the lower cells are displayed as squares and in the upper cells as fourth powers.

Following along a line on the grid is very likely to bring to the surface (an instance of the *variation principle*) that the number added to the product is invariant along a row.

(1)(2)(3)(4) + 1	(2)(3)(4)(5) + 1	(3)(4)(5)(6) + 1	
(1)(2)(3)(4) + 1	(2)(3)(4)(5) + 1	(3)(4)(5)(6) + 1	
25	121	361	

Fig. 2 Starting to fill in bottom row top cells, then inserting bottom row bottom cells

		$(3)(6)(9)(12)$ + 81	
		2025	
	$(2)(4)(6)(8)$ + 16		
	400		
$(1)(2)(3)(4)$ + 1			
25			

$(1)(5)(9)(13)$ + 256			
841			
	$(2)(5)(8)(11)$ + 81		
	961		
		$(3)(5)(7)(9)$ + 16	
		961	

Fig. 3 Using diagonals to form Tunja sequences

Detecting relationships between the upper cells rows is non-trivial and likely to be expressed in multiple ways. There is no need to impose the relationships the teacher has detected, as long as multiple relationships come to the surface. The pedagogic actions *Say What You See* and *Same and Different* may be helpful in focusing learner attention, becoming even more powerful if learners' attention is drawn to their effectiveness during post-activity reflection.

Such prompts could then be faded over time, so that learners internalise them for themselves. Following a column of the grid is likely to bring to the surface that the notion of 'product of consecutive' numbers has been relaxed to include 'the product of consecutive numbers in arithmetic progression'. What remains to be explained is which numbers (from the factored version, the squares of which numbers) are needed in the upper cells and which numbers can appear in the bottom cells. This is an act of *characterisation*, a core theme in mathematics. The factored version may assist in this, depending on the maturity of the learners and their familiarity with large squares.

Being playful with the consecutive terms may eventually generate a conjecture about how, from the four multiplied numbers alone, the final square number can be obtained, and obtained in more than one way. The pedagogic action of asking for *more than one way* serves an important mathematical purpose, because it is through multiplicity of expressions for the same thing that structural awareness is fostered. If used as a prompt

Going With and Across the Grain: 4-fold products

Up	6	$(1)(7)(13)(19)$ + 1296	$(2)(8)(14)(20)$ + 1296	$(3)(9)(15)(21)$ + 1296	$(4)(10)(16)(22)$ + 1296	$(5)(11)(17)(23)$ + 1296	$(6)(12)(18)(24)$ + 1296
		3025	5776	9801	15376	22801	32400
	5	$(1)(6)(11)(16)$ + 625	$(2)(7)(12)(17)$ + 625	$(3)(8)(13)(18)$ + 625	$(4)(9)(14)(19)$ + 625	$(5)(10)(15)(20)$ + 625	$(6)(11)(16)(21)$ + 625
		1681	3481	6241	10201	15625	22801
Show All	4	$(1)(5)(9)(13)$ + 256	$(2)(6)(10)(14)$ + 256	$(3)(7)(11)(15)$ + 256	$(4)(8)(12)(16)$ + 256	$(5)(9)(13)(17)$ + 256	$(6)(10)(14)(18)$ + 256
Hide All		841	1936	3721	6400	10201	15376
Show All	3	$(1)(4)(7)(10)$ + 81	$(2)(5)(8)(11)$ + 81	$(3)(6)(9)(12)$ + 81	$(4)(7)(10)(13)$ + 81	$(5)(8)(11)(14)$ + 81	$(6)(9)(12)(15)$ + 81
Hide All		361	961	2025	3721	6241	9801
Factor	2	$(1)(3)(5)(7)$ + 16	$(2)(4)(6)(8)$ + 16	$(3)(5)(7)(9)$ + 16	$(4)(6)(8)(10)$ + 16	$(5)(7)(9)(11)$ + 16	$(6)(8)(10)(12)$ + 16
Reset		121	400	961	1936	3481	5776
Down	1	$(1)(2)(3)(4)$ + 1	$(2)(3)(4)(5)$ + 1	$(3)(4)(5)(6)$ + 1	$(4)(5)(6)(7)$ + 1	$(5)(6)(7)(8)$ + 1	$(6)(7)(8)(9)$ + 1
Labels		25	121	361	841	1681	3025
Left		1	2	3	4	5	6
		Left					Right

Fig. 4 SVGrid for unfactored products of consecutive terms from APs added to a constant

and then increasingly indirectly, so that learners begin to internalise it for themselves as a worthwhile action, learners may begin to employ it for themselves spontaneously in other situations (*scaffolding and fading; directed–prompted–spontaneous*).

Searching the internet for support on these tasks is not currently helpful, because it seems that number theorists have been more interested in products of consecutive numbers being multiples of a power rather than using an additive constant (see, for example, Saradha 1997).

Going Further

At some point I asked myself a more general question, having noticed that the third constant in the left-hand side was always the sum of the first two. I experimented at first with expressions like

$$n(n+r)(n+r+d)(n+2r+d) + s$$

where I was thinking of n as a column number and r as a row number, and d as some specified difference. My task was to find which constant to add to make a square. My discoveries were immediately turned into another SVGrid, in which the objects were arithmetic calculations, so n is the column number and the difference is controlled by user choice using arrow buttons. My notes of the time mention a connection with Pythagorean triples, but on the next stage of evolution from my notes the connection escaped me! Furthermore, when later I tried to detect structural relationships myself from the grid, it turned out to be more difficult than expected! Tasks that seem obvious at one time can become mysterious at another (Fig. 5).

Comment

The difference d applies to the middle terms for cases when the difference in the first two terms is even; when the difference in the first two terms is odd, the middle terms differ by $2d + 1$.

The SVGrid takes away the need – but also the opportunity – to construct examples for oneself, but affords the opportunity to detect rather more sophisticated relationships and instantiated properties. *Say What You See* and *What Is the Same and What Different* in pairs of cells are useful pedagogic actions to initiate. Many of the relationships that work in one or two cells break down when applied to all the cells.

By making d negative, and by shifting the window down and/or left, learners can be immersed in products involving negative numbers.

Returning to my notes at a later date I expressed the task more algebraically:

How Much More Than?

Write down two numbers a and b . Is there a constant which can be added to the product of n , $n+a$, $n+b$ and $n+a+b$ so that you always (for all n) get a perfect square?

Fig. 5 Arithmetic SVGrid with assigned difference of $d = 4$

Comment

Here, the objects are again polynomials rather than arithmetic expressions. It did not take me long to discover a condition on ab which guarantees it is always possible.

As soon as I discovered this, I built an SVGrid to make it easy to display these examples in a structured format. The grid I constructed has the option to encounter the requisite condition on ab : there is a button (*Fractions*) which displays or blanks out cells requiring fractions. This is, of course, an example of *didactic transposition*, because my pleasure in uncovering the relationship is transformed into a format through which learners might come to an expression of this generality. However, the grid structure can be used in a pedagogically rich manner to make experience on the task mathematically rich. The SVGrid shows nothing of the necessity for the relationship, only the sufficiency (Fig. 6).

Both versions can be used as vehicles for stimulating learners to express generalities based on progressive revealing of cells, whether the uppers, the lowers or both together. The left-hand version invites explanation as to why some cells are blanked out, while the right-hand one requires facility with fractions, but extends the range of permissible coefficients.

Comment

As with any pattern perceived through *going with the grain*, the conjecture has to be justified with mathematical reasoning. *Say What You See*, and *What Is the Same and What Different* are again useful pedagogic actions, so as to enable participants to align what they are attending to with how they are attending to it. It would be interesting to find a geometrical interpretation, so that participants could use geometrical relationships to express and then ‘realise’ the algebraic relationship, but algebra will, of course, suffice.

Notice how the internal structure of the four-fold products is clearer when expressed as a generality when compared with when it is submerged beneath the particularities of arithmetic instances.

Disruptive Remark

In case you want to check your reasoning against mine, the cell in row r and column c in the formula-based grid is:

$$n(n+c)(n+r)(n+c+r) + \left(\frac{rc}{2}\right)^2 = \left(n^2 + n(c+r) + \frac{rc}{2}\right)^2$$

This fails to give integer parameters when both r and c are odd. Hence the blank cells.

The squared quadratic term factors precisely when c and r are legs of a Pythagorean right-angled triangle. Writing $c=2uv$ and r as $u^2 - v^2$ provides a parametrisation for instances that factor as the product of two squares. Notice that whether c or r is assigned to be $2uv$ makes no difference because of symmetry.

$$\begin{aligned} n(n+2uv)(n+u^2-v^2)(n+u^2+2uv-v^2) + u^2v^2(u^2-v^2)^2 \\ = (n+v(u-v))^2(n+u(u+v))^2 \end{aligned}$$

The Pythagorean remark in my notes is explained!

I then constructed a further SVGrid, making use of u and v as the row and column parameters. It proved unsatisfactory because of the way in which u and v parametrise Pythagorean triples, as u and v take on small integer values.

This task, in either its algebraic or SVGrid format, focuses on algebraic manipulation of already perceived generalities in order to find some attractive property. Learners are likely to experience something of what it is like to explore (play) mathematically, using and developing their powers.

Fig. 6 Portions of a formula-based SVGrid

During the next phase of task evolution, I wondered whether the $a, b, a + b$ structure was really necessary:

Four-Fold Products

For what numbers a, b and c is there a constant s such that

$$n(n + a)(n + b)(n + c) + s$$

is a perfect square for all values of n ?

Comment

At first quick reading, *How Much More Than* and *Four-Fold Products* may seem to be the same task, but they differ in the generality of c . They are particularly well suited to gaining facility with algebraic manipulation and the use of a *theorem-in-action* (Vergnaud 1982), that for a polynomial identity to hold for all values of the variable n , coefficients have to match. This learners might intuit for themselves or they could be led to it by a series of examples, perhaps in an SVGrid or a Tunja sequence.

The tasks are also well suited to learners who are using a computer algebra system, because with a simple do-loop they can carry out a large number of special cases (specific examples) from which they can conjecture and then validate polynomial identities.

Notice that *Four-Fold Products* can arise through a desire to show that all possible examples have been located in *How Much More Than*?

Disruptive Remark

Requiring that, for all n ,

$$n(n + a)(n + b)(n + c) + s = (n^2 + pn + q)^2$$

it must be the case that $s = q^2$. Assuming that $a \leq b \leq c$, it turns out that c must be $a + b$.

Personal Narrative

Express for yourself the full space of mathematical objects encompassed by the last generality.

Comment

The full example space of square numbers added to products of four integers being a perfect square has been expressed. Notice the value in trying to express that for yourself.

I was able to locate a necessary and sufficient relationship among the constants on the left to permit factoring into the square of linear terms on the right-hand side. An alternative approach, given a computer algebra system, is to record some examples. Doing this, I found myself with a new task, presented algebraically and in both Tunja and SVGGrid fashion as follows:

Factored Squares

What products of four linear terms added to a constant will factor as the square of the product of two linear terms? In other words, what conditions on a , b , c , s , p and q make

$$n(n+a)(n+b)(n+b) + s = (n+p)^2(n+q)^2$$

for all n ?

Tunja Factoring as Factored Squares

Check, extend and generalise the following sequence of facts:

$$\begin{aligned} n(n+3)(n+4)(n+7) + 6^2 &= (n+1)^2(n+6)^2 \\ n(n+5)(n+12)(n+17) + 30^2 &= (n+2)^2(n+15)^2 \\ n(n+7)(n+24)(n+31) + 84^2 &= (n+3)^2(n+28)^2 \end{aligned}$$

Does your sequence extend backwards as well as forwards?

Comment

Again, this is more relevant to learners concentrating on expressing generality, but also accessible to learners using computer algebra to explore by seeking non-square factorings into two quadratics.

Notice that the use of computer algebra system to generate examples (which here are polynomials) tends to lead to Tunja-type, self-set tasks. If it is from my own initiative, I am more likely to experience a desire to find out than when the task is presented to me by someone more experienced who hopes that I might experience what they experienced (the *didactic transposition*).

Disruptive Remark

In this sequence, I failed to notice the presence of (two terms) of Pythagorean triples on the left, and the two terms on the right whose difference is the third edge of the Pythagorean triangle, even though I had, on a previous phase of exploration, encountered Pythagorean triples (Fig. 7).

Fig. 7 Factored squares SVGGrid

Even More General

During my explorations, I stumbled over instances in which the four-fold product with an added constant factored not as a square but rather as the product of two different quadratic terms. This led me to explore further, resulting in another little frisson of pleasure.

Four-Fold Products Factored

Given a , b and c , under what conditions will

$$n(n+a)(n+b)(n+c) + qs = (n^2 + pn + q)(n^2 + rn + s)$$

for all n . Are there always infinitely many solutions?

Four-Fold Products Factored (Tunja style)

$$\begin{aligned} n(n+1)(n+2)(n+3) + 10 &= (n^2 + 6n + 10)(n^2 + 1) \\ n(n+1)(n+2)(n+3) + 45 &= (n^2 + 7n + 15)(n^2 - n + 3) \\ n(n+1)(n+2)(n+3) + 126 &= (n^2 + 8n + 21)(n^2 - 2n + 6) \\ n(n+1)(n+2)(n+3) + 280 &= (n^2 + 9n + 28)(n^2 - 3n + 10) \\ n(n+1)(n+2)(n+3) + 540 &= (n^2 + 10n + 36)(n^2 - 4n + 15) \end{aligned}$$

Generalise! What happens if you work the sequence upwards?

Are there infinitely many such relationships?

Comment

Switching from finding examples to thinking about what might constrain possible examples is not always easy to do and may require pedagogic prompting. The opportunity to experience mathematical desire (to find out whether or when some property holds) is becoming more intense and closer to my own experience. The temptation to enact the *didactic transposition* also increases! Learners able to embark on such tasks presumably would not be in need of SVGGrids to help them.

Disruptive Remark

It took me several attempts using *Maple* to specify what I was looking for. Given that there are four parameters on the right-hand side and only three given on the left-hand side, there could be whole families of instances, as in the case of $n(n+1)(n+2)(n+3)$. Choosing p as a variable, along with a , b and c , it eventually emerged (*Maple* was invaluable) that the necessary conditions for the factoring to work are that:

$$r = a + b + c - p; s = \frac{(a+b-p)(b+c-p)(c+a-p)}{a+b+c-2p}; q = \frac{(a-p)(b-p)(c-p)}{a+b+c-2p}$$

Both s and q have to be integers, but there is something similar about the relationships. It turns out that $s+q$ is always an integer (*Maple* was very helpful for this) and that q can be rearranged as:

$$q = p^2 - p \left(\frac{a+b+c}{2} \right) + \left(ab + bc + ca - \left(\frac{a+b+c}{2} \right)^2 \right) - \frac{(a+b-c)(b+c-a)(c+a-b)}{8(a+b+c-2p)}$$

(Again, *Maple* was helpful for uncovering and checking such relationships.)

When the last term is not zero, there can only be finitely many values of p to make the last term an integer. This means that q is an integer for all p if and only if $a+b=c$ (assuming $a \leq b \leq c$), in which case $a+b+c$ is even.

This explains why arithmetic progressions work so well to generate infinite sequences.

As is often the case, interesting side explorations emerge involving other areas of mathematics. In this case, polynomials and some number theory. *Maple* made all this easier than trusting my algebraic manipulation. The important thing, though, is to have algebraic awarenesses, which mean conceiving of the conjecture (that s and q are integers together; that q can be re-expressed) and then arranging a convenient way to test it using a computer algebra system.

A number theory problem now emerges: given a , b and c , with $a \leq b \leq c$ and $c \neq a+b$, for what values of p is q an integer? Noting that $a+b+c$ has to be even, putting

$\alpha = (b + c - a)/2$, $\beta = (a + c - b)/2$ and $\gamma = (a + b - c)/2$, means that α , β and γ are to be integers, and since $\alpha + \beta + \gamma = (a + b + c)/2$, q is an integer only when

$$\frac{\alpha\beta\gamma}{2(\alpha + \beta + \gamma - p)}$$

is an integer. There are finitely many values of p for which this can be the case, but how might one determine these efficiently?

A final urge of the *didactic transposition* is to pose the discovery as a problem, perhaps suitable for Olympiad training:

Finite or Infinite?

For which integers a , b and c are there infinitely many integers s for which

$$n(n + a)(n + b)(n + c) + s$$

factors as the product of two non-trivial quadratics in n ?

General Comments

I used *Maple* to do factoring for me, so that I could quickly check results in the SVGGrid. I also constructed a number of different SVGGrids in a search for what might be most promising and most useful pedagogically. I find that constructing applets in and through which to engage and assist learners often requires more and deeper mathematical thinking on my part, and reveals aspects that might be overlooked when simply gazing at a posed problem. By struggling myself to find and express general relationships, I feel that I have a greater sensitivity to learners struggling with admittedly simpler situations in which learners are invited to express detect and express general relationships.

Once the examples are laid out as in an SVGGrid (with all cells displayed), it is mostly quite straight-forward to *go with the grain* following the natural numbers or other familiar sub-sequences, as mentioned earlier. However, what matters most is to *go across the grain*, to seek to justify equalities between the upper and the lower cells. This is where algebra becomes necessary, because it is through algebraic manipulation that the truth of various conjectures can be reasoned out and the equalities justified.

Beyond Four-Fold Products

The task name *Four-Fold Products* explicitly labels an invariant feature of the tasks so far, namely considering the product of four numbers. In line with the observation made by Brown and Walter (1983), that stressing a particular word often triggers a reaction of ‘why that word, why not some other?’, stressing *four* suggests considering products of *five* or more.

Despite having searched in vain in the past, I again tried using Maple and came across the following:

$$n(n+1)(n+2)(n+3)(n+4) + 210 = (n^2 + 6n + 15)(n^3 + 4n^2 - 4n + 14)$$

Looking for some more general pattern, the right-hand side of can be written as

$$\left((n+3)^2 + 6\right)\left((n+1)^3 + (n-3)^2 - (n-4)\right)$$

or as

$$\left((n+3)^2 + 6\right)\left((n+3)^3 - 5(n+3)^2 - (n+3) + 35\right)$$

and the second one is a translation $n \rightarrow n-2$.

Or, more symmetrically perhaps, as

$$(n-2)(n-1)n(n+1)(n+2) + 210 = (n^2 + 2n + 7)(n^3 - 2n^2 - 8n + 30)$$

which can be expressed as

$$\left((n+1)^2 + 7\right)\left((n+1)^3 - 5(n+1)^2 - (n+1) + 35\right)$$

which is simply a translation $n \rightarrow n-2$.

There must be lots of these sorts of relationships, but without clever advanced methods, they are likely to be needles in a haystack. Note that $210 = 5 \times 6 \times 7$, which is certainly intriguing. But are there any five-fold products with infinitely many constants that will afford a factoring?

Disruptive Remark

There appears to be only one other such relationship for five consecutive numbers with a non-zero additive constant, factoring as a quadratic and a cubic, namely

$$n(n+1)(n+2)(n+3)(n+4) + 2160 = (n^2 + 7n + 30)(n^3 + 3n^2 - 16n + 72)$$

This is based on the fact that considering

$$(n-2)(n-1)n(n+1)(n+2) + be = (n^2 + an + b)(n^3 + cn^2 + dn + e)$$

and eliminating c , d and e yields an expression which is bi-quadratic in a and quadratic in b . The discriminant for b has to be a perfect square in order to have integer values for the coefficients. Put

$$D^2 = 5a^4 - 10a^2 + 9$$

which when written as

$$D^2 - 5(a^2 - 1)^2 = 4$$

presents itself in the form of Pell's equation. It turns out that there are integer solutions only when a (assumed positive) takes the values 1, 2 or 3. From these, only two non-trivial factorings appear.

Suddenly, exploration enters the domain quadratic (and cubic) equations, discriminants and number theory. At the very least, it may send some learners off to find out about Pell's equation.

Extending to 6 or more products is of course possible.

Reflections on Dimensions of Possible Variation

The first task invites considering the dimension of different sequences of four consecutive numbers, described and experienced as *going with the grain* (Watson 2000). The mathematically productive move is then to *go across the grain* and to seek justification for why the identity always holds. Other features of the situation were then identified as parameters that could be varied: the number of terms and the separation between the terms. Always in mind is the possibility of presenting some parameter variation as a Tunja sequence or in an SVGrid. A further dimension of possible variation is in the factored form of the right-hand sides, moving from a single square to a further or different factoring.

To begin with, the objects I was working with were numbers and arithmetic relationships. The aim was to generalise to polynomials. The polynomials then became objects of interest, with attention on two ways of writing them, and whether they belonged to a general class of results or whether there were only finitely many such relationships. For some time, the notion of squares (of numbers, of polynomials) remained invariant, and it was only when confidence grew that this condition was relaxed. While a strong mathematician might jump to a very broadly encompassing generality, learners are likely to need to be enculturated into such moves, supported in making smaller steps in order to gain confidence.

Rarely is it possible to say that an exploration is finished or completed. In revising this manuscript for publication, I yet again discovered more questions, enriching still further my sense of the domain of related tasks. It seems there are almost always ways of putting results into a more general context, building a bigger picture.

Summary

By varying and extending aspects of a single task, a domain of tasks opens up. Opportunities for mathematical exploration arise. Those explorations sometimes give rise to alternative tasks with different pedagogic affordances. Availability of a computer algebra system takes exploration to a more complex level.

The *didactic transposition* (Chevallard 1985) warns of the tendency for expert awareness, arising from mathematical exploration, to be transposed into instructions in behaviour for learners, in the fond hope that somehow learners will experience the

epiphanies experienced by the expert. By being alert to alternative pedagogic possibilities in how a task is presented, and how computers are used, this almost inevitable transposition may perhaps sometimes be circumvented.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- Brown, L., & Coles, A. (2000). Same/different: A 'natural' way of learning mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 153–160). Hiroshima: PME.
- Brown, S., & Walter, M. (1983). *The art of problem posing*. Philadelphia: Franklin Press.
- Brown, S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32–41.
- Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.
- Chi, M., & Bassok, M. (1989). Learning from examples via self-explanation. In L. Resnick (Ed.), *Knowing, learning and instruction: Essays in honour of Robert Glaser* (pp. 251–282). Hillsdale: Lawrence Erlbaum Associates.
- Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge & Kegan Paul.
- Love, E., & Mason, J. (1995). Telling and asking. In P. Murphy, M. Sellinger, J. Bourne, & M. Briggs (Eds.), *Subject learning in the primary curriculum: Issues in english, science and mathematics* (pp. 252–270). London: Routledge.
- Marton, F. (2015). *Necessary conditions for learning*. Abingdon: Routledge.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Hillsdale: Lawrence Erlbaum Associates.
- Mason, J. (1992). Researching problem solving from the inside. In J. Ponte, J. Matos, J. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technology: Research in contexts of practice* (pp. 17–36). London: Springer-Verlag.
- Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1(3), 243–267.
- Mason, J. (1999). Incitación al estudiante para que use su capacidad natural de expresar generalidad: Las secuencias de Tunja. *Revista EMA*, 4(3), 232–246.
- Mason, J. (2001). Tunja sequences as examples of employing students' powers to generalise. *Mathematics Teacher*, 94(3), 164–169.
- Mason, J. (2002a). *Researching your own practice: The discipline of noticing*. London: RoutledgeFalmer.
- Mason, J. (2002b). Generalisation and algebra: Exploiting children's powers. In L. Haggerty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 105–120). London: RoutledgeFalmer.
- Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (second extended edition). Harlow: Prentice-Hall (Pearson).
- Pólya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving* (combined edition). New York: Wiley.
- Saradha, N. (1997). On perfect powers in products with terms from arithmetic progressions. *Acta Arithmetica*, 82(2), 147–172.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169.
- van der Veer, R., & Valsiner, J. (1991). *Understanding Vygotsky*. London: Blackwell.
- Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematics*, 3(2), 31–41.

- Vygotsky, L. (1978). *Mind in society: The development of the higher psychological processes*. Cambridge: Harvard University Press.
- Watson, A. (2000). Going across the grain: Mathematical generalisation in a group of low attainers. *Nordisk Matematikdidaktikk (NOMAD – Nordic Studies in Mathematics Education)*, 8(1), 7–22.
- Watson, A., & Mason, J. (2002). Extending example spaces as a learning/teaching strategy in mathematics. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of mathematics education* (Vol. 4, pp. 377–385). Norwich: PME.
- Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, 8(2), 91–111.