**ORIGINAL ARTICLE**

# Analysis of Hamming and Hausdorff 3D distance measures for complex pythagorean fuzzy sets and their applications in pattern recognition and medical diagnosis

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## Abstract

Similarity measures are very effective and meaningful tool used for evaluating the closeness between any two attributes which are very important and valuable to manage awkward and complex information in real-life problems. Therefore, for better handing of fuzzy information in real life, Ullah et al. (Complex Intell Syst 6(1): 15–27, 2020) recently introduced the concept of complex Pythagorean fuzzy set (CPyFS) and also described valuable and dominant measures, called various types of distance measures (DisMs) based on CPyFSs. The theory of CPyFS is the essential modification of Pythagorean fuzzy set to handle awkward and complicated in real-life problems. Keeping the advantages of the CPyFS, in this paper, we first construct an example to illustrate that a DisM proposed by Ullah et al. does not satisfy the axiomatic definition of complex Pythagorean fuzzy DisM. Then, combining the 3D Hamming distance with the Hausdorff distance, we propose a new DisM for CPyFSs, which is proved to satisfy the axiomatic definition of complex Pythagorean fuzzy DisM. Moreover, similarly to some DisMs for intuitionistic fuzzy sets, we present some other new complex Pythagorean fuzzy DisMs. Finally, we apply our proposed DisMs to a building material recognition problem and a medical diagnosis problem to illustrate the effectiveness of our DisMs. Finally, we aim to compare the proposed work with some existing measures is to enhance the worth of the derived measures.

**Keywords** Complex pythagorean fuzzy set · Hamming and Hausdorff distance measures · Pattern recognition · Madical diagnosis

## Introduction

Similarity and dissimilarity are important because they are used by a number of data mining techniques, such as clustering nearest neighbor classification and anomaly detection. The term proximity is used to refer to either similarity or dissimilarity. The similarity between two objects is a numeral measure of the degree to which the two objects are alike. Consequently, similarities are higher for pairs of objects that are more alike. Similarities are usually non-negative and are often between 0 (no similarity) and 1 (complete similar-

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ity). The dissimilarity between two objects is the numerical measure of the degree to which the two objects are different. Dissimilarity is lower for more similar pairs of objects. Frequently, the term distance is used as a synonym for dissimilarity. Dissimilarities sometimes fall in the interval [0, 1, but it is also common for them to range from 0 to ∞. Certain people have diagnosed the theory of similarity measures and distance measures for classical information and because of this reason, we loss a lot of information. For managing with such sort of issues, Zadeh [46] introduced the concept of fuzzy set (FS) by using a function from the universe of discourse to [0, 1], which was called the membership degree function, to describe the importance of an element in the universe of discourse. Then, Zadeh's fuzzy set theory constitutes the basis of fuzzy decision-making [10,11,19]. However, the FS can only deal with the situation containing two opposite responses. Therefore, it failed to deal with the situation with hesitant/neutral state of "this and also that". According to this, Atanassov [8] generalized Zadeh's fuzzy set by proposing the concept of intuitionistic fuzzy sets (IFSs), characterized by a membership function and a non-membership function meeting the condition that their sum at every point is less than or equal to 1. In the theory of IFSs, the condition that the sum of the membership degree and the non-membership degree is less than or equal to 1 induces some decision evaluation information that cannot be expressed effectively. Hence, the range of their applications is limited. To overcome this shortcoming, Yager [42–44] proposed the concepts of Pythagorean fuzzy sets (PFSs) and q-rung orthopair fuzzy sets (q-ROFSs). These sets satisfy the condition that the square sum or the $q$th power sum of the membership degree and the non-membership degree is less than or equal to 1. It is determined from the aforementioned in-depth research and DMPs that their use is restricted to handling only the data's uncertainty while failing to address its fluctuations at a particular point in time. But data derived from "medical research, a database for biometric and facial recognition" are constantly updated in tandem with time. Thus, a range of MD is expanded from a real subset to the unit disc of the complex plane to cope with these kinds of difficulties, which developed the idea of the complex fuzzy sets (CFSs) Ramot et al. [31]. Further, Alkouri and Salleh [7] introduced the concepts of complex intuitionistic fuzzy sets (CIFSs). To enlarge the representing domain, Ullah et al. [36] proposed the concept of complex Pythagorean fuzzy sets (CPyFSs), and Liu et al. [20] introduced the concept of complex q-rung orthopair fuzzy sets (Cq-ROFSs). Since then, CIFSs, CPyFSs, and Cq-ROFSs have been widely applied to various fields, such as MCDM/MADM [1–6,12,15,20–22,24,25,32, 38,40], medical diagnosis [13,27–29,32], pattern recognition [12,13,27,36], cluster analysis [12,47], and image processing [16].

It is necessary to gauge the degree of discriminating between the pairs of sets due to the intricate decision-making process. The most effective tools for this purpose are instant messengers. The decision-maker have the ability to assess the degree of discriminating between the sets among the many measures like entropy, similarity, inclusion, etc. The major goal of this work is to create some exponential-based decision-makers to quantify the information, which is encouraged by the CPyFS model's characteristics and the quality of decision-maker. In order to achieve this, the data was designated under the CPyFS model to quantify the data using the suggested metric for resolving the decision-making procedures. The qualities of a few axioms are studied in detail. Later, an algorithm was developed based on the proposed investigation, to assess the differences for various types of complex fuzzy sets, the normalized distance measure (DisM) and the similarity measure (SimM), being a pair of dual concepts, are important tools for decision-making and pattern recognition under CPyFSs and CIFSs frameworks. For the CIFSs, Rani and Garg [32] presented a few two-dimensional (2D) CIFDisMs by using the Hamming, Euclidean, and Hausdorff distances. Then, Garg and Rani [12] proposed some new CIF information measures, including SimMs, DisMs, entropies, and inclusion measures) and obtained the transformation relationships among them. Meanwhile, they [12] developed a CIF clustering algorithm. For Cq-ROFSs, Garg et al. [13] gave the notion of Cq-ROF dice SimM and weighted Cq-ROF dice SimM and derived some new Cq-ROF dice SimMs. Liu et al. [21] proposed some cosine DisMs and SimMs for Cq-ROFSs and obtained developed a TOPSIS method under Cq-ROFS framework. To distinguish different Cq-ROFSs with high similarity, Mahmood and Ali [25] obtained some new SimMs for Cq-ROFSs. For CPyFSs, Aldring and Ajay [5] developed a MCGDM method by introducing a new CPyF projection measure between the alternatives and the relative CPyF ideal point. Based on the Hamming distance and the Hausdorff distance, Ullah et al. [36] developed two parametric DisMs for CPyFSs and applied to a building material recognition problem. However, because the different weights are assigned to the degrees of membership, non-membership, and hesitancy for the 3D DisM $D^2_{\text{CPyFS}}$ of Ullah et al. [36], the may cause an unreasonable result that the DisM $D^2_{\text{CPyFS}}$ does not satisfy the axiomatic definition of complex Pythagorean fuzzy DisM (see Example 1). The geometrical shape of the proposed work is described in the form of Fig. 1.

To overcome the drawback of Ullah et al.'s DisM $D^2_{\text{CPyFS}}$ in [36], we introduce a new 3D DisM for CPyFSs by combining the 3D Hamming distance with the Hausdorff distance and prove that it satisfies the axiomatic definition of CPyFDisM. Moreover, similarly to the DisMs for IFSs in [14,34,37,39,45], we propose some other new CPyFDisMs.

**Fig. 1** Geometrical representation of the proposed work

1. We construct an example to illustrate that a DisM proposed by Ullah et al. does not satisfy the axiomatic definition of complex Pythagorean fuzzy DisM.

2. We propose a new DisM for CPyFSs, which is proved to satisfy the axiomatic definition of complex Pythagorean fuzzy DisM.

3. We present some other new complex Pythagorean fuzzy DisMs.

4. We apply our proposed DisMs to a building material recognition problem and a medical diagnosis problem to illustrate the effectiveness of our DisMs.

5. We aim to compare the proposed work with some existing measures is to enhance the worth of the derived measures.

Finally, we give the comparative analysis by using our proposed DisMs to a building material recognition problem and a medical diagnosis problem to illustrate the effectiveness of our DisMs. The comparative analysis results also indicate the unreasonableness of Ullah et al.'s DisM $D^2_{\text{CPyFS}}$.

## Preliminaries

This section gives some elements on IFS, PFS, CIFS, and CPyFS. Throughout this paper, let $\mathcal{O}_{\mathbb{C}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

### Intuitionistic fuzzy set (IFS)

**Definition 2.1** ([9, Definition 1.1]). An *intuitionistic fuzzy set* (IFS) $I$ in $\mathfrak{X}$ is defined as

$$I = \left\{ \langle \vartheta, \mathfrak{P}_I(\vartheta), \mathfrak{D}_I(\vartheta) \rangle \mid \vartheta \in \mathfrak{X} \right\}, \tag{1}$$

where the functions $\mathfrak{P}_I : \mathfrak{X} \longrightarrow [0,1]$ and $\mathfrak{D}_I : \mathfrak{X} \longrightarrow [0,1]$ define the *degree of membership* and the *degree of non-membership* of the element $\vartheta \in \mathfrak{X}$ to the set $I$, respectively, and for every $\vartheta \in \mathfrak{X}, \mathfrak{P}_I(\vartheta) + \mathfrak{D}_I(\vartheta) \leq 1$. Moreover, the *hesitancy degree* $\mathfrak{H}_I(\vartheta)$ of an element $\vartheta$ belonging to $I$ is defined by $\mathfrak{H}_I(\vartheta) = 1 - \mathfrak{P}_I(\vartheta) - \mathfrak{D}_I(\vartheta)$. The pair $\langle \mathfrak{P}_I(\vartheta), \mathfrak{D}_I(\vartheta) \rangle$ is called an *intuitionistic fuzzy number* (IFN) by Xu [41]. Let $\Theta$ be the set of all IFNs, i.e., $\Theta = \{\langle \mathfrak{P}, \mathfrak{D} \rangle \in [0,1]^2 \mid \mathfrak{P} + \mathfrak{D} \leq 1\}$.

### Pythagorean fuzzy set (PFS)

**Definition 2.2** ([42]). A *Pythagorean fuzzy set* (PFS) $P$ in $\mathfrak{X}$ is defined as

$$P = \left\{ \langle \vartheta, \mathfrak{P}_P(\vartheta), \mathfrak{D}_P(\vartheta) \rangle \mid \vartheta \in \mathfrak{X} \right\}, \tag{2}$$

where the functions $\mathfrak{P}_P : \mathfrak{X} \longrightarrow [0,1]$ and $\mathfrak{D}_P : \mathfrak{X} \longrightarrow [0,1]$ define the *degree of membership* and the *degree of non-membership* of the element $\vartheta \in \mathfrak{X}$ to the set $P$, respectively, and for every $\vartheta \in \mathfrak{X}, \mathfrak{P}^2_P(\vartheta) + \mathfrak{D}^2_P(\vartheta) \leq 1$. Moreover, the *hesitancy degree* $\mathfrak{H}_P(\vartheta)$ of an element $\vartheta$ belonging to $P$ is defined by $\mathfrak{H}_P(\vartheta) = \sqrt{1 - \mathfrak{P}^2_P(\vartheta) - \mathfrak{D}^2_P(\vartheta)}$.

### Complex intuitionistic fuzzy set (CIFS)

**Definition 2.3** ([7, Definition 3.1], [36, Definition 6]). A *complex intuitionistic fuzzy set* (CIFS) $C$ in $\mathfrak{X}$ is defined as

$$C = \left\{ \langle \vartheta, \mathfrak{P}_C(\vartheta), \mathfrak{D}_C(\vartheta) \rangle \mid \vartheta \in \mathfrak{X} \right\}, \tag{3}$$

where the functions $\mathfrak{P}_C : \mathfrak{X} \longrightarrow \mathcal{O}_{\mathbb{C}}$ and $\mathfrak{D}_C : \mathfrak{X} \longrightarrow \mathcal{O}_{\mathbb{C}}$ define the *degree of membership* and the *degree of non-membership* of the element $\vartheta \in \mathfrak{X}$ to the set $C$, respectively, and for every $\vartheta \in \mathfrak{X}$, $\mathfrak{P}_C(\vartheta) = T_C(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{T_C}(\vartheta)}$ and $\mathfrak{D}_C(\vartheta) = F_C(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{F_C}(\vartheta)}$ satisfying that $0 \leq T_C(\vartheta), F_C(\vartheta) \leq 1, 0 \leq W_{T_C}(\vartheta), W_{F_C}(\vartheta) \leq 1$, and $T_C(\vartheta) + F_C(\vartheta) \leq 1, W_{T_C}(\vartheta) + W_{F_C}(\vartheta) \leq 1$. Moreover, the *hesitancy degree* $\mathfrak{H}_C(\vartheta) = R_C(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{R_C}(\vartheta)}$ of the element $\vartheta$ belonging to $C$ is defined by $R_C(\vartheta) = 1 - T_C(\vartheta) - F_C(\vartheta)$ and $W_{R_C}(\vartheta) = 1 - W_{T_C}(\vartheta) - W_{F_C}(\vartheta)$.

## Complex pythagorean fuzzy set (CPyFS)

**Definition 2.4** ([36, Definition 7]). A *complex Pythagorean fuzzy set* (CPyFS) $\mathfrak{C}$ in $\mathfrak{X}$ is defined as

$$\mathfrak{C} = \left\{ \langle \vartheta, \underbrace{\mathfrak{P}_{\mathfrak{C}}(\vartheta), \mathfrak{O}_{\mathfrak{C}}(\vartheta)}_{\mathfrak{C}(\vartheta)} \rangle \ \middle| \ \vartheta \in \mathfrak{X} \right\}, \tag{4}$$

where the functions $\mathfrak{P}_{\mathfrak{C}} : \mathfrak{X} \longrightarrow \mathscr{O}_{\mathbb{C}}$ and $\mathfrak{O}_{\mathfrak{C}} : \mathfrak{X} \longrightarrow \mathscr{O}_{\mathbb{C}}$ define the *degree of membership* and the *degree of non-membership* of the element $\vartheta \in \mathfrak{X}$ to the set $\mathfrak{C}$, respectively, and for every $\vartheta \in \mathfrak{X}$, $\mathfrak{P}_{\mathfrak{C}}(\vartheta) = T_{\mathfrak{C}}(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}}}(\vartheta)}$ ($\mathbf{i} = \sqrt{-1}$) and $\mathfrak{O}_{\mathfrak{C}}(\vartheta) = F_{\mathfrak{C}}(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}}}(\vartheta)}$ satisfying that

$$0 \leq T_{\mathfrak{C}}(\vartheta), F_{\mathfrak{C}}(\vartheta) \leq 1, \ 0 \leq W_{T_{\mathfrak{C}}}(\vartheta), W_{F_{\mathfrak{C}}}(\vartheta) \leq 1,$$
$$\text{and } T_{\mathfrak{C}}^2(\vartheta) + F_{\mathfrak{C}}^2(\vartheta) \leq 1, \ W_{T_{\mathfrak{C}}}^2(\vartheta) + W_{F_{\mathfrak{C}}}^2(\vartheta) \leq 1. \tag{5}$$

Moreover, the *hesitancy degree* $\mathfrak{H}_{\mathfrak{C}}(\vartheta) = R_{\mathfrak{C}}(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{R_{\mathfrak{C}}}(\vartheta)}$ of the element $\vartheta$ belonging to $\mathfrak{C}$ is defined by $R_{\mathfrak{C}}(\vartheta) = \sqrt{1 - T_{\mathfrak{C}}^2(\vartheta) - F_{\mathfrak{C}}^2(\vartheta)}$ and $W_{R_{\mathfrak{C}}}(\vartheta) = \sqrt{1 - W_{T_{\mathfrak{C}}}^2(\vartheta) - W_{F_{\mathfrak{C}}}^2(\vartheta)}$. Let CPyFS$(\mathfrak{X})$ denote the set of all CPyFSs in $\mathfrak{X}$.

In [36], the pair $(T_{\mathfrak{C}}(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}}}(\vartheta)}, F_{\mathfrak{C}}(\vartheta) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}}}(\vartheta)})$ satisfying the condition in (5) is called a *complex Pythagorean fuzzy set* (CPyFN). For convenience, use $\mathfrak{C} = (T_{\mathfrak{C}} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}}}}, F_{\mathfrak{C}} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}}}})$ to represent a CPyFN $\mathfrak{C}$, which satisfies $0 \leq T_{\mathfrak{C}}, F_{\mathfrak{C}} \leq 1, 0 \leq W_{T_{\mathfrak{C}}}, W_{F_{\mathfrak{C}}} \leq 1$, and $T_{\mathfrak{C}}^2 + F_{\mathfrak{C}}^2 \leq 1$, $W_{T_{\mathfrak{C}}}^2 + W_{F_{\mathfrak{C}}}^2 \leq 1$. Let $\mathscr{C}_{\mathrm{PyFN}}$ denote the set of all CPyFNs, i.e., $\mathscr{C}_{\mathrm{PyFN}} = \{(T_{\mathfrak{C}} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}}}}, F_{\mathfrak{C}} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}}}}) \mid 0 \leq T_{\mathfrak{C}}, F_{\mathfrak{C}} \leq 1, 0 \leq W_{T_{\mathfrak{C}}}, W_{F_{\mathfrak{C}}} \leq 1$, and $T_{\mathfrak{C}}^2 + F_{\mathfrak{C}}^2 \leq 1, W_{T_{\mathfrak{C}}}^2 + W_{F_{\mathfrak{C}}}^2 \leq 1\}$.

Meanwhile, Ullah et al. [36] introduced the following basic operations for CPyFSs and CPyFNs.

**Definition 2.5** ([36, Definition 8]). (1) Let $\mathfrak{C}_1 = (T_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_1}}}, F_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_1}}})$ and $\mathfrak{C}_2 = (T_{\mathfrak{C}_2} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_2}}}, F_{\mathfrak{C}_2} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_2}}})$ be two CPyFNs. Define

- (i) (Inclusion) $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ if and only if $T_{\mathfrak{C}_1} \leq T_{\mathfrak{C}_2}$, $F_{\mathfrak{C}_1} \geq F_{\mathfrak{C}_2}$, and $W_{T_{\mathfrak{C}_1}} \leq W_{T_{\mathfrak{C}_2}}$, $W_{F_{\mathfrak{C}_1}} \geq W_{F_{\mathfrak{C}_2}}$;
- (ii) $\mathfrak{C}_1 = \mathfrak{C}_2$ if and only if $T_{\mathfrak{C}_1} = T_{\mathfrak{C}_2}$, $F_{\mathfrak{C}_1} = F_{\mathfrak{C}_2}$, and $W_{T_{\mathfrak{C}_1}} = W_{T_{\mathfrak{C}_2}}$, $W_{F_{\mathfrak{C}_1}} = W_{F_{\mathfrak{C}_2}}$;
- (iii) (Complement) $(\mathfrak{C}_1)^{\complement} = (F_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_1}}}, T_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_1}}})$.

(2) Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CPyFSs in $\mathfrak{X}$. Define

- (i) (Inclusion) $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ if and only if, for any $\vartheta \in \mathfrak{X}$, $\mathfrak{C}_1(\vartheta) \subseteq \mathfrak{C}_2(\vartheta)$;
- (ii) $\mathfrak{C}_1 = \mathfrak{C}_2$ if and only if, for any $\vartheta \in \mathfrak{X}$, $\mathfrak{C}_1(\vartheta) = \mathfrak{C}_2(\vartheta)$;
- (iii) (Complement) $(\mathfrak{C}_1)^{\complement} = \{\langle \vartheta, (\mathfrak{C}_1(\vartheta))^{\complement} \mid \vartheta \in \mathfrak{X}\rangle\}$.

## Distance and similarity measures on CPyFSs

**Definition 2.6** A function $D : \mathrm{CPyFS}(\mathfrak{X}) \times \mathrm{CPyFS}(\mathfrak{X}) \longrightarrow \mathbb{R}$ is a *normalized distance measure* (DisM) on CPyFS$(\mathfrak{X})$ if it satisfies the following conditions: for any $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \in \mathrm{CPyFS}(\mathfrak{X})$,

- (1) $0 \leq D(\mathfrak{C}_1, \mathfrak{C}_2) = D(\mathfrak{C}_2, \mathfrak{C}_1) \leq 1$;
- (2) $D(\mathfrak{C}_1, \mathfrak{C}_2) = 0$ if and only if $\mathfrak{C}_1 = \mathfrak{C}_2$;
- (3) $D(\mathfrak{C}_1, \mathfrak{C}_3) \leq D(\mathfrak{C}_1, \mathfrak{C}_2) + D(\mathfrak{C}_2, \mathfrak{C}_3)$;
- (4) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$, then $D(\mathfrak{C}_1, \mathfrak{C}_3) \geq D(\mathfrak{C}_1, \mathfrak{C}_2)$ and $D(\mathfrak{C}_1, \mathfrak{C}_3) \geq D(\mathfrak{C}_2, \mathfrak{C}_3)$.

**Definition 2.7** A function $\mathbf{S} : \mathrm{CPyFS}(\mathfrak{X}) \times \mathrm{CPyFS}(\mathfrak{X}) \longrightarrow \mathbb{R}$ is a *similarity measure* (SimM) on CPyFS$(\mathfrak{X})$ if it satisfies the following conditions: for any $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \in \mathrm{CPyFS}(\mathfrak{X})$,

- (1) $0 \leq \mathbf{S}(\mathfrak{C}_1, \mathfrak{C}_2) = \mathbf{S}(\mathfrak{C}_2, \mathfrak{C}_1) \leq 1$;
- (2) $\mathbf{S}(\mathfrak{C}_1, \mathfrak{C}_2) = 1$ if and only if $\mathfrak{C}_1 = \mathfrak{C}_2$;
- (3) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$, then $\mathbf{S}(\mathfrak{C}_1, \mathfrak{C}_3) \leq \mathbf{S}(\mathfrak{C}_1, \mathfrak{C}_2)$ and $\mathbf{S}(\mathfrak{C}_1, \mathfrak{C}_3) \leq \mathbf{S}(\mathfrak{C}_2, \mathfrak{C}_3)$.

## Drawback of DisM of Ullah et al. [36]

Let $\mathfrak{X} = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_\ell\}$ and $\mathfrak{C}_1 = \{(T_{\mathfrak{C}_1}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_1}}(\vartheta_j)}, F_{\mathfrak{C}_1}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_1}}(\vartheta_j)}) \mid 1 \leq j \leq \ell\}$ and $\mathfrak{C}_2 = \{(T_{\mathfrak{C}_2}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_2}}(\vartheta_j)}, F_{\mathfrak{C}_2}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_2}}(\vartheta_j)}) \mid 1 \leq j \leq \ell\}$ be two CPyFSs on $\mathfrak{X}$. Recently, Ullah et al. [36] introduced two DisMs $D_{\mathrm{CPyFS}}^1$ and $D_{\mathrm{CPyFS}}^2$ for CPyFSs as follows (see [36, Definition 12]):

$$\begin{aligned} D_{\mathrm{CPyFS}}^1(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{1}{2\ell} \sum_{j=1}^{\ell} \big[ & a_1 \cdot |T_{\mathfrak{C}_1}^2(x_j) - T_{\mathfrak{C}_2}^2(x_j)| \\ & + b_1 \cdot |F_{\mathfrak{C}_1}^2(x_j) - F_{\mathfrak{C}_2}^2(x_j)| + c_1 \cdot \max\{|T_{\mathfrak{C}_1}^2(x_j) \\ & - T_{\mathfrak{C}_2}^2(x_j)|, |F_{\mathfrak{C}_1}^2(x_j) - F_{\mathfrak{C}_2}^2(x_j)|\} \\ & a_2 \cdot |W_{T_{\mathfrak{C}_1}}^2(x_j) - W_{T_{\mathfrak{C}_2}}^2(x_j)| \\ & + b_2 \cdot |W_{F_{\mathfrak{C}_1}}^2(x_j) - W_{F_{\mathfrak{C}_2}}^2(x_j)| \\ & + c_2 \cdot \max\{|W_{T_{\mathfrak{C}_1}}^2(x_j) - W_{T_{\mathfrak{C}_2}}^2(x_j)|, \\ & |W_{F_{\mathfrak{C}_1}}^2(x_j) - W_{F_{\mathfrak{C}_2}}^2(x_j)|\}\big], \end{aligned} \tag{6}$$

and

$$\begin{aligned} D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{1}{2\ell} \sum_{j=1}^{\ell} \big[ & a_1 \cdot |T_{\mathfrak{C}_1}^2(x_j) - T_{\mathfrak{C}_2}^2(x_j)| \\ & + b_1 \cdot |F_{\mathfrak{C}_1}^2(x_j) - F_{\mathfrak{C}_2}^2(x_j)| \\ & + r_1 \cdot |R_{\mathfrak{C}_1}^2(x_j) - R_{\mathfrak{C}_2}^2(x_j)| \end{aligned}$$

$$+ c_1 \cdot \max\{|T_{\mathfrak{C}_1}^2(x_j) - T_{\mathfrak{C}_2}^2(x_j)|,$$
$$|F_{\mathfrak{C}_1}^2(x_j) - F_{\mathfrak{C}_2}^2(x_j)|,$$
$$\times |R_{\mathfrak{C}_1}^2(x_j) - R_{\mathfrak{C}_2}^2(x_j)|\} a_2 \cdot |W_{T_{\mathfrak{C}_1}}^2(x_j)$$
$$- W_{T_{\mathfrak{C}_2}}^2(x_j)| + b_2 \cdot |W_{F_{\mathfrak{C}_1}}^2(x_j) - W_{F_{\mathfrak{C}_2}}^2(x_j)|$$
$$+ r_2 \cdot |W_{R_{\mathfrak{C}_1}}^2(x_j) - W_{R_{\mathfrak{C}_2}}^2(x_j)|$$
$$+ c_2 \cdot \max\{|W_{T_{\mathfrak{C}_1}}^2(x_j) - W_{T_{\mathfrak{C}_2}}^2(x_j)|,$$
$$|W_{F_{\mathfrak{C}_1}}^2(x_j) - W_{F_{\mathfrak{C}_2}}^2(x_j)|,$$
$$|W_{R_{\mathfrak{C}_1}}^2(x_j) - W_{R_{\mathfrak{C}_2}}^2(x_j)|\}], \tag{7}$$

where $0 \le a_i, b_i, c_i, r_i \le 1$ and $a_i + b_i + c_i + r_i = 1$.

Meanwhile, Ullah et al. [36] proved that DisM $D_{\mathrm{CPyFS}}^2$ has the following property.

**Property** *If* $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$, *then* $D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_3) \ge D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_2)$ *and* $D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_3) \ge D_{\mathrm{CPyFS}}^2(\mathfrak{C}_2, \mathfrak{C}_3)$.

The following example shows that Property 3 does not hold, and thus the DisM $D_{\mathrm{CPyFS}}^2$ does not satisfy the axiomatic definition of complex Pythagorean fuzzy DisM.

**Example 1** Let $\mathfrak{X} = \{\vartheta\}$ and $\mathfrak{C}_1 = (0 \cdot e^{2\pi\mathbf{i}\cdot 0}, 1 \cdot e^{2\pi\mathbf{i}\cdot 0})$, $\mathfrak{C}_2 = (0 \cdot e^{2\pi\mathbf{i}\cdot 0}, 0 \cdot e^{2\pi\mathbf{i}\cdot 0})$, and $\mathfrak{C}_3 = (1 \cdot e^{2\pi\mathbf{i}\cdot 0}, 0 \cdot e^{2\pi\mathbf{i}\cdot 0})$ be three CPyFSs on $\mathfrak{X}$ and choose $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0.1$ and $r_1 = r_2 = 0.7$. Clearly, $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$. By direct calculation, we have

$$D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_2) = 0.1 \times 0 + 0.1 \times 1 + 0.7 \times 1 + 0.1 \times 1 = 0.9,$$

and

$$D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_3) = 0.1 \times 1 + 0.1 \times 1 + 0.7 \times 0 + 0.1 \times 1 = 0.3,$$

implying that $D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_3) < D_{\mathrm{CPyFS}}^2(\mathfrak{C}_1, \mathfrak{C}_2)$. This means that Property 3 does not hold because $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$.

## A new complex Pythagorean fuzzy DisM

### A novel DisM on $\mathscr{C}_{\mathbf{PyFN}}$

Let $\mathfrak{C}_1 = (T_{\mathfrak{C}_1} \cdot e^{2\pi\mathbf{i}\cdot W_{T_{\mathfrak{C}_1}}}, F_{\mathfrak{C}_1} \cdot e^{2\pi\mathbf{i}\cdot W_{F_{\mathfrak{C}_1}}})$ and $\mathfrak{C}_2 = (T_{\mathfrak{C}_2} \cdot e^{2\pi\mathbf{i}\cdot W_{T_{\mathfrak{C}_2}}}, F_{\mathfrak{C}_2} \cdot e^{2\pi\mathbf{i}\cdot W_{F_{\mathfrak{C}_2}}})$ be two CPyFNs. Define the DisM $D_{\mathrm{Wu}}$ between $\mathfrak{C}_1$ and $\mathfrak{C}_2$ by

$$D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{1}{2}\left[\frac{1}{4} \cdot \left(|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|\right.\right.$$
$$\left. + |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|\right)$$
$$+ \frac{1}{2} \cdot \max\{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|, |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|,$$
$$|R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|\}$$

$$+ \frac{1}{4} \cdot \left(|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2| + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|\right.$$
$$\left. + |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2|\right)$$
$$+ \frac{1}{2} \cdot \max\{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|, |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|,$$
$$\left. |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2|\}\right]. \tag{8}$$

**Proposition 4.1** $0 \le D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) \le 1$.

**Proof** From $T_{\mathfrak{C}_1}^2 + F_{\mathfrak{C}_1}^2 + R_{\mathfrak{C}_1}^2 = 1$, $T_{\mathfrak{C}_2}^2 + F_{\mathfrak{C}_2}^2 + R_{\mathfrak{C}_2}^2 = 1$, $W_{T_{\mathfrak{C}_1}}^2 + W_{F_{\mathfrak{C}_1}}^2 + W_{R_{\mathfrak{C}_1}}^2 = 1$, and $W_{T_{\mathfrak{C}_2}}^2 + W_{F_{\mathfrak{C}_2}}^2 + W_{R_{\mathfrak{C}_2}}^2 = 1$, it follows that $|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2| + |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2| \le 2$, $\max\{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|, |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|, |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|\} \le 1$, $|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2| + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2| + |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2| \le 2$, and $\max\{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|, |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|, |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2|\} \le 1$, and thus $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) \le 1$ by Eq. (8). □

**Proposition 4.2** (1) $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = D_{\mathrm{Wu}}(\mathfrak{C}_2, \mathfrak{C}_1)$.
(2) $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = 0$ *if and only if* $\mathfrak{C}_1 = \mathfrak{C}_2$.

**Proof** It follows directly from Eq. (8). □

**Proposition 4.3** *For* $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3 \in \mathscr{C}_{\mathrm{PyFN}}$, $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \le D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) + D_{\mathrm{Wu}}(\mathfrak{C}_2, \mathfrak{C}_3)$.

**Proof** It follows directly from Eq. (8) and triangle inequality. □

**Proposition 4.4** *Let* $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3 \in \mathscr{C}_{\mathrm{PyFN}}$. *If* $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$, *then* $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \ge D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2)$ *and* $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \ge D_{\mathrm{Wu}}(\mathfrak{C}_2, \mathfrak{C}_3)$.

**Proof** By $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \mathfrak{C}_3$, it follows that $T_{\mathfrak{C}_1}^2 \le T_{\mathfrak{C}_2}^2 \le T_{\mathfrak{C}_3}^2$ and $F_{\mathfrak{C}_1}^2 \ge F_{\mathfrak{C}_2}^2 \ge F_{\mathfrak{C}_3}^2$. To prove $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \ge D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2)$, we consider the following four cases:
(i) If $R_{\mathfrak{C}_2}^2 \ge R_{\mathfrak{C}_1}^2$ and $R_{\mathfrak{C}_3}^2 \ge R_{\mathfrak{C}_1}^2$, then

$$|R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2| = (1 - T_{\mathfrak{C}_2}^2 - F_{\mathfrak{C}_2}^2) - (1 - T_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_1}^2)$$
$$= (F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2) + (T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2)$$
$$\le F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2,$$
$$|R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_3}^2| = (1 - T_{\mathfrak{C}_3}^2 - F_{\mathfrak{C}_3}^2) - (1 - T_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_1}^2)$$
$$= (F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_3}^2) + (T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_3}^2)$$
$$\le F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_3}^2,$$
$$|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2| + |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|$$
$$= T_{\mathfrak{C}_2}^2 - T_{\mathfrak{C}_1}^2 + F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2 + R_{\mathfrak{C}_2}^2 - R_{\mathfrak{C}_1}^2$$
$$= T_{\mathfrak{C}_2}^2 - T_{\mathfrak{C}_1}^2 + F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2 + (1 - T_{\mathfrak{C}_2}^2 - F_{\mathfrak{C}_2}^2)$$
$$- (1 - T_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_1}^2)$$
$$= 2(F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2),$$

and

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|$$
$$= T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1} + F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3} + R^2_{\mathfrak{C}_3} - R^2_{\mathfrak{C}_1}$$
$$= T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1} + F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3} + \left(1 - T^2_{\mathfrak{C}_3} - F^2_{\mathfrak{C}_3}\right)$$
$$\quad - \left(1 - T^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_1}\right)$$
$$= 2\left(F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}\right).$$

These, together with $T^2_{\mathfrak{C}_1} \leq T^2_{\mathfrak{C}_2} \leq T^2_{\mathfrak{C}_3}$ and $F^2_{\mathfrak{C}_1} \geq F^2_{\mathfrak{C}_2} \geq F^2_{\mathfrak{C}_3}$, imply that

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|$$
$$\leq |T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|$$
$$\quad + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|,$$

and

$$\max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|\}$$
$$= \max\{T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}\}$$
$$\leq \max\{T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}\}$$
$$= \max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|\}.$$

And thus, $\frac{1}{4} \cdot (|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|)$ $+ \frac{1}{2} \cdot \max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|\} \leq \frac{1}{4} \cdot$ $(|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|) + \frac{1}{2} \cdot \max\{|T^2_{\mathfrak{C}_1}$ $- T^2_{\mathfrak{C}_3}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|\}$. Similarly, it can be verified that $\frac{1}{4} \cdot (|W^2_{T_{\mathfrak{C}_1}} - W^2_{T_{\mathfrak{C}_2}}| + |W^2_{F_{\mathfrak{C}_1}} - W^2_{F_{\mathfrak{C}_2}}| + |W^2_{R_{\mathfrak{C}_1}} - W^2_{R_{\mathfrak{C}_2}}|)$ $+ \frac{1}{2} \cdot \max\{|W^2_{T_{\mathfrak{C}_1}} - W^2_{T_{\mathfrak{C}_2}}|, |W^2_{F_{\mathfrak{C}_1}} - W^2_{F_{\mathfrak{C}_2}}|, |W^2_{R_{\mathfrak{C}_1}} - W^2_{R_{\mathfrak{C}_2}}|\}$ $\leq \frac{1}{4} \cdot (|W^2_{T_{\mathfrak{C}_1}} - W^2_{T_{\mathfrak{C}_3}}| + |W^2_{F_{\mathfrak{C}_1}} - W^2_{F_{\mathfrak{C}_3}}| + |W^2_{R_{\mathfrak{C}_1}} - W^2_{R_{\mathfrak{C}_3}}|)$ $+ \frac{1}{2} \cdot \max\{|W^2_{T_{\mathfrak{C}_1}} - W^2_{T_{\mathfrak{C}_3}}|, |W^2_{F_{\mathfrak{C}_1}} - W^2_{F_{\mathfrak{C}_3}}|, |W^2_{R_{\mathfrak{C}_1}} - W^2_{R_{\mathfrak{C}_3}}|\}$. Therefore,

$$D_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \geq D_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2).$$

(ii) If $R^2_{\mathfrak{C}_2} \leq R^2_{\mathfrak{C}_1}$ and $R^2_{\mathfrak{C}_3} \leq R^2_{\mathfrak{C}_1}$, similarly to the proof of (i), it can be verified that

$$|R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}| \leq T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}| \leq T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1},$$
$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|$$
$$= 2(T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}),$$

and

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|$$
$$= 2\left(T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}\right).$$

These, together with $T^2_{\mathfrak{C}_1} \leq T^2_{\mathfrak{C}_2} \leq T^2_{\mathfrak{C}_3}$ and $F^2_{\mathfrak{C}_1} \geq F^2_{\mathfrak{C}_2} \geq F^2_{\mathfrak{C}_3}$, imply that

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|$$
$$\leq |T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|$$
$$\quad + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|,$$

and

$$\max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|\}$$
$$= \max\{T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}\}$$
$$\leq \max\{T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}\}$$
$$= \max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|\}.$$

And thus, it can be similarly verified that

$$D_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \geq D_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2).$$

(iii) If $R^2_{\mathfrak{C}_2} \geq R^2_{\mathfrak{C}_1}$ and $R^2_{\mathfrak{C}_3} \leq R^2_{\mathfrak{C}_1}$, i.e., $1 - T^2_{\mathfrak{C}_2} - F^2_{\mathfrak{C}_2} \geq 1 - T^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_1}$ and $1 - T^2_{\mathfrak{C}_3} - F^2_{\mathfrak{C}_3} \leq 1 - T^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_1}$, then, by $F^2_{\mathfrak{C}_1} \geq F^2_{\mathfrak{C}_2} \geq F^2_{\mathfrak{C}_3}$, we have $F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2} \geq T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}$ and $F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2} \leq F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3} \leq T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}$. Meanwhile, by (i) and (ii), we have

$$|R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}| \leq F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2},$$
$$|R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}| \leq T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1},$$
$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|$$
$$= 2\left(F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}\right),$$

and

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|$$
$$= 2(T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}).$$

These, together with $F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2} \leq T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}$ and $T^2_{\mathfrak{C}_1} \leq T^2_{\mathfrak{C}_2} \leq T^2_{\mathfrak{C}_3}$ and $F^2_{\mathfrak{C}_1} \geq F^2_{\mathfrak{C}_2} \geq F^2_{\mathfrak{C}_3}$, imply that

$$|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}| + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}| \leq |T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|$$
$$\quad + |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}| + |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|,$$

and

$$\max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_2}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_2}|\}$$
$$= \max\{T^2_{\mathfrak{C}_2} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_2}\}$$
$$\leq \max\{T^2_{\mathfrak{C}_3} - T^2_{\mathfrak{C}_1}, F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}\}$$
$$= \max\{|T^2_{\mathfrak{C}_1} - T^2_{\mathfrak{C}_3}|, |F^2_{\mathfrak{C}_1} - F^2_{\mathfrak{C}_3}|, |R^2_{\mathfrak{C}_1} - R^2_{\mathfrak{C}_3}|\}.$$

And thus, it can be similarly verified that

$$D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \geq D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2).$$

(iv) If $R_{\mathfrak{C}_2}^2 \leq R_{\mathfrak{C}_1}^2$ and $R_{\mathfrak{C}_3}^2 \geq R_{\mathfrak{C}_1}^2$, similarly to the proof of (iii), it is not difficult to check that $D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_3) \geq D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2)$. $\qquad\square$

Summing Propositions 4.1–4.4, we have the following result.

**Theorem 4.1** (1) *The function $D_{\mathrm{Wu}}$ defined by Eq. (8) is a normalized distance measure on $\mathscr{C}_{\mathrm{PyFN}}$.*
(2) *The function $\mathbf{S}_{\mathrm{Wu}}$ defined by $\mathbf{S}_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = 1 - D_{\mathrm{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2)$ is similarity measure on $\mathscr{C}_{\mathrm{PyFN}}$.*

***Remark 1*** (1) Park et al. [30] propose a similar distance formula to Eq. (8) for IFNs as follows: for $\alpha_1 = \langle \mathfrak{P}_1, \mathfrak{O}_1 \rangle$, $\alpha_2 = \langle \mathfrak{P}_2, \mathfrak{O}_2 \rangle \in \Theta$,

$$d_{\mathrm{P}}(\alpha_1, \alpha_2) = \frac{1}{4}(|\mathfrak{P}_1 - \mathfrak{P}_2| + |\mathfrak{O}_1 - \mathfrak{O}_2| + |\mathfrak{H}_1 - \mathfrak{H}_2|)$$
$$+ \frac{1}{2}\max\{|\mathfrak{P}_1 - \mathfrak{P}_2|, |\mathfrak{O}_1 - \mathfrak{O}_2|, |\mathfrak{H}_1 - \mathfrak{H}_2|\},$$

where $\mathfrak{H}_1 = 1 - \mathfrak{P}_1 - \mathfrak{O}_1$ and $\mathfrak{H}_2 = 1 - \mathfrak{P}_2 - \mathfrak{O}_2$, and proved that $d_{\mathrm{P}}$ is a DisM on $\Theta$ (see [30, Theorem 1]). In the proof of [30, Theorem 1], they claimed that, for $\alpha_1, \alpha_2, \alpha_3 \in \Theta$ with $\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3$, one has $\mathfrak{H}_1 \geq \mathfrak{H}_2 \geq \mathfrak{H}_3$. However, the claim does not hold, leading to the result that the proof of [30, Theorem 1] is not right. In fact, choose $\alpha_1 = \langle 0, 1 \rangle$, $\alpha_2 = \langle 0, 0 \rangle$, and $\alpha_3 = \langle 1, 0 \rangle$. Clearly, $\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3$ and $\mathfrak{H}_2 > \mathfrak{H}_1 = \mathfrak{H}_3$, which contradicts the claim.

(2) If we consider the imaginary part as zero in Eq. (8) and replace the constraint $\mathfrak{P}^2 + \mathfrak{O}^2 \leq 1$ by $\mathfrak{P} + \mathfrak{O} \leq 1$, then the distance $D_{\mathrm{Wu}}$ reduces to IF environment. By Theorem 4.1, we know that $d_{\mathrm{P}}$ is a DisM on $\Theta$.

Motivated by the DisMs for IFNs in [14,34,37,39,45], we introduced the following new DisMs for CPyFNs.

Let $\mathfrak{C}_1 = (T_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_1}}}, F_{\mathfrak{C}_1} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_1}}})$ and $\mathfrak{C}_2 = (T_{\mathfrak{C}_2} \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_2}}}, F_{\mathfrak{C}_2} \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_2}}})$ be two CPyFNs. Then, define

(1) Szmidt and Kacprzyk's complex Pythagorean fuzzy DisMs ( [34]):

$$D_{\mathrm{SK}}^{\mathrm{H}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2| + |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|}{4}$$
$$+ \frac{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2| + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2| + |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2|}{4},$$

(9)

and

$$D_{\mathrm{SK}}^{\mathrm{E}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{\sqrt{\frac{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|^2 + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|^2}{2}}}{2}$$
$$+ \frac{\sqrt{\frac{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|^2 + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|^2}{2}}}{2}.$$

(10)

(2) Wang and Xin's complex Pythagorean fuzzy DisMs ([37]):

$$D_{\mathrm{WX1}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{1}{2}\left[\frac{1}{4} \cdot \left(|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|\right)\right.$$
$$+ \frac{1}{2} \cdot \max\{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|, |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|\}$$
$$+ \frac{1}{4} \cdot \left(|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2| + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|\right)$$
$$\left.+ \frac{1}{2} \cdot \max\{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|, |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|\}\right],$$

(11)

and

$$D_{\mathrm{WX2}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2| + |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|}{4}$$
$$+ \frac{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2| + |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|}{4}.$$

(12)

(3) Grzegorzewski's complex Pythagorean fuzzy DisM ([14]):

$$D_{\mathrm{G}}(\mathfrak{C}_1, \mathfrak{C}_2) = \frac{1}{2}\left[\max\{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|, |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|\}\right.$$
$$\left.+ \max\{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|, |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|\}\right],$$

(13)

(4) Yang and Chiclana's complex Pythagorean fuzzy DisM ( [45]):

$$D_{\mathrm{YC}}(\mathfrak{C}_1, \mathfrak{C}_2)$$
$$= \frac{\max\{|T_{\mathfrak{C}_1}^2 - T_{\mathfrak{C}_2}^2|, |F_{\mathfrak{C}_1}^2 - F_{\mathfrak{C}_2}^2|, |R_{\mathfrak{C}_1}^2 - R_{\mathfrak{C}_2}^2|\}}{2}$$
$$+ \frac{\max\{|W_{T_{\mathfrak{C}_1}}^2 - W_{T_{\mathfrak{C}_2}}^2|, |W_{F_{\mathfrak{C}_1}}^2 - W_{F_{\mathfrak{C}_2}}^2|, |W_{R_{\mathfrak{C}_1}}^2 - W_{R_{\mathfrak{C}_2}}^2|\}}{2}.$$

(14)

(5) Wu et al.'s complex Pythagorean fuzzy DisM ([39]):

$$D_{\text{Wu}}^{(1)}(\mathfrak{C}_1, \mathfrak{C}_2)$$

$$= \frac{\sqrt{\frac{1}{2}\left(L\left(1-T_{\mathfrak{C}_1}^2, 1-T_{\mathfrak{C}_2}^2\right) + L\left(F_{\mathfrak{C}_1}^2, F_{\mathfrak{C}_2}^2\right)\right)}}{2}}{\frac{\sqrt{\frac{1}{2}\left(L\left(1-W_{T_{\mathfrak{C}_1}}^2, 1-W_{T_{\mathfrak{C}_2}}^2\right) + L\left(W_{F_{\mathfrak{C}_1}}^2, W_{F_{\mathfrak{C}_2}}^2\right)\right)}}{2}},$$

$$(15)$$

where $L(p, q) = p \cdot \log_2 \frac{2p}{p+q} + q \cdot \log_2 \frac{2q}{p+q}$.

According to the proof in [14,34,37,39,45], by Propositions 4.1–4.4, it can be verified that the above DisMs $D_{\text{SK}}^{\text{H}}$, $D_{\text{SK}}^{\text{E}}$, $D_{\text{WX1}}$, $D_{\text{WX2}}$, $D_{\text{G}}$, $D_{\text{YC}}$, and $D_{\text{Wu}}^{(1)}$ are normalized distance measures on $\mathscr{C}_{\text{PyFN}}$.

## A novel DisM on CPyFSs

Let $\mathfrak{X} = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_\ell\}$ and $\mathfrak{C}_1 = \{\mathfrak{C}_1(\vartheta_j) = (T_{\mathfrak{C}_1}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_1}}(\vartheta_j)}, F_{\mathfrak{C}_1}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_1}}(\vartheta_j)}) \mid 1 \leq j \leq \ell\}$ and $\mathfrak{C}_2 = \{\mathfrak{C}_2(\vartheta_j) = (T_{\mathfrak{C}_2}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{T_{\mathfrak{C}_2}}(\vartheta_j)}, F_{\mathfrak{C}_2}(\vartheta_j) \cdot e^{2\pi \mathbf{i} \cdot W_{F_{\mathfrak{C}_2}}(\vartheta_j)}) \mid 1 \leq j \leq \ell\}$ be two CPyFSs on $\mathfrak{X}$. Define a DisM $\tilde{D}_{\text{Wu}}$ for CPyFSs as follows:

$$\tilde{D}_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = \sum_{j=1}^{\ell} \omega_j \cdot D_{\text{Wu}}(\mathfrak{C}_1(\vartheta_j), \mathfrak{C}_2(\vartheta_j)), \quad (16)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ is the weight vector of $\omega_j$ $(j = 1, 2, \ldots, \ell)$ with $\omega_j \in (0, 1]$ and $\sum_{j=1}^{\ell} \omega_j = 1$.

By Theorem 4.1, we have the following result.

**Theorem 4.2** (1) *The function $\tilde{D}_{\text{Wu}}$ defined by Eq. (16) is a normalized distance measure on* CPyFS$(\mathfrak{X})$.
(2) *The function $\tilde{S}_{\text{Wu}}$ defined by* $S_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2) = 1 - \tilde{D}_{\text{Wu}}(\mathfrak{C}_1, \mathfrak{C}_2)$ *is similarity measure on* CPyFS$(\mathfrak{X})$.

## Applications

### Building material recognition problem

*Example 2* ([36, Example 1]). Consider the following 4 building materials: $\mathscr{M}_1$–sealant; $\mathscr{M}_2$–floor varnish; $\mathscr{M}_3$–wall paint, and $\mathscr{M}_4$–polyvinyl chloride flooring, which are expressed by CPyFNs for attribute set denoted by $\mathfrak{A} = \{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7\}$ in Table 1. Given an unknown building material $\mathscr{M}$, which is expressed by CPyFNs for the above 7 attributes in Table 1. We want to determine to which building material the unknown material $\mathscr{M}$ belongs.

If we take the weight vector $\omega$ of 7 attributes as $\omega = (0.11, 0.14, 0.1, 0.18, 0.21, 0.10, 0.16)^\top$, the SimMs calculated by Eq. (16) are given in Table 2. By the principle of the

maximum degree of SimMs, the unknown building material belongs to the class $\mathscr{M}_4$–polyvinyl chloride flooring.

The pattern classification results by using different DisMs/SimMs are listed in Table 3. Observing from Table 3, we know that (1) our result listed in Table 1 is consistent with the results obtained by the DisM $\text{WD}_{\text{CPyFS}}^1$ in [36] and DisMs defined by Eqs. (9)–(15); (2) because the DisM $\text{WD}_{\text{CPyFS}}^2$ in [36] does not satisfy the axiomatic definition of complex Pythagorean fuzzy DisM, the pattern classification result obtained by this DisM is $\mathscr{M}_1$–sealant, which is unreasonable.

Observing from Table 3, we notice that the existing measures are provided the different ranking results such as $\mathscr{M}_1$ and $\mathscr{M}_4$. But, we also notice that the proposed all measures are given the same ranking measures. Further, we aim to consider some measures which was proposed by different scholars, for this, Rani and Garg [32] presented a few two-dimensional (2D) CIFDisMs by using the Hamming, Euclidean, and Hausdorff distances. Then, Garg and Rani [12] proposed some new CIF information measures, including SimMs, DisMs, entropies, and inclusion measures and obtained the transformation relationships among them. Meanwhile, they [12] developed a CIF clustering algorithm. But these all measures are not able to resolve our selected information because of their limitations, where the computed measures of Rani and Garg [32] have been failed because these measures are the particular case of the proposed measures.

### A medical diagnosis problem

The medical diagnostic data in the following example comes from [32, Example 4.2], which were equivalently expressed by using CIFNs.

*Example 3* ([32, Example 4.2]). Consider a medical diagnosis problem for a patient $\mathbb{P}$ with the symptoms $\mathfrak{S} = \{\text{Temperature, Headache, Stomach pain, Cough}\}$ represented by using CPyFNs, as listed in Table 4. The symptom characteristics for diagnosis $\mathfrak{D} = \{\text{Viral fever, Malaria, Typhoid, Stomach problem}\}$ are represented by using CPyFNs, as shown in Table 5.

By the principle of the maximum degree of SimMs, observing from Table 6, we know that the diagnostic result is that $\mathbb{P}$ suffers from '**Viral fever**', which is consistent with the result in [32, Example 4.2].

Observing from Table 6, we notice that the existing measures are provided the same ranking results such as Vf. But, we also notice that the proposed all measures are given the same ranking measures. Further, we aim to consider some measures which was proposed by different scholars, for this, Rani and Garg [32] presented a few two-dimensional (2D)

**Table 1** CPF information about known and unknown materials in Example 2

|  | $\mathscr{M}_1$ | $\mathscr{M}_2$ | $\mathscr{M}_3$ | $\mathscr{M}_4$ | $\mathscr{M}$ |
|---|---|---|---|---|---|
| $\mathfrak{A}_1$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.7)}, 0.3e^{2\pi\mathbf{i}\cdot(0.4)})$ | $(0.7e^{2\pi\mathbf{i}\cdot(0.5)}, 0.4e^{2\pi\mathbf{i}\cdot(0.7)})$ | $(0.5e^{2\pi\mathbf{i}\cdot(0.6)}, 0.7e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.7e^{2\pi\mathbf{i}\cdot(0.6)}, 0.3e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_2$ | $(0.7e^{2\pi\mathbf{i}\cdot(0.6)}, 0.5e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.7)}, 0.6e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.5)}, 0.5e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.5)}, 0.4e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_3$ | $(0.9e^{2\pi\mathbf{i}\cdot(0.4)}, 0.2e^{2\pi\mathbf{i}\cdot(0.7)})$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.8)}, 0.3e^{2\pi\mathbf{i}\cdot(0.3)})$ | $(0.3e^{2\pi\mathbf{i}\cdot(0.6)}, 0.3e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(0.7e^{2\pi\mathbf{i}\cdot(0.6)}, 0.5e^{2\pi\mathbf{i}\cdot(0.3)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_4$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.6)}, 0.5e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.9)}, 0.3e^{2\pi\mathbf{i}\cdot(0.1)})$ | $(0.5e^{2\pi\mathbf{i}\cdot(0.7)}, 0.5e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.4e^{2\pi\mathbf{i}\cdot(0.8)}, 0.7e^{2\pi\mathbf{i}\cdot(0.2)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_5$ | $(0.5e^{2\pi\mathbf{i}\cdot(0.3)}, 0.6e^{2\pi\mathbf{i}\cdot(0.7)})$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.3)}, 0.5e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.7)}, 0.6e^{2\pi\mathbf{i}\cdot(0.3)})$ | $(0.7e^{2\pi\mathbf{i}\cdot(0.6)}, 0.5e^{2\pi\mathbf{i}\cdot(0.2)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_6$ | $(0.4e^{2\pi\mathbf{i}\cdot(0.6)}, 0.7e^{2\pi\mathbf{i}\cdot(0.6)})$ | $(0.2e^{2\pi\mathbf{i}\cdot(0.7)}, 0.8e^{2\pi\mathbf{i}\cdot(0.2)})$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.6)}, 0.3e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.6e^{2\pi\mathbf{i}\cdot(0.8)}, 0.5e^{2\pi\mathbf{i}\cdot(0.3)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |
| $\mathfrak{A}_7$ | $(0.2e^{2\pi\mathbf{i}\cdot(0.2)}, 0.5e^{2\pi\mathbf{i}\cdot(0.8)})$ | $(0.1e^{2\pi\mathbf{i}\cdot(0.6)}, 0.9e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.2e^{2\pi\mathbf{i}\cdot(0.5)}, 0.8e^{2\pi\mathbf{i}\cdot(0.5)})$ | $(0.8e^{2\pi\mathbf{i}\cdot(0.7)}, 0.3e^{2\pi\mathbf{i}\cdot(0.4)})$ | $(1\cdot e^{2\pi\mathbf{i}}, 0)$ |

**Table 2** SimMs between $\mathscr{M}_i$ $(i = 1, 2, 3, 4)$ and $\mathscr{M}$ in Example 2 using $\tilde{\mathbf{S}}_{\text{Wu}}$

| $\tilde{\mathbf{S}}_{\text{Wu}}(\mathscr{M}_1, \mathscr{M})$ | $\tilde{\mathbf{S}}_{\text{Wu}}(\mathscr{M}_2, \mathscr{M})$ | $\tilde{\mathbf{S}}_{\text{Wu}}(\mathscr{M}_3, \mathscr{M})$ | $\tilde{\mathbf{S}}_{\text{Wu}}(\mathscr{M}_4, \mathscr{M})$ | Ranking | Classification |
|---|---|---|---|---|---|
| 0.3031 | 0.3981 | 0.3278 | **0.4532** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |

**Table 3** Comparative analysis by different SimMs/DisMs in Example 2

| Method | Similarity measure | | | | Ranking | Classification |
|---|---|---|---|---|---|---|
| | $1 - \text{dis}(\mathscr{M}_1, \mathscr{M})$ | $1 - \text{dis}(\mathscr{M}_2, \mathscr{M})$ | $1 - \text{dis}(\mathscr{M}_3, \mathscr{M})$ | $1 - \text{dis}(\mathscr{M}_4, \mathscr{M})$ | | |
| $\text{WD}^1_{\text{CPyFS}}$ in [36] | 0.432 | 0.923 | 0.921 | **0.933** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| $\text{WD}^2_{\text{CPyFS}}$ in [36] | **0.730** | 0.718 | 0.725 | 0.712 | $\mathscr{M}_1 \succ \mathscr{M}_3 \succ \mathscr{M}_2 \succ \mathscr{M}_4$ | $\mathscr{M}_1$ |
| DisM $D^{\text{H}}_{\text{SK}}$ | 0.3031 | 0.3981 | 0.3278 | **0.4532** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D^{\text{E}}_{\text{SK}}$ | 0.5319 | 0.6125 | 0.5917 | **0.7346** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D_{\text{WX1}}$ | 0.0717 | 0.1706 | 0.0819 | **0.2948** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D_{\text{WX2}}$ | 0.4861 | 0.5594 | 0.5207 | **0.6330** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D_{\text{G}}$ | 0.3031 | 0.3981 | 0.3278 | **0.4532** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D_{\text{YC}}$ | 0.3031 | 0.3981 | 0.3278 | **0.4532** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| DisM $D^{(1)}_{\text{Wu}}$ | 0.2956 | 0.3597 | 0.3144 | **0.4034** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |
| SimM $\tilde{\mathbf{S}}_{\text{Wu}}$ | 0.3031 | 0.3981 | 0.3278 | **0.4532** | $\mathscr{M}_4 \succ \mathscr{M}_2 \succ \mathscr{M}_3 \succ \mathscr{M}_1$ | $\mathscr{M}_4$ |

**Table 4** CPF representation of symptoms for the patient

| Patient | Temperature | Headache | Stomach pain | Cough |
|---|---|---|---|---|
| $\mathbb{P}$ | $\left(\sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.6}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ | $\left(\sqrt{0.9}e^{2\pi\mathbf{i}\cdot\sqrt{0.7}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ | $\left(\sqrt{0.7}e^{2\pi\mathbf{i}\cdot\sqrt{0.8}}, \sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ | $\left(\sqrt{0.6}e^{2\pi\mathbf{i}\cdot\sqrt{0.5}}, \sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.4}}\right)$ |

**Table 5** CPF representation of symptom characteristics for diagnosis

| Disease | Temperature | Headache | Stomach pain | Cough |
|---|---|---|---|---|
| Viral fever (Vf) | $\left(\sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.7}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ | $\left(\sqrt{0.9}e^{2\pi\mathbf{i}\cdot\sqrt{0.6}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ | $\left(\sqrt{0.7}e^{2\pi\mathbf{i}\cdot\sqrt{0.8}}, \sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ | $\left(\sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.7}}, \sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ |
| Malaria (Ma) | $\left(\sqrt{0.6}e^{2\pi\mathbf{i}\cdot\sqrt{0.4}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.5}}\right)$ | $\left(\sqrt{0.4}e^{2\pi\mathbf{i}\cdot\sqrt{0.9}}, \sqrt{0.5}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ | $\left(\sqrt{0.5}e^{2\pi\mathbf{i}\cdot\sqrt{0.5}}, \sqrt{0.3}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}\right)$ | $\left(\sqrt{0.4}e^{2\pi\mathbf{i}\cdot\sqrt{0.9}}, \sqrt{0.5}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ |
| Typhoid (Ty) | $\left(\sqrt{0.3}e^{2\pi\mathbf{i}\cdot\sqrt{0.8}}, \sqrt{0.3}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}\right)$ | $\left(\sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.6}}\right)$ | $\left(\sqrt{0.7}e^{2\pi\mathbf{i}\cdot\sqrt{0.6}}, \sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ | $\left(\sqrt{0.2}e^{2\pi\mathbf{i}\cdot\sqrt{0.7}}, \sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.2}}\right)$ |
| Stomach problem (Sp) | $\left(\sqrt{0.5}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}, \sqrt{0.4}e^{2\pi\mathbf{i}\cdot\sqrt{0.6}}\right)$ | $\left(\sqrt{0.3}e^{2\pi\mathbf{i}\cdot\sqrt{0.1}}, \sqrt{0.6}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}\right)$ | $\left(\sqrt{0.8}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}, \sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.5}}\right)$ | $\left(\sqrt{0.1}e^{2\pi\mathbf{i}\cdot\sqrt{0.3}}, \sqrt{0.6}e^{2\pi\mathbf{i}\cdot\sqrt{0.5}}\right)$ |

**Table 6** Diagnostic results by different SimMs/DisMs with $\omega = (0.25, 0.25, 0.25, 0.25)^\top$ in Example 3

| Method | Similarity measure | | | | Ranking | Classification |
|---|---|---|---|---|---|---|
| | $1 - \mathrm{dis}\,(\mathrm{Vf}, \mathbb{P})$ | $1 - \mathrm{dis}\,(\mathrm{Ma}, \mathbb{P})$ | $1 - \mathrm{dis}\,(\mathrm{Ty}, \mathbb{P})$ | $1 - \mathrm{dis}\,(\mathrm{Sp}, \mathbb{P})$ | | |
| $\mathrm{WD}^1_{\mathrm{CPyFS}}$ in [36] | **0.9313** | 0.7375 | 0.7625 | 0.6312 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D^{\mathrm{H}}_{\mathrm{SK}}$ | **0.9125** | 0.7000 | 0.7250 | 0.6000 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D^{\mathrm{E}}_{\mathrm{SK}}$ | **0.9832** | 0.8948 | 0.8833 | 0.7967 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D_{\mathrm{WX1}}$ | **0.9031** | 0.5688 | 0.5938 | 0.4219 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D_{\mathrm{WX2}}$ | **0.9437** | 0.7562 | 0.7750 | 0.6625 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D_{\mathrm{G}}$ | **0.9125** | 0.7000 | 0.7250 | 0.6000 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D_{\mathrm{YC}}$ | **0.9125** | 0.7000 | 0.7250 | 0.6000 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| DisM $D^{(1)}_{\mathrm{Wu}}$ | **0.9254** | 0.7190 | 0.7649 | 0.6633 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |
| SimM $\tilde{\mathbf{S}}_{\mathrm{Wu}}$ | **0.9125** | 0.7000 | 0.7250 | 0.6000 | Vf ≻ Ty ≻ Ma ≻ Sp | Vf |

CIFDisMs by using the Hamming, Euclidean, and Hausdorff distances. Then, Garg and Rani [12] proposed some new CIF information measures, including SimMs, DisMs, entropies, and inclusion measures) and obtained the transformation relationships among them. Meanwhile, they [12] developed a CIF clustering algorithm. But these all measures are not able to resolve our selected information because of their limitations, where the computed measures of Rani and Garg [32] have been failed because these measures are the particular case of the proposed measures.

## Conclusion

Complex Pythagorean fuzzy set is very effective and realistic for evaluating awkward and complex information in real-life problems. Keeping the advantages of the CPyFS, the main theme of this analysis is described below:

(1) Ullah et al. [36] introduced the notion of CPyFS to deal with uncertain information and proposed a 2D CPyFDisM $D^1_{\mathrm{CPyFS}}$ and a 3D CPyFDisM $D^2_{\mathrm{CPyFS}}$.

(2) The DisM $D^2_{\mathrm{CPyFS}}$ does not satisfy the axiomatic definition of CPyFDisM (see Example 1).

(3) To overcome the drawback of the DisM $D^2_{\mathrm{CPyFS}}$ in [36], we constructed a novel 3D DisM $\tilde{D}_{\mathrm{Wu}}$ for CPyFSs based on the 3D Hamming distance and the Hausdorff distance for IFSs and show that it satisfies the axiomatic definition of CPyFDisM.

(4) Based on the DisMs for IFSs in [14,34,37,39,45], we proposed some other new CPyFDisMs.

(5) We illustrated the effectiveness of our DisMs, we give the comparative analysis by using our proposed DisMs to a building material recognition problem and a medical diagnosis problem.

In the future, we aim to develop some new ideas such as averaging or geometric aggregation operators [26], Maclaurin symmetric mean operators [35], Aczel–Alsina operators [18], information measures [33], similarity measures [23], Einstein aggregation operators, Hamacher aggregation operators, improved Dombi aggregation operators [17], interaction aggregation operators, similarity measures, distance measures, many techniques based on CPyFSs and try to justify it with the help of some suitable applications such as road signals, computer networks, game theory, artificial intelligence, and decision-making theory.

## References

1. Akram M, Peng X, Al-Kenani AN, Sattar A (2020) Prioritized weighted aggregation operators under complex Pythagorean fuzzy information. J Intell Fuzzy Syst 39(3):4763–4783

2. Akram M, Khan A, Borumand Saeid A (2021) Complex Pythagorean Dombi fuzzy operators using aggregation operators and their decision-making. Expert Syst 38(2):e12626

3. Akram M, Peng X, Sattar A (2021) Multi-criteria decision-making model using complex Pythagorean fuzzy Yager aggregation operators. Arab J Sci Eng 46(2):1691–1717

4. Akram M, Zahid K, Alcantud JCR (2022) A new outranking method for multicriteria decision making with complex Pythagorean fuzzy information. Neural Comput Appl 34(10):8069–8102

5. Aldring J, Ajay D (2022) Multicriteria group decision making based on projection measures on complex Pythagorean fuzzy sets. Granular Comput:1–19

6. Ali Z, Mahmood T (2020) Maclaurin symmetric mean operators and their applications in the environment of complex q-rung orthopair fuzzy sets. Comput Appl Math 39(3):1–27

7. Alkouri AS, Salleh AR (2012) Complex intuitionistic fuzzy sets. In: International Conference on Fundamental and Applied Sciences, volume 1482, pages 464–470. American Institute of Physics

8. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):96–97

9. Atanassov KT (1999) Intuitionistic fuzzy sets: theory and applications, vol 35. Studies in fuzziness and soft computing. Springer, Berlin Heidelberg

10. Bellman RE, Zadeh LA (1970) Decision-making in a fuzzy environment. Manage Sci 17(4):141–164

11. Bustince H, Barrenechea E, Pagola M, Fernandez J, Xu Z, Bedregal B, Montero J, Hagras H, Herrera F, De Baets B (2016) A historical account of types of fuzzy sets and their relationships. IEEE Trans Fuzzy Syst 24(1):179–194

12. Garg H, Rani D (2019) Some results on information measures for complex intuitionistic fuzzy sets. Int J Intell Syst 34(10):2319–2363

13. Garg H, Ali Z, Mahmood T (2021) Generalized dice similarity measures for complex q-rung orthopair fuzzy sets and its application. Complex Intell Syst 7(2):667–686

14. Grzegorzewski P (2004) Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. Fuzzy Sets Syst 148(2):319–328

15. Janani K, Veerakumari KP, Vasanth K, Rakkiyappan R (2022) Complex Pythagorean fuzzy Einstein aggregation operators in selecting the best breed of Horsegram. Expert Syst Appl 187:115990 (**24 pages**)

16. Karakose M, Yaman O (2020) Complex fuzzy system based predictive maintenance approach in railways. IEEE Trans Ind Informat 16(9):6023–6032

17. Khan T, Qaisar M, Ullah K (2021) Applications of improved spherical fuzzy Dombi aggregation operators in decision support system. Soft Comput 14:9097–9119

18. Khan MR, Wang H, Ullah K, Karamti H (2022) Construction material selection by using multi-attribute decision making based on q-rung orthopair fuzzy Aczel-Alsina aggregation operators. Appl Sci 2022(12):8537–8545

19. Li H, Yen VC (1995) Fuzzy sets and fuzzy decision-making. CRC Press

20. Liu P, Mahmood T, Ali Z (2020) Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. Information 11(1):27

21. Liu P, Ali Z, Mahmood T (2021) Some cosine similarity measures and distance measures between complex q-rung orthopair fuzzy sets and their applications. Int J Comput Intell Syst 14(1):1653–1671

22. Liu P, Mahmood T, Ali Z (2022) The cross-entropy and improved distance measures for complex q-rung orthopair hesitant fuzzy sets and their applications in multi-criteria decision-making. Complex Intell Syst 8(2):1167–1186

23. Liu P, Munir M, Mahmood T, Ullah K (2022) Some similarity measures for interval-valued picture fuzzy sets and their applications in decision making. Information 2022(12):369–397

24. Ma X, Akram M, Zahid K, Alcantud JCR (2021) Group decision-making framework using complex Pythagorean fuzzy information. Neural Comput Appl 33(6):2085–2105

25. Mahmood T, Ali Z (2021) Entropy measure and TOPSIS method based on correlation coefficient using complex q-rung orthopair fuzzy information and its application to multi-attribute decision making. Soft Comput 25(2):1249–1275

26. Mahmood KUQK, Tahir Jan N (2019) An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. Neural Comput Appl 31(11):7041–7053

27. Mahmood T, Ur Rehman U, Ali Z, Mahmood T (2021) Hybrid vector similarity measures based on complex hesitant fuzzy sets and their applications to pattern recognition and medical diagnosis. J Intell Fuzzy Syst 40(1):625–646

28. Ngan RT, Ali M, Fujita H, Abdel-Basset M, Giang NL, Manogaran G, Priyan MK (2019) A new representation of intuitionistic fuzzy systems and their applications in critical decision making. IEEE Intell Syst 35(1):6–17

29. Ngan RT, Son LH, Ali M, Tamir DE, Rishe ND, Kandel A (2020) Representing complex intuitionistic fuzzy set by quaternion numbers and applications to decision making. Appl Soft Comput 87:105961

30. Park J-H, Lim K-M, Kwun Y-C (2009) Distance measure between intuitionistic fuzzy sets and its application to pattern recognition. J Korean Inst Intell Syst 19(4):556–561

31. Ramot D, Milo R, Friedman M, Kandel A (2002) Complex fuzzy sets. IEEE Trans Fuzzy Syst 10(2):171–186

32. Rani D, Garg H (2017) Distance measures between the complex intuitionistic fuzzy sets and their applications to the decision-making process. Int J Uncertain Quantif 7(5):423–439

33. Shen X, Sakhi S, Ullah K, Abid MN, Jin Y (2022) Information measures based on T-spherical fuzzy sets and their applications in decision making and pattern recognition. Axioms 11(7):23

34. Szmidt E, Kacprzyk J (2000) Distances between intuitionistic fuzzy sets. Fuzzy Sets Syst 114(3):505–518

35. Ullah K (2021) Picture fuzzy maclaurin symmetric mean operators and their applications in solving multiattribute decision-making problems. Math Probl Eng 2021:13

36. Ullah K, Mahmood T, Ali Z, Jan N (2020) On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex Intell Syst 6(1):15–27

37. Wang W, Xin X (2005) Distance measure between intuitionistic fuzzy sets. Pattern Recog. Lett. 26(13):2063–2069

38. Wang H, Zhang F (2022) Complex Pythagorean uncertain linguistic group decision-making model based on Heronian mean aggregation operator considering uncertainty, interaction and interrelationship. Complex Intell Syst. https://doi.org/10.1007/s40747-022-00749-y

39. Wu X, Wang T, Chen G, Zhu Z, Liu P (2022) Strict intuitionistic fuzzy distance/similarity measures based on Jensen-Shannon divergence. IEEE Trans Syst. Man Cybern. Syst. arxiv:2207.06980

40. Wu X, Zhu Z, Chen C, Chen G, Liu P (2022) A monotonous intuitionistic fuzzy TOPSIS method under general linear orders via admissible distance measures. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2022.3205435

41. Xu Z, Cai X (2012) Intuitionistic fuzzy information aggregation: theory and applications, volume 20 of Mathematics Monograph Series. Science Press

42. Yager RR (2014) Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst 22(4):958–965

43. Yager RR (2017) Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst 25(5):1222–1230

44. Yager RR, Alajlan N (2017) Approximate reasoning with generalized orthopair fuzzy sets. Inf Fus 38:65–73
45. Yang Y, Chiclana F (2012) Consistency of 2D and 3D distances of intuitionistic fuzzy sets. Expert Syst Appl 39(10):8665–8670
46. Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353
47. Zhang X (2018) Pythagorean fuzzy clustering analysis: a hierarchical clustering algorithm with the ratio index-based ranking methods. Int J Intell Syst 33(9):1798–1822