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ORIGINAL ARTICLE

A new approach to investigate the effects of artificial neural networks based on bipolar complex spherical fuzzy information

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Abstract

Artificial neural network is revolutionizing business and everyday life, bringing us to the next level in artificial intelligence. It has a unique ability to extract meaning from complex data to find patterns and detect trends that are too convoluted for the human brain. This paper analyzes the artificial neural network impact on different computational organizations by using the innovative structure of bipolar complex spherical fuzzy relation which is any subset of the Cartesian product of two bipolar complex spherical fuzzy sets. This notion has a comprehensive structure that consists of membership grade, abstinence grade, and non-membership grade. Furthermore, various kinds of bipolar complex spherical fuzzy relation with suitable examples are given and some authentic results also have been proved. These newly defined structures are used to investigate the impact of artificial neural network work on a variety of organizations. The innovative framework is also compared with the existing structure in the field of fuzzy set theory to prove its superiority.

Keywords Artificial neural network · Bipolar complex spherical fuzzy set · Uncertainty · Fuzzy relation · Complex fuzzy set

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Introduction

The ambiguity involved in any problem-solving condition gives a result in form of some data inadequacy. It is also found in the working of different organizations. To resolve this problem and make the work more reliable and effective, a new innovation in the history of mathematics was proposed named the fuzzy set (FS) by Zadeh [1] in 1965. Each element of an FS can be assigned a function whose range is in the unit interval [0,1] which is called the membership grade (MG) of an element. Klir and Folger [2] announced the concept of relations in classical set theory. Classical relation only discusses yes and no situations. Mendel [3] conceived the idea of fuzzy relation (FR) which is used to examine the relationship between two or more FSs. FRs do not just answer yes or no; they also determine the grade of a good relationship between any two FSs. If the MG grade value is closer to 1, then they specify a good relationship, and the MG value closer to 0 represents a poor-quality relationship. Torra [4] developed the idea of FSs and introduced some basic operators. Laengle et al. [5] suggested a bibliometric assessment of FSs by using the evolutionary algorithm. Remot et al. [6] proposed the novel notion of the complex fuzzy set (CFS), in which MG is determined by values from the complex plane. Since the s of MG in a CFS are complex numbers, they are divided into two parts: amplitude and phase term. Furthermore, they also defined the complex FR (CFR). Chen et al. [7] investigated a CFS-based neuro-fuzzy architecture. Shami et al. [8] discussed the (2, 1)-Fuzzy sets: properties, weighted aggregated operators and their applications to multi-criteria decision-making methods. Ibrahim et al. [9] defined the (3, 2)-Fuzzy sets and their applications to topology and optimal choices. Shami et al. [10] introduced the innovative generalization of fuzzy soft set. Shami et al. [11] explained the soft relation (SR)-fuzzy sets and their applications to weighted aggregated operators in decision making.

Atanassov [12] proposed the idea of the intuitionistic fuzzy set (IFS). The IFS is a generalization form of FS because FS discussed only the MG but IFS discussed both the MG and non-membership grade (non-MG). The sum of the MG and non-MG contains the interval [0,1]. Vlachos and Sergiadis [13] used the IFS in pattern recognition. Burillo and Bustince [14] give the idea of intuitionistic FR (IFR). Bustince [15] concocted the construction of IFR with predetermined properties. Alkouri et al. [16] proposed the concept of a complex intuitionistic fuzzy set (CIFS). The CIFS shows the MG and non-MG in the form of a complex number. Ngan et al. [17] represented the CIFS with application to decision making. Nasir et al. [18] used the complex intuitionistic FR with the application of cyber security and cybercrime in the oil and gas industries. Kumar et al. [19] defined the intuitionistic fuzzy solid assignment difficulties. Kumar et al. [20] discussed the method for solving type-1 and type-3 fuzzy transportation harms. Kumar et al. [21] came up with a new approach to resolve solid assignment difficulties under intuitionistic fuzzy environment.

Yager [22] created the new ideas of the Pythagorean fuzzy set (PyFS) by changing the condition of IFSs. The PyFS removed the limitation of IFS which extends the variety to select MG and non-MG by applying a new limit, i.e., the summation of the square of MG and non-MG must be a closed unit interval, i.e., [0,1]. Ullah et al. [23] proposed the new notion of the complex Pythagorean fuzzy set (CPyFS) with application in pattern recognition. The CPyFS gives the value of MG and non-MG in the form of a complex number. Dick et al. [24] gave the idea of CPyFS operations. Nasir et al. [25] defined the idea of complex Pythagorean FR (CPyFR) with the application of economic relationships. Yager et al. [26] developed the idea of q-rung orthopair fuzzy set (qROFSs). These set theories remove the constraints imposed by IFS and PyFS theories, allowing to choose freely MG and non-MG. In qROFSs, the sum of the nth power of MG and non-MG must lie in [0,1], where n is a natural number. Jan et al. [27] generalized qROFS with similarity measure application. Garg et al. [28] introduced the complex q-rung orthopair fuzzy set (CqROFS) with an aggregation operator. Nasir et al. [29] proposed a novel concept of complex q-rung orthopair FR (CqROFR) using the investigation of the financial track. Cuong et al. [30] initiated the novel concept of picture fuzzy set (PFS) which is the broader form of FS and IFS.

The PFS discussed the MG, AG, and non-MGs. The sum of all of MG, AG, and non-MG lies in between the interval [0,1]. Wang [31] defines the picture hesitant fuzzy set with multiple criteria for decision making. Akram et al. [32] proposed the complex picture fuzzy set (CPFS). The CPFS is the more generalized form of CIFS. CPFS defines three stages of an element with both amplitude term (AT) and phase term (PT). Nasir et al. [33] introduced the complex picture FR (CPFR) using the Cartesian product (CP) of two CPFS with the application of network security. Mahmood et al. [34] established the concept of spherical fuzzy set (SFS) with the application of decision-making and medical diagnosis problems. The SFS increased the space of PFS and removed some limitations for choosing the MG, AG, and non-MG. In SFS, the sum of the square of MG, AG, and non-MG ranges in the unit interval [0,1]. Guleria and Bajaj [35] used the concept of SFS in the problem of decision making. Ali et al. [36] discussed the complex spherical fuzzy set (CSFS). The theory of CSFS is a mixture of two theories, i.e., CFS and SFS. Nasir et al. [37] introduced the complex spherical FR (CSFR) with the application of economic relationships and international trade.

A new concept in fuzzy algebra was conducted by Zhang et al. [38] who proposed the new concept of bipolar fuzzy set (BFS) and bipolar FR (BFR). BFS is a more extensive version of FS, with MG ranging from [0,1] to [− 1,1]. The MG of BFS indicates the level of satisfaction with a quality equivalent to a fuzzy set and its associated property. The elements fulfill the corresponding property when their MG is (0,1], while some elements satisfy the equivalent counter property when their MG is [− 1,0). In a BFS, positive MG denotes what is guaranteed to be true, whereas negative MG denotes what is certain to be false. Lee [39] presented the bipolar valued fuzzy sets with some operations. Chen et al. [40] generalized the extension of BFS. Dudziak et al. [41] explained the equivalent bipolar FRs. Mahmood et al. [42] proposed the novel structure of bipolar complex fuzzy set (BCFS), by expanding the range of BFS to the domain of complex numbers. In BCFS, both positive and negative MG s are discussed in both amplitude term (AT) and phase term (PT). Ezhilmaran and Sankar [43] presented the notion of a bipolar intuitionistic fuzzy set (BIFS), which describes the both possibility and the impossibility of MG and non-MG. Mandal [44] introduced the bipolar Pythagorean fuzzy set (BPyFS) with the application. The BPyFS is the generalization form of BIFS because the BPyFS increased the space. Sindhu et al. [45] developed the concept of a bipolar picture fuzzy set (BPFS). The BPFS is the extended form of BFS and BIFS. The BPFS also discussed the AG with possibility and impossibility. Princy and Mohana [46] proposed the novelty concept of bipolar spherical fuzzy set (BSFS) with the application. The BSFS removed the limitations of BPFS and select all stages of MG, AG, and non-MG with both effects of positive and negative.

In this paper, a novel innovative framework of the bipolar complex spherical fuzzy set (BCSFS) and the CP of two BCSFSs is established. Furthermore, discussed the different types of relations such as reflexive, symmetric, transitive, equivalence relation, equivalence class, partial order, and strict order with appropriate examples. Some reliable results have also been proved. The novel structure of BCSFR is superior to all pre-existing structures such as CFS, CIFS, CPyFS, CqROFS, CPFS, CSFS, BFS, BCFS, BIFS, BPyFS, BqROFS, BPFS, and BSFS. The benefit of this newly planned framework is it explains all the s, i.e., MG, AG, and non-MG with possibility and impossibility. It includes all pre-defined components. The BCSFR discussed all the stages in complex form, i.e., they also discussed the periodicity. This article also includes an application of an artificial neural network (ANN), which is a critical component of a digital system and can be used effectively in many organizations. ANN is the fastest in processing information. Different organizations have improved performance and development due to ANN. This new BCSFR structure will be employed in various types of fields in the future, including computer science, physics, sports science, and economics.

The sequential framework of the paper is given as follows: "Preliminaries" discourses the pre-defined structures. "Main result" presents the newly defined BCSFR structures with suitable examples. "Application" defines a relationship between ANN types and their organizations using BCSFRs. "Comparative analysis" compares the BCSFR with pre-existing structures. The last section explains the conclusion of the article.

Preliminaries

This section discusses some basic ideas of fuzzy algebra.

Definition 1 [1]: Let S be a universal set. Then, FS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \{(u,\ \vartheta(u) : u \in S)\}. \tag{1}$$

Since $\vartheta(u)$ is a mapping of MG, defined as $\vartheta(u) : S \to [0,\ 1]$.

Definition 2 [6]: Let S be a universal set. Then, a CFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{\left(u,\ \sigma_\vartheta(u)e^{2\pi\rho_\vartheta(u)i}\right) : u \in S\right\}. \tag{2}$$

Since σ_ϑ, ρ_ϑ are AT and PT of MG, respectively, defined as $\sigma_\vartheta : S \to [0,\ 1]$ and $\rho_\vartheta : S \to [0,\ 1]$.

Definition 3 [16]: Let S be a universal set. Then, a CIFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{\left(u,\ \sigma_\vartheta(u)e^{2\pi\rho_\vartheta(u)i},\ \sigma_m(u)e^{2\pi\rho_m(u)i}\right) : u \in S\right\}. \tag{3}$$

Since σ_ϑ, σ_m and ρ_ϑ, ρ_m are AT and PT of MG and non-MG, respectively, and defined as $\sigma_\vartheta,\ \sigma_m : S \to [0,\ 1]$ and $\rho_\vartheta,\ \rho_m : S \to [0,\ 1]$.

Definition 4 [18]: Let $\mathrm{Q} = \left\{\left(u,\ \begin{matrix}\left(\sigma_\vartheta(u)e^{2\pi\rho_\vartheta(u)i}\right),\\ \left(\sigma_m(u)e^{2\pi\rho_m(u)i}\right)\end{matrix}\right) : u \in S\right\}$ and $\mathrm{E} = \left\{\left(b,\ \begin{matrix}\left(\sigma_\vartheta(b)e^{2\pi\rho_\vartheta(b)i}\right),\\ \left(\sigma_m(b)e^{2\pi\rho_m(b)i}\right)\end{matrix}\right) : b \in S\right\}$ be two CIFS on S. Then, their Cartesian product (CP) is defined as

$$Q \times E = \Big\{(u,\ b),\ \left(\sigma_\vartheta(u,\ b)e^{2\pi\rho_\vartheta(u,b)i}\right),\ \left(\sigma_m(u,\ b)e^{2\pi\rho_m(u,b)i}\right) : u \in \mathrm{Q},\ b \in \mathrm{E}\Big\}, \tag{4}$$

where $\sigma_\vartheta(u,\ b) = \min\{\sigma_\vartheta(u),\ \sigma_\vartheta(b)\}$ and $\rho_\vartheta(u,\ b) = \min\{\rho_\vartheta(u),\ \rho_\vartheta(b)\}$, $\sigma_m(u,\ b) = \max\{\sigma_m(u),\ \sigma_m(b)\}$ and $\rho_m(u,\ b) = \max\{\rho_m(u),\ \rho_m(b)\}$.

The complex intuitionistic FR (CIFR) denoted by R is a subset of the CP between two CIFSs.

Definition 5 [22]: Let S be a universal set. Then, a PyFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \{(u,\ \vartheta(u),\ m(u) : u \in S)\}. \tag{5}$$

Since $\vartheta(u)$ and $m(u)$ is a mapping of MG and non-MG, respectively, defined as $\vartheta(u),\ m(u) : S \to [0,\ 1]$. On condition that $0 \le (\vartheta(u))^2 + (m(u))^2 \le 1$.

Definition 6 [32]: Let S be a universal set. Then, a CPFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{\left(\begin{matrix}u,\ \sigma_\vartheta(u)e^{2\pi\rho_\vartheta(u)i},\ \sigma_\lambda(u)e^{2\pi\rho_\lambda(u)i},\\ \sigma_m(u)e^{2\pi\rho_m(u)i}\end{matrix}\right) : u \in S\right\}. \tag{6}$$

Since σ_ϑ, σ_λ, σ_m and ρ_ϑ, ρ_λ, ρ_m are AT and PT of MG, AG and non-MG respectively and defined as $\sigma_\vartheta,\ \sigma_m,\ \sigma_\lambda : S \to [0,\ 1]$ and $\rho_\vartheta,\ \rho_m,\ \rho_\lambda : S \to [0,\ 1]$.

Definition 7 [36]: Let S be a universal set. Then, a CSFS on S, denoted by Q, is expressed as

$$Q = \left\{\left(\begin{matrix}u,\ \sigma_\vartheta(u)e^{2\pi\rho_\vartheta(u)i},\ \sigma_\lambda(u)e^{2\pi\rho_\lambda(u)i},\\ \sigma_m(u)e^{2\pi\rho_m(u)i}\end{matrix}\right) : u \in S\right\}. \tag{7}$$

Since σ_ϑ, σ_λ, σ_m and ρ_ϑ, ρ_λ, ρ_m are AT and PT of MG, AG and non-MG, respectively, and defined as $\sigma_\vartheta,\ \sigma_m,\ \sigma_\lambda : S \to [0,\ 1]$ and $\rho_\vartheta,\ \rho_m,\ \rho_\lambda : S \to [0,\ 1]$.

Definition 8 [38]: Let S be a universal set. Then, a BFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(u,\ \vartheta^{+}(u),\ \vartheta^{-}(u) \right) : u \in S \right\}. \tag{8}$$

Since $\vartheta^{+}(u)$, $\vartheta^{-}(u)$ is a mapping of positive and negative MG and defined as $\vartheta^{+}(u) : S \to [0, 1]$ and $\vartheta^{-}(u)$: $S \to [-1, 0]$.

Definition 9 [42]: Let S be a universal set. Then, a BCFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(\begin{array}{c} u,\ \left(\sigma_{\vartheta}^{+}(u) + i\rho_{\vartheta}^{+}(u)\right), \\ \left(\sigma_{\vartheta}^{-}(u) + i\rho_{\vartheta}^{-}(u)\right) \end{array} \right) : u \in S \right\}. \tag{9}$$

Since σ^{+} and ρ^{+} are known as mappings of AT and PT of positive MG, respectively. σ^{-} and ρ^{-} are known as mappings of AT and PT of negative MG, respectively. These terms are defined as $\sigma^{+} : S \to [0, 1]$ and $\rho^{+} : S \to [0, 1]$, $\sigma^{-} : S \to [-1, 0]$ and $\rho^{-} : S \to [-1, 0]$.

Definition 10 [43]: Let S be a universal set. Then, a BIFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(u,\ \left(\vartheta^{+}(u),\ \vartheta^{-}(u)\right),\ \left(m^{+}(u),\ m^{-}(u)\right) \right) : u \in S \right\}. \tag{10}$$

Since $\vartheta^{+}(u)$, $m^{+}(u)$ is a mapping of positive MG and non-MG, respectively. $\vartheta^{-}(u)$, $m^{-}(u)$ is a mapping of negative MG and non-MG, respectively. These terms are defined as $\vartheta^{+}(u) : S \to [0, 1]$ and $\vartheta^{-}(u){:}S \to [-1, 0]$, $m^{+}(u) : S \to [0, 1]$ and $m^{-}(u){:}S \to [-1, 0]$. On condition that $0 \le \vartheta^{+}(u) + m^{+}(u) \le 1$ and $-1 \le \vartheta^{-}(u) + m^{-}(u) \le 0$.

Definition 11 [44]: Let S be a universal set. Then, a BPyFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(u,\ \left(\vartheta^{+}(u),\ \vartheta^{-}(u)\right),\ \left(m^{+}(u),\ m^{-}(u)\right) \right) : u \in S \right\}. \tag{11}$$

Since $\vartheta^{+}(u)$, $m^{+}(u)$ is a mapping of positive MG and non-MG, respectively. $\vartheta^{-}(u)$, $m^{-}(u)$ is a mapping of negative MG and non-MG, respectively. These terms are defined as $\vartheta^{+}(u) : S \to [0, 1]$ and $\vartheta^{-}(u){:}S \to [-1, 0]$, $m^{+}(u) : S \to [0, 1]$ and $m^{-}(u){:}S \to [-1, 0]$. On condition that $0 \le \left(\vartheta^{+}(u)\right)^{2} + \left(m^{+}(u)\right)^{2} \le 1$ and $-1 \le \left(\vartheta^{-}(u)\right)^{2} + \left(m^{-}(u)\right)^{2} \le 0$.

Definition 12 [45]: Let S be a universal set. Then, a BPFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(\begin{array}{c} u,\ \left(\vartheta^{+}(u),\ \vartheta^{-}(u)\right),\ \left(\lambda^{+}(u),\ \lambda^{-}(u),\right. \\ \left. m^{+}(u),\ m^{-}(u)\right) \end{array} \right) : u \in S \right\}. \tag{12}$$

Since $\vartheta^{+}(u)$, $\lambda^{+}(u)$, $m^{+}(u)$ is a mapping of positive MG, AG and non-MG, respectively. $\vartheta^{-}(u)$, $\lambda^{-}(u)$, $m^{-}(u)$ is a mapping of negative MG, AG and non-MG, respectively. These terms are defined as $\vartheta^{+}(u) : S \to [0, 1]$ and$\vartheta^{-}(u){:}S \to [-1, 0]$, , $\lambda^{+}(u) : S \to [0, 1]$ and$\lambda^{-}(u) : S \to [-1, 0]$, $m^{+}(u) : S \to [0, 1]$ and$m^{-}(u){:}S \to [-1, 0]$. On condition that $0 \le \vartheta^{+}(u) + \lambda^{+}(u) + m^{+}(u) \le 1$ and $-1 \le \vartheta^{-}(u) + \lambda^{-}(u) + m^{-}(u) \le 0$.

Definition 13 [46]: Let S be a universal set. Then, a BSFS on S, denoted by Q, is expressed as

$$\mathrm{Q} = \left\{ \left(\begin{array}{c} u,\ \left(\vartheta^{+}(u),\ \vartheta^{-}(u)\right),\ \left(\lambda^{+}(u),\ \lambda^{-}(u),\right. \\ \left. m^{+}(u),\ m^{-}(u)\right) \end{array} \right) : u \in S \right\}. \tag{13}$$

Since $\vartheta^{+}(u)$, $\lambda^{+}(u)$, $m^{+}(u)$ is a mapping of positive MG, AG and non-MG, respectively. $\vartheta^{-}(u)$, $\lambda^{-}(u)$, $m^{-}(u)$ is a mapping of negative MG, AG and non-MG, respectively. These terms are defined as $\vartheta^{+}(u) : S \to [0, 1]$ and $\vartheta^{-}(u){:}S \to [-1, 0]$, , $\lambda^{+}(u) : S \to [0, 1]$ and $\lambda^{-}(u) : S \to [-1, 0]$, $m^{+}(u) : S \to [0, 1]$ and $m^{-}(u){:}S \to [-1, 0]$. On condition that $0 \le \vartheta^{2+}(u) + \lambda^{2+}(u) + m^{2+}(u) \le 1$ and $-1 \le \vartheta^{2-}(u) + \lambda^{2-}(u) + m^{2-}(u) \le 0$.

Main result

In this section, we discuss the BCSFS and the new idea of BCSFR using the CP of two BCSFSs.

Definition 14 Let S be a universal set. Then, a BCSFS Q on S is defined as

$$Q = \left\{ u,\ \begin{pmatrix} \sigma_{\vartheta}^{+}(u) + i\rho_{\vartheta}^{+}(u), \\ \sigma_{\vartheta}^{-}(u) + i\rho_{\vartheta}^{-}(u) \end{pmatrix},\ \begin{pmatrix} \sigma_{\lambda}^{+}(u) + i\rho_{\lambda}^{+}(u), \\ \sigma_{\lambda}^{-}(u) + i\rho_{\lambda}^{-}(u) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(u) + i\rho_{m}^{+}(u), \\ \sigma_{m}^{-}(u) + i\rho_{m}^{-}(u) \end{pmatrix} : u \in Q \right\}, \tag{14}$$

with a condition $0 \le \sigma_{\vartheta}^{+2} + \sigma_{\lambda}^{+2} + \sigma_{m}^{+2} \le 1$, $0 \le \rho_{\vartheta}^{+2} + \rho_{\lambda}^{+2} + \rho_{m}^{+2} \le 1$, $-1 \le \sigma_{\vartheta}^{-2} + \sigma_{\lambda}^{-2} + \sigma_{m}^{-2} \le 0$ and $-1 \le \rho_{\vartheta}^{-2} + \rho_{\lambda}^{-2} + \rho_{m}^{-2} \le 0$, where σ_{ϑ}^{+}, ρ_{ϑ}^{+}, σ_{λ}^{+}, ρ_{λ}^{+}, σ_{m}^{+}, $\rho_{m}^{+} : S \to [0, 1]$ are ATs and PTs of MG, AG and non-MG of positive mappings, respectively. In addition, σ_{ϑ}^{-}, ρ_{ϑ}^{-}, σ_{λ}^{-}, ρ_{λ}^{-}, σ_{m}^{-}, $\rho_{m}^{-} : S \to [-1, 0]$ are ATs and PTs of MG, AG and non-MG of negative mappings, respectively.

Example 1 The $s = \left\{ \begin{array}{l} \left(s_1, \begin{pmatrix} 0.63 + 0.51i, \\ -0.31 - 0.41i \end{pmatrix}, \begin{pmatrix} 0.43 + 0.61i, \\ -0.36 - 0.24i \end{pmatrix}, \begin{pmatrix} 0.34 + 0.31i, \\ -0.49 - 0.29i \end{pmatrix} \right), \\ \left(s_2, \begin{pmatrix} 0.63 + 0.74i, \\ -0.42 - 0.54i \end{pmatrix}, \begin{pmatrix} 0.53 + 0.32i, \\ -0.36 - 0.14i \end{pmatrix}, \begin{pmatrix} 0.39 + 0.59i, \\ -0.23 - 0.43i \end{pmatrix} \right) \end{array} \right\}$ is a BCSFS.

Definition 15 Take two BCSFSs:

$$B=\left\{b,\begin{pmatrix}\sigma_{\vartheta}^{+}(b)+i\rho_{\vartheta}^{+}(b),\\ \sigma_{\vartheta}^{-}(b)+i\rho_{\vartheta}^{-}(b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b)+i\rho_{m}^{+}(b),\\ \sigma_{m}^{-}(b)+i\rho_{m}^{-}(b)\end{pmatrix}:b\in Q\right\},$$

and

$$C=\left\{c,\begin{pmatrix}\sigma_{\vartheta}^{+}(c)+i\rho_{\vartheta}^{+}(c),\\ \sigma_{\vartheta}^{-}(c)+i\rho_{\vartheta}^{-}(c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c)+i\rho_{\lambda}^{+}(c),\\ \sigma_{\lambda}^{-}(c)+i\rho_{\lambda}^{-}(c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c)+i\rho_{m}^{+}(c),\\ \sigma_{m}^{-}(c)+i\rho_{m}^{-}(c)\end{pmatrix}:c\in Q\right\}.$$

Then, their cartesian product is defined as follows:

$$B\times C=\left\{(b,c)\begin{pmatrix}\sigma_{\vartheta}^{+}(b,c)+i\rho_{\vartheta}^{+}(b,c),\\ \sigma_{\vartheta}^{-}(b,c)+i\rho_{\vartheta}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,c)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b,c)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,c)+i\rho_{m}^{+}(b,c),\\ \sigma_{m}^{-}(b,c)+i\rho_{m}^{-}(b,c)\end{pmatrix}:b\in Q\right\},$$

where $\sigma_{\vartheta}^{+}(b,c)=\min\{\sigma_{\vartheta}^{+}(b),\sigma_{\vartheta}^{+}(c)\}$, $\rho_{\vartheta}^{+}(b,c)=\min\{\rho_{\vartheta}^{+}(b),\rho_{\vartheta}^{+}(c)\}$, $\sigma_{\vartheta}^{-}(b,c)=\max\{\sigma_{\vartheta}^{-}(b),\sigma_{\vartheta}^{-}(c)\}$, $\rho_{\vartheta}^{-}(b,c)=\max\{\rho_{\vartheta}^{-}(b),\rho_{\vartheta}^{-}(c)\}$, $\sigma_{\lambda}^{+}(b,c)=\min\{\sigma_{\lambda}^{+}(b),\sigma_{\lambda}^{+}(c)\}$, $\rho_{\lambda}^{+}(b,c)=\min\{\rho_{\lambda}^{+}(b),\rho_{\lambda}^{+}(c)\}$, $\sigma_{\lambda}^{-}(b,c)=\max\{\sigma_{\lambda}^{-}(b),\sigma_{\lambda}^{-}(c)\}$, $\rho_{\lambda}^{-}(b,c)=\max\{\rho_{\lambda}^{-}(b),\rho_{\lambda}^{-}(c)\}$, $\sigma_{m}^{+}(b,c)=\max\{\sigma_{m}^{+}(b),\sigma_{m}^{+}(c)\}$, $\rho_{m}^{+}(b,c)=\max\{\rho_{m}^{+}(b,c),\rho_{m}^{+}(b,c))\}$, $\sigma_{m}^{-}(b,c)=\min\{\sigma_{m}^{-}(b),\sigma_{m}^{-}(c)\}$, $\rho_{m}^{-}(b,c)=\min\{\rho_{m}^{-}(b),\rho_{m}^{-}(c))\}$.

Example 2 Let a BCSFS T on F be expressed as

$$T=\left\{\begin{array}{l}\left(t_1,\begin{pmatrix}0.69+0.34i,\\-0.21-0.02i\end{pmatrix},\begin{pmatrix}0.45+0.59i,\\-0.33-0.17i\end{pmatrix},\begin{pmatrix}0.38+0.43i,\\-0.23-0.29i\end{pmatrix}\right),\\ \left(t_2,\begin{pmatrix}0.55+0.43i,\\-0.13-0.25i\end{pmatrix},\begin{pmatrix}0.48+0.49i,\\-0.30-0.38i\end{pmatrix},\begin{pmatrix}0.29+0.30i,\\-0.12-0.19i\end{pmatrix}\right),\\ \left(t_3,\begin{pmatrix}0.49+0.43i,\\-0.29-0.34i\end{pmatrix},\begin{pmatrix}0.30+0.46i,\\-0.04-0.33i\end{pmatrix},\begin{pmatrix}0.51+0.25i,\\-0.24-0.34i\end{pmatrix}\right),\\ \left(t_4,\begin{pmatrix}0.88+0.79i,\\-0.23-0i\end{pmatrix},\begin{pmatrix}0.35+0.36i,\\-0.30-0.29i\end{pmatrix},\begin{pmatrix}0.25+0.43i,\\-0.12-0.12i\end{pmatrix}\right)\end{array}\right\}$$

Then, the CP is

$$T \times T = \left\{ \begin{array}{l} \left((t_1, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.59i, \\ -0.33 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_1, t_2), \begin{pmatrix} 0.55 + 0.34i, \\ -0.13 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.49i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_1, t_3), \begin{pmatrix} 0.49 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_1, t_4), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_2, t_1), \begin{pmatrix} 0.55 + 0.34i, \\ -0.13 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.49i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_2, t_2), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.48 + 0.49i, \\ -0.30 - 0.38i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.30i, \\ -0.12 - 0.19i \end{pmatrix} \right), \\ \left((t_2, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.30i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_2, t_4), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.43i, \\ -0.12 - 0.19i \end{pmatrix} \right), \\ \left((t_3, t_1), \begin{pmatrix} 0.49 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_2), \begin{pmatrix} 0.49 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.30i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.29 - 0.34i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.25i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_4), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_4, t_2), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.43i, \\ -0.12 - 0.19i \end{pmatrix} \right), \\ \left((t_4, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_4), \begin{pmatrix} 0.88 + 0.79i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.25 + 0.43i, \\ -0.12 - 0.12i \end{pmatrix} \right) \end{array} \right\}. \quad (15)$$

Definition 16 A BCSFR denoted by R is a subset of the CP of two or more BCSFSs.

Example 3 Take a subset of the CP of the BCSFS from Eq. (15). The BCSFR R is

$$R = \left\{ \begin{array}{l} \left((t_1, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.59i, \\ -0.33 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_2, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.30i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_1), \begin{pmatrix} 0.49 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.29 - 0.34i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.25i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_4), \begin{pmatrix} 0.88 + 0.79i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.25 + 0.43i, \\ -0.12 - 0.12i \end{pmatrix} \right) \end{array} \right\}. \tag{16}$$

Definition 17 A BCSFR R is a BCS reflexive FR on a BCSFS Q if it is

$$\forall \left(b, \begin{pmatrix} \sigma_{\vartheta}^{+}(b) + i\rho_{\vartheta}^{+}(b), \\ \sigma_{\vartheta}^{-}(b) + i\rho_{\vartheta}^{-}(b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b) + i\rho_{\lambda}^{+}(b), \\ \sigma_{\lambda}^{-}(b) + i\rho_{\lambda}^{-}(b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b) + i\rho_{m}^{+}(b), \\ \sigma_{m}^{-}(b) + i\rho_{m}^{-}(b) \end{pmatrix} \right) \in Q$$
$$\Rightarrow \left((b, b), \begin{pmatrix} \sigma_{\vartheta}^{+}(b, b) + i\rho_{\vartheta}^{+}(b, b), \\ \sigma_{\vartheta}^{-}(b, b) + i\rho_{\vartheta}^{-}(b, b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b, b) + i\rho_{\lambda}^{+}(b, b), \\ \sigma_{\lambda}^{-}(b, b) + i\rho_{\lambda}^{-}(b, b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b, b) + i\rho_{m}^{+}(b, b), \\ \sigma_{m}^{-}(b, b) + i\rho_{m}^{-}(b, b) \end{pmatrix} \right) \in R.$$

Definition 18 A BCSFR R is a BCS inverse FR on a BCSFS Q if it is

$$\forall \left(\begin{array}{l} b, \begin{array}{l} \begin{pmatrix} \sigma_{\vartheta}^{+}(b) + i\rho_{\vartheta}^{+}(b), \\ \sigma_{\vartheta}^{-}(b) + i\rho_{\vartheta}^{-}(b) \end{pmatrix}, \\ \begin{pmatrix} \sigma_{\lambda}^{+}(b) + i\rho_{\lambda}^{+}(b), \\ \sigma_{\lambda}^{-}(b) + i\rho_{\lambda}^{-}(b) \end{pmatrix}, \\ \begin{pmatrix} \sigma_{m}^{+}(b) + i\rho_{m}^{+}(b), \\ \sigma_{m}^{-}(b) + i\rho_{m}^{-}(b) \end{pmatrix} \end{array} \end{array} \right), \left(\begin{array}{l} c, \begin{array}{l} \begin{pmatrix} \sigma_{\vartheta}^{+}(c) + i\rho_{\vartheta}^{+}(c), \\ \sigma_{\vartheta}^{-}(c) + i\rho_{\vartheta}^{-}(c) \end{pmatrix}, \\ \begin{pmatrix} \sigma_{\lambda}^{+}(c) + i\rho_{\lambda}^{+}(c), \\ \sigma_{\lambda}^{-}(c) + i\rho_{\lambda}^{-}(c) \end{pmatrix}, \\ \begin{pmatrix} \sigma_{m}^{+}(c) + i\rho_{m}^{+}(c), \\ \sigma_{m}^{-}(c) + i\rho_{m}^{-}(c) \end{pmatrix} \end{array} \end{array} \right) \in Q.$$

$$\text{If} \left((b, c), \begin{pmatrix} \sigma_{\vartheta}^{+}(b, c) + i\rho_{\vartheta}^{+}(b, c), \\ \sigma_{\vartheta}^{-}(b, c) + i\rho_{\vartheta}^{-}(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b, c) + i\rho_{\lambda}^{+}(b, c), \\ \sigma_{\lambda}^{-}(b, c) + i\rho_{\lambda}^{-}(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b, c) + i\rho_{m}^{+}(b, c), \\ \sigma_{m}^{-}(b, c) + i\rho_{m}^{-}(b, c) \end{pmatrix} \right) \in R$$
$$\Rightarrow \left((c, b), \begin{pmatrix} \sigma_{\vartheta}^{+}(c, b) + i\rho_{\vartheta}^{+}(c, b), \\ \sigma_{\vartheta}^{-}(c, b) + i\rho_{\vartheta}^{-}(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c, b) + i\rho_{\lambda}^{+}(c, b), \\ \sigma_{\lambda}^{-}(c, b) + i\rho_{\lambda}^{-}(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c, b) + i\rho_{m}^{+}(c, b), \\ \sigma_{m}^{-}(c, b) + i\rho_{m}^{-}(c, b) \end{pmatrix} \right) \in R^{-1}.$$

Definition 19 A BCSFR R is a BCS irreflexive FR on a BCSFS Q if it is

$$\forall\left(b,\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b)+i\rho_{\vartheta}^{+}(b),\\ \sigma_{\vartheta}^{-}(b)+i\rho_{\vartheta}^{-}(b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b)+i\rho_{m}^{+}(b),\\ \sigma_{m}^{-}(b)+i\rho_{m}^{-}(b)\end{pmatrix}\right)\in Q$$
$$\Rightarrow\left((b,b),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,b)+i\rho_{\vartheta}^{+}(b,b),\\ \sigma_{\vartheta}^{-}(b,b)+i\rho_{\vartheta}^{-}(b,b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,b)+i\rho_{\lambda}^{+}(b,b),\\ \sigma_{\lambda}^{-}(b,b)+i\rho_{\lambda}^{-}(b,b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,b)+i\rho_{m}^{+}(b,b),\\ \sigma_{m}^{-}(b,b)+i\rho_{m}^{-}(b,b)\end{pmatrix}\right)\notin R.$$

Definition 20 A BCSFR R is a BCS symmetric FR on a BCSFS Q if it is

$$\forall\left(\left(b,\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(b)+i\rho_{\vartheta}^{+}(b),\\ \sigma_{\vartheta}^{-}(b)+i\rho_{\vartheta}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(b)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(b)+i\rho_{m}^{+}(b),\\ \sigma_{m}^{-}(b)+i\rho_{m}^{-}(b)\end{pmatrix}\end{matrix}\right),\left(c,\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(c)+i\rho_{\vartheta}^{+}(c),\\ \sigma_{\vartheta}^{-}(c)+i\rho_{\vartheta}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(c)+i\rho_{\lambda}^{+}(c),\\ \sigma_{\lambda}^{-}(c)+i\rho_{\lambda}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(c)+i\rho_{m}^{+}(c),\\ \sigma_{m}^{-}(c)+i\rho_{m}^{-}(c)\end{pmatrix}\end{matrix}\right)\right)\in Q$$
$$\left((b,c),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,c)+i\rho_{\vartheta}^{+}(b,c),\\ \sigma_{\vartheta}^{-}(b,c)+i\rho_{\vartheta}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,c)+i\rho_{\lambda}^{+}(b,c),\\ \sigma_{\lambda}^{-}(b,c)+i\rho_{\lambda}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,c)+i\rho_{m}^{+}(b,c),\\ \sigma_{m}^{-}(b,c)+i\rho_{m}^{-}(b,c)\end{pmatrix}\right)\in$$
$$R\Rightarrow\left((c,b),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(c,b)+i\rho_{\vartheta}^{+}(c,b),\\ \sigma_{\vartheta}^{-}(c,b)+i\rho_{\vartheta}^{-}(c,b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c,b)+i\rho_{\lambda}^{+}(c,b),\\ \sigma_{\lambda}^{-}(c,b)+i\rho_{\lambda}^{-}(c,b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c,b)+i\rho_{m}^{+}(c,b),\\ \sigma_{m}^{-}(c,b)+i\rho_{m}^{-}(c,b)\end{pmatrix}\right)\in R$$

Definition 21 A BCSFR R is a BCS antisymmetric FR on a BCSFS Q if it is

$$\forall\left(\left(b,\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(b)+i\rho_{\vartheta}^{+}(b),\\ \sigma_{\vartheta}^{-}(b)+i\rho_{\vartheta}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(b)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(b)+i\rho_{m}^{+}(b),\\ \sigma_{m}^{-}(b)+i\rho_{m}^{-}(b)\end{pmatrix}\end{matrix}\right),\left(c,\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(c)+i\rho_{\vartheta}^{+}(c),\\ \sigma_{\vartheta}^{-}(c)+i\rho_{\vartheta}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(c)+i\rho_{\lambda}^{+}(c),\\ \sigma_{\lambda}^{-}(c)+i\rho_{\lambda}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(c)+i\rho_{m}^{+}(c),\\ \sigma_{m}^{-}(c)+i\rho_{m}^{-}(c)\end{pmatrix}\end{matrix}\right)\right)\in Q$$
$$\left((b,c),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,c)+i\rho_{\vartheta}^{+}(b,c),\\ \sigma_{\vartheta}^{-}(b,c)+i\rho_{\vartheta}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,c)+i\rho_{\lambda}^{+}(b,c),\\ \sigma_{\lambda}^{-}(b,c)+i\rho_{\lambda}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,c)+i\rho_{m}^{+}(b,c),\\ \sigma_{m}^{-}(b,c)+i\rho_{m}^{-}(b,c)\end{pmatrix}\right)\in R$$
$$\text{and}\left((c,b),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(c,b)+i\rho_{\vartheta}^{+}(c,b),\\ \sigma_{\vartheta}^{-}(c,b)+i\rho_{\vartheta}^{-}(c,b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c,b)+i\rho_{\lambda}^{+}(c,b),\\ \sigma_{\lambda}^{-}(c,b)+i\rho_{\lambda}^{-}(c,b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c,b)+i\rho_{m}^{+}(c,b),\\ \sigma_{m}^{-}(c,b)+i\rho_{m}^{-}(c,b)\end{pmatrix}\right)\in R$$
$$\Rightarrow\left((b,c),\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(b,c)+i\rho_{\vartheta}^{+}(b,c),\\ \sigma_{\vartheta}^{-}(b,c)+i\rho_{\vartheta}^{-}(b,c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(b,c)+i\rho_{\lambda}^{+}(b,c),\\ \sigma_{\lambda}^{-}(b,c)+i\rho_{\lambda}^{-}(b,c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(b,c)+i\rho_{m}^{+}(b,c),\\ \sigma_{m}^{-}(b,c)+i\rho_{m}^{-}(b,c)\end{pmatrix}\end{matrix}\right)=\left((c,b),\ \begin{matrix}\begin{pmatrix}\sigma_{\vartheta}^{+}(c,b)+i\rho_{\vartheta}^{+}(c,b),\\ \sigma_{\vartheta}^{-}(c,b)+i\rho_{\vartheta}^{-}(c,b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(c,b)+i\rho_{\lambda}^{+}(c,b),\\ \sigma_{\lambda}^{-}(c,b)+i\rho_{\lambda}^{-}(c,b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(c,b)+i\rho_{m}^{+}(c,b),\\ \sigma_{m}^{-}(c,b)+i\rho_{m}^{-}(c,b)\end{pmatrix}\end{matrix}\right).$$

Definition 22 A BCSFR R is a BCS transitive FR on a BCSFS Q if it is

$$\forall\left(b,\begin{array}{c}\begin{pmatrix}\sigma_{\vartheta}^{+}(b)+i\rho_{\vartheta}^{+}(b),\\ \sigma_{\vartheta}^{-}(b)+i\rho_{\vartheta}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(b)+i\rho_{\lambda}^{+}(b),\\ \sigma_{\lambda}^{-}(b)+i\rho_{\lambda}^{-}(b)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(b)+i\rho_{m}^{+}(b),\\ \sigma_{m}^{-}(b)+i\rho_{m}^{-}(b)\end{pmatrix}\end{array}\right),\left(c,\begin{array}{c}\begin{pmatrix}\sigma_{\vartheta}^{+}(c)+i\rho_{\vartheta}^{+}(c),\\ \sigma_{\vartheta}^{-}(c)+i\rho_{\vartheta}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(c)+i\rho_{\lambda}^{+}(c),\\ \sigma_{\lambda}^{-}(c)+i\rho_{\lambda}^{-}(c)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(c)+i\rho_{m}^{+}(c),\\ \sigma_{m}^{-}(c)+i\rho_{m}^{-}(c)\end{pmatrix}\end{array}\right)\text{ and }\left(d,\begin{array}{c}\begin{pmatrix}\sigma_{\vartheta}^{+}(d)+i\rho_{\vartheta}^{+}(d),\\ \sigma_{\vartheta}^{-}(d)+i\rho_{\vartheta}^{-}(d)\end{pmatrix},\\ \begin{pmatrix}\sigma_{\lambda}^{+}(d)+i\rho_{\lambda}^{+}(d),\\ \sigma_{\lambda}^{-}(d)+i\rho_{\lambda}^{-}(d)\end{pmatrix},\\ \begin{pmatrix}\sigma_{m}^{+}(d)+i\rho_{m}^{+}(d),\\ \sigma_{m}^{-}(d)+i\rho_{m}^{-}(d)\end{pmatrix}\end{array}\right)\in Q.$$

Then,

$$\left((b,c),\begin{pmatrix}\sigma_{\vartheta}^{+}(b,c)+i\rho_{\vartheta}^{+}(b,c),\\ \sigma_{\vartheta}^{-}(b,c)+i\rho_{\vartheta}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,c)+i\rho_{\lambda}^{+}(b,c),\\ \sigma_{\lambda}^{-}(b,c)+i\rho_{\lambda}^{-}(b,c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,c)+i\rho_{m}^{+}(b,c),\\ \sigma_{m}^{-}(b,c)+i\rho_{m}^{-}(b,c)\end{pmatrix}\right)\in R$$

and

$$\left((c,d),\begin{pmatrix}\sigma_{\vartheta}^{+}(c,d)+i\rho_{\vartheta}^{+}(c,d),\\ \sigma_{\vartheta}^{-}(c,d)+i\rho_{\vartheta}^{-}(c,d)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c,d)+i\rho_{\lambda}^{+}(c,d),\\ \sigma_{\lambda}^{-}(c,d)+i\rho_{\lambda}^{-}(c,d)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c,d)+i\rho_{m}^{+}(c,d),\\ \sigma_{m}^{-}(c,d)+i\rho_{m}^{-}(c,d)\end{pmatrix}\right)\in R$$
$$\Rightarrow\left((b,d),\begin{pmatrix}\sigma_{\vartheta}^{+}(b,d)+i\rho_{\vartheta}^{+}(b,d),\\ \sigma_{\vartheta}^{-}(b,d)+i\rho_{\vartheta}^{-}(b,d)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,d)+i\rho_{\lambda}^{+}(b,d),\\ \sigma_{\lambda}^{-}(b,d)+i\rho_{\lambda}^{-}(b,d)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,d)+i\rho_{m}^{+}(b,d),\\ \sigma_{m}^{-}(b,d)+i\rho_{m}^{-}(b,d)\end{pmatrix}\right)\in R.$$

Definition 23 A BCSFR R is a BCS equivalence FR on a BCSFS Q if

- BCS reflexive FR
- BCS symmetric FR
- BCS transitive FR

Example 4 Take the following relation from Example 2:

$$R = \left\{ \begin{array}{l} \left((t_1, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.59i, \\ -0.33 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_2, t_2), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.48 + 0.49i, \\ -0.30 - 0.38i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.30i, \\ -0.12 - 0.19i \end{pmatrix} \right), \\ \left((t_3, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.29 - 0.34i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.25i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_4), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_4), \begin{pmatrix} 0.88 + 0.79i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.25 + 0.43i, \\ -0.12 - 0.12i \end{pmatrix} \right) \end{array} \right\}, \tag{17}$$

is an BCS equivalence FR.

Definition 24 A BCSFR R is a BCS pre-order FR on a BCSFS Q if it is

- BCS reflexive FR
- BCS transitive FR

Definition 25 A BCSFR R is a BCS partial order FR on a BCSFS Q if

- BCS reflexive FR
- BCS antisymmetric FR
- BCS transitive FR

Example 5 Take a following relation from Example 2:

$$R = \left\{ \begin{array}{l} \left((t_1, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.59i, \\ -0.33 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_1, t_2), \begin{pmatrix} 0.55 + 0.34i, \\ -0.13 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.49i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix} \right), \\ \left((t_2, t_2), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.48 + 0.49i, \\ -0.30 - 0.38i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.30i, \\ -0.12 - 0.19i \end{pmatrix} \right), \\ \left((t_3, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.29 - 0.34i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.25i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_3, t_4), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix} \right), \\ \left((t_4, t_4), \begin{pmatrix} 0.88 + 0.79i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.25 + 0.43i, \\ -0.12 - 0.12i \end{pmatrix} \right) \end{array} \right\}, \tag{18}$$

is an BCS partial order FR.

Definition 26 A BCSFR R is a BCS strict-order FR on a BCSFS Q if it is

- BCS irreflexive FR
- BCS transitive FR

Example 6 Take a relation from Eq. (15):

$$R = \left\{ \begin{array}{l} \left((t_1, t_2), \begin{pmatrix} 0.55 + 0.34i, \\ -0.13 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.49i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix}\right), \\ \left((t_1, t_3), \begin{pmatrix} 0.49 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix}\right), \\ \left((t_1, t_4), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix}\right), \\ \left((t_2, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.30i, \\ -0.24 - 0.34i \end{pmatrix}\right), \\ \left((t_3, t_4), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix}\right) \end{array} \right\}, \tag{19}$$

is an BCS strict-order FR.

Definition 27 A BCSFR R is a BCS complete FR on a BCSFS Q if it is

$$\forall\left(b, \begin{pmatrix} \sigma_{\vartheta}^{+}(b) + i\rho_{\vartheta}^{+}(b), \\ \sigma_{\vartheta}^{-}(b) + i\rho_{\vartheta}^{-}(b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b) + i\rho_{\lambda}^{+}(b), \\ \sigma_{\lambda}^{-}(b) + i\rho_{\lambda}^{-}(b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b) + i\rho_{m}^{+}(b), \\ \sigma_{m}^{-}(b) + i\rho_{m}^{-}(b) \end{pmatrix}\right),$$

$$\left(c, \begin{pmatrix} \sigma_{\vartheta}^{+}(c) + i\rho_{\vartheta}^{+}(c), \\ \sigma_{\vartheta}^{-}(a) + i\rho_{\vartheta}^{-}(c) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c) + i\rho_{\lambda}^{+}(c), \\ \sigma_{\lambda}^{-}(c) + i\rho_{\lambda}^{-}(c) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c) + i\rho_{m}^{+}(c), \\ \sigma_{m}^{-}(c) + i\rho_{m}^{-}(c) \end{pmatrix}\right) \in Q$$

$$\Rightarrow \left((b, c), \begin{pmatrix} \sigma_{\vartheta}^{+}(b, c) + i\rho_{\vartheta}^{+}(b, c), \\ \sigma_{\vartheta}^{-}(b, c) + i\rho_{\vartheta}^{-}(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b, c) + i\rho_{\lambda}^{+}(b, c), \\ \sigma_{\lambda}^{-}(b, c) + i\rho_{\lambda}^{-}(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b, c) + i\rho_{m}^{+}(b, c), \\ \sigma_{m}^{-}(b, c) + i\rho_{m}^{-}(b, c) \end{pmatrix}\right) \in \mathrm{R}.$$

Or

$$\left((c, b), \begin{pmatrix} \sigma_{\vartheta}^{+}(c, b) + i\rho_{\vartheta}^{+}(c, b), \\ \sigma_{\vartheta}^{-}(c, b) + i\rho_{\vartheta}^{-}(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c, b) + i\rho_{\lambda}^{+}(c, b), \\ \sigma_{\lambda}^{-}(c, b) + i\rho_{\lambda}^{-}(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c, b) + i\rho_{m}^{+}(c, b), \\ \sigma_{m}^{-}(c, b) + i\rho_{m}^{-}(c, b) \end{pmatrix}\right) \in \mathrm{R}.$$

Definition 28 A BCSFR R is a BCS linear order FR on a BCSFS Q if

- BCS reflexive FR
- BCS antisymmetric FR
- BCS transitive FR
- BCS complete FR

Definition 32 Let R_1 and R_2 be the BCSFR on a BCSFS. Then, the BCS composite FR $R_1 \circ R_2$ is expressed as

$$\left((b, c), \begin{pmatrix} \sigma_\vartheta^+(b, c) + i\rho_\vartheta^+(b, c), \\ \sigma_\vartheta^-(b, c) + i\rho_\vartheta^-(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_\lambda^+(b, c) + i\rho_\lambda^+(b, c), \\ \sigma_\lambda^-(b, c) + i\rho_\lambda^-(b, c) \end{pmatrix}, \begin{pmatrix} \sigma_m^+(b, c) + i\rho_m^+(b, c), \\ \sigma_m^-(b, c) + i\rho_m^-(b, c) \end{pmatrix}\right) \in \mathrm{R}_1$$

$$\text{and}\left((c, d), \begin{pmatrix} \sigma_\vartheta^+(c, d) + i\rho_\vartheta^+(c, d), \\ \sigma_\vartheta^-(c, d) + i\rho_\vartheta^-(c, d) \end{pmatrix}, \begin{pmatrix} \sigma_\lambda^+(c, d) + i\rho_\lambda^+(c, d), \\ \sigma_\lambda^-(c, d) + i\rho_\lambda^-(c, d) \end{pmatrix}, \begin{pmatrix} \sigma_m^+(c, d) + i\rho_m^+(c, d), \\ \sigma_m^-(c, d) + i\rho_m^-(c, d) \end{pmatrix}\right) \in \mathrm{R}_2$$

$$\Rightarrow \left((b, d), \begin{pmatrix} \sigma_\vartheta^+(b, d) + i\rho_\vartheta^+(b, d), \\ \sigma_\vartheta^-(b, d) + i\rho_\vartheta^-(b, d) \end{pmatrix}, \begin{pmatrix} \sigma_\lambda^+(b, d) + i\rho_\lambda^+(b, d), \\ \sigma_\lambda^-(b, d) + i\rho_\lambda^-(b, d) \end{pmatrix}, \begin{pmatrix} \sigma_m^+(b, d) + i\rho_m^+(b, d), \\ \sigma_m^-(b, d) + i\rho_m^-(b, d) \end{pmatrix}\right) \in \mathrm{R}_1 \circ \mathrm{R}_2.$$

Definition 29 Let BCSF equivalence class of b mod R is defined as

$$R[b] = \left\{ \begin{array}{c} \left(c, \begin{pmatrix} \sigma_\vartheta^+(c) + i\rho_\vartheta^+(c), \\ \sigma_\vartheta^-(c) + i\rho_\vartheta^-(c) \end{pmatrix}, \begin{pmatrix} \sigma_\lambda^+(c) + i\rho_\lambda^+(c), \\ \sigma_\lambda^-(c) + i\rho_\lambda^-(c) \end{pmatrix}, \begin{pmatrix} \sigma_m^+(c) + i\rho_m^+(c), \\ \sigma_m^-(c) + i\rho_m^-(c) \end{pmatrix}\right) : \\ \left((c, b), \begin{pmatrix} \sigma_\vartheta^+(c, b) + i\rho_\vartheta^+(c, b), \\ \sigma_\vartheta^-(c, b) + i\rho_\vartheta^-(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_\lambda^+(c, b) + i\rho_\lambda^+(c, b), \\ \sigma_\lambda^-(c, b) + i\rho_\lambda^-(c, b) \end{pmatrix}, \begin{pmatrix} \sigma_m^+(c, b) + i\rho_m^+(c, b), \\ \sigma_m^-(c, b) + i\rho_m^-(c, b) \end{pmatrix}\right) \in R \end{array} \right\}.$$

Example 7 Take a BCS-equivalence class from Example 2:

$$R = \left\{ \begin{array}{l} \left((t_1, t_1), \begin{pmatrix} 0.69 + 0.34i, \\ -0.21 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.59i, \\ -0.33 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.38 + 0.43i, \\ -0.23 - 0.29i \end{pmatrix}\right), \\ \left((t_2, t_2), \begin{pmatrix} 0.55 + 0.43i, \\ -0.13 - 0.25i \end{pmatrix}, \begin{pmatrix} 0.48 + 0.49i, \\ -0.30 - 0.38i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.30i, \\ -0.12 - 0.19i \end{pmatrix}\right), \\ \left((t_3, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.29 - 0.34i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.46i, \\ -0.04 - 0.33i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.25i, \\ -0.24 - 0.34i \end{pmatrix}\right), \\ \left((t_3, t_4), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix}\right), \\ \left((t_4, t_3), \begin{pmatrix} 0.49 + 0.43i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.30 + 0.36i, \\ -0.04 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.51 + 0.43i, \\ -0.24 - 0.34i \end{pmatrix}\right), \\ \left((t_4, t_4), \begin{pmatrix} 0.88 + 0.79i, \\ -0.23 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.35 + 0.36i, \\ -0.30 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.25 + 0.43i, \\ -0.12 - 0.12i \end{pmatrix}\right) \end{array} \right\}.$$

Then, modulo class of each element specified as

$$R[t_1]=\left\{\left(t_1,\begin{pmatrix}0.69+0.34i,\\-0.21-0.02i\end{pmatrix},\begin{pmatrix}0.45+0.59i,\\-0.33-0.17i\end{pmatrix},\begin{pmatrix}0.38+0.43i,\\-0.23-0.29i\end{pmatrix}\right)\right\}$$

$$R[t_2]=\left\{\left(t_2,\begin{pmatrix}0.55+0.43i,\\-0.13-0.25i\end{pmatrix},\begin{pmatrix}0.48+0.49i,\\-0.30-0.38i\end{pmatrix},\begin{pmatrix}0.29+0.30i,\\-0.12-0.19i\end{pmatrix}\right),\right\}$$

$$R[t_3]=\left\{\begin{array}{l}\left(t_2,\begin{pmatrix}0.55+0.43i,\\-0.13-0.25i\end{pmatrix},\begin{pmatrix}0.48+0.49i,\\-0.30-0.38i\end{pmatrix},\begin{pmatrix}0.29+0.30i,\\-0.12-0.19i\end{pmatrix}\right),\\\left(t_3,\begin{pmatrix}0.49+0.43i,\\-0.29-0.34i\end{pmatrix},\begin{pmatrix}0.30+0.46i,\\-0.04-0.33i\end{pmatrix},\begin{pmatrix}0.51+0.25i,\\-0.24-0.34i\end{pmatrix}\right)\end{array}\right\}$$

$$R[t_4]=\left\{\begin{array}{l}\left(t_3,\begin{pmatrix}0.49+0.43i,\\-0.29-0.34i\end{pmatrix},\begin{pmatrix}0.30+0.46i,\\-0.04-0.33i\end{pmatrix},\begin{pmatrix}0.51+0.25i,\\-0.24-0.34i\end{pmatrix}\right)\\\left(t_4,\begin{pmatrix}0.88+0.79i,\\-0.23-0.00i\end{pmatrix},\begin{pmatrix}0.35+0.36i,\\-0.30-0.29i\end{pmatrix},\begin{pmatrix}0.25+0.43i,\\-0.12-0.12i\end{pmatrix}\right)\end{array}\right\}$$

Theorem 1 *Let F be a BCSFR on X, and Rbe a BCSFR on F. Then, R is a BCS symmetric FR on F iff $R=R^{-1}$.*

Proof Assume that R is a BCS symmetric FR on F,

$$\begin{aligned}&\Rightarrow\left((b,c),\begin{pmatrix}\sigma_\vartheta^+(b,c)+i\rho_\vartheta^+(b,c),\\\sigma_\vartheta^-(b,c)+i\rho_\vartheta^-(b,c)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(b,c)+i\rho_\lambda^+(b,c),\\\sigma_\lambda^-(b,c)+i\rho_\lambda^-(b,c)\end{pmatrix},\begin{pmatrix}\sigma_m^+(b,c)+i\rho_m^+(b,c),\\\sigma_m^-(b,c)+i\rho_m^-(b,c)\end{pmatrix}\right)\in R\\&\quad\Longleftrightarrow\left((c,b),\begin{pmatrix}\sigma_\vartheta^+(c,b)+i\rho_\vartheta^+(c,b),\\\sigma_\vartheta^-(c,b)+i\rho_\vartheta^-(c,b)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(c,b)+i\rho_\lambda^+(c,b),\\\sigma_\lambda^-(c,b)+i\rho_\lambda^-(c,b)\end{pmatrix},\begin{pmatrix}\sigma_m^+(c,b)+i\rho_m^+(c,b),\\\sigma_m^-(c,b)+i\rho_m^-(c,b)\end{pmatrix}\right)\in R.\end{aligned}$$

In addition,

$$\begin{aligned}&\left((c,b),\begin{pmatrix}\sigma_\vartheta^+(c,b)+i\rho_\vartheta^+(c,b),\\\sigma_\vartheta^-(c,b)+i\rho_\vartheta^-(c,b)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(c,b)+i\rho_\lambda^+(c,b),\\\sigma_\lambda^-(c,b)+i\rho_\lambda^-(c,b)\end{pmatrix},\begin{pmatrix}\sigma_m^+(c,b)+i\rho_m^+(c,b),\\\sigma_m^-(c,b)+i\rho_m^-(c,b)\end{pmatrix}\right)\in R^{-1}\\&\quad\Rightarrow R=R^{-1}.\end{aligned}$$

Now, suppose that $R = R^{-1}$, then

$$\left((b,c), \begin{pmatrix} \sigma_{\vartheta}^{+}(b,c) + i\rho_{\vartheta}^{+}(b,c), \\ \sigma_{\vartheta}^{-}(b,c) + i\rho_{\vartheta}^{-}(b,c) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b,c) + i\rho_{\lambda}^{+}(b,c), \\ \sigma_{\lambda}^{-}(b,c) + i\rho_{\lambda}^{-}(b,c) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b,c) + i\rho_{m}^{+}(b,c), \\ \sigma_{m}^{-}(b,c) + i\rho_{m}^{-}(b,c) \end{pmatrix}\right) \in R$$

$$\Longleftrightarrow \left((c,b), \begin{pmatrix} \sigma_{\vartheta}^{+}(c,b) + i\rho_{\vartheta}^{+}(c,b), \\ \sigma_{\vartheta}^{-}(c,b) + i\rho_{\vartheta}^{-}(c,b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c,b) + i\rho_{\lambda}^{+}(c,b), \\ \sigma_{\lambda}^{-}(c,b) + i\rho_{\lambda}^{-}(c,b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c,b) + i\rho_{m}^{+}(c,b), \\ \sigma_{m}^{-}(c,b) + i\rho_{m}^{-}(c,b) \end{pmatrix}\right) \in R^{-1}$$

$$\Longleftrightarrow \left((c,b), \begin{pmatrix} \sigma_{\vartheta}^{+}(c,b) + i\rho_{\vartheta}^{+}(c,b), \\ \sigma_{\vartheta}^{-}(c,b) + i\rho_{\vartheta}^{-}(c,b) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c,b) + i\rho_{\lambda}^{+}(c,b), \\ \sigma_{\lambda}^{-}(c,b) + i\rho_{\lambda}^{-}(c,b) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c,b) + i\rho_{m}^{+}(c,b), \\ \sigma_{m}^{-}(c,b) + i\rho_{m}^{-}(c,b) \end{pmatrix}\right) \in R$$

$\Rightarrow$ $R$is a BCS symmetric FR on F.

Theorem 2 *Let F be a BCSFS on X and Rbe a BCSFR on X. Then, $R$is a BCSF transitive FR on F iff $R \circ R \subseteq R$.*

Proof Assume that R is a BCS transitive FR on a BCSFS F.

Let

$$\left((b,d), \begin{pmatrix} \sigma_{\vartheta}^{+}(b,d) + i\rho_{\vartheta}^{+}(b,d), \\ \sigma_{\vartheta}^{-}(b,d) + i\rho_{\vartheta}^{-}(b,d) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b,d) + i\rho_{\lambda}^{+}(b,d), \\ \sigma_{\lambda}^{-}(b,d) + i\rho_{\lambda}^{-}(b,d) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b,d) + i\rho_{m}^{+}(b,d), \\ \sigma_{m}^{-}(b,d) + i\rho_{m}^{-}(b,d) \end{pmatrix}\right) \in R \circ R$$

For any

$$\left((b,c), \begin{pmatrix} \sigma_{\vartheta}^{+}(b,c) + i\rho_{\vartheta}^{+}(b,c), \\ \sigma_{\vartheta}^{-}(b,c) + i\rho_{\vartheta}^{-}(b,c) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b,c) + i\rho_{\lambda}^{+}(b,c), \\ \sigma_{\lambda}^{-}(b,c) + i\rho_{\lambda}^{-}(b,c) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b,c) + i\rho_{m}^{+}(b,c), \\ \sigma_{m}^{-}(b,c) + i\rho_{m}^{-}(b,c) \end{pmatrix}\right) \in R$$

$$\text{and}\left((c,d), \begin{pmatrix} \sigma_{\vartheta}^{+}(c,d) + i\rho_{\vartheta}^{+}(c,d), \\ \sigma_{\vartheta}^{-}(c,d) + i\rho_{\vartheta}^{-}(c,d) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(c,d) + i\rho_{\lambda}^{+}(c,d), \\ \sigma_{\lambda}^{-}(c,d) + i\rho_{\lambda}^{-}(c,d) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(c,d) + i\rho_{m}^{+}(c,d), \\ \sigma_{m}^{-}(c,d) + i\rho_{m}^{-}(c,d) \end{pmatrix}\right) \in R$$

$$\Rightarrow \left((b,d), \begin{pmatrix} \sigma_{\vartheta}^{+}(b,d) + i\rho_{\vartheta}^{+}(b,d), \\ \sigma_{\vartheta}^{-}(b,d) + i\rho_{\vartheta}^{-}(b,d) \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda}^{+}(b,d) + i\rho_{\lambda}^{+}(b,d), \\ \sigma_{\lambda}^{-}(b,d) + i\rho_{\lambda}^{-}(b,d) \end{pmatrix}, \begin{pmatrix} \sigma_{m}^{+}(b,d) + i\rho_{m}^{+}(b,d), \\ \sigma_{m}^{-}(b,d) + i\rho_{m}^{-}(b,d) \end{pmatrix}\right) \in R$$

$\Rightarrow R \circ R \subseteq R$.

Conversely, suppose that $R \circ R \subseteq R$ then by BCS composite FR,

$$\left((b,\,c),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,\,c)+i\rho_{\vartheta}^{+}(b,\,c),\\ \sigma_{\vartheta}^{-}(b,\,c)+i\rho_{\vartheta}^{-}(b,\,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,\,c)+i\rho_{\lambda}^{+}(b,\,c),\\ \sigma_{\lambda}^{-}(b,\,c)+i\rho_{\lambda}^{-}(b,\,c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,\,c)+i\rho_{m}^{+}(b,\,c),\\ \sigma_{m}^{-}(b,\,c)+i\rho_{m}^{-}(b,\,c)\end{pmatrix}\right)\in R$$

$$\text{and}\left((c,\,d),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(c,\,d)+i\rho_{\vartheta}^{+}(c,\,d),\\ \sigma_{\vartheta}^{-}(c,\,d)+i\rho_{\vartheta}^{-}(c,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c,\,d)+i\rho_{\lambda}^{+}(c,\,d),\\ \sigma_{\lambda}^{-}(c,\,d)+i\rho_{\lambda}^{-}(c,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c,\,d)+i\rho_{m}^{+}(c,\,d),\\ \sigma_{m}^{-}(c,\,d)+i\rho_{m}^{-}(c,\,d)\end{pmatrix}\right)\in R$$

$$\Rightarrow\left((b,\,d),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,\,d)+i\rho_{\vartheta}^{+}(b,\,d),\\ \sigma_{\vartheta}^{-}(b,\,d)+i\rho_{\vartheta}^{-}(b,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,\,d)+i\rho_{\lambda}^{+}(b,\,d),\\ \sigma_{\lambda}^{-}(b,\,d)+i\rho_{\lambda}^{-}(b,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,\,d)+i\rho_{m}^{+}(b,\,d),\\ \sigma_{m}^{-}(b,\,d)+i\rho_{m}^{-}(b,\,d)\end{pmatrix}\right)\in R\circ R.$$

Since $R \circ R \subseteq R$

$$\Rightarrow\left((b,\,d),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,\,d)+i\rho_{\vartheta}^{+}(b,\,d),\\ \sigma_{\vartheta}^{-}(b,\,d)+i\rho_{\vartheta}^{-}(b,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,\,d)+i\rho_{\lambda}^{+}(b,\,d),\\ \sigma_{\lambda}^{-}(b,\,d)+i\rho_{\lambda}^{-}(b,\,d)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,\,d)+i\rho_{\lambda}^{+}(b,\,d),\\ \sigma_{m}^{-}(b,\,d)+i\rho_{m}^{-}(b,\,d)\end{pmatrix}\right)\in R.$$

Thus, R is a BCS transitive FR.

Theorem 3 *Let Qbe a BCSFS on $X$and Rbe a BCS equivalence relation on Q.*

Then, $R \circ R = R$.

Proof Assume R is a BCS equivalence FR on Q, for

$$\left((b,\,c),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,\,c)+i\rho_{\vartheta}^{+}(b,\,c),\\ \sigma_{\vartheta}^{-}(b,\,c)+i\rho_{\vartheta}^{-}(b,\,c)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,\,c)+i\rho_{\lambda}^{+}(b,\,c),\\ \sigma_{\lambda}^{-}(b,\,c)+i\rho_{\lambda}^{-}(b,\,c)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,\,c)+i\rho_{m}^{+}(b,\,c),\\ \sigma_{m}^{-}(b,\,c)+i\rho_{m}^{-}(b,\,c)\end{pmatrix}\right)\in R.$$

Then, by the definition of symmetric

$$\left((c,\,b),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(c,\,b)+i\rho_{\vartheta}^{+}(c,\,b),\\ \sigma_{\vartheta}^{-}(c,\,b)+i\rho_{\vartheta}^{-}(c,\,b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(c,\,b)+i\rho_{\lambda}^{+}(c,\,b),\\ \sigma_{\lambda}^{-}(c,\,b)+i\rho_{\lambda}^{-}(c,\,b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(c,\,b)+i\rho_{m}^{+}(c,\,b),\\ \sigma_{m}^{-}(c,\,b)+i\rho_{m}^{-}(c,\,b)\end{pmatrix}\right)\in R.$$

Now, by the definition of transitivity,

$$\left((b,\,b),\ \begin{pmatrix}\sigma_{\vartheta}^{+}(b,\,b)+i\rho_{\vartheta}^{+}(b,\,b),\\ \sigma_{\vartheta}^{-}(b,\,b)+i\rho_{\vartheta}^{-}(b,\,b)\end{pmatrix},\begin{pmatrix}\sigma_{\lambda}^{+}(b,\,b)+i\rho_{\lambda}^{+}(b,\,b),\\ \sigma_{\lambda}^{-}(b,\,b)+i\rho_{\lambda}^{-}(b,\,b)\end{pmatrix},\begin{pmatrix}\sigma_{m}^{+}(b,\,b)+i\rho_{m}^{+}(b,\,b),\\ \sigma_{m}^{-}(b,\,b)+i\rho_{m}^{-}(b,\,b)\end{pmatrix}\right)\in R.$$

But, according to the definition of BCS composite FR,

$$\left((b,b),\begin{pmatrix}\sigma_\vartheta^+(b,b)+i\rho_\vartheta^+(b,b),\\ \sigma_\vartheta^-(b,b)+i\rho_\vartheta^-(b,b)\end{pmatrix},\right.$$
$$\left.\times\begin{pmatrix}\sigma_\lambda^+(b,b)+i\rho_\lambda^+(b,b),\\ \sigma_\lambda^-(b,b)+i\rho_\lambda^-(b,b)\end{pmatrix},\begin{pmatrix}\sigma_m^+(b,b)+i\rho_m^+(b,b),\\ \sigma_m^-(b,b)+i\rho_m^-(b,b)\end{pmatrix}\right)$$
$$\in R\circ R.\Rightarrow R\subseteq R\circ R. \tag{20}$$

Conversely, suppose that

$$\left((b,d),\begin{pmatrix}\sigma_\vartheta^+(b,d)+i\rho_\vartheta^+(b,d),\\ \sigma_\vartheta^-(b,d)+i\rho_\vartheta^-(b,d)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(b,d)+i\rho_\lambda^+(b,d),\\ \sigma_\lambda^-(b,d)+i\rho_\lambda^-(b,d)\end{pmatrix},\right.$$
$$\left.\times\begin{pmatrix}\sigma_m^+(b,d)+i\rho_m^+(b,d),\\ \sigma_m^-(b,d)+i\rho_m^-(b,d)\end{pmatrix}\right)\in R\circ R.$$

Then, $\exists\, c\in X \ni$

$$\left((b,c),\begin{pmatrix}\sigma_\vartheta^+(b,c)+i\rho_\vartheta^+(b,c),\\ \sigma_\vartheta^-(b,c)+i\rho_\vartheta^-(b,c)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(b,c)+i\rho_\lambda^+(b,c),\\ \sigma_\lambda^-(b,c)+i\rho_\lambda^-(b,c)\end{pmatrix},\begin{pmatrix}\sigma_m^+(b,c)+i\rho_m^+(b,c),\\ \sigma_m^-(b,c)+i\rho_m^-(b,c)\end{pmatrix}\right)\in R$$
$$\text{and}\left((c,d),\begin{pmatrix}\sigma_\vartheta^+(c,d)+i\rho_\vartheta^+(c,d),\\ \sigma_\vartheta^-(c,d)+i\rho_\vartheta^-(c,d)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(c,d)+i\rho_\lambda^+(c,d),\\ \sigma_\lambda^-(c,d)+i\rho_\lambda^-(c,d)\end{pmatrix},\begin{pmatrix}\sigma_m^+(c,d)+i\rho_m^+(c,d),\\ \sigma_m^-(c,d)+i\rho_m^-(c,d)\end{pmatrix}\right)\in R$$
$$\Rightarrow\left((b,d),\begin{pmatrix}\sigma_\vartheta^+(b,d)+i\rho_\vartheta^+(b,d),\\ \sigma_\vartheta^-(b,d)+i\rho_\vartheta^-(b,d)\end{pmatrix},\begin{pmatrix}\sigma_\lambda^+(b,d)+i\rho_\lambda^+(b,d),\\ \sigma_\lambda^-(b,d)+i\rho_\lambda^-(b,d)\end{pmatrix},\begin{pmatrix}\sigma_m^+(b,d)+i\rho_m^+(b,d),\\ \sigma_m^-(b,d)+i\rho_m^-(b,d)\end{pmatrix}\right)\in R,$$

because $R$is a BCS equivalence FR and thus BCS transitive relation, i.e.,

$$R\circ R\subseteq R \tag{21}$$

Equations (20) and (21) imply that $R\circ R=R$.

Application

This section introduces an application based on the unique structure of BCSFR. There is a wide range of the application of the neural network into the different sectors [47–51].

Artificial neural network (ANN)

ANN is an organically stimulated network of artificial neutrons constructed to accomplish definite tasks. It is based on an assembly of connected units called artificial neurons, which loosely model the neurons in a biological brain (See Fig. 1). It also used in machine learning that work in alike fashion to the human nervous system. ANN uses the processing of the brain as a basis to develop procedures that can be used to model complex problems and prediction problems. There is a control unit which monitors all other activities of computing. They have hidden layers and data enter through input nodes and exit through output nodes. Figure 1 shows the work procedures of ANN.

There are several kinds of ANN and is used in business for different purposes. In terms of utility, it has become competitive with traditional regression and statistical models. Many organizations are investing in the ANN to solve problem in variety of industries. Figure 2 briefly explains the algorithm of preceding application.

Some types of ANN are discussed below, each value of ANN defines the three levels i.e., MG describes the effectiveness, AG means no effect and non-MG means ineffectiveness. The positive and negative values show the possibility and impossibility of all stages.

i. **Feedforward neural network (FNN):** FNN is used relations between independent variables, which assist as inputs to the network, and dependent variables that are chosen as crops of the network. The data pass through the input nodes and exit on the output nodes. These are found in computer vision and speech recognition where categorizing the target classes is intricate:

$$\left(n_1,\begin{pmatrix}0.39+0.45i,\\ -0.31-0.12i\end{pmatrix},\begin{pmatrix}0.65+0.49i,\\ -0.39-0.29i\end{pmatrix},\begin{pmatrix}0.33+0.32i,\\ -0.43-0.15i\end{pmatrix}\right).$$

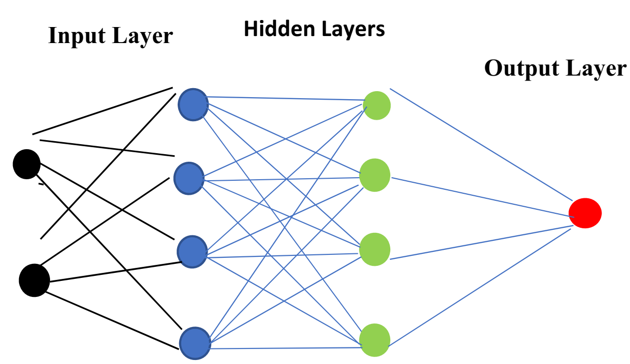

Fig. 1 Procedures of ANN

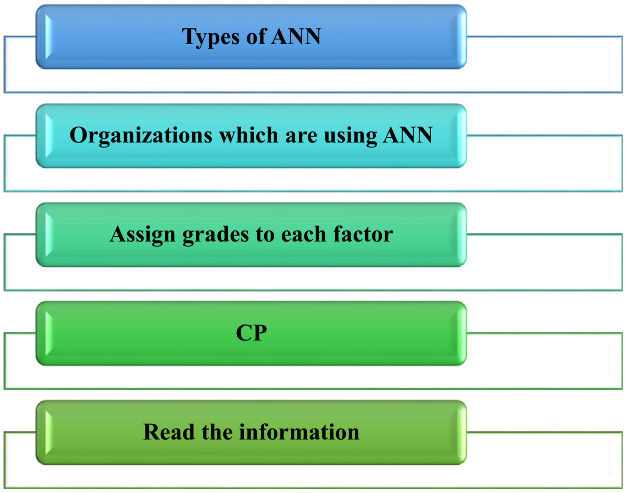

Fig. 2 Algorithm of application

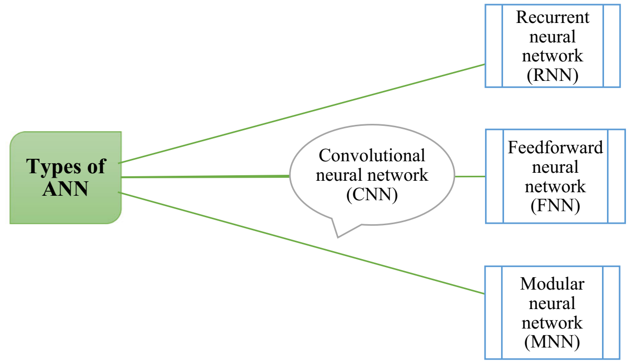

Fig. 3 Summary of ANN types

The value n_1 corresponds to the value of FNN and describes possibility and impossibility of each level.

ii. **Recurrent neural network (RNN):** The RNN works on the principle of saving the output of a layer and feeding this back to the input to help in foreseeing the outcome of the layer. In the computation process, each neuron will act as a memory cell. The RNN can originate in text to speech conversion model.

$$\left(n_2, \begin{pmatrix} 0.29 + 0.22i, \\ -0.33 - 0.39i \end{pmatrix}, \begin{pmatrix} 0.44 + 0.38i, \\ -0.29 - 0.27i \end{pmatrix}, \begin{pmatrix} 0.58 + 0.44i, \\ -0.39 - 0.09i \end{pmatrix}\right).$$

The second value n_2 explains the numerical value of RNN given by experts on the basis of performance and working.

iii. **Modular neural network (MNN):** The MNN has been used to break a large computational process into smaller components by declining the complication. Several independent neural networks are trained concurrently for a specific sub task and their results are united at the end to perform a single task:

$$\left(n_3, \begin{pmatrix} 0.54 + 0.37i, \\ -0.33 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.44 + 0.44i, \\ -0.22 - 0.40i \end{pmatrix}, \begin{pmatrix} 0.28 + 0.01i, \\ -0.49 - 0.34i \end{pmatrix}\right).$$

The above value explains the MG, AG and non-MG with effect of advantage and disadvantage corresponding to MNN.

iv. **Convolutional neural network (CNN):** CNN has been used in signal and image processing which takes over open CV in the field of computer vision. CNN helps the network to recall the images in parts and can calculate the processes. CNN is used to reduce the images into a form which is easier to process, without losing structures which are critical for getting a good calculation. It is widely used for image recognition:

$$\left(n_4, \begin{pmatrix} 0.88 + 0.34i, \\ -0.22 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.59i, \\ -0.41 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.10 + 0.43i, \\ -0.57 - 0.29i \end{pmatrix}\right).$$

Now, summary of all types of ANN is shown in Fig. 3. Let the set corresponding to value of each ANN be

$$G = \left\{\begin{array}{l} \left(n_1, \begin{pmatrix} 0.39 + 0.45i, \\ -0.31 - 0.12i \end{pmatrix}, \begin{pmatrix} 0.65 + 0.49i, \\ -0.39 - 0.29i \end{pmatrix}, \begin{pmatrix} 0.33 + 0.32i, \\ -0.43 - 0.15i \end{pmatrix}\right), \\ \left(n_2, \begin{pmatrix} 0.29 + 0.22i, \\ -0.33 - 0.39i \end{pmatrix}, \begin{pmatrix} 0.44 + 0.38i, \\ -0.29 - 0.27i \end{pmatrix}, \begin{pmatrix} 0.58 + 0.44i, \\ -0.39 - 0.09i \end{pmatrix}\right), \\ \left(n_3, \begin{pmatrix} 0.54 + 0.37i, \\ -0.33 - 0.00i \end{pmatrix}, \begin{pmatrix} 0.44 + 0.44i, \\ -0.22 - 0.40i \end{pmatrix}, \begin{pmatrix} 0.28 + 0.01i, \\ -0.49 - 0.34i \end{pmatrix}\right), \\ \left(n_4, \begin{pmatrix} 0.88 + 0.34i, \\ -0.22 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.59i, \\ -0.41 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.10 + 0.43i, \\ -0.57 - 0.29i \end{pmatrix}\right) \end{array}\right\}.$$

Table 1 Summary of organizations

Organizations	Abbreviations
Security	λ_1
eCommerce	λ_2
Finance	λ_3
Education	λ_4

Organizations that are heavily based on ANN

ANN has the ability to learn and model non-linear relations, that is actually significant in real life. There are many performance developments in various organizations due to ANN. Table 1 shows some organizations that are taken into consideration.

i. **Security** (λ_1)**:** ANN is often the perfect candidate and processes that rely on security. It is widely used for protection from computer viruses, fraud etc. ANN is designed to protect your network for unauthorized access and the latest cyber threats. λ

$$\left(\lambda_1, \begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.26i \end{pmatrix}, \begin{pmatrix} 0.59 + 0.40i, \\ -0.29 - 0.10i \end{pmatrix}, \begin{pmatrix} 0.46 + 0.29i, \\ -0.30 - 0.50i \end{pmatrix}\right).$$

ii. **eCommerce** (λ_2)**:** This technology is used in this organizations for various purposes. ANN in eCommerce is personalizing the purchaser experience. CNN used to resize the images of the items, and analyzes the characteristics of certain items and shows similar ones:

$$\left(\lambda_2, \begin{pmatrix} 0.67 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}, \begin{pmatrix} 0.49 + 0.20i, \\ -0.19 - 0.44i \end{pmatrix}, \begin{pmatrix} 0.37 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}\right).$$

iii. **Finance** (λ_3)**:** ANN helps banks to find solutions for business issues analyzing risk and probable profits. ANN is used for fraud detection, management and forecasting in finance industry.

$$\left(\lambda_3, \begin{pmatrix} 0.80 + 0.43i, \\ -0.19 - 0.05i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.09i, \\ -0.59 - 0.27i \end{pmatrix}, \begin{pmatrix} 0.23 + 0.50i, \\ -0.41 - 0.69i \end{pmatrix}\right).$$

iv. **Education** (λ_4)**:** ANN is a relatively new methodological approach in the areas of learning and education. ANN discusses recommendations for implementing and several considerations for the analysis of academic performance in higher education:

$$\left(\lambda_4, \begin{pmatrix} 0.74 + 0.81i, \\ -0.14 - 0.26i \end{pmatrix}, \begin{pmatrix} 0.33 + 0.49i, \\ -0.60 - 0.32i \end{pmatrix}, \begin{pmatrix} 0.21 + 0.20i, \\ -0.36 - 0.55i \end{pmatrix}\right).$$

Assign MG, AG and non-MG to each organization and their corresponding set is as follows:

$$H = \left\{ \begin{array}{l} \left(\lambda_1, \begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.26i \end{pmatrix}, \begin{pmatrix} 0.59 + 0.40i, \\ -0.29 - 0.10i \end{pmatrix}, \begin{pmatrix} 0.46 + 0.29i, \\ -0.30 - 0.50i \end{pmatrix}\right), \\ \left(\lambda_2, \begin{pmatrix} 0.67 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}, \begin{pmatrix} 0.49 + 0.20i, \\ -0.19 - 0.44i \end{pmatrix}, \begin{pmatrix} 0.37 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}\right), \\ \left(\lambda_3, \begin{pmatrix} 0.80 + 0.43i, \\ -0.19 - 0.05i \end{pmatrix}, \begin{pmatrix} 0.45 + 0.09i, \\ -0.59 - 0.27i \end{pmatrix}, \begin{pmatrix} 0.23 + 0.50i, \\ -0.41 - 0.69i \end{pmatrix}\right), \\ \left(\lambda_4, \begin{pmatrix} 0.74 + 0.81i, \\ -0.14 - 0.26i \end{pmatrix}, \begin{pmatrix} 0.33 + 0.49i, \\ -0.60 - 0.32i \end{pmatrix}, \begin{pmatrix} 0.21 + 0.20i, \\ -0.36 - 0.55i \end{pmatrix}\right) \end{array} \right\}.$$

Then, CP of G and H represented in Table 2 shows the BCSFRs between ANN and different organizations to get the more verified results.

Every ordered pair CP G $\times$ H expresses the impact of one element on the other. Since, $\left((n_4, \lambda_3), \begin{pmatrix} 0.80 + 0.34i, \\ -0.19 - 0.02i \end{pmatrix}, \begin{pmatrix} 0.29 + 0.09i, \\ -0.41 - 0.17i \end{pmatrix}, \begin{pmatrix} 0.23 + 0.50i, \\ -0.57 - 0.69i \end{pmatrix}\right)$ explains that finance organization is more developed using the convolutional neural network, i.e., effectiveness shows higher possibility of 0.80 with duration 0.34, and − 0.19

Table 2 Cartesian product

Ordered pair	MG	AG	Non-MG
(n_1, λ_1)	$\begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.59 + 0.40i, \\ -0.29 - 0.10i \end{pmatrix}$	$\begin{pmatrix} 0.46 + 0.32i, \\ -0.43 - 0.50i \end{pmatrix}$
(n_1, λ_2)	$\begin{pmatrix} 0.39 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.49 + 0.20i, \\ -0.19 - 0.29i \end{pmatrix}$	$\begin{pmatrix} 0.37 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}$
(n_1, λ_3)	$\begin{pmatrix} 0.39 + 0.43i, \\ -0.31 - 0.05i \end{pmatrix}$	$\begin{pmatrix} 0.45 + 0.09i, \\ -0.39 - 0.29i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.50i, \\ -0.43 - 0.69i \end{pmatrix}$
(n_1, λ_4)	$\begin{pmatrix} 0.39 + 0.45i, \\ -0.14 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.49i, \\ -0.39 - 0.29i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.32i, \\ -0.43 - 0.55i \end{pmatrix}$
(n_2, λ_1)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.11 - 0.26i \end{pmatrix}$	$\begin{pmatrix} 0.44 + 0.38i, \\ -0.29 - 0.10i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.44i, \\ -0.39 - 0.50i \end{pmatrix}$
(n_2, λ_2)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.24 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.44 + 0.20i, \\ -0.19 - 0.27i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}$
(n_2, λ_3)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.19 - 0.05i \end{pmatrix}$	$\begin{pmatrix} 0.44 + 0.09i, \\ -0.29 - 0.27i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.50i, \\ -0.41 - 0.69i \end{pmatrix}$
(n_2, λ_4)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.14 - 0.26i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.38i, \\ -0.29 - 0.27i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.44i, \\ -0.39 - 0.55i \end{pmatrix}$
(n_3, λ_1)	$\begin{pmatrix} 0.31 + 0.37i, \\ -0.11 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.40 + 0.40i, \\ -0.22 - 0.10i \end{pmatrix}$	$\begin{pmatrix} 0.46 + 0.29i, \\ -0.49 - 0.50i \end{pmatrix}$
(n_3, λ_2)	$\begin{pmatrix} 0.54 + 0.31i, \\ -0.24 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.40 + 0.20i, \\ -0.19 - 0.40i \end{pmatrix}$	$\begin{pmatrix} 0.37 + 0.57i, \\ -0.49 - 0.73i \end{pmatrix}$
(n_3, λ_3)	$\begin{pmatrix} 0.54 + 0.37i, \\ -0.19 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.40 + 0.09i, \\ -0.22 - 0.27i \end{pmatrix}$	$\begin{pmatrix} 0.28 + 0.50i, \\ -0.49 - 0.69i \end{pmatrix}$
(n_3, λ_4)	$\begin{pmatrix} 0.54 + 0.37i, \\ -0.14 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.44i, \\ -0.22 - 0.32i \end{pmatrix}$	$\begin{pmatrix} 0.28 + 0.20i, \\ -0.49 - 0.55i \end{pmatrix}$
(n_4, λ_1)	$\begin{pmatrix} 0.31 + 0.34i, \\ -0.11 - 0.02i \end{pmatrix}$	$\begin{pmatrix} 0.29 + 0.40i, \\ -0.29 - 0.10i \end{pmatrix}$	$\begin{pmatrix} 0.46 + 0.43i, \\ -0.57 - 0.50i \end{pmatrix}$
(n_4, λ_2)	$\begin{pmatrix} 0.67 + 0.31i, \\ -0.22 - 0.02i \end{pmatrix}$	$\begin{pmatrix} 0.29 + 0.20i, \\ -0.19 - 0.17i \end{pmatrix}$	$\begin{pmatrix} 0.37 + 0.57i, \\ -0.57 - 0.73i \end{pmatrix}$
(n_4, λ_3)	$\begin{pmatrix} 0.80 + 0.34i, \\ -0.19 - 0.02i \end{pmatrix}$	$\begin{pmatrix} 0.29 + 0.09i, \\ -0.41 - 0.17i \end{pmatrix}$	$\begin{pmatrix} 0.23 + 0.50i, \\ -0.57 - 0.69i \end{pmatrix}$
(n_4, λ_4)	$\begin{pmatrix} 0.74 + 0.34i, \\ -0.14 - 0.02i \end{pmatrix}$	$\begin{pmatrix} 0.29 + 0.49i, \\ -0.41 - 0.17i \end{pmatrix}$	$\begin{pmatrix} 0.21 + 0.43i, \\ -0.57 - 0.55i \end{pmatrix}$

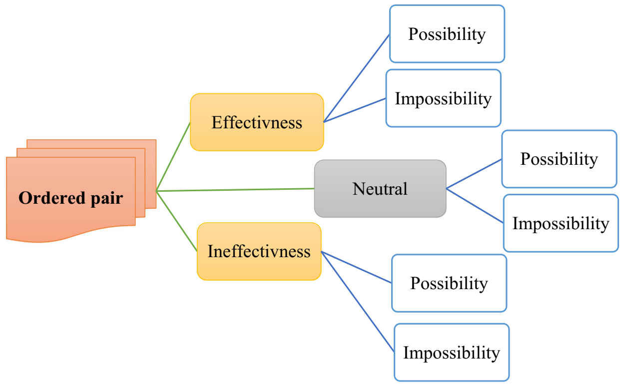

Fig. 4 Information of ordered pair

explains the lower impossibility of effectiveness with − 0.02 duration. The 0.29 shows the possibility of AG with lower time period 0.09, and − 0.41 explain the impossibility of AG with time phase − 0.17. In the same way possibility of higher ineffectiveness 0.23 and higher time duration 0.50, and − 0.14 explain the lower impossibility with − 0.57 duration. This means that finance organizations deals more work effectively using CNN. Finance industry due to ANN finds solutions for every organization problem. Hence, organizations are more improved using the ANN. All ordered pair information are explained in Fig. 4.

Comparative analysis

In this section, the newly structure of BCSFR is compared with the pre-defined structures such as FR, BFR, BCFR, BIFR, BCIFR, BPyFR, BCPyFR, BqROFR, BCqROFR, BPFR, BCPFR, and BSFR.

FR, BFR, BCFR with BCSFR

FR and BFR tells only the MG with only one dimension. They are not capable of solving the multidimensional problem. The FR describes only the positive MG and do not deal the negative factor of MG. The BCFR discusses only MG with the both effect of positive and negative. The BCFR describes the PT. But BCSFR discusses the all levels of stages with both effects. Here, take two sets G and H from application with BCFRs:

$$G = \left\{\left(n_1, \begin{pmatrix} 0.39 + 0.45i, \\ -0.31 - 0.12i \end{pmatrix}\right), \left(n_2, \begin{pmatrix} 0.29 + 0.22i, \\ -0.33 - 0.39i \end{pmatrix}\right), \left(n_3, \begin{pmatrix} 0.54 + 0.37i, \\ -0.33 - 0.00i \end{pmatrix}\right)\right\}.$$

In addition,

Table 3 Cartesian product

Ordered pair	MG
(n_1, λ_1)	$\begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.12i \end{pmatrix}$
(n_1, λ_2)	$\begin{pmatrix} 0.39 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}$
(n_1, λ_3)	$\begin{pmatrix} 0.39 + 0.43i, \\ -0.31 - 0.05i \end{pmatrix}$
(n_2, λ_1)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.11 - 0.26i \end{pmatrix}$
(n_2, λ_2)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.24 - 0.12i \end{pmatrix}$
(n_2, λ_3)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.19 - 0.05i \end{pmatrix}$
(n_3, λ_1)	$\begin{pmatrix} 0.31 + 0.37i, \\ -0.11 - 0.00i \end{pmatrix}$
(n_3, λ_2)	$\begin{pmatrix} 0.54 + 0.31i, \\ -0.24 - 0.00i \end{pmatrix}$
(n_3, λ_3)	$\begin{pmatrix} 0.54 + 0.37i, \\ -0.19 - 0.00i \end{pmatrix}$

$$H = \left\{\left(\lambda_1, \begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.26i \end{pmatrix}\right), \left(\lambda_2, \begin{pmatrix} 0.67 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}\right), \left(\lambda_3, \begin{pmatrix} 0.80 + 0.43i, \\ -0.19 - 0.05i \end{pmatrix}\right)\right\}.$$

Then, their CP is shown in Table 3.

BCFR shows only the MG with both positive and negative possibility. Ordered pair explains the effectiveness of the first factor on the second. Hence, these structures give the incomplete information, and BCSFR provides the complete information.

IFR, BIFR, CIBFR, BCPyFR, and BCqROFR with BCSFR

IFR, BIFR, CIBFR, BCPyFR and BCqROFR describe the MG and non-MG. These structures do no discuss the AG of one factor on the other. The IFR defines the effectiveness and ineffectiveness of only the possibility conditions. The BIFR discusses the both conditions of possibility and impossibility but is not capable to solve the multivariable difficulties. The BCPyFR and BCqROFR increased the space of BCIFR. But these structures do not discuss about the AG. The BCSFR is improved form of all these structures because it also describes the AG. Here, take two sets G and H from "Application" with BCIFRs:

$$G=\left\{\begin{array}{c}\left(n_1,\begin{pmatrix}0.39+0.45i,\\-0.31-0.12i\end{pmatrix}\begin{pmatrix}0.33+0.32i,\\-0.43-0.15i\end{pmatrix}\right),\left(n_2,\begin{pmatrix}0.29+0.22i,\\-0.33-0.39i\end{pmatrix},\begin{pmatrix}0.58+0.44i,\\-0.39-0.09i\end{pmatrix}\right)\\\left(n_3,\begin{pmatrix}0.54+0.37i,\\-0.33-0.00i\end{pmatrix},\begin{pmatrix}0.28+0.01i,\\-0.49-0.34i\end{pmatrix}\right)\end{array}\right\}.$$

In addition,

$$H=\left\{\begin{array}{c}\left(\lambda_1,\begin{pmatrix}0.31+0.42i,\\-0.11-0.26i\end{pmatrix},\begin{pmatrix}0.46+0.29i,\\-0.30-0.50i\end{pmatrix}\right),\left(\lambda_2,\begin{pmatrix}0.67+0.31i,\\-0.24-0.12i\end{pmatrix},\begin{pmatrix}0.37+0.57i,\\-0.48-0.73i\end{pmatrix}\right)\\\left(\lambda_3,\begin{pmatrix}0.80+0.43i,\\-0.19-0.05i\end{pmatrix},\begin{pmatrix}0.23+0.50i,\\-0.41-0.69i\end{pmatrix}\right)\end{array}\right\}.$$

Then, their CP is explained in Table 4.

BCIFR expresses the effectiveness and ineffectiveness of each ordered pair. They have lack of AG, so they give the uncompleted information. But BCSFR also shows the AG of each ordered pair.

PFR, BPFR, BCPFR, and BSFR with BCSFR

These structures discuss the all levels of s, i.e., MG, AG and non-MG. The PFR defines only the positive values of all levels of stages and do not discuss the negative effects. The BPFR, BCPFR and BSFR describe all levels of stages with both effects of positive and negative. BPFR and BSFR are not capable to solve the complex problem. It means that BCSFR is superior to all pre-defined structures because these discuses all levels of stages with both positive and negative effects. Here, take two sets G and H from "Application" with BSFRs:

$$G=\left\{\begin{array}{l}\left(n_1,\begin{pmatrix}0.39,\\-0.31\end{pmatrix},\begin{pmatrix}0.65,\\-0.39\end{pmatrix},\begin{pmatrix}0.33,\\-0.43\end{pmatrix}\right),\\\left(n_2,\begin{pmatrix}0.29,\\-0.33\end{pmatrix},\begin{pmatrix}0.44,\\-0.29\end{pmatrix},\begin{pmatrix}0.58,\\-0.39\end{pmatrix}\right),\\\left(n_3,\begin{pmatrix}0.54,\\-0.33\end{pmatrix},\begin{pmatrix}0.44,\\-0.22\end{pmatrix},\begin{pmatrix}0.28,\\-0.49\end{pmatrix}\right),\end{array}\right\},$$

$$H=\left\{\begin{array}{l}\left(\lambda_1,\begin{pmatrix}0.31,\\-0.11\end{pmatrix},\begin{pmatrix}0.59,\\-0.29\end{pmatrix},\begin{pmatrix}0.46,\\-0.30\end{pmatrix}\right),\\\left(\lambda_2,\begin{pmatrix}0.67,\\-0.24\end{pmatrix},\begin{pmatrix}0.49,\\-0.19\end{pmatrix},\begin{pmatrix}0.37,\\-0.48\end{pmatrix}\right),\\\left(\lambda_3,\begin{pmatrix}0.80,\\-0.19\end{pmatrix},\begin{pmatrix}0.45,\\-0.59\end{pmatrix},\begin{pmatrix}0.23,\\-0.41\end{pmatrix}\right),\end{array}\right\}.$$

Then, CP of G and H represented in Table 5 shows the BSFRs between ANN and different organizations to get the more verified results.

BSFR expresses the effectiveness of no effect and ineffectiveness of each ordered pair. They have lack of time frame, so they give the uncompleted information. But BCSFR also shows the time duration of each ordered pair. Table 6 gives the comparative analysis of BCSFR with pre-existing frameworks.

The advantages of the proposed method

The following are the advantages of the proposed method:

Table 4 Cartesian product of G and H in terms of membership and non-membership

Ordered pair	MG	Non-membership
(n_1, λ_1)	$\begin{pmatrix} 0.31 + 0.42i, \\ -0.11 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.46 + 0.32i, \\ -0.43 - 0.50i \end{pmatrix}$
(n_1, λ_2)	$\begin{pmatrix} 0.39 + 0.31i, \\ -0.24 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.37 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}$
(n_1, λ_3)	$\begin{pmatrix} 0.39 + 0.43i, \\ -0.31 - 0.05i \end{pmatrix}$	$\begin{pmatrix} 0.33 + 0.50i, \\ -0.43 - 0.69i \end{pmatrix}$
(n_2, λ_1)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.11 - 0.26i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.44i, \\ -0.39 - 0.50i \end{pmatrix}$
(n_2, λ_2)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.24 - 0.12i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.57i, \\ -0.48 - 0.73i \end{pmatrix}$
(n_2, λ_3)	$\begin{pmatrix} 0.29 + 0.22i, \\ -0.19 - 0.05i \end{pmatrix}$	$\begin{pmatrix} 0.58 + 0.50i, \\ -0.41 - 0.69i \end{pmatrix}$
(n_3, λ_1)	$\begin{pmatrix} 0.31 + 0.37i, \\ -0.11 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.46 + 0.29i, \\ -0.49 - 0.50i \end{pmatrix}$
(n_3, λ_2)	$\begin{pmatrix} 0.54 + 0.31i, \\ -0.24 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.37 + 0.57i, \\ -0.49 - 0.73i \end{pmatrix}$
(n_3, λ_3)	$\begin{pmatrix} 0.54 + 0.37i, \\ -0.19 - 0.00i \end{pmatrix}$	$\begin{pmatrix} 0.28 + 0.50i, \\ -0.49 - 0.69i \end{pmatrix}$

Table 5 Cartesian product of G and H in terms of MG, AG and non-MG

Ordered pair	MG	AG	Non-MG
(n_1, λ_1)	$\begin{pmatrix} 0.31, \\ -0.11 \end{pmatrix}$	$\begin{pmatrix} 0.59, \\ -0.29 \end{pmatrix}$	$\begin{pmatrix} 0.46, \\ -0.43 \end{pmatrix}$
(n_1, λ_2)	$\begin{pmatrix} 0.39, \\ -0.24 \end{pmatrix}$	$\begin{pmatrix} 0.49, \\ -0.19 \end{pmatrix}$	$\begin{pmatrix} 0.37, \\ -0.48 \end{pmatrix}$
(n_1, λ_3)	$\begin{pmatrix} 0.39, \\ -0.31 \end{pmatrix}$	$\begin{pmatrix} 0.45, \\ -0.39 \end{pmatrix}$	$\begin{pmatrix} 0.33, \\ -0.43 \end{pmatrix}$
(n_2, λ_1)	$\begin{pmatrix} 0.29, \\ -0.11 \end{pmatrix}$	$\begin{pmatrix} 0.44, \\ -0.29 \end{pmatrix}$	$\begin{pmatrix} 0.58, \\ -0.39 \end{pmatrix}$
(n_2, λ_2)	$\begin{pmatrix} 0.29, \\ -0.24 \end{pmatrix}$	$\begin{pmatrix} 0.44, \\ -0.19 \end{pmatrix}$	$\begin{pmatrix} 0.58, \\ -0.48 \end{pmatrix}$
(n_2, λ_3)	$\begin{pmatrix} 0.29, \\ -0.19 \end{pmatrix}$	$\begin{pmatrix} 0.44, \\ -0.29 \end{pmatrix}$	$\begin{pmatrix} 0.58, \\ -0.41 \end{pmatrix}$
(n_3, λ_1)	$\begin{pmatrix} 0.31, \\ -0.11 \end{pmatrix}$	$\begin{pmatrix} 0.40, \\ -0.22 \end{pmatrix}$	$\begin{pmatrix} 0.46, \\ -0.49 \end{pmatrix}$
(n_3, λ_2)	$\begin{pmatrix} 0.54, \\ -0.24 \end{pmatrix}$	$\begin{pmatrix} 0.40, \\ -0.19 \end{pmatrix}$	$\begin{pmatrix} 0.37, \\ -0.49 \end{pmatrix}$
(n_3, λ_3)	$\begin{pmatrix} 0.54, \\ -0.19 \end{pmatrix}$	$\begin{pmatrix} 0.40, \\ -0.22 \end{pmatrix}$	$\begin{pmatrix} 0.28, \\ -0.49 \end{pmatrix}$

- The ability of BCSFSs is to describe two-dimensional facts that involve the distribution of membership, non-membership, and abstinence degrees which is their most important characteristic. Because of this characteristic, BCSFSs are more dominating in expressing the needed information and resolving limitations in other theories, including BCPFS and BSFS.
- For tackling extremely broad decision-making problems, our suggested methodology is more accessible and flexible.
- The membership, abstinence, and non-membership degrees in our proposed BCSF model allow it to represent two-dimensional encrypted information with a variety of skills. The suggested theory now works as a powerful tool to overcome the limitations of the current models of incomplete information.
- By reducing their phase terms to zero, our proposed strategy demonstrates the same accuracy when applied to one-dimensional data, including bipolar spherical fuzzy data and bipolar picture fuzzy data, demonstrating the versatility and decision-making abilities of our approach. As a result, our suggested methodology is a versatile and efficient strategy that can precisely manage both traditional and two-dimensional ambiguous information.

Conclusion

This paper planned the innovative notion of BCSFS and the CP between two BCSFSs. Further, BCSFR and various kinds are also discussed, such as BCS reflexive FR, BCS irreflexive FR, BCS symmetric FR, BCS transitive FR, BCS antisymmetric FR, BCS equivalence FR, BCS partial order FR and many more. In addition, some authentic theorems are also defined. Further, these newly presented concepts of BCSFR are applied in an application of ANN. The proposed structure and unique modeling study is used to analyze the effective working of ANN and its types in organizations. The innovative concept of BCSFR is a generalization of all the pre-existing frameworks, because this structure covers all levels, i.e., MG, AG and non-MG with complex number. They describe the possibilities or impossibilities of all

Table 6 Comparison of BCSFR with pre-defined frameworks

Structure	MG		AG		Non-MG		Multi-dimension	Space
	Possibility	Impossibility	Possibility	Impossibility	Possibility	Impossibility		
CFR	Yes	No	No	no	No	No	Yes	$n = 1$
CIFR	Yes	No	No	no	Yes	No	Yes	$n = 1$
CPyFR	Yes	No	No	no	Yes	No	Yes	$n = 2$
CqROFR	Yes	No	No	No	Yes	No	Yes	$n = n$
CPFR	Yes	No	Yes	No	Yes	No	Yes	$n = 1$
CSFR	Yes	No	Yes	No	Yes	No	Yes	$n = 2$
BCFR	Yes	YES	No	No	No	No	Yes	$n = 1$
BCIFR	Yes	Yes	No	No	Yes	Yes	Yes	$n = 1$
BCPyFR	Yes	Yes	No	No	Yes	Yes	Yes	$n = 2$
BCqROFR	Yes	Yes	No	No	Yes	Yes	Yes	$n = n$
BCPFR	Yes	Yes	Yes	Yes	Yes	Yes	Yes	$n = 1$
BCSFR	Yes	Yes	Yes	Yes	Yes	Yes	Yes	$n = 2$

levels. Despite dealing with ambiguous information effectively, the proposed technique fails when the sum of the squares of the amplitude or phase terms of membership, neutral membership, and non-membership degrees is larger than 1. Hence, our objective is to expand the suggested technique under the complex bipolar type-2 fuzzy set, complex bipolar mandolboart fuzzy set, and CBTSFSs. Further, BCSFR will be employed in a variety of fields in the future, including physics, economics and various field of science and technology. In the future, we will extend the presented approach to the different extension of neural network as convolution neural network, clustering approaches, etc. and present some more generalized approaches [49–51].

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Data availability All the data used are given in the paper.

Declarations

Conflict of interest The authors have no conflicts of any sort.

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