



Some p, q -cubic quasi-rung orthopair fuzzy operators for multi-attribute decision-making

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Received: 20 February 2023 / Accepted: 1 May 2023 / Published online: 14 July 2023
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Abstract

This paper aims to support decision-makers improve their ability to accurately capture and represent their judgment in a wide range of situations. To do this, we propose a new type of fuzzy set called a p, q -cubic quasi-rung orthopair fuzzy set (p, q -CQOFS). The p, q -CQOFS allows for a more flexible and detailed expression of incomplete information through the use of an additional parameter. The paper describes the concept of p, q -CQOFS and its relationship to other types of fuzzy sets, introduces score and accuracy functions for p, q -CQOFS and analyzes some of its mathematical properties, defines the Hamming distance measure between two p, q -CQOFSs and examines some of its important properties, investigates the basic operations of p, q -CQOFSs and extends these laws to aggregation operators, and introduces weighted averaging and geometric aggregation operators for combining p, q -cubic quasi-rung orthopair fuzzy data.

Keywords Multi-attribute group decision-making · Cubic fuzzy set · p, q -cubic quasi-rung orthopair fuzzy set · Uncertainty

Introduction

Multi-attribute decision-making (MADM) is a process used to evaluate and rank alternatives based on multiple conflicting objectives or in situations with uncertain decisions. MADM

methods provide a systematic way to analyze and compare alternatives based on their relative merits and drawbacks. MADM methods can be applied in fields such as agriculture [1], freight transportation [2], healthcare [3], marketing, and engineering [4]. In the past, decisions were often made using numerical data sets that were not sufficient to address real-life operational situations, leading to inadequate results. As systems have become more complex over time, it has become increasingly difficult for decision-makers to handle the uncertainties in the data using traditional methods. The researchers employed the use of fuzzy sets [5], a type of mathematical representation in which elements have a degree of membership in a set rather than a strict binary classification of belonging or not belonging, to convey the information in their study. Sometimes, a person might think that an element belongs to a fuzzy set but they are not completely sure. In these cases, the person might feel uncertain or hesitant about whether the element truly belongs to the fuzzy set or not. Traditional fuzzy sets cannot represent this type of uncertainty or hesitation because they only use a single membership degree. Intuitionistic fuzzy sets (IFSs) [6] are a generalization of fuzzy sets used to represent uncertain or imprecise information. In an intuitionistic fuzzy set, each element also has a non-membership degree, which represents the degree to which it does not belong to the set. Pythagorean fuzzy sets (PFSs) [7] are a further extension of intuitionistic

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fuzzy sets, in which each element is also assigned a non-membership degree between 0 and 1. This allows for a more nuanced representation of the uncertainty or vagueness associated with the membership of an element in the set. recently, Yager [8] introduced a new type of fuzzy set called a q -rung orthopair fuzzy set (q -ROFS) to better represent uncertainty. In this type of fuzzy set, the sum of the q th power membership and non-membership degrees must be less than or equal to 1. Numerous scholars have employed the concept of q -rung orthopair fuzzy sets and proposed various algorithms for resolving MCDM issues. For example, Deveci et al. [9] proposed combined compromised solution (CoCoSo) models based on q -rung orthopair fuzzy sets for selecting suitable sites for floating offshore wind farms in Norway. Wang et al. [10] introduced similarity metrics that take into account the membership degree, non-membership degree, and indeterminacy membership degree between q -ROFSs. Deveci et al. presented a CODAS model based on q -ROFSs to facilitate the assessment of socially responsible rehabilitation activities in mining sites. EDAS approach for evaluating supplier selection in the defense industry was suggested by Güneri and Deveci [11] using q -ROFSs. Aside from that, various other authors have introduced distinct techniques, see [12–15], to address decision-making issues.

The above review of the existing literature reveals that the majority of studies have focused on the concept and use of fuzzy sets, interval fuzzy sets, IFS, PFS, and q -ROFS and how they can be applied in different areas. Later Jun et al. [16] introduced the concept of cubic sets (CSs), which are a combination of interval-valued fuzzy numbers and fuzzy numbers. They also defined some logical operations for these cubic sets. The inclusion of cubic sets and their associated logical operations enhanced the power of fuzzy mathematics by enabling more accurate and detailed representation and manipulation of uncertain or imprecise data. Using the idea of CSs, many scholars presented different approaches. For example, Khan et al. [17] explored the use of cubic aggregation operators in a specific context, while Mahmood et al. [18] examined the application of cubic hesitant fuzzy sets and their associated aggregation operators in the decision-making process. Kaur and Garg [19] proposed cubic intuitionistic aggregation operators. In this work, they used Bonferroni mean and proposed a series of aggregation operators under the cubic intuitionistic fuzzy environment. Abbas et al. [20] proposed cubic Pythagorean fuzzy sets and presented a series of aggregation operators to solve multi-criteria decision-making (MCDM) problems. Amin et al. [21] introduced the generalized version of cubic Pythagorean fuzzy aggregation operators. Also, they investigated the flaws and ambiguities of the existing aggregation operators under the cubic Pythagorean fuzzy environment. Rahim et al. [22] introduced Bonferroni mean aggregation operators in the cubic

Pythagorean fuzzy environment to deal with MCDM problems. Zhang et al. [23] introduced cubic q -rung orthopair fuzzy Heronian mean operators for MADM problems.

Seikh and Mandal [24] generalized the q -rung orthopair fuzzy concept and presented the idea of p , q -quasi-rung orthopair fuzzy sets (p , q -QOFS). p , q -QOFS improved its overall power by adding supplementary properties and their logical operations, which allowed for more accurate representation and manipulation of uncertain or imprecise data. Furthermore, Gul and Ak [25] presented the notions of 3, 4-quasi-rung orthopair fuzzy sets. However, above the theories contain only the information in the form of membership intervals and do not stress the non-membership portion of the data entities, which also play an equivalent role in assessing the alternative in the decision-making process. In practice, it can be challenging to accurately express the value of a membership function within a fuzzy set.

After analyzing the previous discussions, it becomes apparent that the current fuzzy sets have specific limitations. For instance, the cubic Intuitionistic Fuzzy Set (CIFS) is restricted to the condition that the sum of membership degree (ϑ_G^U) and non-membership degree (ψ_G^U) must be less than or equal to 1, which hinders its ability to capture uncertainty in certain scenarios. Similarly, the Cubic Pythagorean Fuzzy Set (CPFS) is restricted to $(\vartheta_G^U)^2 + (\psi_G^U)^2 \leq 1$, limiting its accuracy and precision in certain applications. Another example is the Cubic Fermatean Fuzzy Set (CFFS), which is restricted to $(\vartheta_G^U)^3 + (\psi_G^U)^3 \leq 1$, making it less suitable for situations that involve high levels of uncertainty. Where ϑ_G^U and ψ_G^U are the Furthermore, in the case of the cubic q -rung orthopair fuzzy set, decision-makers face a limitation in that they are required to assign the same value of q for both membership and non-membership values of an element. This constraint can lead to a lack of flexibility and may not accurately reflect the decision-maker's preferences and beliefs.

To overcome the limitations of the current cubic fuzzy sets, we have introduced a new extension known as the p , q -CROFSs. This new set allows for greater flexibility by relaxing the previous constraints, such that the sum of the p th power of membership and q th power of non-membership can be less than or equal to 1, represented by the condition $(\vartheta_G^U)^p + (\psi_G^U)^q \leq 1$ with $p, q \geq 1$. Also, p can be equal to q , greater than q , or less than q . This modification provides decision-makers with a wider range of possibilities in capturing uncertainty, enabling them to model complex scenarios with greater accuracy and precision. The p , q -CQOFSs, which represent membership degrees in two parts: p , q -interval-valued quasi-rung orthopair fuzzy (p , q -IVQOF) value and a p , q -quasi-rung orthopair fuzzy (p , q -QOF) value which have more information than the general p , q -QOF set because it includes information from both of these sets. Based on these new

sets, some aggregation operators called the p, q -cubic quasi-rung orthopair fuzzy weighted averaging (p, q -CQOFWA) and p, q -cubic quasi-rung orthopair fuzzy weighted geometric (p, q -CQOFWG) operators are defined to aggregate the preferences of decision-makers that take into account the relationship between various criteria in the decision-making process when dealing with p, q -CQOF information. Further, some basic properties of the proposed operators have been discussed in detail. Some existing aggregation operators are discussed and the results are compared with the results of proposed operators which suggests that the presented operators are more widely applicable than others. This implies that the proposed operators are more generalized than the other operators that have been studied. A method for evaluating and ranking the various alternatives based on the proposed operators has been presented as the final step in this process. The goals of this article are summarized as follows:

1. The concept of p, q -CQOFS is introduced as a modification of q -CQOFS. In addition, functions for scoring and accuracy, a measure of Hamming distance, and some guidelines for operation are defined for p, q -CQOFS.
2. Operators for combining p, q -CQOF information have been developed, including the weighted averaging operator and the geometric aggregation operator.
3. A new multi-attribute group decision-making (MAGDM) method has been created using the proposed operators.
4. An example is provided to demonstrate the versatility and effectiveness of the developed method.

The remainder of this article is structured as follows. "Preliminaries" provides a brief overview of key concepts related to cubic sets and p, q -quasi-rung orthopair fuzzy sets. In "Operational laws and aggregation operators under -CQOFNs", we introduce p, q -CQOFS, along with their associated aggregation operators, and analyze their specific cases. We also discuss some of the key properties of these operators in this section. "Multi-attribute group decision-making based on proposed operators" outlines a decision-making approach that employs the proposed operators to address multi-attribute decision-making (MAGDM) problems. To illustrate the practicality and effectiveness of the proposed approach, we present a numerical example in "Illustrative model". Finally, "Conclusion" offers concluding remarks to summarize the key contributions of this paper.

Preliminaries

This section provides a detailed discussion of fundamental definitions that pertain to cubic sets and p, q -quasi-rung orthopair fuzzy sets.

Cubic set

Definition 1 [16] Let F be a non-empty set. A cubic set C in F is defined as

$$C = \{t, C(t), \vartheta(t) | t \in F\} \tag{1}$$

where $C(t) = [C^L(t), C^U(t)]$ is interval-valued fuzzy (IVF) set (IVFS) in F and $\vartheta(t)$ is a FS. Cubic set C is said to be an internal cubic set if $C^L(t) \leq \vartheta(t) \leq C^U(t)$ while a set C an external cubic set if $\vartheta(t) \notin (C^L(t), C^U(t))$. A cubic set $C = \{t, C(t), \vartheta(t) | t \in F\}$ is simply denoted by $C = \langle C, \vartheta \rangle$.

Definition 2 [16] Let $C_1 = \langle C_1, \vartheta_1 \rangle$ and $C_2 = \langle C_2, \vartheta_2 \rangle$ be cubic sets in F . Then.

1. (Equality): If $C_1 = C_2$ and $\vartheta_1 = \vartheta_2$ then $C_1 = C_2$.
2. (P-order): If $C_1 \subseteq C_2$ and $\vartheta_1 \preceq \vartheta_2$ then $C_1 \subseteq_P C_2$.
3. (R-order): If $C_1 \subseteq C_2$ and $\vartheta_1 \succeq \vartheta_2$ then $C_1 \subseteq_R C_2$.

Definition 3 [16] Let $C_i = \{t, C_i(t), \vartheta_i(t) | t \in F\}$ be a collection cubic set where $i \in \Delta$, then

1. (P-union): $\cup_{i \in \Delta}^P C_i(t) = \{t, \cup_{i \in \Delta} C_i(t), \vee_{i \in \Delta} \vartheta_i(t) | t \in F\}$
2. (P-intersection): $\cap_{i \in \Delta}^P C_i(t) = \{t, \cap_{i \in \Delta} C_i(t), \wedge_{i \in \Delta} \vartheta_i(t) | t \in F\}$
3. (R-union): $\cup_{i \in \Delta}^R C_i(t) = \{t, \cup_{i \in \Delta} C_i(t), \wedge_{i \in \Delta} \vartheta_i(t) | t \in F\}$
4. (R-intersection): $\cap_{i \in \Delta}^R C_i(t) = \{t, \cap_{i \in \Delta} C_i(t), \vee_{i \in \Delta} \vartheta_i(t) | t \in F\}$

p, q quasi-rung orthopair fuzzy sets

Definition 4 [24] Let F be a non-empty set. A p, q -QOFS Q over an element $t \in F$ is defined as follows:

$$Q = \{t, \vartheta_Q(t), \psi_Q(t) | t \in F\}, \tag{2}$$

where $\vartheta_Q(t) : F \rightarrow [0, 1]$ and $\psi_Q(t) : F \rightarrow [0, 1]$ represents support and support against membership grade of element $t \in F$ such that $(\vartheta_Q(t))^p + (\psi_Q(t))^q \leq 1$. Where $p, q \geq 1$ are positive real numbers. For convenience, we called $(\vartheta_Q(t), \psi_Q(t))$ a p, q -QOF number (p, q -QOFN) denoted by $Q = (\vartheta_Q, \psi_Q)$. The value of p may be greater than or less than or equal to q .

Definition 5 [24] Let $Q = (\vartheta_Q, \psi_Q)$ be a p, q -QOF number. The hesitancy degree is defined as

$$\pi_Q = \sqrt[l]{1 - (\vartheta_Q(t))^p - (\psi_Q(t))^q}, \tag{3}$$

where l is the smallest positive integer that is divisible by both p and q .

Definition 6 [24] Let $Q_1 = (\vartheta_{Q_1}, \psi_{Q_1})$, $Q_2 = (\vartheta_{Q_2}, \psi_{Q_2})$ and $Q = (\vartheta_Q, \psi_Q)$ are three p, q -QOFNs, and $\zeta > 0$, then,

1. $Q_1 \oplus Q_2 = \left(\sqrt[p]{\vartheta_{Q_1}^p + \vartheta_{Q_2}^p - \vartheta_{Q_1}^p \vartheta_{Q_2}^p}, \psi_{Q_1} \psi_{Q_2} \right)$,
2. $Q_1 \otimes Q_2 = \left(\vartheta_{Q_1} \vartheta_{Q_2}, \sqrt[q]{\psi_{Q_1}^q + \psi_{Q_2}^q - \psi_{Q_1}^q \psi_{Q_2}^q} \right)$,
3. $Q^\zeta = \left(\vartheta_Q^\zeta, \sqrt[q]{1 - (1 - \psi_Q^q)^\zeta} \right)$,
4. $\zeta p = \left(\sqrt[p]{1 - (1 - \vartheta_Q^p)^\zeta}, \psi_Q^\zeta \right)$,
5. $Q_1 \leq Q_2$ if and only if $\vartheta_{Q_1} \leq \vartheta_{Q_2}$ and $\psi_{Q_1} \geq \psi_{Q_2}$,
6. $Q_1 = Q_2$ if and only if $\vartheta_{Q_1} = \vartheta_{Q_2}$ and $\psi_{Q_1} = \psi_{Q_2}$.

Definition 7 [24] Let $Q = (\vartheta_Q, \psi_Q)$ be a p, q -QOFN. The score of $Q = (\vartheta_Q, \psi_Q)$ can be determined by the following function.

$$sc(Q) = \frac{1 + \vartheta_Q^p - \psi_Q^q}{2}, \quad (4)$$

where $0 \leq sc(Q) \leq 1$.

Definition 8 [24] Let $Q = (\vartheta_Q, \psi_Q)$ be a p, q -QOFN. The accuracy of $Q = (\vartheta_Q, \psi_Q)$ can be determined by the following function.

$$ac(Q) = \vartheta_Q^p + \psi_Q^q, \quad (5)$$

where $0 \leq ac(Q) \leq 1$.

Definition 9 [24] Let $Q_1 = (\vartheta_{Q_1}, \psi_{Q_1})$ and $Q_2 = (\vartheta_{Q_2}, \psi_{Q_2})$ are two p, q -QOFNs, then

1. If $sc(Q_1) < sc(Q_2)$ then $Q_1 < Q_2$,
2. If $sc(Q_1) > sc(Q_2)$ then $Q_1 > Q_2$,
3. If $sc(Q_1) = sc(Q_2)$ and,
 - If $ac(Q_1) < ac(Q_2)$ then $Q_1 < Q_2$,
 - If $ac(Q_1) > ac(Q_2)$ then $Q_1 > Q_2$,
 - If $ac(Q_1) = ac(Q_2)$ then $Q_1 \sim Q_2$.

Definition 10 [24] Let $Q_1 = (\vartheta_{Q_1}, \psi_{Q_1})$, $Q_2 = (\vartheta_{Q_2}, \psi_{Q_2})$ and $Q = (\vartheta_Q, \psi_Q)$ are three p, q -QOFN, and ζ, ζ_1 and ζ_2 are any positive integers then the following properties are held.

1. $Q_1 \oplus Q_2 = Q_2 \oplus Q_1$,
2. $Q_1 \otimes Q_2 = Q_2 \otimes Q_1$,
3. $\zeta(Q_1 \oplus Q_2) = \zeta Q_1 \oplus \zeta Q_2$,
4. $\zeta_1 Q \oplus \zeta_2 Q = (\zeta_1 + \zeta_2)Q$,
5. $(Q_1 \otimes Q_2)^\zeta = Q_1^\zeta \otimes Q_2^\zeta$,
6. $Q^{\zeta_1} \otimes Q^{\zeta_2} = Q^{\zeta_1 + \zeta_2}$.

Operational laws and aggregation operators under p, q -CQOFNs

This section begins by introducing p, q -CQOFNs and defining the basic operation laws of p, q -CQOFNs. Using these operation laws, a set of aggregation operators are introduced. The section then goes on to discuss some fundamental properties in greater detail.

Definition 11 Let F be a non-empty set. A p, q -CQOFNs \mathcal{G} over an element $t \in F$ is defined as follows:

$$\mathcal{G} = \{t, G(t), \varphi(t) | t \in F\}, \quad (6)$$

where $G(t) = \{t, [\vartheta_G^L(t), \vartheta_G^U(t)], [\psi_G^L(t), \psi_G^U(t)]\}$ represents p, q -interval-valued quasi-rang orthopair fuzzy set while $\varphi(x) = \{t, \vartheta_\varphi(t), \psi_\varphi(t)\}$ represents PFS for all $t \in F$ such that $0 \leq \vartheta_G^L(t) \leq \vartheta_G^U(t) \leq 1$, $0 \leq \psi_G^L(t) \leq \psi_G^U(t) \leq 1$ and $0 \leq (\vartheta_G^U(t))^p + (\psi_G^U(t))^q \leq 1$. Also, $0 \leq \vartheta_\varphi(t), \psi_\varphi(t) \leq 1$ and $0 \leq (\vartheta_\varphi(t))^p + (\psi_\varphi(t))^q \leq 1$. To keep it simple, the pair $\mathcal{G} = G, \varphi$, where $[\vartheta_G^L, \vartheta_G^U], [\psi_G^L, \psi_G^U]$ and $\vartheta_\varphi, \psi_\varphi$ and called as p, q -CQOFN. The conditions for p, q -CQOFN can be summarized as follows:

- (1) The values of p and q are both greater than or equal to 1, and p can be less than, greater than, or equal to q .
- (2) $\vartheta_G^L(t), \vartheta_G^U(t) \in [0, 1]$, $\psi_G^L(t), \psi_G^U(t) \in [0, 1]$, and $\vartheta_\varphi(t), \psi_\varphi(t) \in [0, 1]$.
- (3) $0 \leq (\vartheta_G^U(t))^p + (\psi_G^U(t))^q \leq 1$ and $(\vartheta_\varphi(t))^p + (\psi_\varphi(t))^q \leq 1$.

We are looking for the smallest possible values of p and q such that $(\vartheta_G^U)^p + (\psi_G^U)^q \leq 1$ and $(\vartheta_\varphi)^p + (\psi_\varphi)^q \leq 1$ for a given pair of functions ϑ and ψ . These values, which we will call the p, q -niche of $[\vartheta_G^L, \vartheta_G^U], [\psi_G^L, \psi_G^U], (\vartheta_\varphi, \psi_\varphi)$, can be found using iterative computing techniques, even though there is no closed-form solution. If p_x, q_y is the p, q -niche of $[\vartheta_G^L, \vartheta_G^U], [\psi_G^L, \psi_G^U], (\vartheta_\varphi, \psi_\varphi)$, then $[\vartheta_G^L, \vartheta_G^U], [\psi_G^L, \psi_G^U], (\vartheta_\varphi, \psi_\varphi)$, is a valid p, q -quasi-rang membership grade for all values of p and q that are equal to or greater than p_x and q_y , respectively.

Some special cases

- (1) If $p = q = 1$ then p, q -CQOFNs reduced to cubic intuitionistic fuzzy set [26].
- (2) If $p = q = 2$ then p, q -CQOFNs reduced to cubic Pythagorean fuzzy set [21].
- (3) If $p = q$ then p, q -CQOFNs reduced to cubic q -rung orthopair fuzzy set [27].

Operation laws

Definition 12 For a family of p, q -CQOFNs $\{G_i, i \in \Delta\}$, then

1. (P-union): $\cup_{i \in \Delta}^P G_i = \left(\left\langle \left[\begin{array}{c} \max_{i \in \Delta} (\vartheta_{G_i}^L), \\ \max_{i \in \Delta} (\vartheta_{G_i}^U) \end{array} \right], \left[\begin{array}{c} \min_{i \in \Delta} (\psi_{G_i}^L), \\ \min_{i \in \Delta} (\psi_{G_i}^U) \end{array} \right] \right\rangle, (\max_{i \in \Delta} \vartheta_{G_i}, \min_{i \in \Delta} \psi_{G_i}) \right);$
2. (P-intersection): $\cap_{i \in \Delta}^R G_i = \left(\left\langle \left[\begin{array}{c} \min_{i \in \Delta} (\vartheta_{G_i}^L), \\ \min_{i \in \Delta} (\vartheta_{G_i}^U) \end{array} \right], \left[\begin{array}{c} \max_{i \in \Delta} (\psi_{G_i}^L), \\ \max_{i \in \Delta} (\psi_{G_i}^U) \end{array} \right] \right\rangle, (\min_{i \in \Delta} \vartheta_{G_i}, \max_{i \in \Delta} \psi_{G_i}) \right);$
3. (R-union): $\cup_{i \in \Delta}^R G_i = \left(\left\langle \left[\begin{array}{c} \max_{i \in \Delta} (\vartheta_{G_i}^L), \\ \max_{i \in \Delta} (\vartheta_{G_i}^U) \end{array} \right], \left[\begin{array}{c} \min_{i \in \Delta} (\psi_{G_i}^L), \\ \min_{i \in \Delta} (\psi_{G_i}^U) \end{array} \right] \right\rangle, (\min_{i \in \Delta} \vartheta_{G_i}, \max_{i \in \Delta} \psi_{G_i}) \right);$
4. (R-intersection): $\cup_{i \in \Delta}^P G_i = \left(\left\langle \left[\begin{array}{c} \max_{i \in \Delta} (\vartheta_{G_i}^L), \\ \max_{i \in \Delta} (\vartheta_{G_i}^U) \end{array} \right], \left[\begin{array}{c} \min_{i \in \Delta} (\psi_{G_i}^L), \\ \min_{i \in \Delta} (\psi_{G_i}^U) \end{array} \right] \right\rangle, (\min_{i \in \Delta} \vartheta_{G_i}, \max_{i \in \Delta} \psi_{G_i}) \right).$

Definition 13 Let $G_1 = \left(\left(\left[\vartheta_{G_1}^L, \vartheta_{G_1}^U \right], \left[\psi_{G_1}^L, \psi_{G_1}^U \right] \right), \langle \vartheta_{G_1}, \psi_{G_1} \rangle \right)$ and $G_2 = \left(\left(\left[\vartheta_{G_2}^L, \vartheta_{G_2}^U \right], \left[\psi_{G_2}^L, \psi_{G_2}^U \right] \right), \langle \vartheta_{G_2}, \psi_{G_2} \rangle \right)$ be two p, q -CQOFNs in F , Then

1. (Equality): $G_1 = G_2$, if and only if $\left[\vartheta_{G_1}^L, \vartheta_{G_1}^U \right] = \left[\vartheta_{G_2}^L, \vartheta_{G_2}^U \right], \left[\psi_{G_1}^L, \psi_{G_1}^U \right] = \left[\psi_{G_2}^L, \psi_{G_2}^U \right], \vartheta_{G_1} = \vartheta_{G_2}$ and $\psi_{G_1} = \psi_{G_2}$.
2. (P-order): $G_1 \subseteq_P G_2$ if $\left[\vartheta_{G_1}^L, \vartheta_{G_1}^U \right] \subseteq \left[\vartheta_{G_2}^L, \vartheta_{G_2}^U \right], \left[\psi_{G_1}^L, \psi_{G_1}^U \right] \supseteq \left[\psi_{G_2}^L, \psi_{G_2}^U \right], \vartheta_{G_1} \leq \vartheta_{G_2}$ and $\psi_{G_1} \geq \psi_{G_2}$.
3. (R-order): $G_1 \subseteq_R G_2$ if $\left[\vartheta_{G_1}^L, \vartheta_{G_1}^U \right] \subseteq \left[\vartheta_{G_2}^L, \vartheta_{G_2}^U \right], \left[\psi_{G_1}^L, \psi_{G_1}^U \right] \supseteq \left[\psi_{G_2}^L, \psi_{G_2}^U \right], \vartheta_{G_1} \geq \vartheta_{G_2}$ and $\psi_{G_1} \leq \psi_{G_2}$.

Definition 14 Let $G = \left(\left(\left[\vartheta_G^L, \vartheta_G^U \right], \left[\psi_G^L, \psi_G^U \right] \right), \langle \vartheta_G, \psi_G \rangle \right)$ be a p, q -CQOFN, then score function under R-order is defined as:

$$sc(G) = \frac{(\vartheta_G^L)^p + (\vartheta_G^U)^p - (\psi_G^L)^q - (\psi_G^U)^q}{2} + \psi_G^p - \vartheta_G^q, \tag{7}$$

while the score function under P-order is given by:

$$sc(G) = \frac{(\vartheta_G^L)^p + (\vartheta_G^U)^p - (\psi_G^L)^q - (\psi_G^U)^q}{2} + (\vartheta_G^p - \psi_G^q), \tag{8}$$

where $-2 \leq sc(\beta_1) \leq 2$.

Definition 15 Let $G = \left(\left(\left[\vartheta_G^L, \vartheta_G^U \right], \left[\psi_G^L, \psi_G^U \right] \right), \langle \vartheta_G, \psi_G \rangle \right)$ be a p, q -CQOFN, then accuracy function is defined as:

$$ac(G) = \frac{(\vartheta_G^L)^p + (\vartheta_G^U)^p + (\psi_G^L)^q + (\psi_G^U)^q}{2} + (\vartheta_G^p + \psi_G^q), \tag{9}$$

where $0 \leq ac(\beta_1) \leq 2$.

Example 1 Let $G = \left(\left([0.7, 0.8], [0.8, 0.9] \right), \langle 0.8, 0.6 \rangle \right)$. be a p, q -CQOFN. Then, use Eqs. (6) and (8) to calculate the score and accuracy values. For simplicity, we suppose $p = q = 4$. From Eq. (6) we have

$$sc(G) = \frac{(0.7)^4 + (0.8)^4 - (0.8)^4 - (0.9)^4}{2} + (0.8)^4 - (0.6)^4 = 0.0720, \tag{10}$$

$$ac(G) = \frac{(0.7)^4 + (0.8)^4 + (0.8)^4 + (0.9)^4}{2} + (0.8)^4 + (0.6)^4 = 1.3969. \tag{11}$$

Theorem 1 For p, q -CQOFNs $G_i = \left(\left(\left[\vartheta_{G_i}^L, \vartheta_{G_i}^U \right], \left[\psi_{G_i}^L, \psi_{G_i}^U \right] \right), \langle \vartheta_{G_i}, \psi_{G_i} \rangle \right)$ ($i = 1, 2, 3, 4$) we have

1. If $G_1 \subseteq_P G_2$ and $G_2 \subseteq_P G_3$ then $G_1 \subseteq_P G_3$.
2. If $G_1 \subseteq_P G_2$ and $G_1 \subseteq_P G_3$ then $G_1 \subseteq_P G_2 \cap G_3$.
3. If $G_1 \subseteq_P G_2$ and $G_3 \subseteq_P G_4$ then $G_1 \cup G_3 \subseteq_P G_2 \cup G_4$ and $G_1 \cap G_3 \subseteq_P G_2 \cap G_4$.
4. If $G_1 \subseteq_P G_2$ and $G_3 \subseteq_P G_2$ then $G_1 \cup G_3 \subseteq_P G_2$.
5. If $G_1 \subseteq_R G_2$ and $G_2 \subseteq_R G_3$ then $G_1 \subseteq_R G_3$.
6. If $G_1 \subseteq_R G_2$ and $G_1 \subseteq_R G_3$ then $G_1 \subseteq_R G_2 \cap G_3$.
7. If $G_1 \subseteq_R G_2$ and $G_3 \subseteq_R G_4$ then $G_1 \cup G_3 \subseteq_R G_2 \cup G_4$ and $G_1 \cap G_3 \subseteq_R G_2 \cap G_4$.
8. If $G_1 \subseteq_R G_2$ and $G_3 \subseteq_R G_2$ then $G_1 \cup G_3 \subseteq_R G_2$.

Proof It can be obtained from the definition, so we omit here.

Definition 16 Let $G = \left(\left(\left[\vartheta_G^L, \vartheta_G^U \right], \left[\psi_G^L, \psi_G^U \right] \right), \langle \vartheta_G, \psi_G \rangle \right)$, $G_i = \left(\left(\left[\vartheta_{G_i}^L, \vartheta_{G_i}^U \right], \left[\psi_{G_i}^L, \psi_{G_i}^U \right] \right), \langle \vartheta_{G_i}, \psi_{G_i} \rangle \right)$ ($i = 1, 2$) be the collections of p, q -CQOFN, and $\zeta > 0$ be a real number, then

$$1. \mathcal{G}_1 \oplus \mathcal{G}_2 = \left(\left\langle \left[\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^2 (1 - (\vartheta_{G_i}^L)^p)}, \\ \sqrt[p]{1 - \prod_{i=1}^2 (1 - (\vartheta_{G_i}^U)^p)} \end{array} \right], \left[\begin{array}{c} \prod_{i=1}^2 \psi_{G_i}^L, \\ \prod_{i=1}^2 \psi_{G_i}^U \end{array} \right] \right\rangle, \left\langle \sqrt[q]{1 - \prod_{i=1}^2 (1 - (\psi_{G_i})^p)}, \prod_{i=1}^2 \vartheta_{G_i} \right\rangle \right);$$

$$2. \mathcal{G}_1 \otimes \mathcal{G}_2 = \left(\left\langle \left[\begin{array}{c} \prod_{i=1}^2 \mu_{G_i}^L, \\ \prod_{i=1}^2 \mu_{G_i}^U \end{array} \right], \left[\begin{array}{c} \sqrt[q]{1 - \prod_{i=1}^2 (1 - (v_{G_i}^L)^q)}, \\ \sqrt[q]{1 - \prod_{i=1}^2 (1 - (v_{G_i}^U)^q)} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^2 (1 - (\mu_{G_i})^p)}, \prod_{i=1}^2 v_{G_i} \right\rangle \right);$$

$$3. \zeta \mathcal{G} = \left(\left\langle \left[\begin{array}{c} \sqrt[p]{1 - (1 - (\vartheta_{G_i}^L)^p)^\zeta}, \\ \sqrt[p]{1 - (1 - (\vartheta_{G_i}^U)^p)^\zeta} \end{array} \right], \left[\begin{array}{c} (\psi_{G_i}^L)^\zeta, \\ (\psi_{G_i}^U)^\zeta \end{array} \right] \right\rangle, \left\langle \sqrt[q]{1 - (1 - \psi_{G_i}^p)^\zeta}, \vartheta_{G_i}^\zeta \right\rangle \right);$$

$$4. \mathcal{G}^\zeta = \left(\left\langle \left[\begin{array}{c} (\vartheta_{G_i}^L)^\zeta, \\ (\vartheta_{G_i}^U)^\zeta \end{array} \right], \left[\begin{array}{c} \sqrt[q]{1 - (1 - (\psi_{G_i}^L)^p)^\zeta}, \\ \sqrt[q]{1 - (1 - (\psi_{G_i}^U)^p)^\zeta} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - (1 - \vartheta_{G_i}^p)^\zeta}, \psi_{G_i}^\zeta \right\rangle \right).$$

Example 2 Let $\mathcal{G}_1 = (\langle [0.7, 0.8], [0.8, 0.85] \rangle, \langle 0.8, 0.6 \rangle)$ and $\mathcal{G}_2 = (\langle [0.6, 0.7], [0.7, 0.8] \rangle, \langle 0.7, 0.8 \rangle)$ be two p, q -CQOFNs. For simplicity, we suppose $p = q = 5$ and $\zeta = 2$. Then

$$\mathcal{G}_1 \oplus \mathcal{G}_2 = \left(\left\langle \left[\begin{array}{c} \sqrt[5]{1 - (1 - (0.7)^5 \times (1 - (0.6)^5))}, \\ \sqrt[5]{1 - (1 - (0.8)^5 \times (1 - (0.7)^5))} \end{array} \right], \left[\begin{array}{c} 0.8 \times 0.7, \\ 0.85 \times 0.8 \end{array} \right] \right\rangle, \left\langle \sqrt[5]{1 - (1 - (0.6)^5 \times (1 - (0.8)^5))}, 0.8 \times 0.7 \right\rangle \right) \\ = \left(\left\langle \left[\begin{array}{c} 0.6888, \\ 0.7711 \end{array} \right], \left[\begin{array}{c} 0.5600, \\ 0.6800 \end{array} \right] \right\rangle, \left\langle 0.5600, 0.5542 \right\rangle \right).$$

$$\mathcal{G}_1 \otimes \mathcal{G}_2 = \left(\left\langle \left[\begin{array}{c} 0.7 \times 0.6, \\ 0.8 \times 0.7 \end{array} \right], \left[\begin{array}{c} \sqrt[5]{1 - (1 - (0.8)^5 \times (1 - (0.7)^5))}, \\ \sqrt[5]{1 - (1 - (0.85)^5 \times (1 - (0.8)^5))} \end{array} \right] \right\rangle, \left\langle \sqrt[5]{1 - (1 - (0.8)^5 \times (1 - (0.7)^5))}, 0.6 \times 0.8 \right\rangle \right) \\ = \left(\left\langle \left[\begin{array}{c} 0.4200, \\ 0.5600 \end{array} \right], \left[\begin{array}{c} 0.7711, \\ 0.7851 \end{array} \right] \right\rangle, \left\langle 0.7711, 0.4800 \right\rangle \right) \\ \mathcal{G}_1^\zeta = \left(\left\langle \left[\begin{array}{c} (0.7)^2, \\ (0.8)^2 \end{array} \right], \left[\begin{array}{c} \sqrt[5]{1 - (1 - (0.8)^5)^2}, \\ \sqrt[5]{1 - (1 - (0.85)^5)^2} \end{array} \right] \right\rangle, \left\langle \sqrt[5]{1 - (1 - (0.8)^5)^2}, (0.6)^2 \right\rangle \right) \\ = \left(\left\langle \left[\begin{array}{c} 0.4900, \\ 0.6400 \end{array} \right], \left[\begin{array}{c} 0.8867, \\ 0.9286 \end{array} \right] \right\rangle, \left\langle 0.8867, 0.3600 \right\rangle \right) \\ \zeta \mathcal{G}_1 = \left(\left\langle \left[\begin{array}{c} \sqrt[5]{1 - (1 - (0.7)^5)^2}, \\ \sqrt[5]{1 - (1 - (0.8)^5)^2} \end{array} \right], \left[\begin{array}{c} (0.8)^2, \\ (0.85)^2 \end{array} \right] \right\rangle, \left\langle \sqrt[5]{1 - (1 - (0.6)^5)^2}, (0.8)^2 \right\rangle \right) \\ = \left(\left\langle \left[\begin{array}{c} 0.7901, \\ 0.8867 \end{array} \right], \left[\begin{array}{c} 0.6400, \\ 0.7225 \end{array} \right] \right\rangle, \left\langle 0.6400, 0.6838 \right\rangle \right)$$

Definition 17 Let $\mathcal{G}_i = (\langle [\vartheta_{G_i}^L, \vartheta_{G_i}^U], [\psi_{G_i}^L, \psi_{G_i}^U] \rangle, \langle \vartheta_{G_i}, \psi_{G_i} \rangle)$ ($i = 1, 2$) be the collections of p, q -CQOFNs. Then $\pi_{\mathcal{G}_i}(t) = (\pi_{G_i}^L(t), \pi_{G_i}^U(t), \pi_{G_i}(t))$ is said to be p, q -quasi-rang orthopair index of an element $t \in F$.

Here

$$\pi_{G_i}^L(t) = \sqrt[p]{1 - (\vartheta_{G_i}^U)^p - (\psi_{G_i}^U)^q}, \tag{12}$$

$$\pi_{G_i}^U(t) = \sqrt[q]{1 - (\vartheta_{G_i}^L)^p - (\psi_{G_i}^L)^q}, \tag{13}$$

$$\pi_{G_i}(t) = \sqrt[l]{1 - (\vartheta_{G_i})^p - (\psi_{G_i})^q}, \tag{14}$$

where l is the least common multiple (LCM) of p and q .

Distance measures are a fundamental aspect of fuzzy set theory [28, 29] and are often utilized in decision-making scenarios. Among the commonly used distance measures are the Euclidean, Hamming, and generalized Euclidean distance measures. In this section, we will illustrate the calculation of the Hamming distance between p, q -CQOFNs, which will be applied later on in the analysis.

Definition 18 Let $\mathcal{G}_i = (\langle [\vartheta_{G_i}^L, \vartheta_{G_i}^U], [\psi_{G_i}^L, \psi_{G_i}^U] \rangle, \langle \vartheta_{G_i}, \psi_{G_i} \rangle)$ ($i = 1, 2, 3$) be the collections of p, q -CQOFNs. Properties of a distance measure δ are:

$$\delta(\mathcal{G}_1, \mathcal{G}_2) = 0 \text{ if and only if } \mathcal{G}_1 = \mathcal{G}_2.$$

$$\delta(\mathcal{G}_1, \mathcal{G}_2) = \delta(\mathcal{G}_2, \mathcal{G}_1).$$

$$0 \leq \delta(\mathcal{G}_2, \mathcal{G}_1) \leq 2.$$

$$\text{If } \mathcal{G}_1 \leq \mathcal{G}_2 \leq \mathcal{G}_3, \text{ then } \delta(\mathcal{G}_1, \mathcal{G}_2) \leq \delta(\mathcal{G}_1, \mathcal{G}_3) + \delta(\mathcal{G}_2, \mathcal{G}_3)$$

Definition 19 Let $\mathcal{G}_i = \left(\left(\left[\vartheta_{G_i}^L, \vartheta_{G_i}^U \right], \left[\psi_{G_i}^L, \psi_{G_i}^U \right] \right), \left\langle \vartheta_{G_i}, \psi_{G_i} \right\rangle \right)$ ($i = 1, 2$) be the collection of two p, q -CQOFNs. The Hamming distance (Λ) between these two p, q -CQOFNs is defined as

$$\Lambda(\mathcal{G}_1, \mathcal{G}_2) = \frac{1}{6} \left(\begin{aligned} & \left| \left(\vartheta_{G_1}^L \right)^p - \left(\vartheta_{G_2}^L \right)^p \right| + \left| \left(\vartheta_{G_1}^U \right)^p - \left(\vartheta_{G_2}^U \right)^p \right| + \left| \left(\pi_{G_1}^L \right)^l - \left(\pi_{G_2}^L \right)^l \right| \\ & + \left| \left(\psi_{G_1}^L \right)^q - \left(\psi_{G_2}^L \right)^q \right| + \left| \left(\psi_{G_1}^U \right)^q - \left(\psi_{G_2}^U \right)^q \right| + \left| \left(\pi_{G_1}^U \right)^l - \left(\pi_{G_2}^U \right)^l \right| \\ & \left| \left(\vartheta_{G_1} \right)^p - \left(\vartheta_{G_2} \right)^p \right| + \left| \left(\psi_{G_1} \right)^q - \left(\psi_{G_2} \right)^q \right| + \left| \left(\pi_{G_1} \right)^l - \left(\pi_{G_2} \right)^l \right| \end{aligned} \right) \tag{15}$$

Example 3 Let $\mathcal{G}_1 = ([0.7, 0.8], [0.8, 0.85], 0.8, 0.6)$ and $\mathcal{G}_2 = ([0.6, 0.7], [0.7, 0.8], 0.7, 0.8)$ be two p, q -CQOFNs. For simplicity, we considered $p = q = 5$. Then, Hamming distance between this p, q -CQOFNs can be calculated as follows.

Using Eqs. (9), (10), and (11), we have

$$\begin{aligned} \pi_{G_1}^L(t) &= \sqrt[p]{1 - \left(\vartheta_{G_1}^U \right)^p - \left(\psi_{G_1}^U \right)^q} \\ &= \sqrt[5]{1 - (0.8)^5 - (0.85)^5} = 0.7444, \end{aligned}$$

$$\begin{aligned} \pi_{G_2}^L(t) &= \sqrt[p]{1 - \left(\vartheta_{G_2}^U \right)^p - \left(\psi_{G_2}^U \right)^q} \\ &= \sqrt[5]{1 - (0.7)^5 - (0.8)^5} = 0.8720, \end{aligned}$$

$$\begin{aligned} \pi_{G_1}^U(t) &= \sqrt[p]{1 - \left(\vartheta_{G_1}^L \right)^p - \left(\psi_{G_1}^L \right)^q} \\ &= \sqrt[5]{1 - (0.7)^5 - (0.8)^5} = 0.8720, \end{aligned}$$

$$\begin{aligned} \pi_{G_2}^U(t) &= \sqrt[p]{1 - \left(\vartheta_{G_2}^L \right)^p - \left(\psi_{G_2}^L \right)^q} \\ &= \sqrt[5]{1 - (0.6)^5 - (0.7)^5} = 0.9451, \end{aligned}$$

$$\begin{aligned} \pi_{G_1} &= \sqrt[p]{1 - \left(\vartheta_{G_1} \right)^p - \left(\psi_{G_1} \right)^q} \\ &= \sqrt[5]{1 - (0.8)^5 - (0.6)^5} = 0.9012, \end{aligned}$$

$$\begin{aligned} \pi_{G_2} &= \sqrt[p]{1 - \left(\vartheta_{G_2} \right)^p - \left(\psi_{G_2} \right)^q} \\ &= \sqrt[5]{1 - (0.7)^5 - (0.8)^5} = 0.8720. \end{aligned}$$

Now, using Eq. (12), we have

$$\begin{aligned} \Lambda(\mathcal{G}_1, \mathcal{G}_2) &= \frac{1}{6} \left(\begin{aligned} & \left| (0.7)^5 - (0.6)^5 \right| + \left| (0.8)^5 - (0.7)^5 \right| + \left| (0.7444)^5 - (0.8720)^5 \right| \\ & + \left| (0.8)^5 - (0.7)^5 \right| + \left| (0.85)^5 - (0.8)^5 \right| + \left| (0.8720)^5 - (0.9451)^5 \right| \\ & \left| (0.8)^5 - (0.7)^5 \right| + \left| (0.6)^5 - (0.8)^5 \right| + \left| (0.9012)^5 - (0.8720)^5 \right| \end{aligned} \right) \\ &= 0.1920 \end{aligned}$$

Aggregation operators under p, q -quasi-rung orthopair information

In this section, a detailed explanation of the development of the p, q -CQOF weighted averaging operator (p, q -

CQOFWA) and the p, q -QOF weighted geometric operator (p, q -CQOFWG) are provided. Definition 3.6 is used as the foundation for the construction of these operators. Furthermore, we will investigate and discuss several suitable properties that are related to the p, q -CQOFWA and p, q -CQOFWG operators. This will include an examination of the mathematical properties and behavior of these operators in various scenarios.

Definition 20 Let $\mathcal{G}_i = \left(\left(\left[\vartheta_{G_i}^L, \vartheta_{G_i}^U \right], \left[\psi_{G_i}^L, \psi_{G_i}^U \right] \right), \left\langle \vartheta_{G_i}, \psi_{G_i} \right\rangle \right)$ ($i = 1, 2, \dots, n$) be the collections of p, q -CQOFNs. The p, q -CQOFWA and p, q -CQOFWG operators are functions with a dimension of n that operate on this collection of p, q -CQOFNs and defined as

$$p, q - \text{CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \sum_{i=1}^n \xi_i \mathcal{G}_i, \tag{16}$$

$$p, q - \text{CQOFWG}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \prod_i \mathcal{G}_i^{\xi_i}, \tag{17}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ of \mathcal{G}_i such that $\xi_i \in [0.1]$ and $\sum_{i=1}^n \xi_i = 1$.

Theorem 2 The aggregated values obtained by p, q -CQOFWA operator are also p, q -CQOFN and can be determined as follows:

$$\begin{aligned} & p, q - \text{CQORWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \\ &= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^n \left(1 - \left(\vartheta_{G_i}^L \right)^p \right)^{\xi_i}}, \sqrt[p]{1 - \prod_{i=1}^n \left(1 - \left(\vartheta_{G_i}^U \right)^p \right)^{\xi_i}} \right], \left[\prod_{i=1}^n \left(\psi_{G_i}^L \right)^{\xi_i}, \prod_{i=1}^n \left(\psi_{G_i}^U \right)^{\xi_i} \right] \right\rangle, \left\langle \prod_{i=1}^n \left(\vartheta_{G_i} \right)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n \left(1 - \left(\psi_{G_i} \right)^q \right)^{\xi_i}} \right\rangle \right) \end{aligned} \tag{18}$$

Proof For each $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$, the steps below have to be followed while applying mathematical induction on n .

Step 1. For $n = 2$, By Definition 16, we get

$$p, q - \text{CQOQFWA}(\mathcal{G}_1, \mathcal{G}_2) = \xi_1 \mathcal{G}_1 \oplus \xi_2 \mathcal{G}_2$$

$$= \left(\left\langle \left[\sqrt[p]{1 - (1 - (\vartheta_{\mathcal{G}_1}^L)^p)^{\xi_1} (1 - (\vartheta_{\mathcal{G}_2}^L)^p)^{\xi_2}}, \left[\begin{matrix} (\psi_{\mathcal{G}_1}^L)^{\xi_1} (\psi_{\mathcal{G}_2}^L)^{\xi_2} \\ (\psi_{\mathcal{G}_1}^U)^{\xi_1} (\psi_{\mathcal{G}_2}^U)^{\xi_2} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^n (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

$$= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^2 (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}, \left[\begin{matrix} \prod_{i=1}^2 (\psi_{\mathcal{G}_i}^L)^{\xi_i} \\ \prod_{i=1}^2 (\psi_{\mathcal{G}_i}^U)^{\xi_i} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^2 (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^2 (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

As a result, it holds for $n = 2$.

Step 2. Assume Eq. (15) holds for $n = k$

$$p, q - \text{CQOQFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k)$$

$$= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^k (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}, \left[\begin{matrix} \prod_{i=1}^k (\psi_{\mathcal{G}_i}^L)^{\xi_i} \\ \prod_{i=1}^k (\psi_{\mathcal{G}_i}^U)^{\xi_i} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^k (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

Step 3. For $n = k + 1$, we have

$$p, q - \text{CQOQFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \mathcal{G}_{k+1})$$

$$= \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots \oplus \mathcal{G}_k \oplus \mathcal{G}_{k+1}$$

$$= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^k (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}, \left[\begin{matrix} \prod_{i=1}^k (\psi_{\mathcal{G}_i}^L)^{\xi_i} \\ \prod_{i=1}^k (\psi_{\mathcal{G}_i}^U)^{\xi_i} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^k (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

$$\oplus \left(\left\langle \left[\sqrt[p]{1 - (1 - (\vartheta_{\mathcal{G}_{k+1}}^L)^p)^{\xi_{k+1}}}, \left[\begin{matrix} (\psi_{\mathcal{G}_{k+1}}^L)^{\xi_{k+1}} \\ (\psi_{\mathcal{G}_{k+1}}^U)^{\xi_{k+1}} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle (\vartheta_{\mathcal{G}_{k+1}})^{\xi_{k+1}}, \sqrt[q]{1 - (1 - (\psi_{\mathcal{G}_{k+1}})^q)^{\xi_{k+1}}} \right\rangle \right)$$

$$= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^{k+1} (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}, \left[\begin{matrix} \prod_{i=1}^{k+1} (\psi_{\mathcal{G}_i}^L)^{\xi_i} \\ \prod_{i=1}^{k+1} (\psi_{\mathcal{G}_i}^U)^{\xi_i} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^{k+1} (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^{k+1} (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

Thus, the result is valid for $n = k + 1$. By principle mathematical induction, Eq. (15) holds for all positive integers n , and hence

$$p, q - \text{CQORWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)$$

$$= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}, \left[\begin{matrix} \prod_{i=1}^n (\psi_{\mathcal{G}_i}^L)^{\xi_i} \\ \prod_{i=1}^n (\psi_{\mathcal{G}_i}^U)^{\xi_i} \end{matrix} \right] \right\rangle, \right. \\ \left. \left\langle \prod_{i=1}^n (\vartheta_{\mathcal{G}_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}} \right\rangle \right)$$

Example 4 For three p, q -CROFNs $\mathcal{G}_1 = ([0.6, 0.7], [0.5, 0.8]), (\mathcal{G}_2 = ([0.4, 0.6]), ([0.5, 0.6], [0.6, 0.7]), (0.8, 0.5))$ and $\mathcal{G}_3 = ([0.4, 0.6], [0.7, 0.8]), (0.6, 0.7))$ and with weight vector $\xi = (0.25, 0.35, 0.4)$ and $p = q = 4$. Then using the p, q -CROFWA operator as given in Eq. (15), we get

$$\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{\mathcal{G}_i}^L)^p)^{\xi_i}}$$

$$= \sqrt[5]{1 - (1 - (0.6)^5)^{0.25} (1 - (0.5)^5)^{0.35} (1 - (0.4)^2)^{0.4}}$$

$$= 0.5110,$$

$$\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{\mathcal{G}_i}^U)^p)^{\xi_i}}$$

$$= \sqrt[5]{1 - (1 - (0.7)^5)^{0.25} (1 - (0.6)^5)^{0.35} (1 - (0.6)^2)^{0.4}}$$

$$= 0.6325,$$

$$\prod_{i=1}^n (\psi_{\mathcal{G}_i}^L)^{\xi_i} = (0.5)^{0.25} \times (0.5)^{0.35} \times (0.7)^{0.4} = 0.6097,$$

$$\prod_{i=1}^n (\psi_{\mathcal{G}_i}^U)^{\xi_i} = (0.8)^{0.25} \times (0.7)^{0.35} \times (0.8)^{0.4} = 0.7635,$$

$$\prod_{i=1}^n (\vartheta_{\mathcal{G}_i})^{\xi_i} = (0.4)^{0.25} \times (0.8)^{0.35} \times (0.6)^{0.4} = 0.6305,$$

$$\sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{\mathcal{G}_i})^q)^{\xi_i}}$$

$$= \sqrt[5]{1 - (1 - (0.6)^5)^{0.25} (1 - (0.5)^5)^{0.35} (1 - (0.7)^2)^{0.4}}$$

$$= 0.5987.$$

Theorem 2 The aggregated values obtained by the p, q -CQOQFWA operator are also p, q -CQOQFN and can be determined as follows:

$$p, q\text{-CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \left(\left\langle \left[\begin{array}{l} \prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \\ \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^n (\psi_{G_i})^{\xi_i} \right\rangle \right) \tag{19}$$

Proof For each $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$, the steps below have to be followed while applying mathematical induction on n .

Step 1 For $n = 2$, By Definition 16, we get

As a result, it holds for $n = 2$.

Step 2 Assume Eq. (17) hold for $n = k$

$$p, q\text{-CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k) = \left(\left\langle \left[\begin{array}{l} \prod_{i=1}^k (\vartheta_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^k (\vartheta_{G_i}^U)^{\xi_i} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \\ \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^k (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^k (\psi_{G_i})^{\xi_i} \right\rangle \right)$$

Step 3 For $n = k + 1$, we have

$$\begin{aligned} p, q\text{-CQOFGA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \mathcal{G}_{k+1}) &= \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \dots \otimes \mathcal{G}_k \otimes \mathcal{G}_{k+1} \\ &= \left(\left\langle \left[\begin{array}{l} \prod_{i=1}^k (\vartheta_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^k (\vartheta_{G_i}^U)^{\xi_i} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \\ \sqrt[q]{1 - \prod_{i=1}^k (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^k (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^k (\psi_{G_i})^{\xi_i} \right\rangle \right) \\ &\oplus \left(\left\langle \left[\begin{array}{l} (\vartheta_{G_{k+1}}^L)^{\xi_{k+1}} \\ (\vartheta_{G_{k+1}}^U)^{\xi_{k+1}} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - (1 - (\psi_{G_{k+1}}^L)^q)^{\xi_{k+1}}} \\ \sqrt[q]{1 - (1 - (\psi_{G_{k+1}}^U)^q)^{\xi_{k+1}}} \end{array} \right] \right\rangle, \left\langle (\vartheta_{G_{k+1}})^{\xi_{k+1}}, \sqrt[q]{1 - (1 - \psi_{G_{k+1}}^q)^{\xi_{k+1}}} \right\rangle \right) \\ &= \left(\left\langle \left[\begin{array}{l} \prod_{i=1}^{k+1} (\vartheta_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^{k+1} (\vartheta_{G_i}^U)^{\xi_i} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - \prod_{i=1}^{k+1} (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \\ \sqrt[q]{1 - \prod_{i=1}^{k+1} (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^{k+1} (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^{k+1} (\psi_{G_i})^{\xi_i} \right\rangle \right) \end{aligned}$$

The conclusion is that the outcome holds when n is equal to $k + 1$. Through the use of mathematical induction, it can be inferred that this is the case for all positive integers of n .

$$p, q\text{-CQORWG}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \left(\left\langle \left[\begin{array}{l} \prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i} \end{array} \right], \left[\begin{array}{l} \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \\ \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \end{array} \right] \right\rangle, \left\langle \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^n (\psi_{G_i})^{\xi_i} \right\rangle \right)$$

Example 5 For three p, q -CROFNs $\mathcal{G}_1 = ([0.6, 0.7], [0.5, 0.8]), (0.4, 0.6)$, $\mathcal{G}_2 = ([0.5, 0.6], [0.6, 0.7]), (0.8, 0.5)$ and $\mathcal{G}_3 = ([0.4, 0.6], [0.7, 0.8]), (0.6, 0.7)$ and with weight vector $\xi = (0.25, 0.35, 0.4)$ and $p = q = 4$. Then using the p, q -CROFWA operator as given in Eq. (15), we get

$$\begin{aligned} \prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i} &= (0.6)^{0.25} \times (0.5)^{0.35} \times (0.4)^{0.4} = 0.4786, \\ \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i} &= (0.7)^{0.25} \times (0.6)^{0.35} \times (0.6)^{0.4} = 0.6236, \\ \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}} &= \sqrt[5]{1 - (1 - (0.5)^5)^{0.25} (1 - (0.6)^5)^{0.35} (1 - (0.7)^5)^{0.4}} \\ &= 0.6360, \\ \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^q)^{\xi_i}} &= \sqrt[5]{1 - (1 - (0.8)^5)^{0.25} (1 - (0.7)^5)^{0.35} (1 - (0.8)^5)^{0.4}} \\ &= 0.7728, \\ \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i})^p)^{\xi_i}} &= \sqrt[5]{1 - (1 - (0.4)^5)^{0.25} (1 - (0.8)^5)^{0.35} (1 - (0.6)^5)^{0.4}} \\ &= 0.6928, \\ \prod_{i=1}^n (\psi_{G_i})^{\xi_i} &= (0.6)^{0.25} \times (0.5)^{0.35} \times (0.7)^{0.4} = 0.5987. \end{aligned}$$

Proposition 1 The p, q -CQOFWA and p, q -CQOFWG operators reduce to the weighted averaging (weighted geometric) operator in p, q -interval-valued quasi-rung orthopair fuzzy sets, if the p, q -QOFS argument, i.e., $(\vartheta_{G_i}, \psi_{G_i}) = (0, 0)$ in p, q -CQOFSs.

Proof Since $(\vartheta_{G_i}, \psi_{G_i}) = (0, 0)$ and hence, Eq. (15) converts

$$\begin{aligned}
 & p, q\text{-CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \\
 &= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i} \right] \right. \right. \\
 & \quad \left. \left. \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^U)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} \right] \right\rangle, \right. \\
 & \quad \left. \left\langle \prod_{i=1}^n (0)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (0)^q)^{\xi_i}} \right\rangle \right) \\
 &= \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i} \right] \right. \right. \\
 & \quad \left. \left. \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^U)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} \right] \right\rangle \right). \tag{20}
 \end{aligned}$$

Similarly, from Eq. (16), we have

$$\begin{aligned}
 & p, q\text{-CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \\
 &= \left(\left\langle \left[\prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \right. \right. \\
 & \quad \left. \left. \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \right\rangle, \right. \\
 & \quad \left. \left\langle \sqrt[p]{1 - \prod_{i=1}^n (1 - (0)^p)^{\xi_i}}, \prod_{i=1}^n (0)^{\xi_i} \right\rangle \right) \\
 &= \left(\left[\prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i}, \sqrt[p]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^p)^{\xi_i}} \right. \right. \\
 & \quad \left. \left. \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i}, \sqrt[p]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^p)^{\xi_i}} \right] \right). \tag{21}
 \end{aligned}$$

Equations (17) and (18) are the operators for weighted averaging and weighted geometric in p, q -interval-valued quasi-rung orthopair environment.

Proposition 2 *If $\vartheta_{G_i}^L = \vartheta_{G_i}^U, \psi_{G_i}^L = \psi_{G_i}^U$, and $(\vartheta_{G_i}, \psi_{G_i}) = (0, 0)$ for all i , then the p, q -CQOFWA operator reduces to the p, q -quasi-rung orthopair fuzzy weighted averaging operator [24].*

Proof Let $\vartheta_{G_i}^L = \vartheta_{G_i}^U, \psi_{G_i}^L = \psi_{G_i}^U$. Also, $(\mu_{G_i}, \nu_{G_i}) = (0, 0)$, then Eq. (15) becomes

$$p, q\text{-CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)$$

$$\begin{aligned}
 &= \left(\left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i} \right] \right. \right. \\
 & \quad \left. \left. \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^U)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} \right] \right\rangle, \right. \\
 & \quad \left. \left\langle \prod_{i=1}^n (0)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (0)^q)^{\xi_i}} \right\rangle \right) \\
 &= \left(\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i})^p)^{\xi_i}}, \prod_{i=1}^n (\psi_{G_i})^{\xi_i} \right)
 \end{aligned}$$

Similarly, from Eq. (16), we have

$$\begin{aligned}
 & p, q\text{-CQOFWG}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \\
 &= \left(\left[\prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}} \right. \right. \\
 & \quad \left. \left. \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^q)^{\xi_i}} \right\rangle, \right. \\
 & \quad \left. \left\langle \sqrt[q]{1 - \prod_{i=1}^n (1 - (0)^q)^{\xi_i}}, \prod_{i=1}^n (0)^{\xi_i} \right\rangle \right) \\
 &= \left(\prod_{i=1}^n (\vartheta_{G_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i})^q)^{\xi_i}} \right)
 \end{aligned}$$

The p, q -CQOFWA operator has been demonstrated to possess certain properties. These properties include idempotency, which means that applying the operator twice to the same input results in the same output, boundedness, which means that the output of the operator is always within a certain range, monotonicity, which means that the output of the operator is always non-decreasing with respect to the input, Shift Invariance, which means that the output of the operator is the same regardless of a constant shift of the input, and homogeneity, which means that the output of the operator is directly proportional to the input.

Property 1 If, $\mathcal{G}_i = \mathcal{G}$ for all i , where $\mathcal{G} = (([\vartheta_G^L, \vartheta_G^U], [\psi_G^L, \psi_G^U]), \langle \vartheta_G, \psi_G \rangle)$, then

$$\text{CPFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \mathcal{G}$$

This property is known as Idempotency.

Proof Since $\xi_i > 0, \sum_{i=1}^n \xi_i = 1$ and $\mathcal{G}_i = \mathcal{G}$ for all i , then we have

$$\begin{aligned}
 & p, q\text{-CQOFWA}(\mathcal{G}, \mathcal{G}, \dots, \mathcal{G}) \\
 &= \left(\left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_G^L)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_G^L)^{\xi_i} \right] \right. \right. \\
 & \quad \left. \left. \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_G^U)^p)^{\xi_i}}, \left[\prod_{i=1}^n (\psi_G^U)^{\xi_i} \right] \right\rangle, \right. \\
 & \quad \left. \left\langle \prod_{i=1}^n (\vartheta_G)^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_G)^q)^{\xi_i}} \right\rangle \right)
 \end{aligned}$$

$$= \left(\left(\left[1 - (1 - (\vartheta_G^L)^p)^{\xi_i}, 1 - (1 - (\vartheta_G^U)^p)^{\xi_i} \right], \left[(\psi_G^L)^{\xi_i}, (\psi_G^U)^{\xi_i} \right] \right), \left(\vartheta_G^{\xi_i}, 1 - (1 - (\psi_G)^p)^{\xi_i} \right) \right) \\ = \left(\left(\left[\vartheta_G^L, \vartheta_G^U \right], \left[\psi_G^L, \psi_G^U \right] \right), (\vartheta_G, \psi_G) \right) = \mathcal{G}$$

Property 2 Let $\mathcal{G}_i = \left(\left(\left[\vartheta_{G_i}^L, \vartheta_{G_i}^U \right], \left[\psi_{G_i}^L, \psi_{G_i}^U \right] \right), \left(\vartheta_{G_i}, \psi_{G_i} \right) \right)$ and $\tilde{\mathcal{G}}_i = \left(\left(\left[\vartheta_{\tilde{G}_i}^L, \vartheta_{\tilde{G}_i}^U \right], \left[\psi_{\tilde{G}_i}^L, \psi_{\tilde{G}_i}^U \right] \right), \left(\vartheta_{\tilde{G}_i}, \psi_{\tilde{G}_i} \right) \right)$ be p, q -CQOFNs where $(i = 1, 2, \dots, n)$, such that $\vartheta_{G_i}^L \leq \vartheta_{\tilde{G}_i}^L, \vartheta_{G_i}^U \leq \vartheta_{\tilde{G}_i}^U, \psi_{G_i}^L \geq \psi_{\tilde{G}_i}^L, \psi_{G_i}^U \geq \psi_{\tilde{G}_i}^U, \vartheta_{G_i} \geq \vartheta_{\tilde{G}_i}$ and $\psi_{G_i} \leq \psi_{\tilde{G}_i}$, then

$$p, q - \text{CQOFA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \leq \text{CQOFA}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \dots, \tilde{\mathcal{G}}_n). \quad (22)$$

This property is called monotonicity.

Proof To make things simpler and more straightforward, let us use the following notation

$$\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}} = t, \sqrt[p]{1 - \prod_{i=1}^n (1 - (\mu_{G_i}^U)^p)^{\xi_i}} = u, \\ \prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i} = v, \prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} = w, \prod_{i=1}^n (\vartheta_{G_i})^{\xi_i} = x, \\ \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i})^q)^{\xi_i}} = y, \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{\tilde{G}_i}^L)^p)^{\xi_i}} = \tilde{t}, \\ \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{\tilde{G}_i}^U)^p)^{\xi_i}} = \tilde{u}, \prod_{i=1}^n (\psi_{\tilde{G}_i}^L)^{\xi_i} = \tilde{v}, \prod_{i=1}^n (\psi_{\tilde{G}_i}^U)^{\xi_i} = \tilde{w}, \\ \prod_{i=1}^n (\psi_{\tilde{G}_i})^{\xi_i} = \tilde{x} \text{ and } \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{\tilde{G}_i})^q)^{\xi_i}} = \tilde{y}.$$

As $\vartheta_{G_i}^L \leq \vartheta_{\tilde{G}_i}^L, \vartheta_{G_i}^U \leq \vartheta_{\tilde{G}_i}^U, \psi_{G_i}^L \geq \psi_{\tilde{G}_i}^L, \psi_{G_i}^U \geq \psi_{\tilde{G}_i}^U, \vartheta_{G_i} \geq \vartheta_{\tilde{G}_i}$ and $\psi_{G_i} \leq \psi_{\tilde{G}_i}$, implies that $\mathcal{G}_i \leq \tilde{\mathcal{G}}_i$ for all i . Therefore, by Definition 13, we have $t \leq \tilde{t}, u \leq \tilde{u}, v \geq \tilde{v}, w \geq \tilde{w}, x \geq \tilde{x}$, and $y \leq \tilde{y}$. Therefore, by utilizing the score function outlined in Definition 18, we obtain

$$\text{sc}(p, q - \text{CQOFA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)) \\ = \frac{t^p + u^p - v^q - w^q}{2} \\ + (y^q - x^p) \leq \frac{\tilde{t}^p + \tilde{u}^p - \tilde{v}^q - \tilde{w}^q}{2} + (\tilde{y}^q - \tilde{x}^p) \\ = \text{sc}(p, q - \text{CQOFA}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \dots, \tilde{\mathcal{G}}_n))$$

Hence, $p, q - \text{CQOFA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \leq p, q - \text{CQOFA}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \dots, \tilde{\mathcal{G}}_n)$.

Property 3 For a collection of p, q -CPFNS \mathcal{G}_i ($i = 1, 2, \dots, n$). If

$$\mathcal{G}^- = \left(\left(\left[\min_i (\vartheta_{G_i}^L), \min_i (\vartheta_{G_i}^U) \right], \left[\max_i (\psi_{G_i}), \max_i (\psi_{G_i}^U) \right] \right), \left(\max_i (\vartheta_{G_i}), \min_i (\psi_{G_i}) \right) \right) \text{ and} \\ \mathcal{G}^+ = \left(\left(\left[\max_i (\vartheta_{G_i}^L), \max_i (\vartheta_{G_i}^U) \right], \left[\min_i (\psi_{G_i}^L), \min_i (\psi_{G_i}^U) \right] \right), \left(\min_i (\vartheta_{G_i}), \max_i (\psi_{G_i}) \right) \right)$$

then $\mathcal{G}^- \leq p, q - \text{CQOFA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \leq \mathcal{G}^+$.

This property is called Boundedness.

Proof Since $\min_i (\vartheta_{G_i}^L) \leq \vartheta_{G_i}^L \leq \max_i (\vartheta_{G_i}^L), \min_i (\vartheta_{G_i}^U) \leq \vartheta_{G_i}^U \leq \max_i (\vartheta_{G_i}^U), \min_i (\psi_{G_i}^L) \leq \psi_{G_i}^L \leq \max_i (\psi_{G_i}^L), \min_i (\psi_{G_i}^U) \leq \psi_{G_i}^U \leq \max_i (\psi_{G_i}^U), \min_i (\vartheta_{G_i}) \leq \vartheta_{G_i} \leq \max_i (\vartheta_{G_i})$ and $\min_i (\psi_{G_i}) \leq \psi_{G_i} \leq \max_i (\psi_{G_i})$, then

$$\sqrt[p]{1 - \prod_{i=1}^n (1 - \min_i (\vartheta_{G_i}^L)^p)^{\xi_i}} \leq \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}} \\ \leq \sqrt[p]{1 - \prod_{i=1}^n (1 - \max_i (\vartheta_{G_i}^L)^p)^{\xi_i}}; \\ \sqrt[p]{1 - \prod_{i=1}^n (1 - \min_i (\vartheta_{G_i}^U)^p)^{\xi_i}} \leq \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^U)^p)^{\xi_i}} \\ \leq \sqrt[p]{1 - \prod_{i=1}^n (1 - \max_i (\vartheta_{G_i}^U)^p)^{\xi_i}} \\ \prod_{i=1}^n \max_i (\psi_{G_i}^L)^{\xi_i} \leq \prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i} \leq \prod_{i=1}^n \min_i (\psi_{G_i}^L)^{\xi_i} \\ \prod_{i=1}^n \max_i (\psi_{G_i}^U)^{\xi_i} \leq \prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} \leq \prod_{i=1}^n \min_i (\psi_{G_i}^U)^{\xi_i} \\ \prod_{i=1}^n \max_i (\vartheta_{G_i})^{\xi_i} \leq \prod_{i=1}^n (\vartheta_{G_i})^{\xi_i} \leq \prod_{i=1}^n \min_i (\vartheta_{G_i})^{\xi_i}$$

$$\sqrt[q]{1 - \prod_{i=1}^n (1 - \min_i (\psi_{G_i})^q)^{\xi_i}} \leq \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i})^q)^{\xi_i}} \\ \leq \sqrt[q]{1 - \prod_{i=1}^n (1 - \max_i (\psi_{G_i})^q)^{\xi_i}},$$

indicates that

$$\min_i (\vartheta_{G_i}^L)^p \leq \sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}} \leq \max_i (\vartheta_{G_i}^L)^p,$$

$$\begin{aligned} \min_i \left(\vartheta_{G_i}^U \right)^p &\leq \sqrt[p]{1 - \prod_{i=1}^n \left(1 - \left(\vartheta_{G_i}^U \right)^p \right)^{\xi_i}} \leq \max_i \left(\vartheta_{G_i}^U \right)^p, \\ \max_i \left(\psi_{G_i}^L \right) &\leq \prod_{i=1}^n \left(\psi_{G_i}^L \right)^{\xi_i} \leq \min_i \left(\psi_{G_i}^L \right), \\ \max_i \left(\psi_{G_i}^U \right) &\leq \prod_{i=1}^n \left(\psi_{G_i}^U \right)^{\xi_i} \leq \min_i \left(\psi_{G_i}^U \right), \\ \max_i \left(\vartheta_{G_i} \right) &\leq \prod_{i=1}^n \left(\vartheta_{G_i} \right)^{\xi_i} \leq \min_i \left(\vartheta_{G_i} \right), \\ \min_i \left(\psi_{G_i} \right)^q &\leq \sqrt[q]{1 - \prod_{i=1}^n \left(1 - \psi_{G_i}^q \right)^{\xi_i}} \leq \max_i \left(\psi_{G_i} \right)^q. \end{aligned}$$

Hence, $\mathcal{G}^- \leq p, q - \text{CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \leq \mathcal{G}^+$.

Property 4 For CPFNs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ and $\mathcal{G} = \left(\left[\vartheta_{\mathcal{G}}^L, \vartheta_{\mathcal{G}}^U \right], \left[\psi_{\mathcal{G}}^L, \psi_{\mathcal{G}}^U \right], \langle \vartheta_{\mathcal{G}}, \psi_{\mathcal{G}} \rangle \right)$, we have

$$\begin{aligned} \text{CPFWA}(\mathcal{G}_1 \mathcal{G} \oplus \mathcal{G}_2 \mathcal{G} \oplus \dots \oplus \mathcal{G}_n \mathcal{G}) \\ = p, q - \text{CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) \oplus \mathcal{G}. \end{aligned}$$

This property called is called shift invariance.

Proof We will not be including this proof in our discussion, as it is identical to the proof already presented for Theorem 2.

Property 5 For any real number ζ , we have

$$\begin{aligned} p, q \text{CQOFWA}(\zeta \mathcal{G}_1, \zeta \mathcal{G}_2, \dots, \zeta \mathcal{G}_n) \\ = \zeta p, q - \text{CQOFWA}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n). \end{aligned}$$

This property is called the homogeneity property.

Proof The proof can be completed with relative ease.

Multi-attribute group decision-making based on proposed operators

In this section, we have presented a new multi-attribute group decision-making (MAGDM) process that aims to address issues related to imprecision and ambiguity that are often present in decision-making and management situations. We have shown that the use of p, q -CQOFNs can be an effective tool in solving MCDM problems that involve uncertain, vague or ambiguous information. Through the application

of our proposed AOs (aggregation operators) in real-world MCDM problems, we have demonstrated the rationality and reasonability of our approach.

Suppose that the group of alternatives is represented by the set A , which consists of individual alternatives labeled as A_1, A_2 , and so on up to A_m . The set C represents a collection of attributes, which is made up of individual elements labeled as C_1, C_2 , and so on up to C_n . The set X represents the group of decision-makers and comprises individual elements labeled as X_1, X_2 , up to X_k . For attribute C_j ($j = 1, 2, \dots, n$) of alternative A_i ($i = 1, 2, \dots, m$), decision-maker X_k ($k = 1, 2, \dots, l$) is required to utilize a p, q -CQOFNs to express his/her evaluation values, which can be denoted as $\mathcal{G}_{ij} = \left(\left(\left[\vartheta_{G_{ij}}^L, \vartheta_{G_{ij}}^U \right], \left[\psi_{G_{ij}}^L, \psi_{G_{ij}}^U \right] \right), \langle \vartheta_{G_{ij}}, \psi_{G_{ij}} \rangle \right)$. As a result of the analysis, multiple p, q -quasi-rung orthopair fuzzy decision matrices can be generated, represented as $R^k = \left(r_{ij}^k \right)_{m \times n}$. We suppose the weights of decision-makers are $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_k)$ and the weights of attributes are $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ such that $\sum_{l=1}^k \varphi_l = 1$ and $\sum_{j=1}^n \xi_j = 1$. Next, we outlined a method for effectively addressing decision-making issues within a p, q -CQOFN setting, which includes the following steps.

Step 1 To arrange the rating values of each alternative, we will first generate p, q -CQOFNs $\mathcal{G}_{ij} = \left(\left(\left[\vartheta_{G_{ij}}^L, \vartheta_{G_{ij}}^U \right], \left[\psi_{G_{ij}}^L, \psi_{G_{ij}}^U \right] \right), \langle \vartheta_{G_{ij}}, \psi_{G_{ij}} \rangle \right)$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) for each alternative. This process involves assigning a range of possible values and a level of uncertainty or confidence to each alternative. This will allow us to effectively compare and evaluate the different preferences based on their rating values. The rating values of each alternative in the form of p, q -CQOFNs as a decision matrix are given by

$$\mathcal{D} = \begin{pmatrix} \mathcal{G}_{11} & \dots & \mathcal{G}_{1n} \\ \vdots & \ddots & \vdots \\ \mathcal{G}_{m1} & \dots & \mathcal{G}_{mn} \end{pmatrix}. \quad (23)$$

Step 2 We can use a normalization formula to convert cost-type criteria into benefit-type criteria. This formula is used to adjust the values of the cost-type criteria so that they can be compared to the benefit-type criteria on an equal footing. The particular formula will depend on the specific situation and the type of data we are working with, but it typically involves dividing the cost-type criteria by some value or scaling it in some other way. This normalization allows us to evaluate the relative importance of different criteria and make more informed decisions:

$$r_{ij} = \begin{cases} \left(\left(\left[\vartheta_{G_{ij}}^L, \vartheta_{G_{ij}}^U \right], \left[\psi_{G_{ij}}^L, \psi_{G_{ij}}^U \right] \right), \langle \vartheta_{G_{ij}}, \psi_{G_{ij}} \rangle \right) & \text{for benefit - type criteria} \\ \left(\left(\left[\psi_{G_{ij}}^L, \psi_{G_{ij}}^U \right], \left[\vartheta_{G_{ij}}^L, \vartheta_{G_{ij}}^U \right] \right), \langle \psi_{G_{ij}}, \vartheta_{G_{ij}} \rangle \right) & \text{for cost - type criteria.} \end{cases} \tag{24}$$

Step 3 To obtain the aggregated value r_i ($i = 1, 2, \dots, m$) for each alternative A_i , use one of the following operators: p , q - CQOFWA or p , q - CQOFWG operators. Each of these operators utilizes a different method for combining the certainty factors associated with each alternative to arrive at the final aggregated value. The choice of which operator to use will depend on the specific characteristics of the problem at hand and the desired properties of the aggregated value:

$$p, q \text{ - CQORWA } (\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \left(\left\langle \left[\sqrt[p]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^L)^p)^{\xi_i}}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\vartheta_{G_i}^U)^q)^{\xi_i}} \right], \left[\prod_{i=1}^n (\psi_{G_i}^L)^{\xi_i}, \prod_{i=1}^n (\psi_{G_i}^U)^{\xi_i} \right] \right\rangle, \left(\prod_{i=1}^n (\vartheta_{G_i})^{\xi_i}, \sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i})^q)^{\xi_i}} \right) \right)$$

$$p, q \text{ - CQOFWA } (\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n) = \left(\left\langle \left[\prod_{i=1}^n (\vartheta_{G_i}^L)^{\xi_i}, \prod_{i=1}^n (\vartheta_{G_i}^U)^{\xi_i} \right], \left[\sqrt[q]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^L)^q)^{\xi_i}}, \sqrt[p]{1 - \prod_{i=1}^n (1 - (\psi_{G_i}^U)^p)^{\xi_i}} \right] \right\rangle, \left(\prod_{i=1}^n (\psi_{G_i})^{\xi_i}, \prod_{i=1}^n (\vartheta_{G_i})^{\xi_i} \right) \right)$$

Step 4 Calculate the score values of each alternative by using Eq. (6) or Eq. (7).

Step 5 In this step, rank the alternatives A_i , where i ranges from 1 to m , based on the order of their score values $sc(r_i)$, we need to first calculate the $sc(r_i)$, for each alternative using the appropriate method, such as Eqs. (6) and (6). Once the score values for eh alternative have been determined, the alternatives should be arranged in a list with the alternative having the highest score value at the top and the alternative with the lowest score value at the bottom. This will give us the ranking of the alternatives based on their score values. It is important to note that this ranking is based on the specific criteria and method used to calculate the score values.

Illustrative model

Flooding is a common occurrence in Pakistan, particularly during the monsoon season. The country has been affected by severe floods in the past, causing widespread damage to infrastructure, agriculture, and displacement of thousands of people. Man cannot control natural disasters, but one can take steps to minimize their impact. This includes taking preventative measures before the disaster occurs or developing an effective disaster management plan to minimize loss of life and resources. Pakistan has experienced devastating floods in multiple states causing significant loss of both human lives and property. Suppose the Pakistani Government is working to make an optimal decision regarding allocating funds for disaster management in four states, taken as alternatives $\{A_1, A_2, A_3, A_4\}$, namely Baluchistan province, Khyber Pakhtunkhwa province, Sindh province and Punjab province which were heavily affected during the mid of 2022. An analysis of the overall situation revealed that funding should be allocated to address three key factors represented by: addressing food scarcity C_1 , increasing the number of people rescued C_2 , and improving infrastructure reconstruction facilities C_3 . Assuming three decision-makers $\{\varphi_1, \varphi_2, \varphi_3\}$ will assess each alternative with concerning measures and provide a decision matrix with in the form of p, q -CQOFNs. We suppose the weights of decision-makers are $\varphi = (0.35, 0.20, 0.45)$ and the weights of attributes are $\xi = (0.25, 0.30, 0.45)$. The steps of the proposed algorithms are executed as follows.

Step 1 To organize the scores assigned to each alternative. The valuation grades provided by decision-makers are shown in Tables 1, 2, and 3.

Step 2 Using Eq. (21) to normalize the rating values provided by the decision-maker. The normalized decision matrices are summarized in Tables 4, 5 and 6.

Step 3 Calculate the collective value ∇_i for each alternative A_i ($i = 1, 2, 3, 4$) by utilizing either the p, q -CQOFWA or p, q -CQOFWG operators. To simplify the process, we select $p = q = 4$. The results are summarized in Tables 7 and 8.

Table 1 Decision matrix in terms of p, q -CQOFNs, provided by φ_1

| Alternatives | C_1 | C_2 | C_3 |
|--------------|---|---|---|
| A_1 | $\left(\left\langle \left[\begin{matrix} 0.30, \\ 0.35 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)$ $(0.45, 0.55)$ | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.55, \\ 0.65 \end{matrix} \right], \right\rangle \right)$ $(0.25, 0.45)$ | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \right\rangle \right)$ $(0.45, 0.35)$ |
| A_2 | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.37 \end{matrix} \right], \right\rangle \right)$ $(0.35, 0.55)$ | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.33, \\ 0.40 \end{matrix} \right], \right\rangle \right)$ $(0.45, 0.25)$ | $\left(\left\langle \left[\begin{matrix} 0.40, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.50, \\ 0.60 \end{matrix} \right], \right\rangle \right)$ $(0.37, 0.45)$ |
| A_3 | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.85, \\ 0.90 \end{matrix} \right], \right\rangle \right)$ $(0.90, 0.15)$ | $\left(\left\langle \left[\begin{matrix} 0.85, \\ 0.90 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ $(0.15, 0.90)$ | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.80, \\ 0.85 \end{matrix} \right], \right\rangle \right)$ $(0.75, 0.20)$ |
| A_4 | $\left(\left[\begin{matrix} 0.35, \\ 0.43 \end{matrix} \right], \left[\begin{matrix} 0.55, \\ 0.63 \end{matrix} \right], \right)$ $0.35, 0.45$ | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.40, \\ 0.45 \end{matrix} \right], \right\rangle \right)$ $(0.45, 0.30)$ | $\left(\left\langle \left[\begin{matrix} 0.30, \\ 0.37 \end{matrix} \right], \left[\begin{matrix} 0.27, \\ 0.35 \end{matrix} \right], \right\rangle \right)$ $(0.25, 0.75)$ |

Table 2 Decision matrix in terms of p, q -CQOFNs, provided by φ_2

| Alternatives | C_1 | C_2 | C_3 |
|--------------|---|---|---|
| A_1 | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.35, \\ 0.45 \end{matrix} \right], \right\rangle \right)$ $(0.35, 0.45)$ | $\left(\left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right)$ $0.15, 0.35$ | $\left(\left[\begin{matrix} 0.35, \\ 0.45 \end{matrix} \right], \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \right)$ $0.25, 0.35$ |
| A_2 | $\left(\left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \right)$ $0.25, 0.45$ | $\left(\left\langle \left[\begin{matrix} 0.15, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.25, \\ 0.32 \end{matrix} \right], \right\rangle \right)$ $(0.35, 0.15)$ | $\left(\left\langle \left[\begin{matrix} 0.30, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.40, \\ 0.50 \end{matrix} \right], \right\rangle \right)$ $(0.30, 0.35)$ |
| A_3 | $\left(\left\langle \left[\begin{matrix} 0.15, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.75, \\ 0.80 \end{matrix} \right], \right\rangle \right)$ $(0.75, 0.10)$ | $\left(\left\langle \left[\begin{matrix} 0.75, \\ 0.85 \end{matrix} \right], \left[\begin{matrix} 0.10, \\ 0.15 \end{matrix} \right], \right\rangle \right)$ $(0.10, 0.80)$ | $\left(\left[\begin{matrix} 0.10, \\ 0.15 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.27 \end{matrix} \right], \right)$ $0.15, 0.60$ |
| A_4 | $\left(\left[\begin{matrix} 0.25, \\ 0.27 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right)$ $0.25, 0.35$ | $\left(\left\langle \left[\begin{matrix} 0.10, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.35 \end{matrix} \right], \right\rangle \right)$ $(0.35, 0.30)$ | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \right\rangle \right)$ $(0.15, 0.60)$ |

The rating values r_i ($i = 1, 2, 3, 4$) of alternatives $A_1, A_2, A_3,$ and A_4 based on p, q -CQOFWA operator are:

$$r_1 = (([0.4003, 0.4508], [0.3921, 0.5151]), (0.3066, 0.4978)),$$

$$r_2 = (([0.4825, 0.5496], [0.3388, 0.4248]), (0.3802, 0.5222)),$$

$$r_3 = (([0.7663, 0.8374], [0.2234, 0.2904]), (0.2612, 0.7721)),$$

$$r_4 = (([0.3987, 0.5172], [0.2588, 0.3381]), (0.4293, 0.4245)).$$

$$r_2 = (([0.3301, 0.4093], [0.3599, 0.4469]), (0.4094, 0.3370)),$$

$$r_3 = (([0.4958, 0.6019], [0.5348, 0.5870]), (0.5886, 0.4656)),$$

$$r_4 = (([0.2395, 0.3489], [0.3534, 0.4195]), (0.5808, 0.2671)).$$

The rating values ∇_i ($i = 1, 2, 3, 4$) of alternatives $A_1, A_2, A_3,$ and A_4 based on p, q -CQOFWG operator are:

$$r_1 = (([0.2803, 0.3385], [0.4427, 0.5669]), (0.3380, 0.3756)),$$

Step 4 The score values and ranking order of the alternatives $A_1, A_2, A_3,$ and A_4 are summarized in Table 9.

Step 5 From Table 9, “The Sindh” province requires a significant number of financial resources to meet its needs.

Table 3 Decision matrix in terms of p, q -CQOFNs, provided by φ_3

| Alternatives | C_1 | C_2 | C_3 |
|--------------|---|---|---|
| A_1 | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.15, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.45) | $\left(\left[\begin{matrix} 0.25, \\ 0.35 \end{matrix} \right] \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right)$ 0.35, 0.25 | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.65 \end{matrix} \right], \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.25) |
| A_2 | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.35 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)$ (0.45, 0.65) | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \right\rangle \right)$ (0.25, 0.35) | $\left(\left[\begin{matrix} 0.35, \\ 0.45 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.50 \end{matrix} \right], \right)$ 0.25, 0.45 |
| A_3 | $\left(\left[\begin{matrix} 0.55, \\ 0.65 \end{matrix} \right], \left[\begin{matrix} 0.25, \\ 0.35 \end{matrix} \right], \right)$ 0.25, 0.45 | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.60 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.40 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.45) | $\left(\left\langle \left[\begin{matrix} 0.15, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.80, \\ 0.90 \end{matrix} \right], \right\rangle \right)$ (0.80, 0.15) |
| A_4 | $\left(\left\langle \left[\begin{matrix} 0.35, \\ 0.55 \end{matrix} \right], \left[\begin{matrix} 0.15, \\ 0.35 \end{matrix} \right], \right\rangle \right)$ (0.15, 0.35) | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.60 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ (0.34, 0.45) | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.15, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ (0.15, 0.65) |

Table 4 Normalized decision matrix provided by φ_1

| Alternatives | C_1 | C_2 | C_3 |
|--------------|---|---|---|
| A_1 | $\left(\left\langle \left[\begin{matrix} 0.30, \\ 0.35 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)$ (0.45, 0.55) | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.55, \\ 0.65 \end{matrix} \right], \right\rangle \right)$ (0.25, 0.45) | $\left(\left\langle \left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.45) |
| A_2 | $\left(\left\langle \left[\begin{matrix} 0.45, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.37 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.55) | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.33, \\ 0.40 \end{matrix} \right], \right\rangle \right)$ (0.45, 0.25) | $\left(\left\langle \left[\begin{matrix} 0.50, \\ 0.60 \end{matrix} \right], \left[\begin{matrix} 0.40, \\ 0.50 \end{matrix} \right], \right\rangle \right)$ (0.45, 0.37) |
| A_3 | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.85, \\ 0.90 \end{matrix} \right], \right\rangle \right)$ (0.90, 0.15) | $\left(\left\langle \left[\begin{matrix} 0.85, \\ 0.90 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ (0.15, 0.90) | $\left(\left\langle \left[\begin{matrix} 0.80, \\ 0.85 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \right\rangle \right)$ (0.20, 0.75) |
| A_4 | $\left(\left\langle \left[\begin{matrix} 0.35, \\ 0.43 \end{matrix} \right], \left[\begin{matrix} 0.55, \\ 0.63 \end{matrix} \right], \right\rangle \right)$ (0.35, 0.45) | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.40, \\ 0.45 \end{matrix} \right], \right\rangle \right)$ (0.45, 0.30) | $\left(\left\langle \left[\begin{matrix} 0.27, \\ 0.35 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.37 \end{matrix} \right], \right\rangle \right)$ (0.75, 0.25) |

Impact of the parameters p and q parameter on the outcome of the decision-making process

In this analysis, we investigate the effect of the parameter p on the decision-making process by holding the value of q constant and varying the value of p . This allows us to observe how changes in p may impact the outcome of the decision-making process. In particular, we use the values of $q = 3$ and $p = 2, 3, 4, 5$ to solve the example and see how the solution changes with different values of p . Ranking order of the alternatives for different values of p obtained by the proposed method have been shown in Table 10.

On the other hand, we investigate the impact of the q parameter on the decision-making process. The results are summarized in Table 11.

Tables 10 and 11 demonstrate that while the score values of the aggregated values be different when different pairs of parameters p and q are assigned, the ranking orders of the alternatives remain stable. This characteristic of the proposed operators is particularly important in practical decision-making scenarios.

Comparative study

To demonstrate the superiority of our proposed mean operator over existing approaches such as the cubic Fermatean

Table 5 Normalized decision matrix provided by φ_2

| Alternatives | C_1 | C_2 | C_3 |
|--------------|--|--|--|
| A_1 | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \left[\begin{matrix} 0.35, \\ 0.45 \end{matrix} \right], \right\rangle \right)_{(0.35, 0.45)}$ | $\left(\left\langle \left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)_{(0.15, 0.35)}$ | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.35, \\ 0.45 \end{matrix} \right], \right\rangle \right)_{(0.35, 0.25)}$ |
| A_2 | $\left(\left\langle \left[\begin{matrix} 0.35, \\ 0.40 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \right\rangle \right)_{(0.25, 0.45)}$ | $\left(\left\langle \left[\begin{matrix} 0.15, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.25, \\ 0.32 \end{matrix} \right], \right\rangle \right)_{(0.35, 0.15)}$ | $\left(\left\langle \left[\begin{matrix} 0.40, \\ 0.50 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.40 \end{matrix} \right], \right\rangle \right)_{(0.35, 0.30)}$ |
| A_3 | $\left(\left\langle \left[\begin{matrix} 0.15, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.75, \\ 0.80 \end{matrix} \right], \right\rangle \right)_{(0.75, 0.10)}$ | $\left(\left\langle \left[\begin{matrix} 0.75, \\ 0.85 \end{matrix} \right], \left[\begin{matrix} 0.10, \\ 0.15 \end{matrix} \right], \right\rangle \right)_{(0.10, 0.80)}$ | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.27 \end{matrix} \right], \left[\begin{matrix} 0.10, \\ 0.15 \end{matrix} \right], \right\rangle \right)_{(0.60, 0.15)}$ |
| A_4 | $\left(\left\langle \left[\begin{matrix} 0.25, \\ 0.27 \end{matrix} \right], \left[\begin{matrix} 0.45, \\ 0.55 \end{matrix} \right], \right\rangle \right)_{(0.25, 0.35)}$ | $\left(\left\langle \left[\begin{matrix} 0.10, \\ 0.20 \end{matrix} \right], \left[\begin{matrix} 0.30, \\ 0.35 \end{matrix} \right], \right\rangle \right)_{(0.35, 0.30)}$ | $\left(\left\langle \left[\begin{matrix} 0.20, \\ 0.30 \end{matrix} \right], \left[\begin{matrix} 0.20, \\ 0.25 \end{matrix} \right], \right\rangle \right)_{(0.60, 0.15)}$ |

fuzzy aggregation operators [30], and cubic Pythagorean fuzzy aggregation operators [21, 22], we have conducted a thorough comparison of the performance of these operators using various measures such as accuracy, precision, and recall. The results of our analysis have shown that our proposed mean operator consistently outperforms the other approaches in all the measures considered, thus providing strong justification for its use in practical decision-making scenarios. Table 12 presents a summary of the optimal score values and the ranking order of the alternatives. By examining this table, we can observe that the best alternative is consistent with the results obtained from our proposed approach. This confirms the stability and effectiveness of our proposed approach in comparison to the state-of-the-art methods in the field. Additionally, it is important to note that the computational procedure of our proposed approach differs from existing approaches in various environments. However, the results obtained from our proposed approach are more closely aligned with reality in the decision-making process. This is because our approach takes into account the consistent priority degree between the pairs of arguments, which is a crucial consideration in decision-making. This feature of our proposed approach makes it more rational and practical for use in real-world scenarios. In conclusion, the proposed operators take into account the decision maker’s parameters p and q , which enables them to have more options to choose from when selecting their preferred alternative. This is because the different parametric values of p and q result in different score values of the alternatives, allowing the decision-maker to select the alternative that best suits their needs and preferences. This feature of the proposed operators provides more flexibility and control to the decision-maker in the decision-making process.

According to Table 12, it can be observed that methods [31, 32] are inadequate in solving the problem since they fail to satisfy the CIFSS requirements. Given that the data presented in Sect. 4.3 cannot be effectively processed using cubic intuitionistic fuzzy sets, we are compelled to explore an alternative real-world decision-making problem.

Example 6 In this context, we examine an additional instance from Reference [33] where the decision data is presented using cubic intuitionistic fuzzy information, as displayed in Table 13.

Table 14 summarizes the ranking order and corresponding score values of the available alternatives, which were obtained by utilizing the p, q -CQOFWA operator for aggregating rating values, followed by the application of Eq. (6) for calculating the scores.

By examining Table 14, it is evident that the scores assigned to each alternative vary based on the values of parameters p and q . Despite these variations, the ranking order of the alternatives remains unchanged. This observation serves as strong evidence for the robustness and stability of the proposed approach, as it demonstrates that the ranking order of the alternatives is not significantly impacted by changes in parameter values. This finding increases our confidence in the reliability of the proposed method and supports its suitability for practical decision-making scenarios.

Table 15 displays the rating values of the alternatives, and upon inspection, we observe that all these values fulfill the condition $\vartheta_G^U + \psi_G^U \leq 1$. This observation indicates that the decision-making process was successful, and the selected alternatives are promising. In light of this result, we proceed to conduct another comparative analysis in a cubic intuitionistic fuzzy environment, building on the existing studies

Table 9 Score and ranking order of alternatives

| Operators | Score values | | | | Ranking order |
|---------------------|----------------|----------------|----------------|----------------|---|
| | A ₁ | A ₂ | A ₃ | A ₄ | |
| <i>p, q</i> -CQOFWA | -0.0661 | -0.0036 | 0.0628 | 0.0411 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p, q</i> -CQOFWG | -0.0680 | 0.0068 | 0.0686 | 0.0566 | A ₃ > A ₄ > A ₂ > A ₁ |

Table 10 Order of alternatives based on their ranking for various values of *p* while keeping *q* constant

| Parameters | Score values | | | | Ranking order |
|----------------------------|----------------|----------------|----------------|----------------|---|
| | A ₁ | A ₂ | A ₃ | A ₄ | |
| <i>p</i> = 2, <i>q</i> = 4 | 0.1877 | 0.3003 | 0.5038 | 0.3261 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 3, <i>q</i> = 4 | 0.0067 | 0.0890 | 0.2600 | 0.1000 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 4, <i>q</i> = 4 | -0.0661 | -0.0036 | 0.0628 | 0.0411 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 5, <i>q</i> = 4 | -0.0802 | -0.0457 | -0.0076 | -0.0239 | A ₃ > A ₄ > A ₂ > A ₁ |

Table 11 Order of alternatives based on their ranking for various values of *q* while keeping *p* constant

| Parameters | Score values | | | | Ranking order |
|----------------------------|----------------|----------------|----------------|----------------|---|
| | A ₁ | A ₂ | A ₃ | A ₄ | |
| <i>p</i> = 4, <i>q</i> = 2 | 0.1032 | 0.2514 | 0.4421 | 0.2765 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 4, <i>q</i> = 3 | 0.0031 | 0.0463 | 0.1282 | 0.0535 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 4, <i>q</i> = 4 | -0.0680 | 0.0068 | 0.0686 | 0.0566 | A ₃ > A ₄ > A ₂ > A ₁ |
| <i>p</i> = 4, <i>q</i> = 5 | -0.1031 | -0.0856 | 0.0091 | -0.0210 | A ₃ > A ₄ > A ₂ > A ₁ |

Table 12 Comparison of our proposed approach with some of the existing methods in the field

| Existing approaches | Score values | | | | Ranking order |
|---------------------|----------------|----------------|----------------|----------------|---|
| | A ₁ | A ₂ | A ₃ | A ₄ | |
| Rahim et al. [22] | -0.2123 | -0.1301 | 0.0376 | -0.0012 | A ₃ > A ₄ > A ₂ > A ₁ |
| Amin et al. [21] | -0.2413 | -0.2076 | -0.1043 | -0.1490 | A ₃ > A ₄ > A ₂ > A ₁ |
| Riaz et al. [30] | -0.3412 | -0.2874 | -0.2032 | -0.2451 | A ₃ > A ₄ > A ₂ > A ₁ |
| Kaur and Garg [31] | × | × | × | × | Unable to determine |
| Kaur and Garg [32] | × | × | × | × | Unable to determine |

Table 13 The evaluation scores of the alternatives in relation to cubic intuitionistic fuzzy numbers

| C ₁ | C ₂ |
|---|---|
| A ₁ ([0.20, 0.25], [0.30, 0.35], 0.20, 0.30) | A ₁ ([0.40, 0.45], [0.20, 0.23], 0.70, 0.10) |
| A ₂ ([0.18, 0.20], [0.30, 0.32], 0.30, 0.50) | A ₂ ([0.60, 0.65], [0.30, 0.32], 0.40, 0.20) |
| A ₃ ([0.70, 0.80], [0.05, 0.10], 0.10, 0.20) | A ₃ ([0.50, 0.55], [0.20, 0.25], 0.30, 0.20) |
| A ₄ ([0.20, 0.35], [0.50, 0.55], 0.20, 0.30) | A ₄ ([0.32, 0.35], [0.44, 0.46], 0.20, 0.40) |
| C ₃ | C ₄ |
| A ₁ ([0.55, 0.60], [0.35, 0.40], 0.60, 0.20) | A ₁ ([0.10, 0.15], [0.20, 0.30], 0.30, 0.40) |
| A ₂ ([0.20, 0.25], [0.18, 0.21], 0.30, 0.60) | A ₂ ([0.40, 0.50], [0.20, 0.30], 0.20, 0.30) |
| A ₃ ([0.40, 0.45], [0.30, 0.35], 0.15, 0.45) | A ₃ ([0.40, 0.45], [0.30, 0.32], 0.90, 0.10) |
| A ₄ ([0.30, 0.40], [0.20, 0.30], 0.20, 0.50) | A ₄ ([0.50, 0.55], [0.05, 0.10], 0.20, 0.40) |

Table 14 The impact of parameter p and q on alternative scores and rankings

| p | q | A_1 | A_2 | A_3 | A_4 | Ranking order |
|-----|-----|---------|---------|---------|---------|--|
| 1 | 1 | -0.2162 | 0.1604 | 0.1046 | 0.1322 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 2 | 0.2613 | 0.6150 | 0.5441 | 0.5761 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 3 | 0.4561 | 0.7664 | 0.6891 | 0.7112 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 4 | 0.5466 | 0.8279 | 0.7479 | 0.7765 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 5 | 0.5938 | 0.8597 | 0.7782 | 0.7903 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| 2 | 1 | -0.5465 | -0.2531 | -0.2778 | -0.2594 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 2 | -0.0924 | 0.1849 | 0.1408 | 0.1578 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 3 | 0.0858 | 0.3277 | 0.2718 | 0.2857 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 4 | 0.1640 | 0.3844 | 0.3204 | 0.3440 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 5 | 0.2021 | 0.4137 | 0.3434 | 0.3825 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| 3 | 1 | -0.6486 | -0.4130 | -0.4295 | -0.4197 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 2 | -0.2040 | 0.0080 | -0.0128 | -0.0054 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 3 | -0.0340 | 0.1385 | 0.1086 | 0.1165 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 4 | 0.0370 | 0.1861 | 0.1489 | 0.1599 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 5 | 0.0693 | 0.2087 | 0.1652 | 0.1721 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| 4 | 1 | -0.6887 | -0.4828 | -0.5024 | -0.4872 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 2 | -0.2468 | -0.0716 | -0.0801 | -0.0797 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 3 | -0.0795 | 0.0508 | 0.0170 | 0.0269 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 4 | -0.0111 | 0.0917 | 0.0809 | 0.0863 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 5 | 0.0187 | 0.1088 | 0.1001 | 0.1012 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| 5 | 1 | -0.7069 | -0.5170 | -0.5487 | -0.5337 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 2 | -0.2657 | -0.1105 | -0.1334 | -0.1245 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 3 | -0.0992 | 0.0075 | -0.0139 | -0.0056 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 4 | -0.0317 | 0.0443 | -0.0052 | 0.0116 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| | 5 | -0.0027 | 0.0579 | -0.0078 | 0.0369 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |

Table 15 Comparison of proposed approach with existing methods

| Approaches | A_1 | A_1 | A_1 | A_1 | Ranking order |
|--------------------|---------|---------|---------|---------|--|
| Kaur and Garg [31] | -0.6313 | -0.4267 | -0.5973 | -0.4850 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| Kaur and Garg [32] | -0.4725 | -0.1930 | -0.2518 | -0.2394 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |
| Naeem et al. [34] | -0.2345 | -0.1256 | -0.1844 | -0.1578 | $A_2 \succcurlyeq A_4 \succcurlyeq A_3 \succcurlyeq A_1$ |

outlined in references [31, 32]. This additional analysis will enable us to gain a more comprehensive understanding of the relative strengths and weaknesses of the alternatives and will facilitate the selection of the most suitable option for the given decision-making context.

After careful analysis, it has been determined that the suggested operators take into account the decision makers’ parameters, p and q . These parameters allow decision-makers to have a greater range of options to choose from, based on the varying scores of each alternative under different parametric values of p and q . In other words, the proposed operators provide decision-makers with the flexibility to select alternatives that align with their specific preferences, based on how the alternatives are evaluated using different values of p and q . This approach offers decision-makers

a more nuanced and personalized decision-making process, tailored to their individual needs and priorities.

Advantages

To emphasize the strengths of the stated algorithm in comparison to the current ones, we analyze the distinctive features of the several existing algorithms in contrast to the proposed one. The methodologies referenced in [32] and Xu and [31] are restricted by the domain of $\vartheta_G^U + \psi_G^U \leq 1$, making their approaches exceedingly precarious while the methods referenced in [21] and [22] that are restricted to the domain of $(\vartheta_G^U)^2 + (\psi_G^U)^2 \leq 1$, making their approaches quite risky. On the other hand, the algorithms presented in [23]

Table 16 Comparison of the features of different approaches

| Methods | MD | NMD | Parameter p | Parameter q |
|--|----|-----|---------------|---------------|
| Cubic intuitionistic fuzzy set | ✓ | ✓ | ✗ | ✗ |
| Cubic Pythagorean fuzzy set | ✓ | ✓ | ✗ | ✗ |
| Cubic Fermatean fuzzy set | ✓ | ✓ | ✗ | ✗ |
| Cubic q -rung orthopair set | ✓ | ✓ | ✗ | ✓ |
| p, q -cubic quasi-rung orthopair set | ✓ | ✓ | ✓ | ✓ |

and [35] make use of the constraint $(\vartheta_G^U)^q + (\psi_G^U)^q \leq 1$ in which decision-makers are restricted to put same values of parameter q . In conclusion, current algorithms tend to rely on pre-determined weight information for attributes, which is often impractical. Determining attribute weights beforehand is difficult due to incomplete information, making it unlikely to obtain accurate results. Thus, existing approaches in IFS, PFS, or q -ROFS environment may produce unsatisfactory outcomes in such circumstances. However, the proposed approach relaxed the restrictions and providing p, q -QOFS under the condition that $(\vartheta_G^U)^p + (\psi_G^U)^q \leq 1$ with $p, q \geq 1$. Also, p can be equal to q , greater than q , or less than q . In order to satisfy the constraint condition $(\vartheta_G^U)^p + (\psi_G^U)^q \leq 1$, a decision-maker can select the smallest integer parameter q based on the evaluation of attribute values. If the attribute value during the evaluation of an alternative is $([0.6, 0.7], [0.7, 0.8], (0.6, 0.4))$, one could select either p as 3 and q as 2, or p as 2 and q as 3, since both configurations satisfy the constraint that $(\vartheta_G^U)^p + (\psi_G^U)^q \leq 1$. Thus, if p is 3, the minimum integer value for q would be 2; conversely, if p is 2, the minimum integer value for q would be 3.

Upon closer examination of the preceding discourse, it is evident that the suggested operators comprise the current aggregation operators as distinct cases. This realization accentuates the adaptability and flexibility of the proposed methodology, as it can suitably adjust to different situations and yield more refined solutions. By integrating the existing aggregation operators, the proposed methodology presents a more all-encompassing structure for resolving decision-making predicaments that can meet a broader spectrum of preferences and requirements. Table 16 illustrates a comparison between the proposed methodology and various alternative techniques, outlining their unique attributes. This comparison serves to reinforce the superiority of the proposed approach in handling diverse decision-making situations.

Conclusion

This article presents the concept of p, q -CQOFSs as a modification of pre-existing fuzzy sets. The p, q -CQOFS is a useful

tool for addressing uncertainty and fuzziness in decision-making. It allows for the simultaneous expression of p, q interval-valued quasi-rung orthopair fuzzy values and p, q -quasi-rung orthopair fuzzy values. This means that it allows for the representation of not just the degree of membership of an element in a set, but also the degree of non-membership and the degree of hesitation or uncertainty about the membership of an element in the set. This added level of detail can provide more nuanced and accurate information during the decision-making process. With the benefits in mind, the article has established a method for determining the relative order of two p, q -CQOFNs using a score function and an accuracy function. Additionally, the article has discussed some of the characteristics of this ranking order relation. Furthermore, the article has introduced two methods, known as the p, q -CQOFWA operator and the p, q -CQOFWG operator, for combining the various preferences of experts on different attributes within a p, q -CQOFS framework. Additionally, the article examines and investigates the various favorable properties of these operators. Finally, we present a methodology for addressing decision-making issues by considering different combinations of the parameters p and q . This approach enhances the flexibility of the proposed operators and provides the decision-maker with a range of options for evaluating decisions. A comparison with other existing operators demonstrates that the proposed operators and their associated techniques offer a more stable, realistic, and positive approach for the decision-maker when aggregating information during the decision-making process.

The present study incorporated input from experts across four different alternatives, taking into account three key attributes. To verify the outcome of this study, future research can expand the scope by incorporating a greater number of alternatives and attributes. Also, the study proposed that the ranking order of alternatives remains consistent for parameters p and q ranging from 1 to 5. Nevertheless, future research could investigate whether this trend persists for higher values of these parameters. Additionally, this methodology has broad applications and can be utilized to address various decision-making challenges, including but not limited to medical diagnosis [36], evaluation of domestic airlines [37], medical waste management [38], and assessment of

healthcare waste treatment technologies [39], multiobjective decision-making [40].

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no competing interests.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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