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An approach for reaching consensus in large-scale group decision-making focusing on dimension reduction

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Abstract
Group decision-making and consensus modeling have always been important research topics. With the widespread use of the Internet, group decisions can be made online, in which a large number of decision-makers participate. Most of the existing studies on large-scale group decision-making consider 20–50 decision-makers. Therefore, there is a need for a framework that focuses on situations where thousands of decision-makers exist. As dimension reduction is one of the five primary challenges in large-scale group decision-making, in this study, after reviewing the existing approaches, a new model is presented using a statistical approach along with complex network analysis techniques. The opinions are generalized first, and then the network of opinions is built. This new method reduces the dimensions of the problem by considering a hierarchy of opinions. Different scenarios were designed for the evaluation. The results show that the effect of this generalization on dimension reduction depends on the parameters of the problem. We have shown that in a group decision scenario with 3000 decision-makers and 6 alternatives, 99% of the data was reduced. As dimension reduction is the main focus of the current research, the effect of consistency on the diversity of opinions has also been investigated, and the results show that opinion consistency affects opinion generalization, which in turn affects dimension reduction. In addition, in the performed simulations, three types of functions were used to calculate similarity. The aim was to determine the best similarity function for the decision problems whose purpose was to rank the available alternatives. The results show that Euclidean similarity is a strict criterion compared with Cosine similarity.

Keywords Large-scale group decision-making · Consensus · Uncertainty · Multi-layer networks · Dimension reduction

Introduction

Decision-making has always been an important research subject. Most of the prediction mechanisms [1] and decision support systems [2] aim to help humans in the decision-making process. Group decision-making is a step further where all decision-makers having their own opinion need to reach a consensus. Therefore, group decision-making consists of a group of decision-makers expressing their opinions about a set of alternatives to make a decision. After expressing their initial opinions, decision-makers enter into an iterative and dynamic process to increase the agreement between group members. One of the main challenges in any group decision-making scenario is to reach a perfect agreement among all decision-makers, which in most cases, is not possible. There are two types of consensus: (1) hard and (2) soft. In hard consensus, at the end of the process, the preferences of all decision-makers converge into one. So, the hard consensus seems impossible in most situations. In soft consensus, the agreement between decision-makers is measured with a fuzzy number, which can be considered an acceptable level of agreement. The core idea of fuzzy membership has also been used in many fields such as control theory [3] and security [4]. Group decision-making has traditionally been seen as a process in which a small number of experts interact with each other to make a selection from available alternatives. The consensus process is defined as dynamic and repetitive group discussions aimed at bringing together the opinions of decision-makers [5]. Recently, the

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number of groups participating in the decision-making process has increased, ranging from tens to thousand [6]. Due to technological advancements, there is a strong tendency among researchers to propose consensus-reaching or opinion dynamics models that involve a large number of DMs who are assumed to interact through a social network platform Therefore, there is a need for a framework that focuses on situations where thousands of decision-makers are attending. A classic view of large-scale group decision-making considered the problem as a decision-making process in which there are more than 20 decision-makers. Based on this view, many researchers have presented their method for groups of approximately 20 decision-makers. Tang et al. criticized the fact that most of the available studies involve 20–50 decision-makers and emphasized that in large-scale group decision-making, the performance of models in larger groups with thousands of decision-makers should be examined [7]. In group decision-making, in which the goal is to increase the level of agreement, the following main processes are needed:

1. *Consensus-reaching process (CRP)* First, the opinions of individuals are collected, and the degree of agreement is calculated. If the degree of agreement is not reached, this means that some decision-makers have to change their opinions. One of the termination conditions is that the level of agreement reaches a predetermined threshold [8].
2. *Selection process* When the similarity of decision-makers' opinions reaches a predetermined consensus threshold value, the selection process is applied to obtain the final ranking of the alternatives [9, 10].

Large-scale group decision-making scenarios are fundamentally different from small groups, and specific key elements appear in large-scale group decision-making models [6, 11]. In the work of Du et al. [12], clustering, which can be a solution for reducing the dimension, is also mentioned as one of the main processes of large-scale decision-making. As the number of decision-makers increases, the processing time of clustering increases as well. Therefore, dimension reduction using a simpler approach can affect the speed of the consensus process. García-Zamora et al. indicated that classic consensus-reaching processes are not suitable for large-scale group decision-making problems [13]. Most of the existing methods are time-consuming in such situations, which makes them unsuitable. So, in large-scale group decision-making, the performance of models in larger groups with thousands of decision-makers should be examined. Most of the studies reviewed in [7] take advantage of clustering for dimension reduction, but selecting the clustering algorithms and their parameters is also challenging in group decision-making. So, to provide a specific consensus process for large-scale group decision-making, a framework is proposed using opinion transformation and a two-layer network. The main idea is to create a hierarchy of preferences and reduce the dimension by considering the transformed form of preferences, which are much smaller than the original preferences. Thus, here, the dimension means the number of preferences, and it can be applied in situations in which a large number of decision-makers take part. This generalization is similar to roll-upping in data mining techniques. Rolling-up the data is about data summarization and is the process of aggregating data elements from a lower level data structure into a higher level one. Therefore, fast and efficient dimension reduction is an advantage of the proposed generalization. It is fast because it does not have the overhead of clustering. It is efficient because it summarizes the data. In the proposed model, consistency and certainty can affect the size of the opinion data. Therefore, in this study, the effects of consistency and certainty on opinion size were investigated for the first time. The results in the evaluation section show that depending on the number of decision-makers and alternatives, the opinion data can be reduced by up to 99%. In the evaluation section, three types of similarity criteria are used in the consensus process, and it is shown that Euclidean similarity is a strict criterion for decision problems whose purpose is to rank the available alternatives. The rest of the paper is organized as follows. Related work is reviewed in "Related work". "Background knowledge" presents the concepts and the main mathematical prerequisites of the domain. The proposed approach is presented in "Problem statement". In "The proposed approach", the proposed approach is evaluated using several group decision-making simulation scenarios. Finally, in "Conclusion", the paper concludes and some pointers regarding possible future directions are provided.

Related work

We have divided the processes in various consensus frameworks into three phases: (1) preprocessing, (2) consensus process, and (3) selection process. The preprocessing phase includes all tasks performed before the consensus-reaching process. By adding this phase to the framework, we can separate the consensus processes from the preprocessing and simplify the framework. According to [14], preprocessing consists of data cleaning, data integration, data reduction, and data transformation. Some studies focus on approaches to fill-out incomplete opinions. These techniques can be considered data-cleaning techniques [15–19]. In some studies, data integration and data transformation techniques have been used to provide a framework in which decision-makers can express their opinions in their desired way so that all opinions are then integrated into a single form [9, 20, 21]. Data reduction techniques are widely used in large-scale group decision-making frameworks. Some studies, such as [11, 12, 22–27], have used clustering and grouping of opinions to reduce the number of decision-makers. Some studies, such as [28–30], have used network partitioning based on network structures. In some articles, the goal was not to reduce the dimension. But, because only some of the more important nodes were considered, other nodes were excluded from the calculations. The next phase is the consensus process. In this phase, the consensus level is calculated, and if it is below a certain threshold, feedback is generated. The feedback mechanism is responsible for generating suggestions for decision-makers, and if decision-makers accept these suggestions, their opinions converge. The feedback mechanism consists of two steps. The first step is selecting the decision-makers for changing their opinions, and the second step is to generate suggestions for the selected decision-makers and determine opinion changes. Based on our review, there are four approaches for decision-maker selection, which are explained below:

1. *Selecting all of the decision-makers* In this approach, all opinions must change.
2. *Selecting a subgroup of decision-makers* In this approach, decision-makers are divided into several groups, and at each iteration, the opinion of one or several groups changes [5, 17, 31].
3. *Selecting a decision-maker* Individuals whose opinions are not similar to others are identified and individual feedback is generated [9, 19, 32–34].
4. *Adaptive* This approach is a combination of the second and third approaches. In this way, the higher the consensus degree is, the fewer opinions have to change [7, 12].

Three approaches for suggestion generation exist, which are explained below:

1. *According to the collective opinion* In these studies, a change of opinion is performed by considering a collective opinion [19, 22, 30, 33–35].
2. *Using opinion dynamics* Opinion dynamics describes the process of forming opinions among a group of interactive decision-makers [10]. Rules that are used in related research are divided into two general categories: (1) linear and (2) nonlinear rules [28, 36, 37].
3. *Considering the decision-maker's network* Another way to change opinions is to use the network of decision-makers. In some solutions, a network is created based on input information [5, 32, 36, 38–40]. In other studies, it was assumed that additional information is available from another network, such as trust networks or social networks, which is separate from the consensus problem space [17, 23, 30, 33, 41–46].

The third phase is the selection process. Since the goal is to reach a consensus, this stage has not been addressed in some studies such as [12, 22]. In most studies, opinions are aggregated to determine the final opinion. Simple weight-average functions can be used for aggregation [15, 30, 31].

Different types of research on group decision-making have been reviewed to obtain a complete understanding of the problem. Table 1 shows the details and key characteristics of the reviewed approaches. In the table, it is specified which research used a network as its application context or was only presented for large-scale decision-making groups Here network means any network such as a social network or similarity network. Table 2 summarizes the reviewed related work.

Researches in the field of large-scale decision-making are more recent. A new definition of large-scale decision-making proposed in [13] uses an m-LSGDM representation format. So, as mentioned before, and based on the classical view of large-scale decision-making, the scope of most of the existing studies is considered to be 20-LSGDM or at most 50-LSGDM. Based on the performed review, we conclude that the following research challenges should be addressed:

1. Current consensus-reaching models mainly focus on decision situations in which only 20–50 decision-makers are involved, and there is a need for a framework that focuses on group decision-making with hundreds or thousands of decision-makers, e.g., 1000-LSGDM
2. The main method for calculating consensus level is opinion similarity. Many researchers have not mentioned the reasons for using a specific similarity measure. However, this study searched for the best similarity measure for a consensus-reaching problem.

By considering previous approaches and their limitations, a new framework is proposed using opinion transformation and a two-layer network. The selection approach is subgroup based and the suggestion approach is network based. As previously mentioned, the main idea is to create a hierarchy of preferences and reduce the dimension by considering the transformed forms of preferences. To the best of our knowledge, this is the first time that opinion generalization is used to reduce the number of preferences. In this study, we aim to propose an approach for reaching a consensus in large-scale group decision-making focusing on dimension reduction.

Background knowledge

In this section, preference relations, consistency, consensus, and network as the required basic notions for understanding this paper are defined.

Table 1 Key characteristics of the reviewed related work in the GDM domain

Article	Uses a network?	LSGDM	Description
Urena et al. [5]	✓		A social network is created based on similarity, confidence, and consistency. The main idea is that some opinions with high similarity, confidence, and consistency have a greater influence on the opinions of others
Chao et al. [9]		✓	A clustering method is used to detect non-cooperative behavior, and a weighting process is used to manage this behavior. The number of clusters should be determined, which is different for different group-decision scenarios
Liao et al. [11]		✓	K-means is used for clustering, which requires the k parameter initialization
Du et al. [12]		✓	Opinion punishment and weight punishment are used to manage non-cooperative behaviors. The number of clusters should be determined which is different in various group decision scenarios
Wu et al. [15]	✓		A trust network was used to estimate the unknown preference values and experts' weight determination
Wu et al. [16]	✓		A trust network was used to estimate the unknown preference values and extract the reputation between experts as historic actions
Taghavi et al. [17]	✓		A feedback-based influence network was proposed, in which the influence between agents was calculated by trust, self-confidence, and similarity
Herrera-Viedma et al. [18]			Using the additive-consistency concept, a procedure is provided to estimate the missing information in an expert's incomplete preference
Cheng et al. [19]	✓		A weight allocation method is provided by analyzing the tie strength and topology structure of social networks
Xu et al. [20]			Prospect theory was used to solve the group decision-making problem
Zhang et al. [21]			Heterogeneous preference structures were accepted as inputs
Wu et al. [22]		✓	K-means was used for clustering and clusters were allowed to change. To use K-means, the parameter k should be determined. Changing the clustering approach has computational overhead in LSGDM
Lu et al. [23]	✓	✓	K-means is used for clustering. To use K-means, the parameter k should be determined
Wu et al. [24]	✓	✓	K-means is used for clustering. To use K-means, the parameter k should be determined
Trillo et al. [25]		✓	NLP techniques were used to detect the degree of positivity and aggressiveness of experts. The main idea revolves around sentiment analysis
Zhong et al. [26]		✓	A combination of similarities in the evaluation information is used for clustering. The K-means algorithm is used for clustering. To use K-means, the parameter k should be determined
Liu et al. [27]		✓	The probabilistic K-means clustering algorithm is used to improve the selection of the initial centroids. However, the k parameter should be determined
Dong et al. [28]	✓		Leaders and their followers are detected in the network. Leaders influence the opinion of their followers
Zhang et al. [29]	✓		A feedback mechanism is provided for each expert by considering the leadership and bounded confidence levels of experts
Chu et al. [30]	✓	✓	A two-stage consensus-reaching method is proposed in which cluster preferences can change
Ding et al. [31]	✓	✓	A negative conflict relationship is considered between DMs. Feedback is generated for each DM which may not be suitable in LSGDM
Wu et al. [33]	✓		Trust-based recommendation mechanism is applied
Gavrilets et al. [35]	✓		The dynamics of consensus building in groups is investigated which is composed of individuals who are heterogeneous in preferences and have different personality traits (agreeability and persuasiveness) and reputation
Li et al. [36]	✓		Extracts the influence network from expert opinions and social networks

Table 1 (continued)

Article	Uses a network?	LSGDM	Description
Zhang et al. [38]	✓		A signed network is used, in which both positive and negative relations can be considered
Triantaphyllou et al. [39]	✓		A post-consensus analysis was used in the log file to identify any dynamics that may exist in the way experts make ranking decisions
He et al. [40]	✓	✓	The shadowed theory is used for preference presentation and clustering. The network construction complexity is $O(n^2)$ as it requires pairwise analysis
Xiao et al. [42]	✓		Centrality, consistency, and similarity indices were used for weighting experts. Using similarity makes the complexity $O(n^2)$ as it requires pairwise analysis
Zhang et al. [43]	✓	✓	Leadership and non-cooperative behaviors were detected in the trust network. A social network is needed for the input
Li et al. [44]		✓	A fast unfolding algorithm was used to reduce the dimension of the large-scale DMs and the experts' weights were obtained by social network analysis techniques. This approach needs a social network as input
Liao et al. [46]	✓	✓	Two different roles are considered
Chao et al. [47]		✓	A two-layer network was used with an inner layer consisting of participants whose preference similarities and trust relations were known. The outside layer includes participants whose trust relations cannot be determined. The complexity of the proposed method is approximately $O(n^3)$
Li et al. [48]	✓		An interaction network is used to detect and manage manipulative behaviors
Xiong et al. [49]	✓	✓	A clustering method with historical data to support large-scale consensus-reaching process

Preference relations

The pairwise comparison has been introduced as preference relations and it is widely used in this area. Preference relation P on a set X is a binary relation $\mu_P : X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision-maker. Therefore, a preference relation P constitutes a matrix $P = (p_{ij})$ of dimension $\#X$ (number of items in the set X set), in which $p_{ij} = \mu_P\,(x_i, x_j)$ is the degree or intensity of preference for alternative x_i over x_j. The elements of P could be numeric or linguistic depending on the type of decision-making process that is being carried out [5].

Single-valued neutrosophic preference relations

A single-valued neutrosophic preference relation on X is expressed by a matrix $P = (p_{ij})_{n\times n}$ such that $p_{ij} = \langle t(x_i, x_j), i(x_i, x_j), f(x_i, x_j)\rangle$ is a single-valued neutrosophic preference relation with the following conditions [5]:

$$t(x_i, x_j) \to [0, 1],\ i(x_i, x_j) \to [0, 1],\ f(x_i, x_j) \to [0, 1] \tag{1}$$

$$0 \le t(x_i, x_j) + i(x_i, x_j) + f(x_i, x_j) \le 3 \tag{2}$$

The parameter t_{ij} indicates the truth membership that x_i is preferred to x_j. The parameter f_{ij} is interpreted as falsity membership that x_i is preferred to x_j. Finally, i_{ij} indicates indeterminacy of decision-making about x_i and x_j.

Consistency of decision-maker's preference

Preference consistency means rationality of preferences. In preference relations, there are three levels of relation rationality: (1) *the first level:* requires indifference between any alternative x_i and itself, (2) *the second level:* tequires that if an expert prefers x_i to x_j, that expert should not simultaneously prefer x_j to x_i, (3) *third level:* associated with the transitivity in the pairwise comparison among any three alternatives. If an expert prefers x_i to x_j and x_j to x_k, then alternative x_i should be preferred to x_k. A preference relation that verifies the third level of rationality is referred to as a consistent preference relation. Various formalisms have been developed to examine the transitivity of a preference relation [50, 51].

Consensus level

Different methods are used to measure the degree of consensus among decision-makers. In [22], preference similarity was used to calculate the consensus level. In [31], the dissimilarity of preferences was considered as a criterion for

Table 2 A summarization of the reviewed related work

	Article	Pros and cons
Selection approach		
All	[35]	No need for DM identification for suggestions One rule for all DMs Simple
Subgroup	[5, 9, 17, 22, 23, 26–28, 31, 36, 43, 46–48]	Accelerates the consensus process as the preferences change within a group Needs more calculations for subgroup detection and subgroup suggestion
Decision-maker	[15–17, 19–21, 29, 31, 33, 49]	Slows down the consensus process as the preferences change one by one Used in small groups
Adaptive	[12, 30]	Speeds up the consensus process as the preferences change within a group at first Needs more calculations for subgroup or DM selection
Suggestion generation approach		
Collective opinion	[9, 11, 12, 15–17, 20–23, 26, 27, 29, 33, 43, 46–48]	A proper aggregation function is needed
Opinion dynamics	[28, 36]	The network's existence is mandatory
Network-based feedback	The network is created [5, 32, 36, 38–40]	A network is created based on the input information
	There is a predefined network [17, 23, 30, 33, 41–46]	It is assumed that additional information is available from another network

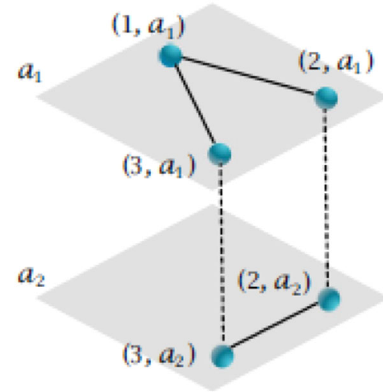


Fig. 1 Example of a multiplex network [56]

calculating the consensus level. In [52] a statistical criterion based on Shannon's entropy and a probability distribution was presented. The distance functions commonly used in modeling the consensus process are Manhattan, Euclidean, Cosine, Dice, and Jacquard [53]. Studies have shown that the speed of the consensus process is significantly influenced by the different aggregation operators and the distance functions [54, 55].

Graphs and networks

A graph is a structure consisting of a set of nodes and edges. There are different types of graphs. For example, a directional graph is a graph in which a direction is assigned to each edge, or in a weighted graph, a number is assigned to each edge. Multiplex networks are a kind of network that is used to represent a network system in which there are different types of interactions between components. In this type of network, the nodes in each layer are the same, and only the concepts of connections created between the nodes are different in each layer. Figure 1 shows a sample multiplex network.

Problem statement

In this study, we aim to propose an approach for reaching a consensus in large-scale group decision-making. Group decision-making is a situation when decision-makers collectively choose from a set of existing alternatives. Large-scale group decision-making refers to situations where a large number of decision-makers participate. Decision-makers provide their opinions by preferences. A group decision-making framework is proposed for reaching a consensus in such situations. As dimension reduction is one of the primary challenges in large-scale group decision-making, the focus of this study is on dimension reduction. Dimension reduction means reducing the number of input data. Questions that this research aims to answer are:

1. How can the dimension reduction be performed without clustering?
2. How can opinions be generalized?
3. What affects opinion generalization?
4. How can generalized data be changed to achieve a consensus?
5. What are the best similarity measures in the context of group decision-making?

We will tackle the above questions by meeting the following objectives:

1. Reducing the data dimension using opinion generalization.
2. Investigating the effect of opinion generalization on data reduction.
3. Investigating the effect of consistency and certainty on opinion generalization.
4. Introducing a new algorithm for the consensus process that works with generalized opinions.
5. Comparing the results of different similarity measures to select the most effective one.

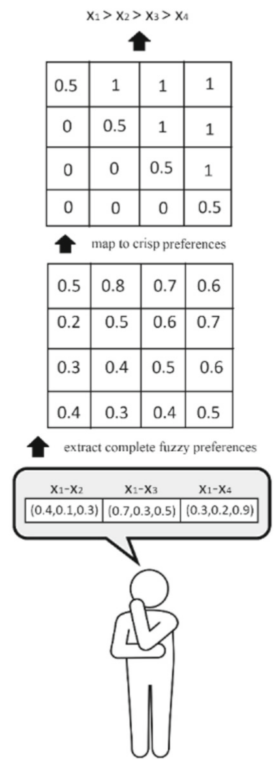


Fig. 2 Example of preference mapping

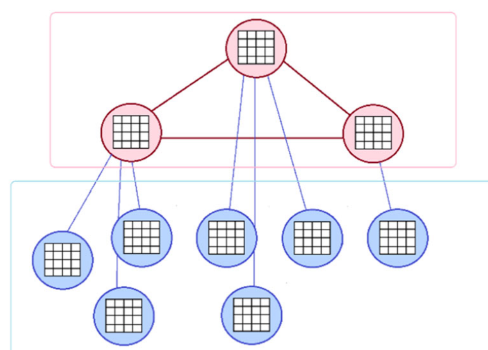


Fig. 3 Symbolic representation of preference mapping

The proposed approach

The main idea of the proposed approach is data generalization. Suppose there are four alternatives. Then there are $4! = 24$ choices for sorted alternatives (e.g., one choice is $x_1 > x_2 > x_3 > x_4$). It means that the final decision is one of these 24 choices. As sorted alternatives can be represented by a certain consistent crisp preference, the proposed approach maps the decision-maker's preferences to crisp preferences and then changes the preferences to reach a consensus. An example of such mapping is illustrated in Fig. 2. As shown in the figure, the decision-maker compares only one alternative with the others and gives preferences in the form of an incomplete single-valued neutrosophic preference. Then the uncertainty matrix and the complete fuzzy preferences are extracted from the original preferences. The fuzzy preferences are illustrated in Fig. 2. Mapping the decision-maker's preferences to crisp preferences is the next step. Using single-valued neutrosophic preferences allows decision-makers to express their preferences without limitation. For example, when we ask an expert's opinion, he or she may say that the possibility that the statement is true is 0.5, the possibility that the statement is false is 0.6, and the degree that he or she is not sure is 0.2 [57]. Receiving incomplete preferences from decision-makers is proposed for inconsistency prevention. Consistent complete preferences are extracted in the preferences process component. Then a two-layer network of crisp and fuzzy preferences is created in the network component.

A symbolic representation is illustrated in Fig. 3. Red nodes represent crisp preferences and blue nodes represent fuzzy preferences. In large-scale group decision-making, the number of blue nodes is large. There is no connection in the blue layer to eliminate the similarity calculation overhead. García-Zamora et al. indicated that the classic idea of the consensus model as an iterative discussion process should be replaced by an automatic algorithm in LSGDM [13]. So, the proposed consensus process is automatic. The selection process is only added for evaluation purposes and is an optional

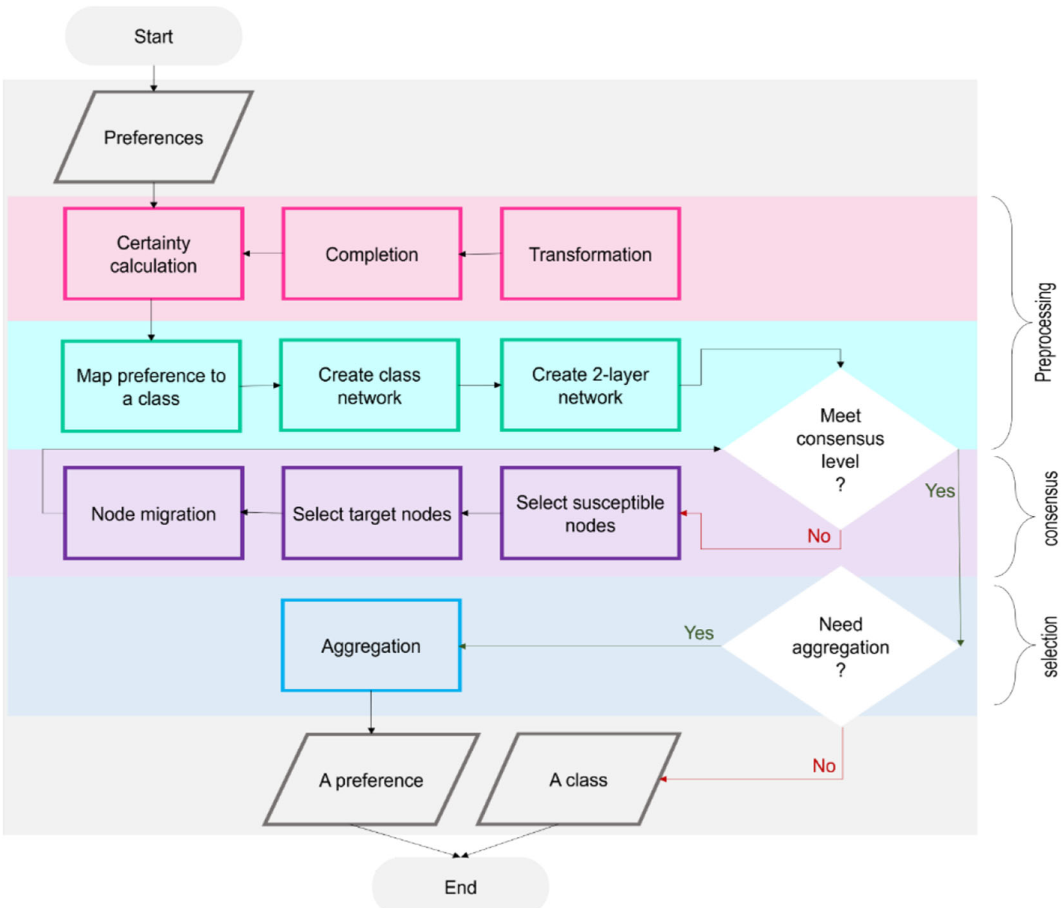


Fig. 4 The overall architecture of the proposed approach

process. So, in the proposed approach, we have three components: (1) the preprocessing, (2) the consensus process, and (3) the selection process. The preprocessing component has two subcomponents: (1) opinion processing and (2) network. Figure 4 shows the overall architecture of the proposed approach. This approach will be discussed in detail in the rest of this section.

In the proposed framework, decision-makers present their opinions in the form of a single-valued neutrosophic preference and in a way that they only compare one alternative with all of the other alternatives. The reason for using single-valued neutrosophic preferences is to obtain certainty from them. Decision-makers can express their preferences as fuzzy preferences, but they should provide the certainty matrix. The reason for receiving an incomplete preference is that the decision-maker's preference is automatically completed by mathematical modeling equations. The reason for insisting on consistency is explained in detail in the evaluation section. After processing the opinions, a two-layer network of preference and preference classes is formed. Preference classes are crisp, consistent, and certain. After the formation of the network, if decision-makers have not reached a consensus feedback is generated. The preferences of people in less important classes change as a group. This change is performed such that preferences are transferred to a new class. This change in classes is performed repeatedly to reach a consensus in one class.

Preprocessing

The first phase of the approach is preprocessing. Suppose that the decision-maker compares all alternatives with the alternative x. Then six steps should be taken which are explained in the subsequent sub-sections. The decision-makers can give their preferences using fuzzy preference matrices. But, the decision-maker has to provide the certainty matrix too. In this case, the first three preprocessing steps are ignored. We suppose the decision-makers provide single-value neutrosophic preferences because it is easier for them to express their preferences without limitation. In addition, if there are m alternatives, using this approach, m comparisons will be required instead of $m^2 - m$. The decision-makers can also express their certainty with indeterminacy numbers as well.

So, if we suppose that the preferences will be expressed in a matrix form, the matrix can be defined as below and every entry is a single-value neutrosophic number:

$$D_{matrix} = \begin{array}{c} 1 \\ \dots \\ x \\ \dots \\ m \end{array} \begin{bmatrix} - & - & - & - \\ \dots & \dots & \dots & \dots \\ <t_{x1}, f_{x1}, i_{x1}> & <t_{x1}, f_{x1}, i_{x1}> & \dots & <t_{xm}, f_{xm}, i_{xm}> \\ \dots & \dots & \dots & \dots \\ - & - & - & - \end{bmatrix}$$

So, it can be considered as a vector:

$$D_{vector} = \left[<t_{x1}, f_{x1}, i_{x1}> <t_{x1}, f_{x1}, i_{x1}> \dots <t_{xm}, f_{xm}, i_{xm}> \right]$$

Opinion transformation

As mentioned before, the values of $t_{ix} = t(x_i, x_x)$, $f_{ix} = f(x_i, x_x)$ and $i_{ix} = i(x_i, x_x)$ are in the range between zero and one independently. But, it is rational that there should be a limit $t_{ix} + f_{ix} = 1$ in a preference relation. To make this limitation, we use min–max normalization. Formula (3) shows the transformation rules.

$$t_{ix}{}' = \frac{t_{ix}}{t_{ix} + f_{ix}}, \; f_{ix}{}' = \frac{f_{ix}}{t_{ix} + f_{ix}} \tag{3}$$

Figure 5 shows the transformation process. It provides all opinion vectors and performs a transformation on them. In this figure, the t and f parameters of each cell in the last vector are transformed into new numbers.

The complement process

The proposed solution in [18] has been used to create a consistent preferences matrix. At this step, the decision maker's fuzzy preference matrix (r_{ij_n}) and determinacy matrix (d_{ij_n}) are extracted from opinion vectors. This is calculated using the following equations:

$$r_{ij_n} = \begin{cases} t_{ix} & \text{if } j = x \\ t_{xj} = f_{jx} & \text{if } i = x \text{ for } i, j \in \{1, m\} \\ max\left\{0, \min\left\{t_{ix} - t_{jx} + 0.5, 1\right\}\right\} & else \end{cases} \tag{4}$$

$$i_{ij_n} = \begin{cases} i_{ix} & \text{if } j = x \\ i_{xj} = i_{jx} & \text{if } i = x \text{ for } i, j \in \{1, m\} \\ min\left\{i_{ix}, i_{jx}\right\} & else \end{cases} \tag{5}$$

$$d_{ij_n} = 1 - i_{ij_n} \text{ for } i, j \in \{1, m\} \tag{6}$$

Figures 6 and 7 show an example of the complement process. In this step, each vector is converted into a complete matrix, and the determinacy matrix is calculated.

Calculating the decision-maker's certainty

In this section, the certainty matrix is calculated based on the determinacy matrix obtained in the previous step. Uncertainty is defined by Eq. (7) where d_{ij}^c represents the d_{ij} complement and is defined by Eq. (8). Finally, decision-maker uncertainty is calculated using Eq. (9):

$$Certainty_{ij} = \left| d_{ij} - d_{ij}^c \right| \tag{7}$$

$$d_{ij}^c = \left| 1 - d_{ij} \right| \tag{8}$$

$$Certainty_n = \frac{\sum_{i \in I, j \in J} Certainty_{ij}}{n^2 - n} \tag{9}$$

Figure 8 shows an example of the certainty calculation process. In this step, the certainty matrix is calculated from the determinacy matrix.

The preference pre-possessing algorithm is shown as Algorithm 1. If the number of decision-makers is n, then the preferences process algorithm's complexity order is $O(3n)$. In this algorithm, for the complement process, the number 0.5 is considered as not preferring one alternative over another, 1 is the maximum which means an alternative is completely preferred over another, and 0 is the minimum which means an alternative is completely not preferred over another in Eq. (4).

Determining the preference's class

The proposed network is a two-layer network. In the first layer, there are nodes related to the preference classes. In the second layer, the preference nodes exist. To create a two-layer network, the first layer nodes (i.e., the class nodes) must first be defined. Equation (10) is used to calculate the class of each preference. Equation (11) defines certain classes (we have assumed the value of 0.5 for the diameter).

$$C_{ij_n} = \begin{cases} 1 & \text{if } r_{ij_n} > 0.5 \\ 0 & \text{if } r_{ij_n} < 0.5 \\ 0.5 & else \end{cases} \tag{10}$$

$$UC_l = \left\{ C_n | C_{ij_n} \neq 0.5, i, j \in \{1, n\} \right\} \tag{11}$$

Figure 9 shows an example of the class determination process. In this step, the class of each preference is determined. The class of a preference is shown in the figure. The similarity between classes is defined by Eq. (12). It is inspired by the equation for calculating similarity in [5].

Here, $\#I^{ij}$ is the number of the common entry of the two matrices defined by Eq. (13), and m is the number of all entries. The similarity between all classes is calculated and

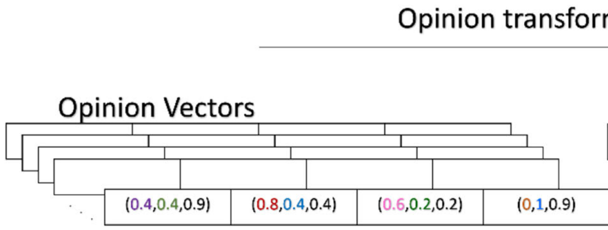
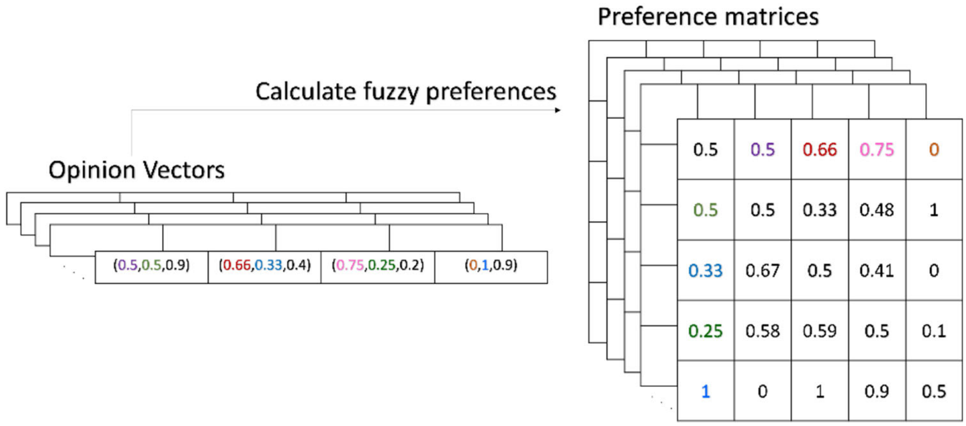


Fig. 5 An example of the transformation process

Fig. 6 An example of fuzzy preferences complementing process



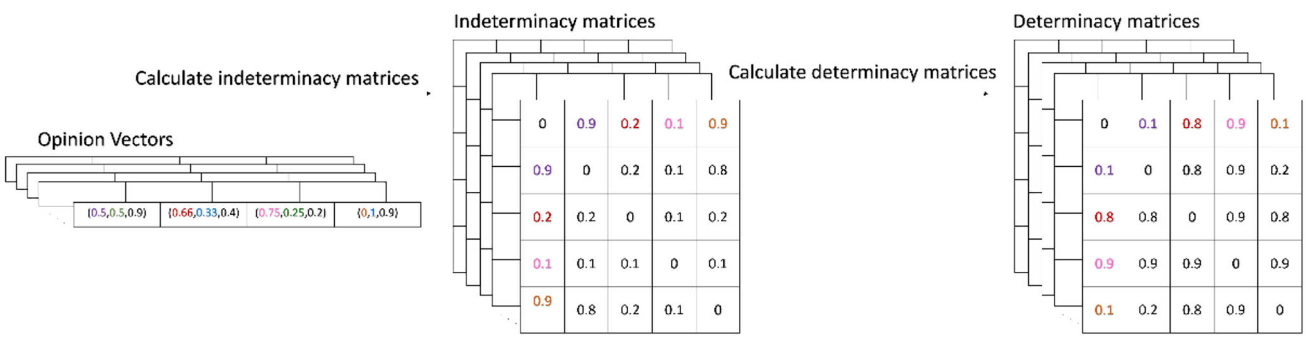
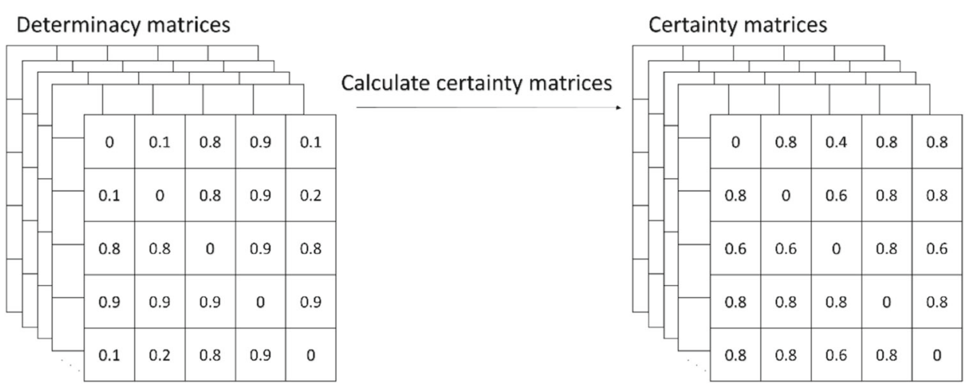


Fig. 7 An example of determinacy calculation

Fig. 8 An example of the certainty calculation process



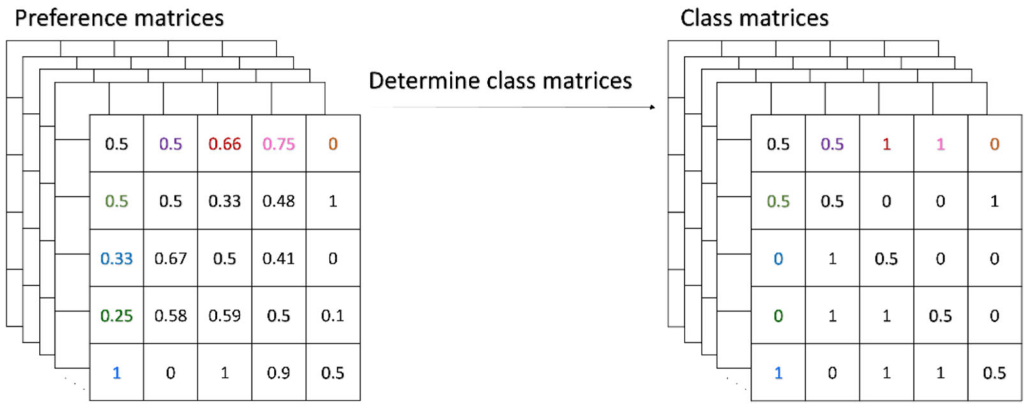


Fig. 9 An example of class determination

is represented as a matrix called similarity matrix defined by Eq. (14).

$$Similarity(UC_i, UC_j) = \frac{\#I^{ij}}{m} \tag{12}$$

Input:
Incomplete preferences of Decision Makers IP

Output:
complete preferences of Decision Makers CP
Complete Indeteminacies of Decision Makers CI
Complete Deteminacies of Decision Makers CD
A Vertor for reperesenting the certainty of Decision Makers $E^{[DM]}$

PROCEDURE
FOR *preference in IP:*
    Calculate *TransformedPreferences using equation (3)*
END FOR
FOR *preference in TransformedPreferences:*
    Calculate *CP using equation (4)*
    Calculate *CI using equation (5)*
    Calculate *CD using equation (6)*
END FOR
FOR *preference in CD:*
    Calculate $E^{[DM]}$ *using equation (9)*
END FOR
END PROCEDURE

Algorithm 1: Preferences process

$$I^{ij} = \left\{k|(UC_i(k) = UC_j(k) = 1) \vee (UC_i(k) = UC_j(k) = 0),\ k \in \{1,\ n\}\right\} \tag{13}$$

$$SimilarityMatrix = \begin{bmatrix} Similarity(UC_1, UC_1) & \cdots & Similarity(UC_1, UC_m) \\ \vdots & \ddots & \vdots \\ Similarity(UC_m, UC_1) & \cdots & Similarity(UC_m, UC_m) \end{bmatrix} \tag{14}$$

If some of the preferences of an uncertain decision-maker were unknown, 0.5 is inserted in the corresponding entry in their class's preference matrix. It needs to be converted into 0 or 1 because the classes are crisp. Converting the uncertain classes (C_u) into certain classes (UC_k) will be performed using Eq. (15). It is based on the maximum similarity with other classes.

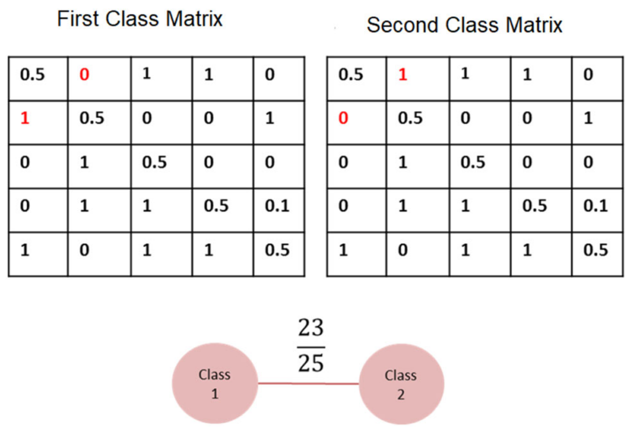


Fig. 10 An example of a class network construction process

$$C_u = \{UC_k | Similarity(C_u, C_k) = \max(Similarity(C_u, UC_k), k \in \{1, 2, \ldots, l\}\} \tag{15}$$

Creating the class graph/network

Each node in the class network represents a class and each edge represents the similarity of classes. The adjacency matrix is obtained after computing the similarity matrix using Eq. (16). So, a connecting component will exist where the edge between nodes indicates the highest similarity that a node has.

$$A_{ij} = \max_{j \in J}\left(s_{ij}\right), \; i \in \{1, 2, \ldots, n\} \tag{16}$$

A standard graph is often described by $G = \langle V, E\rangle$ where V is defined as the set of vertices and E as the set of links. Therefore, the class's graph is defined by Eq. (17).

$$V_c = \{UC_1, UC_2, \ldots, UC_l\}, \quad E_c = \left\{(UC_i, UC_j) | a_{ij} = 1 \forall a_{ij} \in A_{ij}\right\} \tag{17}$$

Figure 10 shows an example of two classes and their similarity. In this step, each class presents a node in the network, and the edges are constructed.

Creating the two-layer network

A multiplex network is represented by the quadruple in Eq. (18). The parameter V represents the network nodes, P is defined as a pair that represents the nodes in each layer, M is also the set of links in each network, and L is the network layer.

$$G = \langle P, M, V, L\rangle, \; P \subseteq V \times L, \; M \subseteq P \times P \tag{18}$$

The proposed two-layer network is defined by the below equations. The network has two layers and they are defined using Eq. (19). The network nodes are defined by Eq. (20). $NodeMaping$ is a set that contains the (preference number, preference's class number) tuple. The second layer nodes are defined by Eq. (22) and the first-layer nodes are defined by Eq. (23). It should be noted that $PreferenceNode$ is a multi-set.

$$L = \{class, preference\} \tag{19}$$

$$V = \{1, 2, \ldots l\}, \; l = |UC| \tag{20}$$

$$NodeMaping = \{(n, l) | C_n \in UC_l\} \tag{21}$$

$$PreferenceNode = \{(n, preference) | n \in NodeMaping\}, \; PreferenceNode \in P \tag{22}$$

$$ClassNode = \{(l, class) | l \in NodeMaping\}, \; ClassNode \in P \tag{23}$$

Each preference node has a weight denoted by W_{DM}, which is the importance of preference in decision-making. In the proposed framework, the importance is only related to the certainty of a preference. Equation (24) shows how it is calculated.

$$W_{DM} = Certainty_{dm} \tag{24}$$

Each class node has a weight defined by W_c notation, which is the importance of the class in decision-making. Equation (25) shows how it is calculated. In the proposed framework, the importance of a class is related to its certainty, population, and eigenvector which are defined by Eqs. (26), (27), and (28), respectively. The eigenvector represents the importance of a node in the network.

$$W_c = ClassCertainty_{[c]} + Population_{[c]} + Eigenvector_{[c]}, \; w_c \in [0, 3] \, \forall c \tag{25}$$

$$ClassCertainty_{[c]} = \frac{\sum_{dm \epsilon C} Certainty(dm)}{\sum_{dm \epsilon DM} Certainty(dm)} \tag{26}$$

$$Population_{[c]} = \frac{|ClassMembers|}{|PreferenceNode|}, \quad Class\ Members = \{(c, preference) | c \in UC_l\} \tag{27}$$

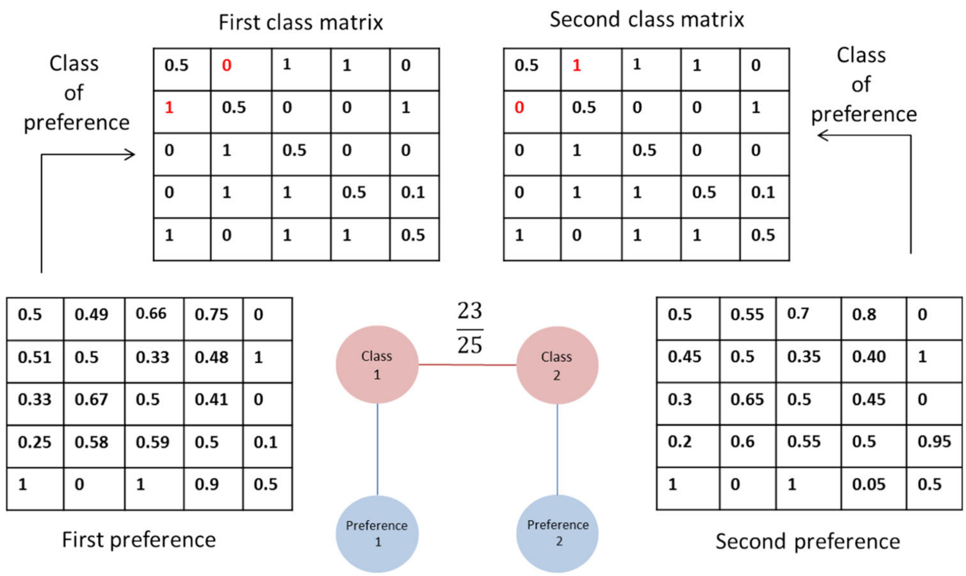


Fig. 11 An example of a two-layer network construction process

$$Eigenvector_{[c]}(t+1) = \sum_{j=1}^{n} A_{ij} \times Eigenvector_{[j]}(t) \tag{28}$$

Figure 11 shows an example of a two-layer network construction process. In this step, the edges between preferences and classes are built and the weights of the class nodes are calculated.

The two-layer network is constructed as the class network in its first layer and the preferences nodes in the second layer. Each preferences node connects its class with an interlayer edge. The two-layer network creation algorithm is as follows. The following steps are needed to construct the proposed two-layer network. If the number of decision-makers is denoted by n and the number of classes is represented by m, then the two-layer network creation algorithm's complexity order is $O(n + m^2)$. For determining the preference's class, the number 0.5 is considered as not preferring one alternative over another in Eq. (11).

Step 1: Determine the class of each preference.

Step 2: Create the network of classes.

Step 3: Create a two-layer network of classes and preferences.

Consensus process

The consensus level is measured by calculating the similarity of class nodes. If preference nodes in the second layer are connected to more than one class, it means that the similarity of preference is not enough. In other words, the consensus level does not meet the threshold, and the feedback is generated for preferences. In the feedback step, we have identified two categories of nodes we call susceptible and target nodes. It then describes how the opinion should change.

Feedback

The susceptible nodes are the nodes that belong to a specific class and this class has the minimum weight. So, this class is selected for change and merge. Susceptible classes are defined mathematically by Eq. (29).

$$Susceptible = \left\{C_s \middle| weight\,(c) = \min\left(weight_c\right),\ c \in \{1, 2, \ldots, l\}\right\} \tag{29}$$

Target nodes are the nodes that are selected to be merged with susceptible nodes. Target nodes are the neighbors of the susceptible node such that they have the highest similarity. The reason for selecting the target node according to similarity is to change the susceptible class with minimum changes. The target class is defined mathematically by Eq. (30).

$$target = \left\{C_t \middle| weight(c) = \max\left(weight_c\right) \wedge A_{st} = 1\right\} \tag{30}$$

To change the class, if the preference belongs to C_i and C_j is selected for transfer, the change equation will be in the form of Eqs. (31) to (33). The aim is to change the class of a group of preferences. The reason for considering the numbers 0.51 and 0.49 is to change the preference as little as possible. As the preferences change, it is rational that the degree of determinacy of the preference is reduced. In Eq. (32), the diameter of the network is used.

Input:
CompletePreference of Decision Makers cp ,
Complete Deteminacies of Decision Makers cd

Output:
Adjansency matrix of Different Classes $A^{[C]}$
Similarity matrix of Different Classes $S^{[C]}$
$W^{[DM]}$: *node weights in preference layer*
Weights in Class network $W^{[C]}$
A Set for reperesenting Different Classes $UC^{[C]}$
Preference's Classes $C^{[DM]}$
Prederence Node Tuples PN
Class Node Tuples CN
Population of A Class Node $P^{[C]}$
Eigenvector of A Class Node $EV^{[C]}$
Certainty of A Class Node $C^{[C]}$
A Tuple Set that Shows (node? layers) P

PROCEDURE
FOR *opinion in cp :*
  Calculate $W^{[DM]}$: *using equation (24)*
END FOR
Calculate $UC^{[C]}$: *using equation (11)*
Calculate *PN*: *using equation (22)*
FOR *class in* $UC^{[C]}$ *:*
  Calculate *CN*: *using equation (23)*
  FOR *class in* $UC^{[C]}$ *:*
    Calculate $S^{[C]}$ *using equation (14)*
END FOR
END FOR
Calculate $C^{[DM]}$: *using formula (15)*
Calculate $A^{[C]}$: *using formula (16)*
Calculate $C^{[C]}$: *using formula (26)*
Calculate $P^{[C]}$: *using formula (27)*
Calculate $EV^{[C]}$: *using formula (28)*
Calculate $W^{[C]}$: *using formula (25)*
Calculate *P*: *using formulas (22) and (23)*
END PROCEDURE

Algorithm 2: Two-layer network creation

The reason for using the diameter is that if the node traverses the entire graph, it will not be considered.

$$r_{ij_k} = \begin{cases} lower\ bound = 0.49\ if\left(C_{Susceptible} \neq C_{target} = 0\right), \forall k \in DM \\ upper\ bound = 0.51\ if\left(C_{Susceptible} \neq C_{target} = 1\right), \forall k \in DM \end{cases} \quad (31)$$

$$d_{ij_k} = \min\left\{0,\ d_{ij_k} - \left(\frac{1}{Diameter}\right)\right\} \quad (32)$$

$$w_c = Classcertainty_{[t]} + Population_{[t]} + w_c \quad (33)$$

Figure 12 shows an example feedback process. In this example, *class 1* is susceptible and *class 2* is the target.

The consensus process algorithm is shown as Algorithm 3. If the number of classes is denoted by m, then the consensus process algorithm's complexity order is $O(m)$.

Selection process

If the goal is to calculate the final preference (an aggregation of all preferences), the selection process is performed. But, if the goal is to just rank the alternatives, there is no need to do this step. As the classes are consistent and certain, the final class is consistent and certain as well. So, we can rank the alternative without doing the selection process. If the final preference is not needed, there is no need for this step, and there is no need to create the preference nodes in the second layer in the network construction step. Because we only need them in the selection process. We have added this step just for the sake of the framework's completion. So, if the final preference (r'_{ij}) is needed, it is calculated using Eq. (34).

The selection algorithm is given as Algorithm 4. If the number of decision-makers is denoted by m, then the consensus process algorithm's complexity order is $O(n)$. In the consensus algorithm, the numbers 0.51 and 0.49 are considered as the lower bound and upper bound in Eq. (31). The upper bound can be any number between 0.5 and 1, and the lower bound can be any number between 0 and 0.5. Therefore, the lower and upper bounds could change, and the effect of this change would be on the final preference. As mentioned, if the goal is to rank the alternatives, there is no need to calculate the final preference. So, changing both bounds will not affect the outcome in this case.

$$r'_{ij} = \frac{\sum_{dm \in DM} d_{ij_{dm}} \times r_{ij}}{\sum_{dm \in DM} d_{ij_{dm}}} \quad (34)$$

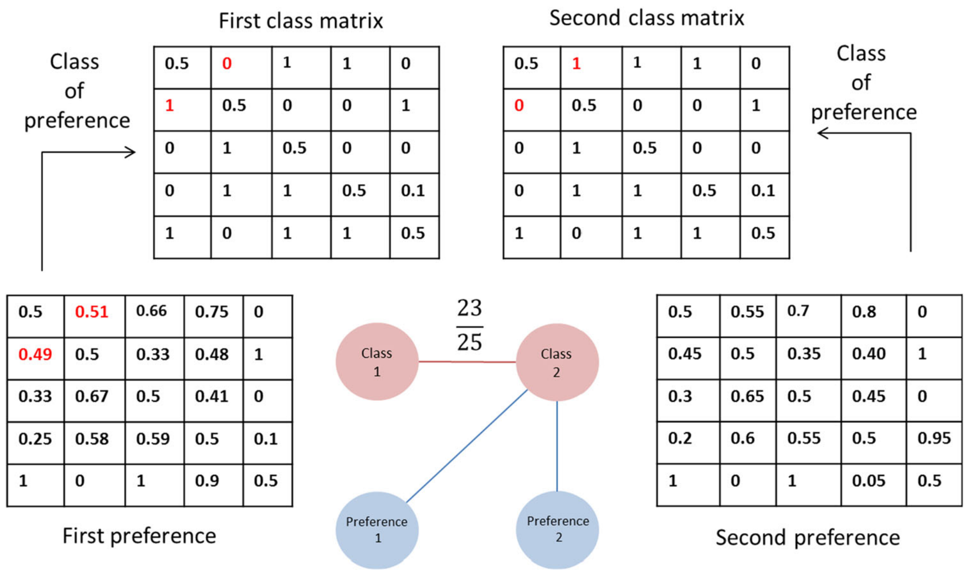


Fig. 12 An example of the feedback process

```
Input:
Deterninacy Matrices d_ij_k
Complete Preference Matrices r_ij_k
Weights in classes network W^[C]
Weights in preference network W^[DM]
A Tuple that Shows (node□ layers) P

Output:
Deterninacy Matrices d_ij_k
Complete Opinions Matrices r_ij_k
Weights in classes network W^[C]
Weights in preferences network W^[DM]
A Tuple that Shows (node□ layers) P

While all nodes is not proccessed
 | Calculate Susceptible nodes using equation (29)
 | IF "Susceptible" is not processed:
 |  |   FOR each Susceptible nodes:
 |  |   |   Calculate r_ij_k using equation (31)
 |  |   |   Calculate d_ij_n using equation (32)
 |  |   |   Calculate W^[C] using equation (33)
 |  |   |   update P
 |  |   End FOR
 | End IF
 | Else
 |  |   Add "Susceptible" to processed
 |  |   Calculate target using equation (30)
 |  |   While target is processed:
 |  |   |   target ← first node in depth first search
 |  |   End while
 |  |   Add (Susceptible□ target) to Migrate
 | End Else
End while
```

Algorithm 3: The consensus process

Evaluations

The proposed framework was evaluated using various simulation scenarios. The performance of the framework is also evaluated in the simulation section. There are three simulation scenarios in the first section. The first simulation investigates the effects of preference consistency and certainty on dimension reduction. The second one simulates a group decision-making scenario to demonstrate how the proposed algorithm works.

```
Input:
Deterninacy Matrices d_ij_k
Complete preference Matrices r_ij_k

Output:
Final Opinions Matrix r_ij_k'

FOR x in X :
 | FOR matrix in CP :
 |  |   Calculate Final Ranking using equation (34)
 | END FOR
END FOR
```

Algorithm 4: The selection process

The third one simulates a set of group decision-making scenarios with different numbers of decision-makers, alternatives, and similarity measures to investigate the performance of algorithms in different scenarios. Comparison is not a common evaluation approach in the field. Previous studies have demonstrated the steps of their algorithm through simulations or examples. However, as synthetic data are used for simulation, a comparison section is added to apply the proposed

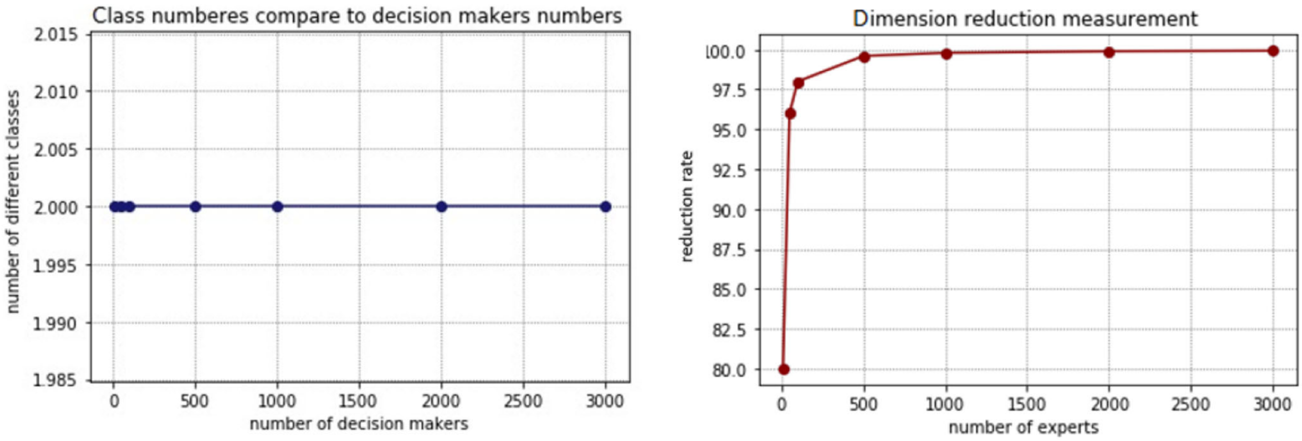


Fig. 13 The effect of consistent preferences in dimension reduction when the number of alternatives is 2

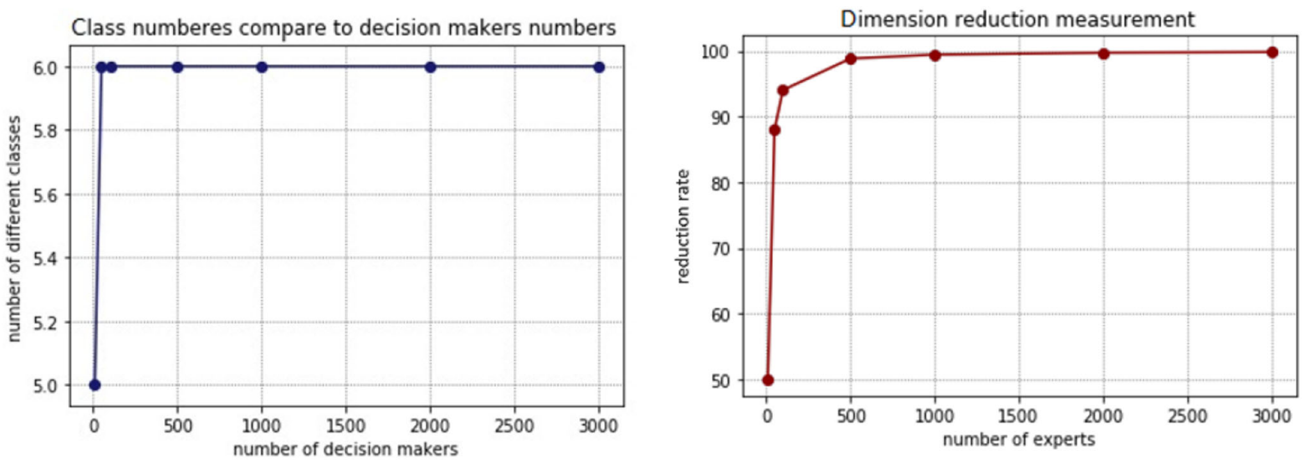


Fig. 14 The effect of consistent preferences in dimension reduction when the number of alternatives is 3

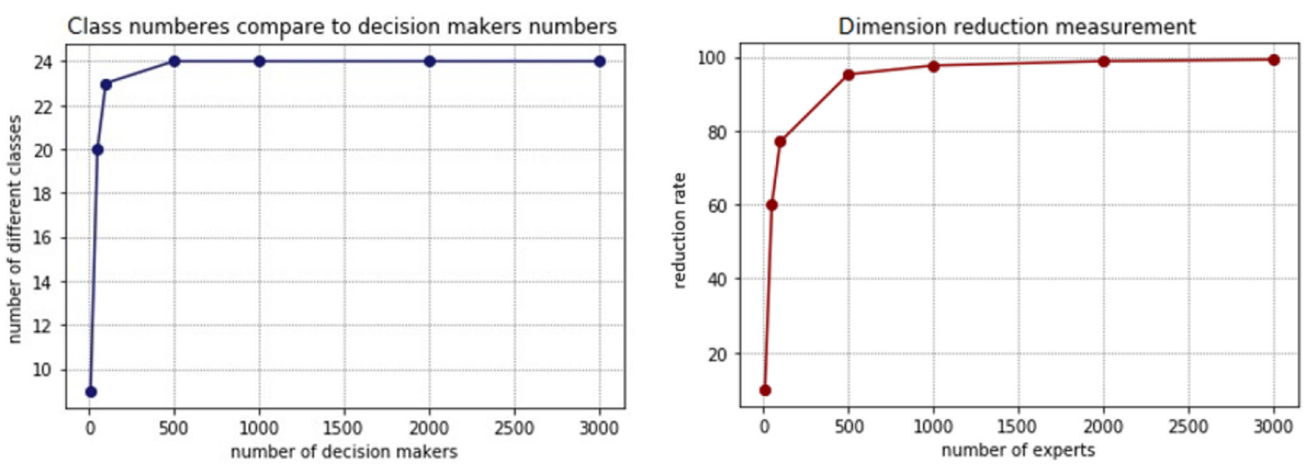


Fig. 15 The effect of consistent preferences in dimension reduction when the number of alternatives is 4

algorithm to the data used by other studies to compare the results. The candidate studies for comparison are selected from related works where the authors claim that in their research, large-scale group decision-making is addressed and their data are publicly accessible. It is expected that, after running the algorithm on the data of the selected studies, similar results compared to the simulation outcomes will be obtained.

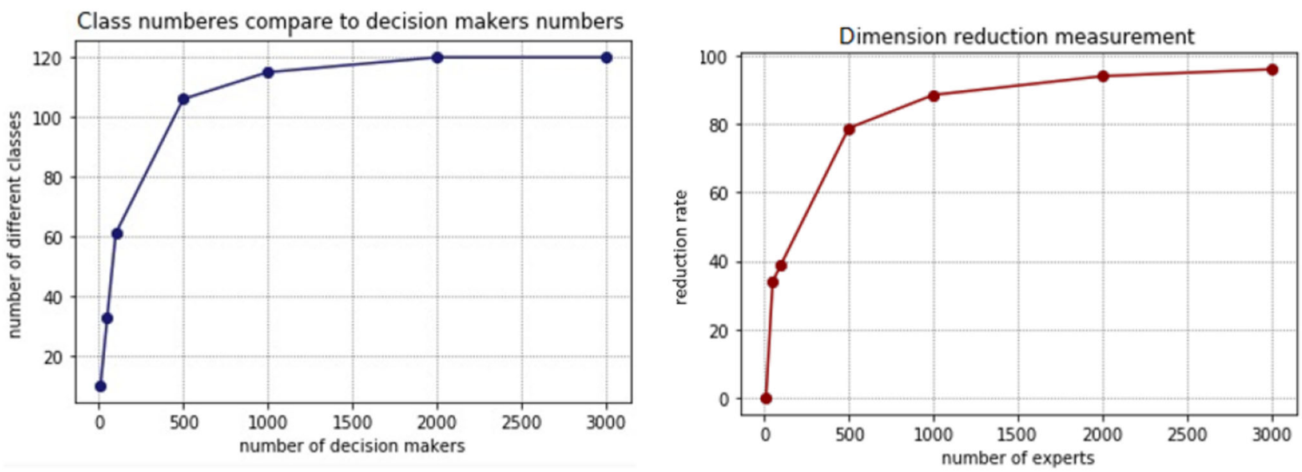


Fig. 16 The effect of consistent preferences in dimension reduction when the number of alternatives is 5

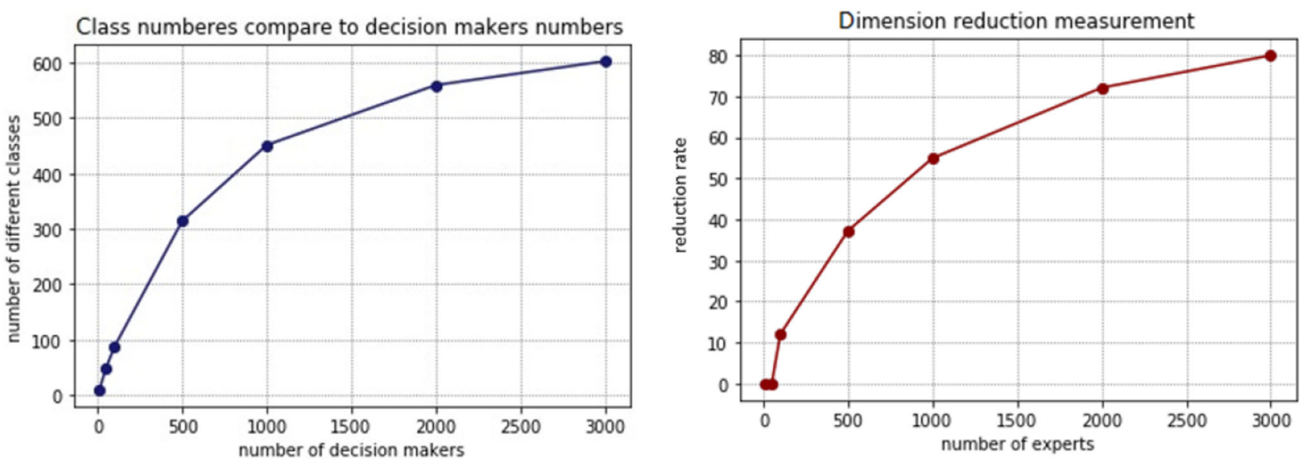


Fig. 17 The effect of consistent preferences in dimension reduction when the number of alternatives is 6

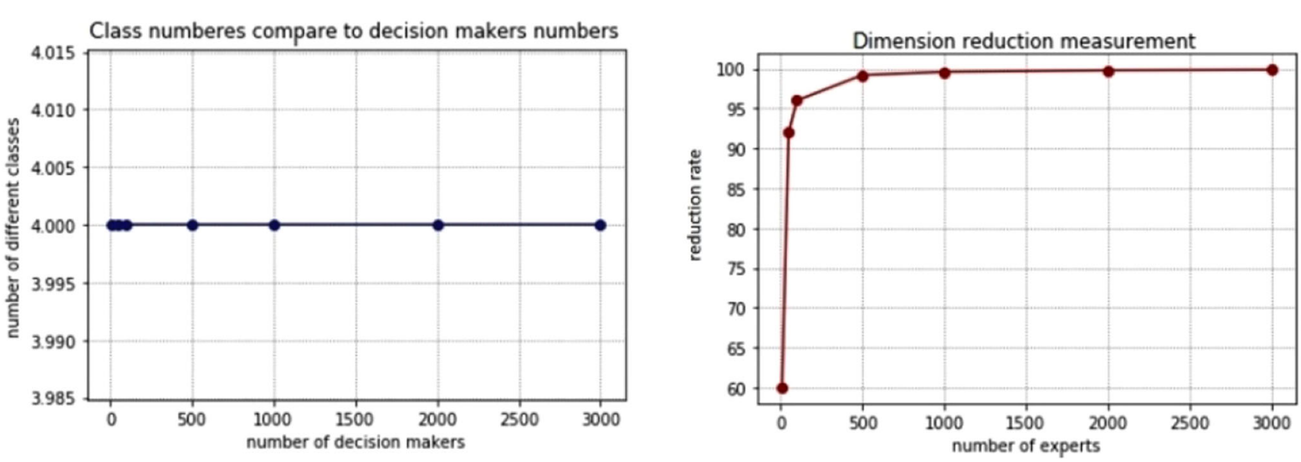


Fig. 18 The effect of inconsistent preferences in dimension reduction when the number of alternatives is 2

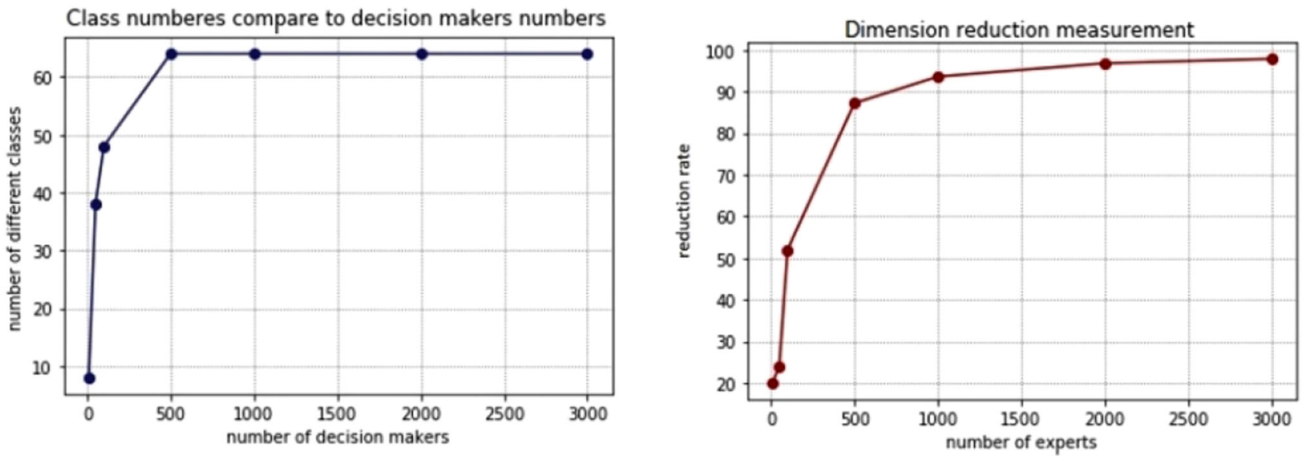


Fig. 19 The effect of inconsistent preferences in dimension reduction when the number of alternatives is 3

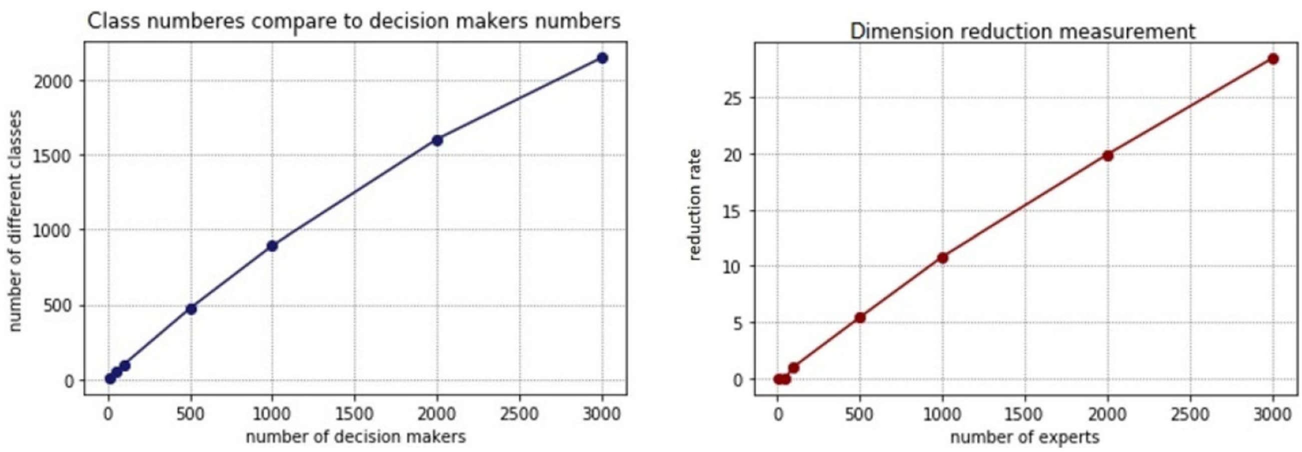


Fig. 20 The effect of inconsistent preferences in dimension reduction when the number of alternatives is 4

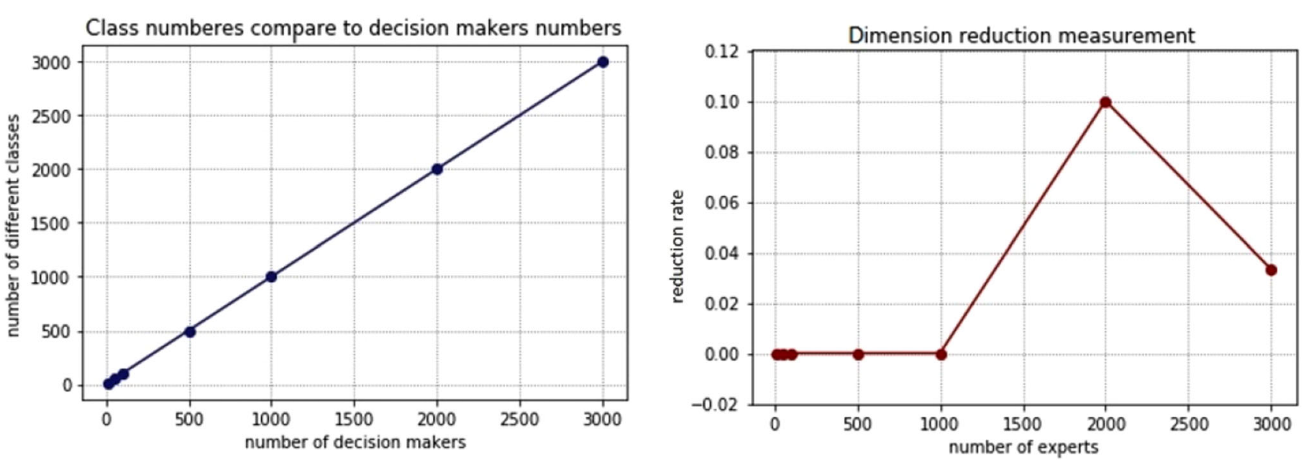


Fig. 21 The effect of inconsistent preferences in dimension reduction when the number of alternatives is 5

Simulation scenarios

In this section, we will examine the consistency and certainty of preferences and their effect on dimension reduction. Here, the effect of considering consistent preferences and their mapping to certain preferences in reducing the dimensions of the problem is examined. As it was mentioned before, if the goal of the group decision process is alternative ranking,

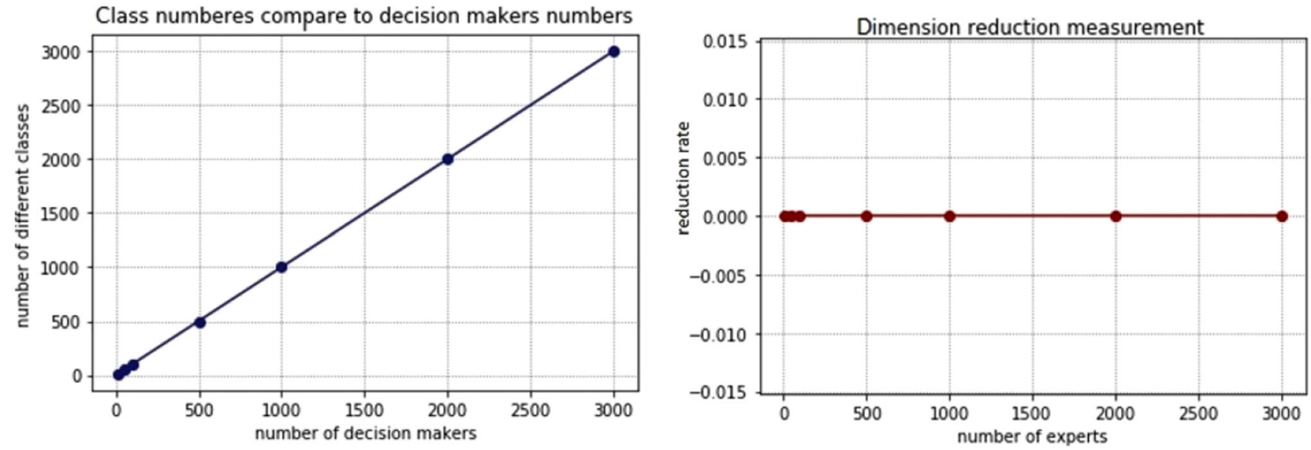


Fig. 22 The effect of inconsistent preferences in dimension reduction when the number of alternatives is 6

there is no need to create the preference nodes in the second layer. So, only the class nodes in the first layer are considered. Since they are fewer than the preference nodes, the dimension of the problem is reduced. An example of an alternative ranking is to choose a person for a job from several candidates. In this section, we investigate the differences between the number of classes and preference nodes and show that if the preferences are consistent and certain, the dimension reduction by opinion generalization can be applied effectively. For this purpose, decision-making data are generated with several different alternatives and decision-makers. In the simulation scenarios with different alternatives, two scenarios were defined in which in one scenario only consistent preferences exist, and in another scenario, inconsistent and consistent preferences are present. The result of the dimension reduction for different parameters is shown in Figs. 13, 14, 15, 16 and 17 and Figs. 18, 19, 20 and 21. In these figures, the blue line shows the number of different classes in each scenario and the red line shows the rate of reduction in each scenario. As the diagrams show, by increasing the alternatives in the scenario where there are also inconsistent preferences, the reduction of dimensions is reduced. It should be noted that for the production of simulation data, the worst-case scenario is considered in which the preferences are generated from any possible class. In real-world decisions, these classes do not need to be as much heterogeneous as possible.

If the decision-making is performed to reach a consensus on the ranking of alternatives, the second layer, in which the preference nodes are located, can be removed. Thus, the consensus is reached only with the changes made in the first layer. This implies that the selection process can be ignored. But, in the evaluation, we did not remove the second layer nodes and the selection process. The aim was to track changes in preferences and to report preference changes. The advantage of using the proposed approach under this condition is that the

Table 3 The specifications of the generated data

Number of decision-makers	Number of alternatives	Possibility of a fully consistent preference	Possibility of a certain preference
1000	4	1	0.1

dimensions of the problem can be significantly reduced. As shown in the simulations, depending on the problem parameters, the number of decision-makers can be reduced by up to 99% by considering the number of classes. This reveals that if the consensus process runs on consistent and certain preferences, the consensus-reaching process will be faster. To evaluate the performance of the proposed consensus model, in the next section, a simulation is described in detail and other simulation scenarios are briefly reviewed (Fig. 22).

Simulating a group decision-making scenario

To investigate the steps of the algorithms in the proposed model, the steps of the proposed algorithms were executed using simulated data. First, preference data were simulated. This data was generated using the specifications given in Tables 3 and 4.

It is worth mentioning that all the generated simulation data for the evaluated scenarios are publicly available on GitHub.[1] Figure 23a shows the histogram related to the Euclidean similarity distribution of preferences and shows that in the generated data, the Euclidean similarity distribution of preferences follows a normal distribution. The network was created after preprocessing the preference data. There are 24 different class preferences. Therefore, there are

[1] https://github.com/FtmhBkhsh/DataOfPreferences.

Table 4 Changes in the consensus level during the feedback process in simulation scenarios with different parameters

<table>
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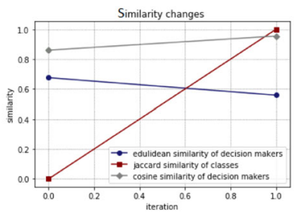
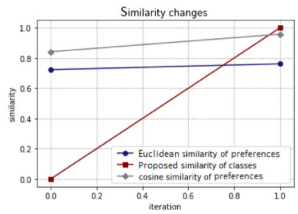
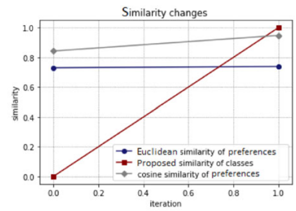
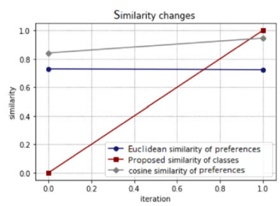
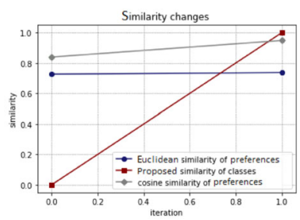
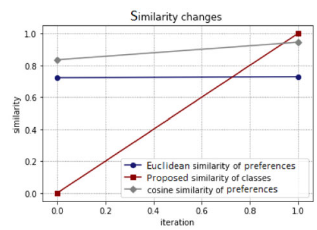
<tr><td rowspan="4">2</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>500</td><td>1000</td><td>3000</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
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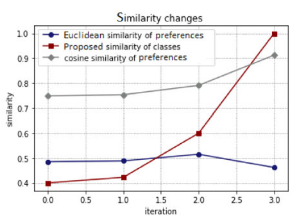
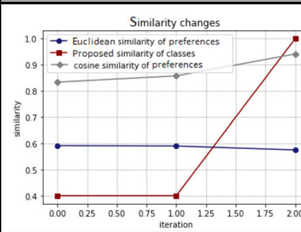
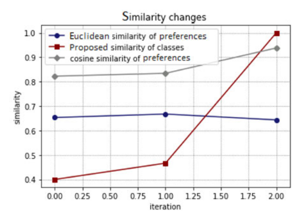
<tr><td>
</td><td>
</td><td>
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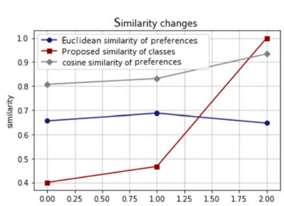
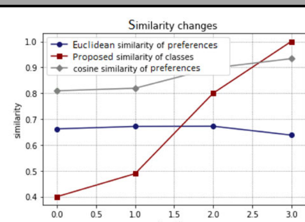
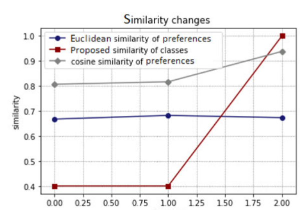
<tr><td>
</td><td>
</td><td>
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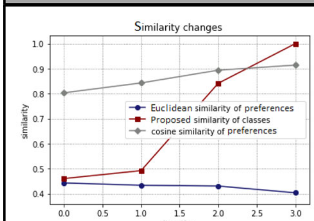
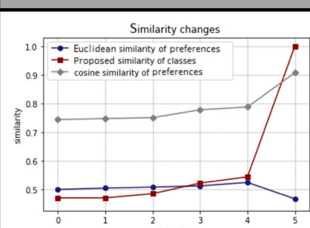
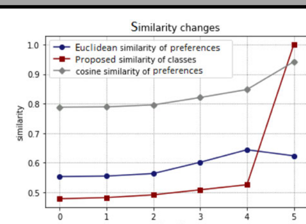
<tr><td>
</td><td>
</td><td>
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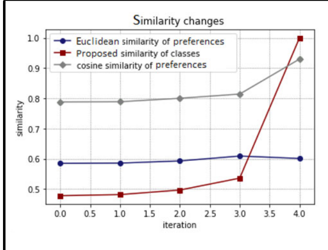
<td>
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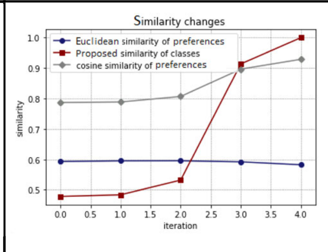
<td>
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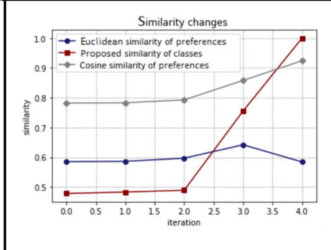
<td>
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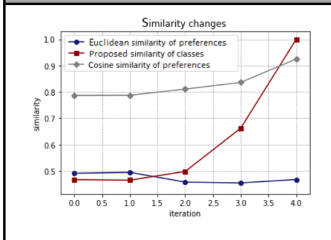
<td>
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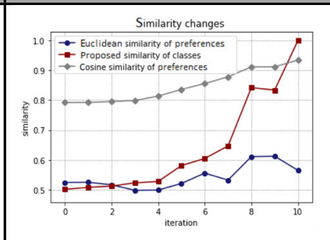
<td>
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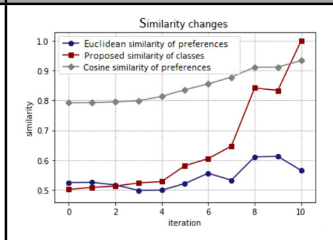
<td>
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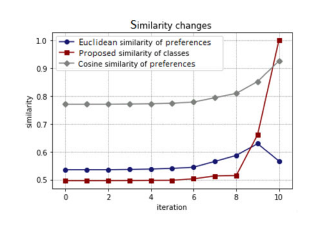
<td>
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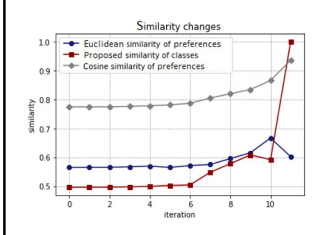
<td>
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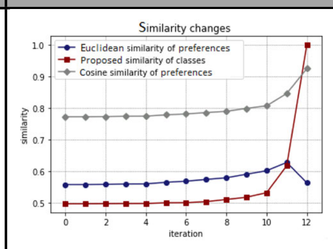
<td>
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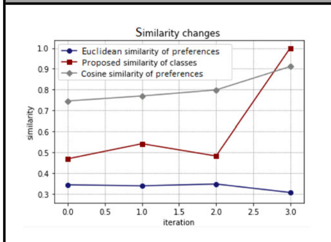
<td>
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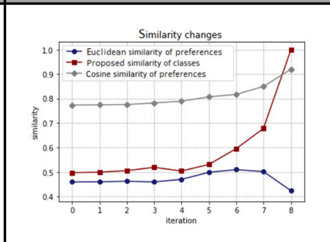
<td>
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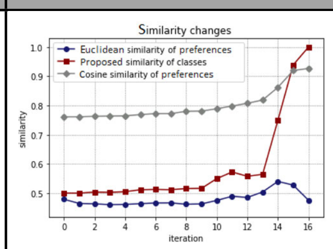
<td>
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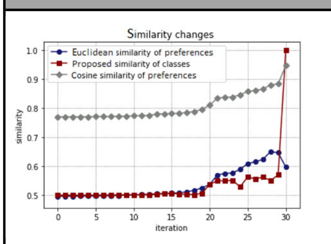
<td>
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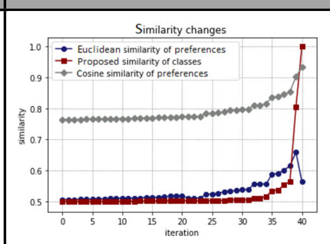
<td>
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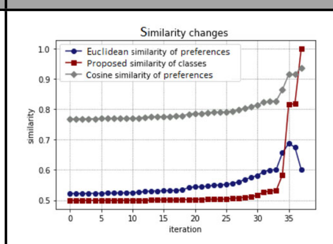
<td>
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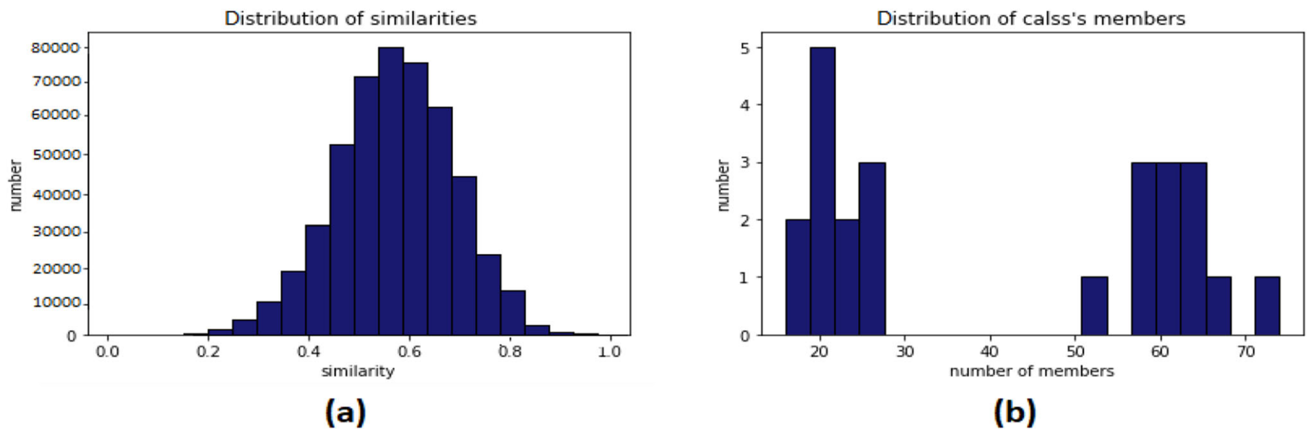

Fig. 23 Euclidean similarity distribution and the distribution of the number of members in each class

24 nodes in the first layer of the network. Figure 23b shows a histogram related to the class size distribution.

Figure 24a shows the class network. The network in the first layer is a connected component. Blue nodes in Fig. 24b represent preferences. There is no edge in the second layer between blue nodes as no edges were created in this layer. Because the number of decision-makers is large, calculating the similarity of preference pairs causes a computational overhead. Figure 24c shows the two-layer network. An interlayer edge indicates that a preference belongs to a particular class. All nodes in the second layer are connected to first-layer nodes. This representation is a new model of visual preference representation in group decision-making. Figure 25 shows how the network changes its status after each feedback step. In each feedback step, only the links between the two layers are changed. In the performed simulation, decision-makers reach a consensus after receiving four feedbacks. During this phase, some second-layer nodes change their interlayer connections. Changing links means that the class associated with them has the least importance. During the feedback process, the amount of uncertainty will change, and the indeterminacy degree of the preferences decreases. Changes in the degree of determinacy of the preferences are shown in the diagram in Fig. 26a. In addition, during the feedback process, the degree of similarity between preferences and classes changes. The similarity changes based on the proposed similarity measure (i.e., red line) and Euclidean similarity of the preferences (i.e., blue line) are shown in Fig. 26b. As the diagram shows, the Jacquard similarity is maximized, and consensus is reached whereas the Euclidean distance has a lower value.

Simulation with different numbers of decision-makers, alternatives, and similarity measures

Tang et al. emphasize that in large-scale group decision-making, the performance of models in larger groups with thousands of decision-makers should be examined [7]. According to the aforementioned critique, in the previous simulation, there were 1000 preferences. However, to examine the performance of the proposed approach with different parameters, simulations with a different number of decision-makers and different alternatives have been performed, and the consensus level at each stage of the feedback is given in Table 5. In addition, three similarity measures were examined during the consensus process. Preference data are available at [58].

The results obtained from examining the diagrams in Table 5 can be analyzed as discussed below:

- It can be concluded that if the number of decision-makers is larger than $(m)!$ the number of decision-makers does not have a significant effect on the number of iterations of the algorithm. In that case, the structural features of the class's network and the number of preferences belonging to a class determine the number of iterations. The proof of this is shown in Appendix A.
- In some diagrams (e.g., simulation number 32), a decrease in the proposed similarity occurred in the consensus process. This decrease occurs because the selected nodes for integration at this stage have a lower eigenvector. In other words, the population or certainty of some classes was high enough to overcome the eigenvector criterion, and the preferences shifted to classes with more population or more certainty, rather than moving to more similar classes.
- In most diagrams, the degree of similarity in the final stage changes sharply. This is because in the early stages, nodes that have a smaller population are integrated and their changes do not cause a significant change in the similarity. On the other hand, with the migration of nodes in each stage, the population of nodes in the final stages increases, and consequently, the number of links that change in further stages increases. Another reason is the dissimilarity in preference data. Data generation is performed in such a

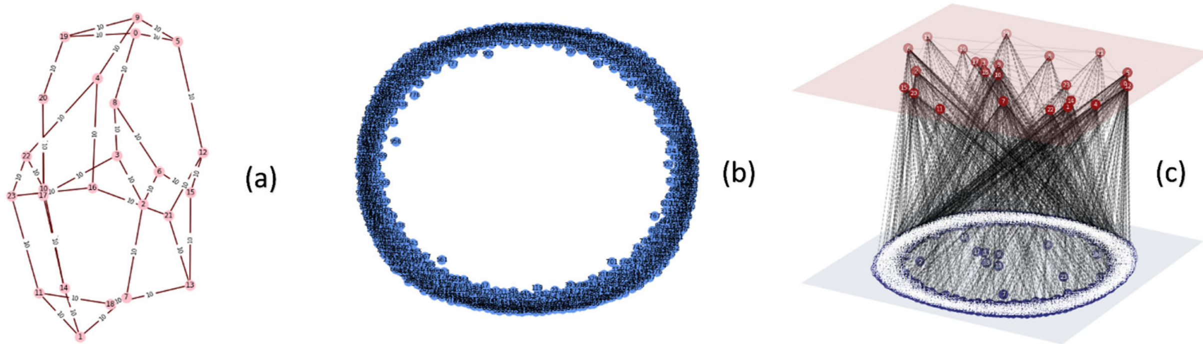

Fig. 24 The two-layer network

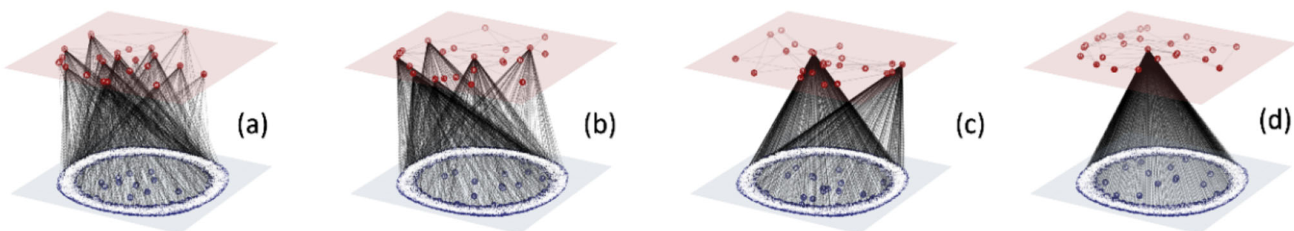

Fig. 25 The network changes

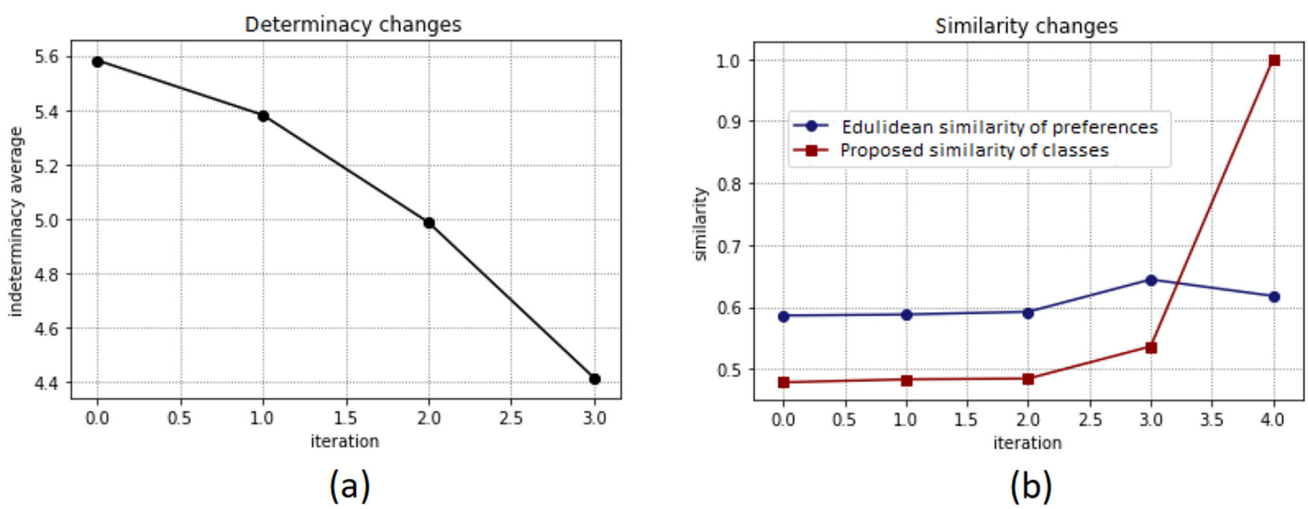

Fig. 26 Determinacy of preferences and similarity of preference changes

Table 5 The characteristics of the examined approaches

Approach	Dimension reduction	Has *large-scale decision-making* in the title	Considering uncertainty	Considering consistency	Diversity of decision-makers
[22]	✓	✓	✓	–	–
[5]	–	✓	✓	✓	✓
[48]	–	✓	–	–	–
[11]	✓	✓	–	–	–
[49]	✓	✓	–	–	–
The proposed approach	✓	✓	✓	✓	–

way that it is produced from any kind of consistent and certain class, which means that there are nodes in the network that have no similarities, and the preferences may converge to two opposite preferences. Therefore, the latest change is for the migration of the last two opposite classes which sharply increases the similarity.

– As it turns out, at the end of the process, when the class of preferences is the same, the Euclidean similarity is still low. Therefore, it can be concluded that the use of Euclidean similarity is not a suitable measure for the problems with the goal of ranking the alternatives where there is no need for Euclidean similarities to be high. In other words, the low Euclidean similarity between preferences does not mean that decision-makers have not reached a consensus on the ranking of available alternatives. Although Cosine distance has been used in fewer studies, it provides better results than Euclidean distance.

Comparisons

In this section, the proposed approach is compared to other approaches. Because the proposed approach focuses on large-scale decision-making groups, it would be better to compare it with methods in which thousands of decision-makers' preference data exist. However, to the best of our knowledge, no published large-scale dataset of preferences can be used for comparison. Most of the proposed models use toy examples and there is no information about the performance of these models in the presence of a large decision group [13]. These examples are datasets that researchers manually created for model testing. To test the proposed model using data from existing works, we have selected the studies where the evaluation data were publicly available. Chao et al. are among the few researchers who consider a large number of decision-makers (i.e., 1861) in their case study [47]. Unfortunately, their input data are heterogeneous and cannot be used directly in the proposed model. The characteristics of the selected approaches are summarized in Table 6. The reason for choosing these works is that the solutions proposed in these approaches are different from each other. By comparison, we confirm that the proposed approach works well with other examples in this context and investigates different similarity measures. In addition, they did not consider uncertainty, and the uncertainty of all opinions was assumed to be the same in the simulation. It should be noted that from the results, it is not possible to determine which approach is better or more suited for all application contexts because the objectives of these methods are different. The proposed consensus algorithm is applied to the preference data used in each of the selected works, and the results are presented in Table 6. These candidate studies for comparison are selected from related works in which the authors claim that in their research, large-scale group decision-making is addressed.

In [22], clustering was used for dimension reduction. In their approach, the k-means clustering method was used. The authors considered the value of k to be 3, thus reducing the number of decision-maker entities from 20 to 3 by creating three clusters. However, manually determining the parameter k is a disadvantage of this method. The authors conducted experiments to determine the k parameter that may create a computational overhead in a large-scale group decision scenario. The clusters can change during each iteration. Hence, clustering is performed in each iteration, which may cause a computational overhead. In this work, there is no discussion about the consistency of the preferences, but in their example, all preferences are consistent. By applying the proposed approach to the preferences in their illustrative example section, the number of different classes was 12, and the similarity changes in each iteration are reported in the first row of Table 6. In the evaluation section, we showed that for deciding on four alternatives, if the preferences are consistent, the maximum number of classes is 24. In this example, the opinions were consistent, and there were four alternatives. However, the number of different classes was 12. This indicates that, in decision-making problems, the diversity of different classes is not maximized, and the reduction may be greater than what is estimated in this study.

In [5], the focus is on creating a social network. The main idea is that an opinion can affect and be influenced by the opinions of others. In this research, preferences' consistencies are considered, and it is shown that if inconsistent preferences are received, their consistency increases during the consensus process. In their work, several thresholds were used, and each may be different for each decision problem. The value of each threshold may have changed the results. In each iteration of the algorithm, the preferences of a large number of decision-makers (i.e., all that have medium and low knowledge degrees) change. In this study, simulated data were used for the evaluation. The proposed approach was applied to the data. As there are inconsistent preferences in the data, the number of different classes was 20, and the similarity changes in each iteration are reported in the second row of Table 6.

In [48], the authors focused on detecting and dealing with manipulative and non-cooperative behaviors in the social network context. They used opinion evolution equations. In this work, there is no discussion about the consistency of preferences. However, in their example, there exist inconsistent preferences. As there were inconsistent preferences in the data, by applying the proposed approach to the preferences stated in their appendix section, the number of different classes was 35. The similarity changes in each iteration are presented in the third row of Table 6.

Table 6 The comparison results

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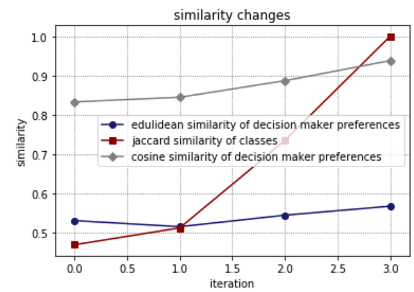
<tr><td>[22]</td><td>
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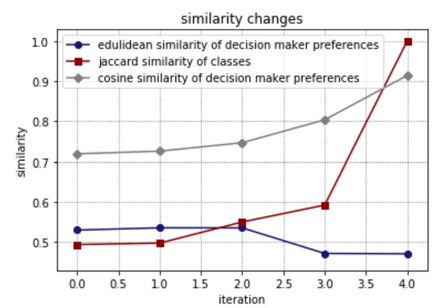
<tr><td>[5]</td><td>
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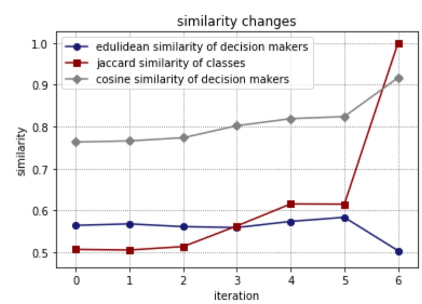
<tr><td>[48]</td><td>
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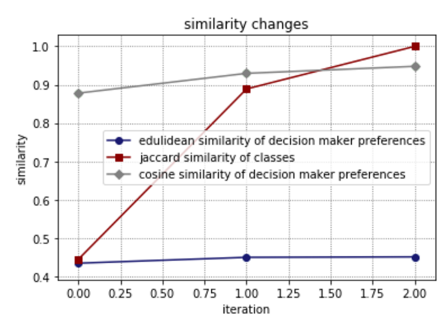
<tr><td>[11]</td><td>
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Number of alternatives</td><td>20
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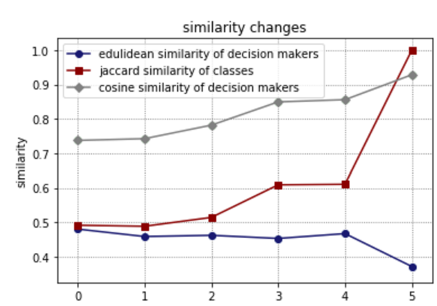
<tr><td>[49]</td><td>
</td><td>Number of decision-makers
Number of alternatives</td><td>20
5</td></tr>
</table>

In [11], a minimum cost consensus model was introduced, in which excessive modifications of the original preferences were avoided. In this study, there was no discussion about the consistency of preferences. However, in their example, all the preferences are consistent. They used k-means for clustering, and the value of k was considered to be 5 in their illustrative example. Thus, they reduced the number of decision-maker entities from 20 to 5 by creating 5 clusters. However, by applying the proposed approach to these preferences, there were four different classes. This comparison demonstrates the effectiveness of the proposed opinion generalization in the case of consistent preferences. The similarity changes in each iteration are presented in the fourth row of Table 6.

In [49] an extended k-means method based on historical preference data was introduced. Similar to the first selected approach, preferences are modified in each iteration, and clustering is performed in each iteration, which may cause computational overhead in large-scale group decision-making. The authors did not mention consistency, and the preferences in their example did not seem to have a third level of consistency. They considered the value of k to be 3 in their illustrative example. Thus, they reduced the number of decision-maker entities from 20 to 3 by creating 3 clusters. However, when applying the proposed approach to these preferences, the number of different classes was 17. The similarity changes in each iteration are presented in the fifth row of Table 6.

As mentioned in the simulation section, the main advantage of the proposed approach is that a larger number of decision-makers participate in the decision-making. As explained in the simulation section, by applying the proposed approach to these data, the similarity criterion related to classes reached its maximum, while no significant changes occurred in the Euclidean similarity criterion.

Conclusion

In this study, a new approach is proposed for large-scale group decision-making. The goal of this study is to provide a new approach for reducing the dimensions in decision-making environments with hundreds and thousands of decision-makers. This approach takes advantage of opinion generalization.

The proposed approach is an attempt to investigate the effectiveness of data reduction by generalization to use summarized data instead of the initial data. The results show that, if the number of alternatives is small, using the opinion generalization, increasing the number of decision-makers does not affect the number of generalized opinions. The results also show that for problems where the goal is to rank the alternatives, only generalized opinions, which are fewer than the actual opinions, can be used. Simulation results show that the proposed generalization method effectively reduces data dimensions in the case of consistent and certain preferences. For example, in one case with 3000 decision-makers, the preferences were reduced to 120. If the number of decision-makers is 30,000 or 300,000, again the maximum number of certain and consistent preferences is reduced to 120. It means that opinion generalization is a possible solution for handling large-scale group decision-making and making the decision-making process faster. Finally, the outcome of similarity changes evaluations demonstrates that Euclidean distance is a strict criterion for calculating the consensus level of decision-makers. If the goal is to reach a consensus about ranking the available alternatives, there is no need for Euclidean similarities to be high among decision-makers' preferences. Previous studies have examined large-scale group decision-making, but have not focused on group decision-making with more than a thousand decision-makers. In addition, there should be a logical reason to select a similarity measure. A summary of the innovations of this research is as follows:

1. Opinion generalization was used for the first time for dimension reduction in group decision-making. Simulation results show that it has a considerable effect on data reduction.
2. As opinion generalization is a recently introduced idea in the field, a new approach for consensus reaching in large-scale group decision-making for generalized opinion is presented. A two-layer network is used in the group decision-making process and for visualizing the preferences of decision-makers. As opinion generalization is a new idea in the field, a new approach for consensus reaching in large-scale group decision-making for generalized opinions is presented. A two-layer network is used in the group decision-making process and for visualizing the preferences of decision-makers.
3. The effects of consistency and certainty on opinion generalization were investigated. The results show that if preferences are consistent and certain, generalization effectively reduces the data dimension.
4. Three types of similarity measures were examined during the consensus process. Simulation results show that at the end of the process when the Cosine similarity is maximum, the Euclidean similarity is still low. So, it can be concluded that the use of Cosine similarity is a more suitable measure for the problems in which ranking the alternatives is important.

Group decision-making has been widely studied, and different researches have considered different goals. As the proposed model is the first in which opinion generalization is used, the focus is mainly on dimension reduction. Of course, the proposed approach has its limitations which are explained below:

1. The proposed approach may not be practical for non-rank-based group decision-making applications. However, some of the concepts introduced in this research, such as opinion generalization or the effect of consistency, can be used in future research to design new approaches based on opinion generalization. Also, based on the results of this research, a more accurate similarity criterion could be found for the decision-making process.
2. Unfortunately, in this context, there are no global metrics that allow researchers to do a fair balancing by showing both the positive and negative aspects of the model [9]. As this research is the first attempt at m-large-scale group decision-making models, it is not comparable to previous research. As almost all the existing studies do not use any form of comparison for evaluation, this cannot be considered a major limitation. However, as a comparison explains how studies relate, this could be insightful. The performed evaluations in this research are publicly available on GitHub for further studies.

Furthermore, the future work of this research intends to expand the proposed approach in the following ways:

1. The proposed approach, which works well for dimension reduction, is not effective in group decision-making problems with many alternatives. Assume that there are m alternatives, and the maximum number of classes is $m!$. The proposed approach is designed for situations in which the number of preferences is much larger than the number of classes (e.g., 4 alternatives and 40,000 decision-makers). So, providing a solution to reduce the number of alternatives can be performed as a continuation of this research.
2. The authors of [13] indicated that the term "expert" should be replaced by "decision maker" in the context of LSGDM. This is because it does not seem reasonable to consider a large number of decision-makers as experts. However, weighting can be applied in this situation. Some decision-makers are experts in a specific domain. As the main focus of this research is data reduction, the diversity of decision-makers is not considered for simplicity. The identification of experts and the weighting process will be addressed in the future. By creating concepts such as expertise, decision-makers can be classified, and a more complex framework can be proposed.
3. The proposed approach is automatic. This means that the preferences of the decision-makers are received once, and the necessary changes are applied automatically. In interactive models, changes that cause the convergence of preferences are presented to decision-makers, who later decide whether to accept the change or reject it. Creating convincing advice for decision-makers in LSGDM is also a challenge and can be viewed as an interesting topic for future research. In interactive models, as decision-makers can reject the advice, their non-cooperative behaviors should be examined, and a solution should be provided to deal with this issue.
4. The proposed approach only receives the preferences of decision-makers and alternatives as inputs. The proposed framework could be extended by collecting additional data. For example, the social network of decision-makers can be another significant input.

Funding This study has received no funding from any organization.

Data availability The generated data in this study is publicly available in a Github repository [58].

Declarations

Conflict of interest All of the authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A: Investigating the effect of consistency and certainty of decision-makers' preferences on the diversity of preferences

In this section, we prove that the consistency of preferences limits the number of possible classes. The number of possible classes depends only on the number of alternatives and is independent of the number of decision-makers. The maximum number of classes (i.e., first-layer nodes) is studied in two scenarios. In the first scenario, all the preferences are consistent. In the second scenario, there are consistent and inconsistent preferences, and it is proved that if the preferences are only consistent, the number of possible classes would be the least. The maximum number of nodes in the first layer in these two scenarios is defined by Eq. (35). In Eqs. (36) and (37), the growth of these two functions is compared. If an individual's preference is consistent and certain, the preference elements of the individual are not defined independently of each other, and there is a relationship between them. This represents the number of possible ways to arrange m alternatives into m locations. This is a permutation that does not involve repetition. Therefore, the maximum number of preferences is $(m)!$. If an individual's preference is inconsistent, this means that the individual's preference elements are defined independently of each other. Therefore, except for the diameter elements, in the decision-maker's preference matrix, the other elements can be 0 or 1, independent of each other. Because the number of non-diameter entries is $(m)(m-1)$, the maximum number of different preferences will be $2^{(m)(m-1)}$.

$$Maximum\ number\ of\ nodes\ in\ the\ first\ layer = \begin{cases} f = (m)\,!\ if\ opinions\ are\ consistent \\ g = 2^{(m)(m-1)}\ if\ opinions\ are\ are\ not\ consistent \end{cases} \tag{35}$$

$$f(x) \in \Omega(g(x)) \tag{36}$$

$$\begin{aligned} \lim_{n \to \infty}\left(\frac{f(x)}{g(x)}\right) &= \left(\frac{\log_2 f(x)}{\log_2 g(x)}\right) = \left(\frac{\log_2 (\#X)\,!}{\log_2\left(2^{(\#X)(\#X-1)}\right)}\right) \\ &= \left(\frac{(\#X)\log_2(\#X)}{((\#X)(\#X-1))\log_2(2)}\right) \\ &= \left(\frac{(\#X)\log_2(\#X)}{((\#X)(\#X-1))}\right) = \left(\frac{\log_2(\#X)}{(\#X-1)}\right) = 0 \end{aligned} \tag{37}$$

Therefore, as the number of alternatives increases, the growth of the f is less than g. Although the growth of the $f(x)$ function is from a factorial degree, its advantage is that it is independent of the number of decision-makers, and in most decision-making scenarios in reviewed studies, the number of alternatives is assumed to be 3, 4, and 5.

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