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ORIGINAL ARTICLE

(2,1)-Fuzzy sets: properties, weighted aggregated operators and their applications to multi-criteria decision-making methods

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Abstract
Orthopair fuzzy sets are fuzzy sets in which every element is represented by a pair of values in the unit interval, one of which refers to membership and the other refers to non-membership. The different types of orthopair fuzzy sets given in the literature are distinguished according to the proposed constrain for membership and non-membership grades. The aim of writing this manuscript is to familiarize a new class of orthopair fuzzy sets called "(2,1)-Fuzzy sets" which are good enough to control some real-life situations. We compare (2,1)-Fuzzy sets with IFSs and some of their celebrated extensions. Then, we put forward the fundamental set of operations for (2,1)-Fuzzy sets and investigate main properties. Also, we define score and accuracy functions which we apply to rank (2,1)-Fuzzy sets. Moreover, we reformulate aggregation operators to be used with (2,1)-Fuzzy sets. Finally, we develop the successful technique "aggregation operators" to handle multi-criteria decision-making (MCDM) problems in the environment of (2,1)-Fuzzy sets. To show the effectiveness and usability of the proposed technique in MCDM problems, an illustrative example is provided.

Keywords (2,1)-Fuzzy set · Score and accuracy functions · (2,1)-Aggregation operators · Multi-criteria decision-making

Introduction

In the real world, we deal with ideas that are loaded with uncertainties and imprecision in several territories such as engineering, medicinal science, economics, and natural science. To handle this scenario Zadeh [34], in 1965, familiarized the concept of fuzzy set that extensively applied in many areas of multi-criteria decision-making (MCDM). Zadeh allotted a membership degree for each element in the domain; however, there are various real-life cases, the non-membership degree is not come from the membership degree. To overcome this shortcoming, Atanassov [5] proposed an extension of fuzzy sets called an intuitionistic fuzzy set (IFS) which was successfully applied in various areas like medical diagnosis and decision-making [1,8].

Then, for sake of enlarging the domain of membership and non-membership degrees, Yager [28] defined a Pythagorean fuzzy set (PFS) as a generalization of intuitionistic fuzzy set. It efficiently deals with the situations which the sum of their membership and non-membership degrees of a specified attribute is greater than one. To made a general umbrella of the generalization class of intuitionistic fuzzy set, Yager [29] presented the idea of q-rung orthopair fuzzy set (q-ROFS). In 2019, Senapati and Yager [22] discussed a Fermatean fuzzy sets (FFS) as a special case of q-rung orthopair fuzzy sets obtained by putting $q = 3$. Recently, Ibrahim et al. [9] have brought a new class of fuzzy sets which lies between the grade spaces of Pythagorean and Fermatean fuzzy sets called (3,2)-Fuzzy sets. They applied to establish new kinds of weighted aggregation operators and address more uncertainty situations than Pythagorean fuzzy sets. Then, Al-shami et al. [3] have investigated the concept of SR-fuzzy sets as a new extension of fuzzy sets and applied to generate new aggregated operators.

Since vagueness is a noteworthy issue in numerous territories and its complexity increases day by day, some improvements for fuzzy theory become necessary to keep up with these developments. In this regard, study fuzziness with bipolarity view was investigated in some published literature like [18,19]. Also, hybridization of fuzzy sets with some uncertainty tools such as rough and fuzzy soft was the goal of some articles such as [2,6,7,31,32]. Other classes of fuzzy sets were established and investigated in many manuscripts

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such as [10,11]. Besides all of these, abstract structures like topologies and their main properties were studied in fuzzy settings; see, for example, [4,20].

Decision-making, as a widely used concept of human daily life, gets more complicated with the progression of communication and technology. One of the major issues for decision-makers is how to obtain a unique result from the collective information given by different sources. To do this, different types of aggregation operators have been introduced which reduce the set of finite values in the decision-making process into a single value. Under intuitionistic fuzzy environment Xu [24] initiated a weighted averaging aggregation operator, and Xu and Yager [27] studied a weighted geometric aggregation operator. Lately, several types of aggregation operators have been explored in the environment of intuitionistic fuzzy sets in the published literature; see, [13,16,25,26,33]. Also, these operations have been investigated in the frame of Pythagorean fuzzy sets as given by Khan et al. [12], Peng and Yuan [15], Shahzadi et al. [23], Rahman et al. [17], and Yager and Abbasov [30]. Investigation of aggregation operators in the frame of Fermatean fuzzy sets was conducted in [21].

Multi-attribute decision-making (MADM) problems are constructed of a finite set of options/alternatives and a finite set of criteria/attributes. In this type of problems, it is important to evaluate the quality of the input data. But it's not only about selecting the environment (FS,IFS,PFS,FFS,etc.), it's also about how you are modeling the problem. In other words, which one of these environments frames the phenomena or problem under study? That is, it is not possible to use some types of fuzzy sets to model some actual problems because the information form (with respect to their membership and non-membership grades) in this problem does not satisfy these types of fuzzy sets (with respect to their constraints); hence, the comparison between the effectiveness or who is the best of these types of fuzzy sets is meaningless.

The motivation of doing this research is, first, to define a new generalization of intuitionistic fuzzy set, namely, (2,1)-Fuzzy sets. This generalization enlarges the space of membership and non-membership degrees more than intuitionistic fuzzy sets. As we see this class does not obtain from the class of q-rung orthopair fuzzy sets since the difference of the values q of membership and non-membership grades. Second, to establish a new kind of weighted aggregation operators which can be employed to handle some practical problems; especially, those that are evaluated with different importance of their membership and non-membership grades. Finally, to display a multi-criteria decision-making methods based on the introduced operators for choosing the optimal alternative. It worthily noting that the grades space of our class is smaller than the grades space of all types of q-rung orthopair fuzzy sets; however, it provides another frame more convenient to represent the input data for some real-life issues.

The rest of this manuscript is arranged as follows:

(1) In "Preliminaries", we recall some definitions to make this article self-contained.
(2) We devote "(2,1)-Fuzzy sets" to introduce a new family of generalized IFSs called (2,1)-Fuzzy sets. We display a set of operations for (2,1)-Fuzzy sets and scrutinize main properties.
(3) In "Aggregation of (2,1)-fuzzy sets with applications", the concepts of weighted aggregated operators via (2,1)-Fuzzy sets are investigated and characterized.
(4) In "Application of (2,1)-FSs to MCDM problems", we describe an MCDM method under these operators and present a practical example to show how it carries out.
(5) Ultimately, we outline the main achievements of the paper and propose some upcoming works in "Conclusions".

Preliminaries

To make this study self-sufficient, we briefly present a few concepts engaged in the remaining parts of this study. We also present some interpretations for the beyond motivations to initiating the extensions of fuzzy sets.

Definition 1 [5] The intuitionistic fuzzy set (IFS) is defined over a universal set B as follows.

$\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$, where the functions δ_Ω and λ_Ω from B into $[0, 1]$ respectively represent the membership and non-membership degrees of every $\nu \in B$ to Ω under the constraint $0 \leq \delta_\Omega(\nu) + \lambda_\Omega(\nu) \leq 1$.

The indeterminacy degree of each $\nu \in B$ with respect to an IFS is given by

$$\zeta_\Omega(\nu) = 1 - (\delta_\Omega(\nu) + \lambda_\Omega(\nu)).$$

Remember that if $\delta_\Omega(\nu) = 1 - \lambda_\Omega(\nu)$ for every element $\nu \in B$, then an intuitionistic fuzzy set Ω becomes a fuzzy set.

The natural question that puts itself is why the non-membership degree is not the complement of membership degree in all cases? To our best knowledge, the membership and non-membership degrees are calculated with respect to independent criteria, or sometimes they are evaluated by two independent groups of experts, one specifies the membership and the other specifies the non-membership. That is, the standards of a membership degree need not be the complement of the standards of a non-membership degree. To explain this matter, the example below is provided.

Example 1 Consider B is a group of students is examined in Mathematics. They are evaluated by 50 questions. Every student has two options, answer (correctly or incorrectly) the question or does not answer the question. The followed technique of evaluating the students' performance is given as an IFS $\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$ such that $\delta_\Omega(\nu) = \frac{c}{50}$ and $\lambda_\Omega(\nu) = \frac{d}{50}$, where c and d denote the number of correct answers and the number of incorrect answers, respectively. Assume that Mustafa is a student of this group, and his performance in the exam is as follows, he correctly answered 30 questions, incorrectly answered 15 questions, and did not answer five questions. The corresponding IFS of his performance is $\Omega = \left\langle Mustafa, \frac{3}{5}, \frac{3}{10}\right\rangle$. It is clear that $\delta_\Omega(Mustafa) \neq 1 - \lambda_\Omega(Mustafa)$.

Definition 2 [28] The Pythagorean fuzzy set (PFS) is defined over a universal set B as follows.

$\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$, where the functions δ_Ω and λ_Ω from B into $[0, 1]$ respectively represent the membership and non-membership degrees of every $\nu \in B$ to Ω under the constraint $0 \leq (\delta_\Omega(\nu))^2 + (\lambda_\Omega(\nu))^2 \leq 1$.

The indeterminacy degree of each $\nu \in B$ with respect to a PFS is given by

$$\zeta_\Omega(\nu) = \sqrt{1 - ((\delta_\Omega(\nu))^2 + (\lambda_\Omega(\nu))^2)}.$$

It can be seen that any Pythagorean fuzzy set is an intuitionistic fuzzy set, but the converse fails as the next example shows.

Example 2 Let $\Omega = \{\langle \nu, 0.8, 0.5\rangle, \langle \mu, 0.6, 0.3\rangle\}$ be defined over $B = \{\nu, \mu\}$. Then, Ω is not an intuitionistic fuzzy set because $\delta_\Omega(\nu) + \lambda_\Omega(\nu) = 1.3 \nleq 1$. On the other hand, Ω is a Pythagorean fuzzy set because $(\delta_\Omega(\nu))^2 + (\lambda_\Omega(\nu))^2 = 0.89 \leq 1$ and $(\delta_\Omega(\mu))^2 + (\lambda_\Omega(\mu))^2 = 0.45 \leq 1$.

To enlarge the grades space of membership and non-membership degrees, Senapati and Yager [22] defined the concept of Fermatean fuzzy set as follows.

Definition 3 [22] The Fermatean fuzzy set (FFS) is defined over a universal set B as follows.

$\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$, where the functions δ_Ω and λ_Ω from B into $[0, 1]$ respectively represent the membership and non-membership degrees of every $\nu \in B$ to Ω under the constraint $0 \leq (\delta_\Omega(\nu))^3 + (\lambda_\Omega(\nu))^3 \leq 1$.

The indeterminacy degree of each $\nu \in B$ with respect to a FFS is given by

$$\zeta_\Omega(\nu) = \sqrt[3]{1 - ((\delta_\Omega(\nu))^3 + (\lambda_\Omega(\nu))^3)}.$$

With the aid of example below, we demonstrate that some Fermatean fuzzy sets fail to be Pythagorean fuzzy sets.

Example 3 Let $\Omega = \{\langle \nu, 0.9, 0.5\rangle, \langle \mu, 0.6, 0.7\rangle\}$ be defined over $B = \{\nu, \mu\}$. Then, Ω is not a Pythagorean fuzzy set because $(\delta_\Omega(\nu))^2 + (\lambda_\Omega(\nu))^2 = 1.06 \nleq 1$. On the other hand, Ω is a Fermatean fuzzy set because $(\delta_\Omega(\nu))^3 + (\lambda_\Omega(\nu))^3 = 0.854 \leq 1$ and $(\delta_\Omega(\mu))^3 + (\lambda_\Omega(\mu))^3 = 0.559 \leq 1$.

(2,1)-Fuzzy Sets

The core concept of this manuscript called "(2,1)-Fuzzy Sets" is introduced herein. The aim of presenting this concept are to extend the grade space of intuitionistic fuzzy sets and create a suitable environment to model some real-life issues. We elucidate that this concept lies between the classes of intuitionistic fuzzy sets and Pythagorean fuzzy sets. Then, We define the main set of operations for (2,1)-Fuzzy sets and find out their master features.

Definition 4 The (2,1)-Fuzzy set (briefly, (2,1)-FS) Ω over the universal set B is defined as follows.

$\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$, where the functions δ_Ω and λ_Ω from B into $[0, 1]$ respectively represent the membership and non-membership degrees of every $\nu \in B$ to Ω under the constraint $0 \leq (\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) \leq 1$.

The indeterminacy degree with respect to a (2,1)-FS Ω is a function $\zeta_\Omega : B \rightarrow [0, 1]$ given by

$$\zeta_\Omega(\nu) = (1 - ((\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu)))^{\frac{2}{3}} \text{ for each } \nu \in B.$$

It is obvious that $(\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) + (\zeta_\Omega(\nu))^{\frac{3}{2}} = 1$. Note that $\zeta_\Omega(\nu) = 0$ whenever $(\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) = 1$.

For the sake of simplicity, we denote the (2,1)-FS $\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\}$ by the symbol $\Omega = (\delta_\Omega, \lambda_\Omega)$. The family of all (2,1)-FSs defined over B is denoted by $I^{(2,1)-FS}$.

In Fig. 1, we display the grades space of (2,1)-Fuzzy membership and (2,1)-Fuzzy non-membership.

In what follows, we compare (2,1)-FS with IFS and PFS.

Proposition 1 *1. Every IFS is a (2,1)-FS.*
2. Every (2,1)-FS is a PFS.

Proof Let $\Omega = (\delta_\Omega, \lambda_\Omega)$ be an IFS over B. Then, for each $\nu \in B$, we have the following implement.

$$\begin{aligned} 0 \leq \delta_\Omega(\nu) + \lambda_\Omega(\nu) \leq 1 &\Rightarrow 0 \leq (\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) \leq 1 \\ &\Rightarrow 0 \leq (\delta_\Omega(\nu))^2 + (\lambda_\Omega(\nu))^2 \leq 1 \end{aligned}$$

Hence, the proof is completed. □

The converses of the assertions furnished in Proposition 1 fail as the next example illustrates.

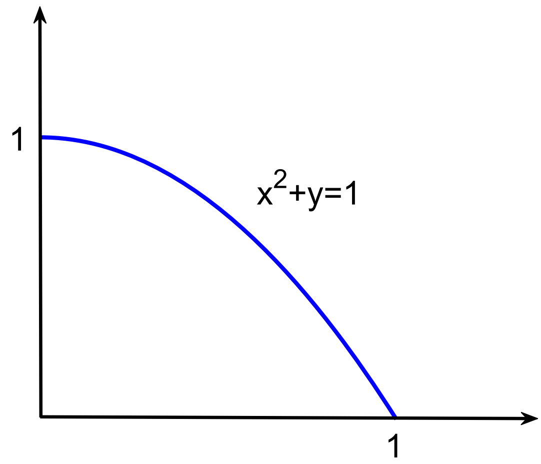

Fig. 1 The grades space of (2,1)-FSs

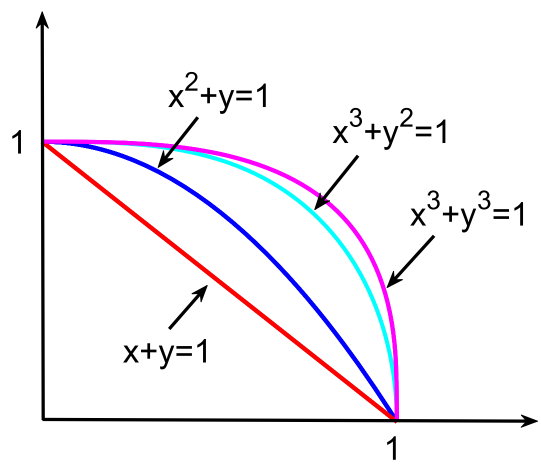

Fig. 2 The relationships among the grades spaces of IFS, (2,1)-FS, PFS and FFS

Example 4 Let $\Omega = (0.7, 0.5)$ and $\Gamma = (0.9, 0.2)$ be defined over $B = \{\nu\}$. Then, Ω is a (2,1)-FS because $(0.7)^2 + 0.5 = 0.99 \leq 1$, but it is not an IFS because $0.7 + 0.5 = 1.2 > 1$. Also, Γ is a PFS because $(0.9)^3 + (0.2)^3 = 0.737 \leq 1$, but it is not a (2,1)-FS because $(0.9)^2 + 0.2 = 1.01 > 1$.

Note that $\zeta_\Omega(\nu) \approx 0.04641589$.

Remark 1 From Proposition 1, we summarize the relationships among the IFS, (2,1)-FS, PFS and FFS in Fig. 2 which illustrates that

1. the grades space of intuitionistic membership is smaller than the space of (2,1)-Fuzzy membership.
2. the grades space of (2,1)-Fuzzy membership is smaller than the space of Pythagorean membership.

Definition 5 Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-Fuzzy sets on B. Then

1. $\Omega_1 \cup \Omega_2 = (\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\})$.
2. $\Omega_1 \cap \Omega_2 = (\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\})$.
3. $\Omega_1^c = (\sqrt{\lambda_{\Omega_1}}, (\delta_{\Omega_1})^2)$.

Note that $(\sqrt{\lambda_{\Omega_1}})^2 + (\delta_{\Omega_1})^2 = \lambda_{\Omega_1} + (\delta_{\Omega_1})^2 \leq 1$, so Ω_1^c is a (2,1)-Fuzzy set. It is obvious that $(\Omega^c)^c = (\sqrt{\lambda_\Omega}, (\delta_\Omega)^2)^c = (\delta_\Omega, \lambda_\Omega)$.

Remark 2 The family of (2,1)-Fuzzy sets is closed under the operators of $\cup$ and $\cap$.

The next example shows how these operators are calculated.

Example 5 Assume that $\Omega_1 = (0.75, 0.25)$ and $\Omega_2 = (0.8, 0.36)$ are both (2,1)-FSs on B. Then

1. $$\begin{aligned}\Omega_1 \cup \Omega_2 &= (\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \\ &= (\max\{0.75, 0.8\}, \min\{0.25, 0.36\}) \\ &= (0.8, 0.25).\end{aligned}$$
2. $$\begin{aligned}\Omega_1 \cap \Omega_2 &= (\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \\ &= (\min\{0.75, 0.8\}, \max\{0.25, 0.36\}) \\ &= (0.75, 0.36).\end{aligned}$$
3. $\Omega_1^c = (0.5, 0.5625)$ and $\Omega_2^c = (0.6, 0.64)$.

Proposition 2 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B. Then*

1. $\Omega_1 \cup \Omega_2 = \Omega_2 \cup \Omega_1$.
2. $\Omega_1 \cap \Omega_2 = \Omega_2 \cap \Omega_1$.

Proof Straightforward. □

Proposition 3 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$, $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ and $\Omega_3 = (\delta_{\Omega_3}, \lambda_{\Omega_3})$ be (2,1)-FSs on B. Then*

1. $\Omega_1 \cup (\Omega_2 \cup \Omega_3) = (\Omega_1 \cup \Omega_2) \cup \Omega_3$.
2. $\Omega_1 \cap (\Omega_2 \cap \Omega_3) = (\Omega_1 \cap \Omega_2) \cap \Omega_3$.

Proof Consider Ω_1, Ω_2 and Ω_3 as (2,1)-FSs on B. Then, according to Definition 5, we obtain,

1. $$\begin{aligned}&\Omega_1 \cup (\Omega_2 \cup \Omega_3) = (\delta_{\Omega_1}, \lambda_{\Omega_1}) \\ &\cup(\max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}) \\ &= (\max\{\delta_{\Omega_1}, \max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\}, \min\{\lambda_{\Omega_1}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\}) \\ &= (\max\{\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\}, \min\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\}) \\ &= (\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \cup (\delta_{\Omega_3}, \lambda_{\Omega_3}) \\ &= (\Omega_1 \cap \Omega_2) \cup \Omega_3.\end{aligned}$$

2. Similar to 1 above. □

Theorem 1 *Let* $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$, $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ *and* $\Omega_3 = (\delta_{\Omega_3}, \lambda_{\Omega_3})$ *be* (m, n)*-FSs. Then*

1. $(\Omega_1 \cup \Omega_2) \cap \Omega_3 = (\Omega_1 \cap \Omega_3) \cup (\Omega_2 \cap \Omega_3)$.

2. $(\Omega_1 \cap \Omega_2) \cup \Omega_3 = (\Omega_1 \cup \Omega_3) \cap (\Omega_2 \cup \Omega_3)$.

Proof Consider Ω_1, Ω_2 and Ω_3 as (2,1)-FSs on B. Then, according to Definition 5, we obtain,

1. $(\Omega_1 \cup \Omega_2) \cap \Omega_3 = (\max\{\lambda_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \cap (\delta_{\Omega_3}, \lambda_{\Omega_3})$
$= (\min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\}, \max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\})$.And,
$(\Omega_1 \cap \Omega_3) \cup (\Omega_2 \cap \Omega_3) = (\min\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}) \cup (\min\{\delta_{\Omega_2}, \delta_{\Omega_3}\}, \max\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\})$
$= (\max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \min\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\}, \min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \max\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\})$.

Then,

$$\min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\} = \begin{cases} \delta_{\Omega_2} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_2} \le \delta_{\Omega_3}, \\ \delta_{\Omega_1} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_1} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_3} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_1} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_3} \le \delta_{\Omega_1}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_2} \le \delta_{\Omega_1}, \end{cases}$$

$$\max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\} = \begin{cases} \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_2} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_1} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_3} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_1} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_1} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_3} \le \lambda_{\Omega_1}, \\ \lambda_{\Omega_2} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_2} \le \lambda_{\Omega_1}, \end{cases}$$

$$\max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \min\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\} = \begin{cases} \delta_{\Omega_2} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_2} \le \delta_{\Omega_3}, \\ \delta_{\Omega_1} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_1} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_3} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_1} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_3} \le \delta_{\Omega_1}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_2} \le \delta_{\Omega_1}, \end{cases}$$

$$\min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \max\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\} = \begin{cases} \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_2} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_1} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_3} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_1} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_1} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_3} \le \lambda_{\Omega_1}, \\ \lambda_{\Omega_2} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_2} \le \lambda_{\Omega_1}. \end{cases}$$

Thus, $\min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\} = \max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \min\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\}$ and $\max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\} = \min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \max\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\}$. Hence, $(\Omega_1 \cup \Omega_2) \cap \Omega_3 = (\Omega_1 \cap \Omega_3) \cup (\Omega_2 \cap \Omega_3)$.

2. $(\Omega_1 \cap \Omega_2) \cup \Omega_3 = (\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \cup (\delta_{\Omega_3}, \lambda_{\Omega_3})$
$= (\max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\}, \min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\})$.And,
$(\Omega_1 \cup \Omega_3) \cap (\Omega_2 \cup \Omega_3) = (\max\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}) \cap (\max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\})$
$= (\min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\}, \max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\})$.

Then,

$$\max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\} = \begin{cases} \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_2} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_1} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_3} \le \delta_{\Omega_2}, \\ \delta_{\Omega_1} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_1} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_3} \le \delta_{\Omega_1}, \\ \delta_{\Omega_2} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_2} \le \delta_{\Omega_1}, \end{cases}$$

$$\min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\} = \begin{cases} \lambda_{\Omega_2} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_2} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_1} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_1} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_3} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_1} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_3} \le \lambda_{\Omega_1}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_2} \le \lambda_{\Omega_1}, \end{cases}$$

$$\min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\} = \begin{cases} \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_2} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_1} \le \delta_{\Omega_3}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_1} \le \delta_{\Omega_3} \le \delta_{\Omega_2}, \\ \delta_{\Omega_1} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_1} \le \delta_{\Omega_2}, \\ \delta_{\Omega_3} & \text{if } \delta_{\Omega_2} \le \delta_{\Omega_3} \le \delta_{\Omega_1}, \\ \delta_{\Omega_2} & \text{if } \delta_{\Omega_3} \le \delta_{\Omega_2} \le \delta_{\Omega_1}, \end{cases}$$

$$\max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\} = \begin{cases} \lambda_{\Omega_2} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_2} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_1} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_1} \le \lambda_{\Omega_3}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_1} \le \lambda_{\Omega_3} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_1} \le \lambda_{\Omega_2}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_2} \le \lambda_{\Omega_3} \le \lambda_{\Omega_1}, \\ \lambda_{\Omega_3} & \text{if } \lambda_{\Omega_3} \le \lambda_{\Omega_2} \le \lambda_{\Omega_1}. \end{cases}$$

Thus, $\max\{\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \delta_{\Omega_3}\} = \min\{\max\{\delta_{\Omega_1}, \delta_{\Omega_3}\}, \max\{\delta_{\Omega_2}, \delta_{\Omega_3}\}\}$ and $\min\{\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}, \lambda_{\Omega_3}\} = \max\{\min\{\lambda_{\Omega_1}, \lambda_{\Omega_3}\}, \min\{\lambda_{\Omega_2}, \lambda_{\Omega_3}\}\}$. Hence, $(\Omega_1 \cap \Omega_2) \cup \Omega_3 = (\Omega_1 \cup \Omega_3) \cap (\Omega_2 \cup \Omega_3)$. □

Theorem 2 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B. Then*

1. $(\Omega_1 \cup \Omega_2)^c = \Omega_1^c \cap \Omega_2^c$.

2. $(\Omega_1 \cap \Omega_2)^c = \Omega_1^c \cup \Omega_2^c$.

Proof 1. Take Ω_1 and Ω_2 as (2,1)-FSs. Then, according to Definition 5, we obtain

$$\begin{aligned}(\Omega_1 \cup \Omega_2)^c &= (\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\})^c \\ &= (\min\{\sqrt{\lambda_{\Omega_1}}, \sqrt{\lambda_{\Omega_2}}\}, \max\{(\delta_{\Omega_1})^2, (\delta_{\Omega_2})^2\}) \\ &= (\sqrt{\lambda_{\Omega_1}}, (\delta_{\Omega_1})^2) \cap (\sqrt{\lambda_{\Omega_2}}, (\delta_{\Omega_2})^2) \\ &= \Omega_1^c \cap \Omega_2^c.\end{aligned}$$

2. Similar to 1.

□

The operators of ∪ and ∩, given in Definition 5, are generalized for arbitrary numbers of (2,1)-FSs as follows.

Definition 6 Let $\{\Omega_i = (\delta_{\Omega_i}, \lambda_{\Omega_i}) : i \in I\}$ be a family of (2,1)-FSs on B. Then

1. $\cup_{i \in I} \Omega_i = (\sup\{\delta_{\Omega_i} : i \in I\}, \inf\{\lambda_{\Omega_i} : i \in I\})$.

2. $\cap_{i \in I} \Omega_i = (\inf\{\lambda_{\Omega_i} : i \in I\}, \max\{\lambda_{\Omega_1}, \sup\{\delta_{\Omega_i} : i \in I\})$.

We close this section by defining the score and accuracy functions of (2,1)-FSs which will be helpful later to rank (2,1)-FSs.

Proposition 4 *For any (2,1)-FS $\Omega = (\delta_\Omega, \lambda_\Omega)$ on B, the value of $\delta_\Omega^2 - \lambda_\Omega$ lies in the closed interval $[-1, 1]$.*

Proof For any (2,1)-FS Ω, we have $\delta_\Omega^2 + \lambda_\Omega \leq 1$. This implies that $\delta_\Omega^2 - \lambda_\Omega \leq \delta_\Omega^2 \leq 1$ and $\delta_\Omega^2 - \lambda_\Omega \geq -\lambda_\Omega \geq -1$. Hence, $-1 \leq \delta_\Omega^2 - \lambda_\Omega \leq 1$, as required. □

Definition 7 The score function $score : I^{(2,1)-FS} \to [-1, 1]$ is given by the formula $score(\Omega) = \delta_\Omega^2 - \lambda_\Omega$ for every (2,1)-FS $\Omega = (\delta_\Omega, \lambda_\Omega)$.

Definition 8 Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs. We say that

(i) If $score(\Omega_1) > score(\Omega_2)$, then $\Omega_1 \succ \Omega_2$.
(ii) If $score(\Omega_1) < score(\Omega_2)$, then $\Omega_1 \prec \Omega_2$.
(iii) If $score(\Omega_1) = score(\Omega_2)$, then $\Omega_1 \simeq \Omega_2$.

Example 6 Let $\Omega_1 = (0.76, 0.42)$ and $\Omega_2 = (0.8, 0.25)$ be (2,1)-FSs. We obtain $score(\Omega_1) = 0.1576$ and $acc(\Omega_2) = 0.39$. Hence,

In some cases, the score function is not a sufficient tool to determine which better (2,1)-FSs can be chosen. This occurs, in particular, for every two (2,1)-FSs satisfy that non-membership degree equals to the root of membership degree, i.e. $\delta_\Omega = \sqrt{\lambda_\Omega}$. But we know that these (2,1)-FSs may not match with each other. So that, comparison depending on the score function is not acceptable (or appropriate) to address these cases.

To efficiently make a comparison of (2,1)-FSs, we introduce the concept of accuracy function for (2,1)-FSs as follows.

Definition 9 The accuracy function $acc : I^{(2,1)-FS} \to [0, 1]$ is given by the formula $acc(\Omega) = \delta_\Omega^2 + \lambda_\Omega$ for every (2,1)-FS $\Omega = (\delta_\Omega, \lambda_\Omega)$.

We make use of the score and accuracy functions to compare between (2,1)-FSs.

Definition 10 Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs, where $score(\Omega_k)$ and $acc(\Omega_k)$ $(k = 1, 2)$ are respectively their score functions and accuracy functions. We say that

(i) If $score(\Omega_1) > score(\Omega_2)$, then $\Omega_1 \succ \Omega_2$.
(ii) If $score(\Omega_1) < score(\Omega_2)$, then $\Omega_1 \prec \Omega_2$.
(iii) If $score(\Omega_1) = score(\Omega_2)$, then

1. If $acc(\Omega_1) > acc(\Omega_2)$, then $\Omega_1 \succ \Omega_2$.
2. If $acc(\Omega_1) < acc(\Omega_2)$, then $\Omega_1 \prec \Omega_2$.
3. If $acc(\Omega_1) = acc(\Omega_2)$, then $\Omega_1 = \Omega_2$.

Example 7 Consider $\Omega_1 = (\sqrt{0.45}, 0.45)$, $\Omega_2 = (0.5, 0.25)$, $\Omega_3 = (0.6, 0.35)$ and $\Omega_4 = (0.7, 0.48)$ are (2,1)-FSs on $B = \{\nu\}$. Obviously, $score(\Omega_1) = score(\Omega_2) = 0$ and $score(\Omega_3) = score(\Omega_4) = 0.01$. Then, according to the above definition, we find that

1. $\Omega_1 \succ \Omega_2$ because $\delta_{\Omega_1}^2 + \lambda_{\Omega_1} = 0.9 > \delta_{\Omega_2}^2 + \lambda_{\Omega_2} = 0.5$.
2. $\Omega_4 \succ \Omega_3$ because $\delta_{\Omega_4}^2 + \lambda_{\Omega_4} = 0.97 > \delta_{\Omega_3}^2 + \lambda_{\Omega_3} = 0.71$.

Definition 11 Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B. A natural quasi-ordering on the (2,1)-FSs is defined as follows.

$\Omega_1 \geq \Omega_2$ iff $\delta_{\Omega_1} \geq \delta_{\Omega_2}$ and $\lambda_{\Omega_1} \leq \lambda_{\Omega_2}$.

Aggregation of (2,1)-fuzzy sets with applications

In this section, we first introduce some new operations on (2,1)-Fuzzy sets and explore their main properties. Then, we initiate novel types of aggregation operators with respect to (2,1)-Fuzzy sets and scrutinize the interrelations between them. We display some elucidative examples.

Some operations on (2,1)-FSs

Herein, we define some operations over the family of (2,1)-Fuzzy sets, and explore the interrelations between them.

Definition 12 Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B, and ξ be a positive real number ($\xi > 0$). We define the following operations.

1. $\Omega_1 \oplus \Omega_2 = \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right).$
2. $\Omega_1 \otimes \Omega_2 = \left(\delta_{\Omega_1}\delta_{\Omega_2}, \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}\right).$
3. $\xi\Omega_1 = \left(\sqrt{1-(1-\delta^2_{\Omega_1})^\xi}, \lambda^\xi_{\Omega_1}\right).$
4. $\Omega_1^\xi = \left(\delta^\xi_{\Omega_1}, 1-(1-\lambda_{\Omega_1})^\xi\right).$

Example 8 Suppose that $\Omega_1 = (0.35, 0.85)$ and $\Omega_2 = (0.5, 0.7)$ are (2,1)-FSs on $B = \{\nu\}$, and $\xi = 3$. Then

1. $$\begin{aligned}\Omega_1 \oplus \Omega_2 &= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right)\\ &= \left(\sqrt{0.35^2 + 0.5^2 - (0.35)^2(0.5)^2}, (0.85)(0.7)\right)\\ &\approx (0.5847, 0.595).\end{aligned}$$
2. $$\begin{aligned}\Omega_1 \otimes \Omega_2 &= \left(\delta_{\Omega_1}\delta_{\Omega_2}, \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}\right)\\ &= ((0.35)(0.5), 0.85 + 0.7 - (0.85 \times 0.7))\\ &= (0.175, 0.955).\end{aligned}$$
3. $$\begin{aligned}3\Omega_1 &= \left(\sqrt{1-(1-\delta^2_{\Omega_1})^3}, \lambda^3_{\Omega_1}\right)\\ &= \left(\sqrt{1-(1-0.35^2)^3}, 0.85^3\right)\\ &\approx (0.32432, 0.614125).\end{aligned}$$
4. $$\begin{aligned}\Omega_1^3 &= \left(\delta^3_{\Omega_1}, 1-(1-\lambda_{\Omega_1})^3\right)\\ &= \left(0.35^3, 1-(1-0.85)^3\right)\\ &= (0.0042875, 0.996625).\end{aligned}$$

Theorem 3 *If $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ are (2,1)-FSs on B, then $\Omega_1 \oplus \Omega_2$ and $\Omega_1 \otimes \Omega_2$ are (2,1)-FSs.*

Proof For (2,1)-FSs $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$, we obtain

$$0 \le (\delta_{\Omega_1})^2 + \lambda_{\Omega_1} \le 1 \text{ and } 0 \le (\delta_{\Omega_2})^2 + \lambda_{\Omega_2} \le 1.$$

Then, we have

$$\delta^2_{\Omega_1} \ge \delta^2_{\Omega_1}\delta^2_{\Omega_2},\ \delta^2_{\Omega_2} \ge \delta^2_{\Omega_1}\delta^2_{\Omega_2},\ 0 \le \delta^2_{\Omega_1}\delta^2_{\Omega_2} \le 1$$

and

$$\lambda_{\Omega_1} \ge \lambda_{\Omega_1}\lambda_{\Omega_2}, \lambda_{\Omega_2} \ge \lambda_{\Omega_1}\lambda_{\Omega_2}, 0 \le \lambda_{\Omega_1}\lambda_{\Omega_2} \le 1.$$

This implies that

$$\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}} \ge 0, \text{ and } \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2} \ge 0.$$

Since $\delta^2_{\Omega_2} \le 1$ and $0 \le 1 - \delta^2_{\Omega_1}$, $\delta^2_{\Omega_2}(1-\delta^2_{\Omega_1}) \le (1-\delta^2_{\Omega_1})$ which means that $\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2} \le 1$. Hence, $\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}} \le 1$.

Following similar arguments, we obtain

$$\lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2} \le 1.$$

It is clear that $0 \le \lambda_{\Omega_1} \le 1-\delta^2_{\Omega_1}$ and $0 \le \lambda_{\Omega_2} \le 1-\delta^2_{\Omega_2}$.

Now, $\left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}}\right)^2 + \lambda_{\Omega_1}\lambda_{\Omega_2} \le \delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2} + (1-\delta^2_{\Omega_1})(1-\delta^2_{\Omega_2}) = 1$.

Thus, $0 \le \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}}\right)^2 + \lambda_{\Omega_1}\lambda_{\Omega_2} \le 1$ which means that $\Omega_1 \oplus \Omega_2$ is a (2,1)-FS.

Following similar arguments, we obtain

$$\begin{aligned}&0 \le \delta_{\Omega_1}\delta_{\Omega_2} \le 1, 0 \le \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2} \le 1 \text{ and}\\ &0 \le (\delta_{\Omega_1}\delta_{\Omega_2})^2 + \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2} \le 1.\end{aligned}$$

Hence, $\Omega_1 \otimes \Omega_2$ is a (2,1)-FS. □

Theorem 4 *Let $\Omega = (\delta_\Omega, \lambda_\Omega)$ be a (2,1)-FS on B and ξ be a positive real number. Then, $\xi\Omega$ and Ω^ξ are (2,1)-FSs.*

Proof Since $0 \le \delta^2_\Omega \le 1$, $0 \le \lambda_\Omega \le 1$ and $0 \le (\delta_\Omega)^2 + \lambda_\Omega \le 1$, we find

$$\begin{aligned}&0 \le \lambda_\Omega \le 1 - \delta^2_\Omega\\ &\quad\Rightarrow 0 \le (1-\delta^2_\Omega)^\xi\\ &\quad\Rightarrow 1-(1-\delta^2_\Omega)^\xi \le 1\\ &\quad\Rightarrow 0 \le \sqrt{1-(1-\delta^2_\Omega)^\xi} \le \sqrt{1} = 1.\end{aligned}$$

It is clear that $0 \le \lambda^\xi_\Omega \le 1$, then we get

$$\begin{aligned}0 \le \left(\sqrt{1-(1-\delta^2_\Omega)^\xi}\right)^2 + \lambda^\xi_\Omega \le 1\\ -(1-\delta^2_\Omega)^\xi + (1-\delta^2_\Omega)^\xi = 1.\end{aligned}$$

Following similar arguments, we obtain

$$0 \leq (\delta_{\Omega}^{\xi})^2 + (1-(1-\lambda_{\Omega})^{\xi})^2 \leq 1.$$

Hence, $\xi\Omega$ and Ω^{ξ} are (2,1)-FSs. □

Theorem 5 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B. Then*

1. $\Omega_1 \oplus \Omega_2 = \Omega_2 \oplus \Omega_1$.

2. $\Omega_1 \otimes \Omega_2 = \Omega_2 \otimes \Omega_1$.

Proof From Definition 12, we obtain:

1. $$\Omega_1 \oplus \Omega_2 = \left(\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_2}^2 - \delta_{\Omega_1}^2\delta_{\Omega_2}^2}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right)$$
$$\left(\sqrt{\delta_{\Omega_2}^2 + \delta_{\Omega_1}^2 - \delta_{\Omega_2}^2\delta_{\Omega_1}^2}, \lambda_{\Omega_2}\lambda_{\Omega_1}\right) = \Omega_2 \oplus \Omega_1.$$

2. $$\Omega_1 \otimes \Omega_2 = \left(\delta_{\Omega_1}\delta_{\Omega_2}, \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}\right) = \left(\delta_{\Omega_2}\delta_{\Omega_1}, \lambda_{\Omega_2} + \lambda_{\Omega_1} - \lambda_{\Omega_2}\lambda_{\Omega_1}\right) = \Omega_2 \otimes \Omega_1.$$

□

Theorem 6 *Let $\Omega = (\delta_{\Omega}, \lambda_{\Omega})$, $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B. Then*

1. $\xi(\Omega_1 \oplus \Omega_2) = \xi\Omega_1 \oplus \xi\Omega_2$ *for* $\xi > 0$.
2. $(\xi_1 + \xi_2)\Omega = \xi_1\Omega \oplus \xi_2\Omega$ *for* $\xi_1, \xi_2 > 0$.
3. $(\Omega_1 \otimes \Omega_2)^{\xi} = \Omega_1^{\xi} \otimes \Omega_2^{\xi}$ *for* $\xi > 0$.
4. $\Omega^{(\xi_1+\xi_2)} = \Omega^{\xi_1} \otimes \Omega^{\xi_2}$ *for* $\xi_1, \xi_2 > 0$.

Proof 1.

$$\begin{aligned}\xi(\Omega_1 \oplus \Omega_2) &= \xi\left(\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_2}^2 - \delta_{\Omega_1}^2\delta_{\Omega_2}^2}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega_1}^2 - \delta_{\Omega_2}^2 + \delta_{\Omega_1}^2\delta_{\Omega_2}^2)^{\xi}}, (\lambda_{\Omega_1}\lambda_{\Omega_2})^{\xi}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega_1}^2)^{\xi}(1-\delta_{\Omega_2}^2)^{\xi}}, \lambda_{\Omega_1}^{\xi}\lambda_{\Omega_2}^{\xi}\right).\end{aligned}$$

And

$$\begin{aligned}\xi\Omega_1 \oplus \xi\Omega_2 &= \left(\sqrt{1-(1-\delta_{\Omega_1}^2)^{\xi}}, \lambda_{\Omega_1}^{\xi}\right) \oplus \left(\sqrt{1-(1-\delta_{\Omega_2}^2)^{\xi}}, \lambda_{\Omega_2}^{\xi}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega_1}^2)^{\xi} + 1-(1-\delta_{\Omega_2}^2)^{\xi} - (1-(1-\delta_{\Omega_1}^2)^{\xi})(1-(1-\delta_{\Omega_2}^2)^{\xi})}, \lambda_{\Omega_1}^{\xi}\lambda_{\Omega_2}^{\xi}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega_1}^2)^{\xi}(1-\delta_{\Omega_2}^2)^{\xi}}, \lambda_{\Omega_1}^{\xi}\lambda_{\Omega_2}^{\xi}\right) = \xi(\Omega_1 \oplus \Omega_2).\end{aligned}$$

2.

$$\begin{aligned}(\xi_1+\xi_2)\Omega = (\xi_1+\xi_2)(\delta_{\Omega}, \lambda_{\Omega}) &= \left(\sqrt{1-(1-\delta_{\Omega}^2)^{\xi_1+\xi_2}}, \lambda_{\Omega}^{\xi_1+\xi_2}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega}^2)^{\xi_1}(1-\delta_{\Omega}^2)^{\xi_2}}, \lambda_{\Omega}^{\xi_1+\xi_2}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega}^2)^{\xi_1} + 1-(1-\delta_{\Omega}^2)^{\xi_2} - (1-(1-\delta_{\Omega}^2)^{\xi_1})(1-(1-\delta_{\Omega}^2)^{\xi_2})}, \lambda_{\Omega}^{\xi_1}\lambda_{\Omega}^{\xi_2}\right)\\ &= \left(\sqrt{1-(1-\delta_{\Omega}^2)^{\xi_1}}, \lambda_{\Omega}^{\xi_1}\right) \oplus \left(\sqrt{1-(1-\delta_{\Omega}^2)^{\xi_2}}, \lambda_{\Omega}^{\xi_2}\right) = \xi_1\Omega \oplus \xi_2\Omega.\end{aligned}$$

3. $$\begin{aligned}(\Omega_1 \otimes \Omega_2)^{\xi} &= \left(\delta_{\Omega_1}\delta_{\Omega_2}, \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}\right)^{\xi}\\ &= \left((\delta_{\Omega_1}\delta_{\Omega_2})^{\xi}, 1-(1-\lambda_{\Omega_1} - \lambda_{\Omega_2} + \lambda_{\Omega_1}\lambda_{\Omega_2})^{\xi}\right)\\ &= \left(\delta_{\Omega_1}^{\xi}\delta_{\Omega_2}^{\xi}, 1-(1-\lambda_{\Omega_1})^{\xi}(1-\lambda_{\Omega_2})^{\xi}\right)\\ &= \left(\delta_{\Omega_1}^{\xi}, 1-(1-\lambda_{\Omega_1})^{\xi}\right) \otimes \left(\delta_{\Omega_2}^{\xi}, 1-(1-\lambda_{\Omega_2})^{\xi}\right)\\ &= \Omega_1^{\xi} \otimes \Omega_2^{\xi}.\end{aligned}$$

4. $$\begin{aligned}\Omega^{\xi_1} \otimes \Omega^{\xi_2} &= \left(\delta_{\Omega}^{\xi_1}, 1-(1-\lambda_{\Omega})^{\xi_1}\right) \otimes \left(\delta_{\Omega}^{\xi_2}, 1-(1-\lambda_{\Omega})^{\xi_2}\right)\\ &= \left(\delta_{\Omega}^{\xi_1+\xi_2}, 1-(1-\lambda_{\Omega})^{\xi_1} + 1-(1-\lambda_{\Omega})^{\xi_2} - (1-(1-\lambda_{\Omega})^{\xi_1})(1-(1-\lambda_{\Omega})^{\xi_2})\right)\\ &= \left(\delta_{\Omega}^{\xi_1+\xi_2}, 1-(1-\lambda_{\Omega})^{\xi_1+\xi_2}\right)\\ &= \Omega^{(\xi_1+\xi_2)}.\end{aligned}$$

□

Theorem 7 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ and $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ be (2,1)-FSs on B, and $\xi > 0$. Then*

1. $\xi(\Omega_1 \cup \Omega_2) = \xi\Omega_1 \cup \xi\Omega_2$.

2. $(\Omega_1 \cup \Omega_2)^\xi = \Omega_1^\xi \cup \Omega_2^\xi$.

Proof For the two (2,1)-FSs Ω_1 and Ω_2, and $\xi > 0$, according to Definitions 5 and 12, we obtain

1. $$\xi(\Omega_1 \cup \Omega_2) = \xi(\max\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \min\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) = \left(\sqrt{1-(1-\max\{\delta_{\Omega_1}^2, \delta_{\Omega_2}^2\})^\xi}, \min\{\lambda_{\Omega_1}^\xi, \lambda_{\Omega_2}^\xi\}\right).$$

   And

$$\xi\Omega_1 \cup \xi\Omega_2 = \left(\sqrt{1-(1-\delta_{\Omega_1}^2)^\xi}, \lambda_{\Omega_1}^\xi\right) \cup \left(\sqrt{1-(1-\delta_{\Omega_2}^2)^\xi}, \lambda_{\Omega_2}^\xi\right)$$
$$= \left(\max\{\sqrt{1-(1-\delta_{\Omega_1}^2)^\xi}, \sqrt{1-(1-\delta_{\Omega_2}^2)^\xi}\}, \min\{\lambda_{\Omega_1}^\xi, \lambda_{\Omega_2}^\xi\}\right)$$
$$= \left(\sqrt{1-(1-\max\{\delta_{\Omega_1}^2, \delta_{\Omega_2}^2\})^\xi}, \min\{\lambda_{\Omega_1}^\xi, \lambda_{\Omega_2}^\xi\}\right)$$
$$= \xi(\Omega_1 \cup \Omega_2).$$

2. Similar to 1. □

Theorem 8 *Let* $\Omega = (\delta_\Omega, \lambda_\Omega)$, $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$ *and* $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ *be (2,1)-FSs on B, and* $\xi > 0$. *Then*

1. $(\Omega_1 \oplus \Omega_2)^c = \Omega_1^c \otimes \Omega_2^c$.

2. $(\Omega_1 \otimes \Omega_2)^c = \Omega_1^c \oplus \Omega_2^c$.

3. $(\Omega^c)^\xi = (\xi\Omega)^c$.

4. $\xi(\Omega)^c = (\Omega^\xi)^c$.

Proof 1.

$$(\Omega_1 \oplus \Omega_2)^c = \left(\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_2}^2 - \delta_{\Omega_1}^2\delta_{\Omega_2}^2}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right)^c$$
$$= \left(\sqrt{\lambda_{\Omega_1}\lambda_{\Omega_2}}, \left(\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_2}^2 - \delta_{\Omega_1}^2\delta_{\Omega_2}^2}\right)^2\right)$$
$$= \left(\sqrt{\lambda_{\Omega_1}}\sqrt{\lambda_{\Omega_2}}, \delta_{\Omega_1}^2 + \delta_{\Omega_2}^2 - \delta_{\Omega_1}^2\delta_{\Omega_2}^2\right)$$
$$= (\sqrt{\lambda_{\Omega_1}}, (\delta_{\Omega_1})^2) \otimes (\sqrt{\lambda_{\Omega_2}}, (\delta_{\Omega_2})^2)$$
$$= \Omega_1^c \otimes \Omega_2^c.$$

2. $$(\Omega_1 \otimes \Omega_2)^c = \left(\delta_{\Omega_1}\delta_{\Omega_2}, \lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}\right)^c$$
$$= \left(\sqrt{\lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}}, (\delta_{\Omega_1}\delta_{\Omega_2})^2\right)$$
$$= \left(\sqrt{\lambda_{\Omega_1} + \lambda_{\Omega_2} - \lambda_{\Omega_1}\lambda_{\Omega_2}}, (\delta_{\Omega_1})^2(\delta_{\Omega_2})^2\right)$$
$$= (\sqrt{\lambda_{\Omega_1}}, (\delta_{\Omega_1})^2) \oplus (\sqrt{\lambda_{\Omega_2}}, (\delta_{\Omega_2})^2)$$
$$= \Omega_1^c \oplus \Omega_2^c.$$

3. $$(\Omega^c)^\xi = (\sqrt{\lambda_\Omega}, (\delta_\Omega)^2)^\xi$$
$$= \left((\sqrt{\lambda_\Omega})^\xi, 1-(1-\delta_\Omega^2)^\xi\right)$$
$$= \left(\sqrt{1-(1-\delta_\Omega^2)^\xi}, \lambda_\Omega^\xi\right)^c$$
$$= (\xi\Omega)^c.$$

4. $$\xi(\Omega)^c = \xi(\sqrt{\lambda_\Omega}, (\delta_\Omega)^2)$$
$$= \left(\sqrt{1-(1-\lambda_\Omega)^\xi}, ((\delta_\Omega)^2)^\xi\right)$$
$$= \left(\delta_\Omega^\xi, 1-(1-\lambda_\Omega)^\xi\right)^c$$
$$= (\Omega^\xi)^c.$$

□

Theorem 9 *Let* $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$, $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ *and* $\Omega_3 = (\delta_{\Omega_3}, \lambda_{\Omega_3})$ *be (2,1)-FSs on B. Then*

1. $(\Omega_1 \cap \Omega_2) \oplus \Omega_3 = (\Omega_1 \oplus \Omega_3) \cap (\Omega_2 \oplus \Omega_3)$.

2. $(\Omega_1 \cup \Omega_2) \oplus \Omega_3 = (\Omega_1 \oplus \Omega_3) \cup (\Omega_2 \oplus \Omega_3)$.

3. $(\Omega_1 \cap \Omega_2) \otimes \Omega_3 = (\Omega_1 \otimes \Omega_3) \cap (\Omega_2 \otimes \Omega_3)$.

4. $(\Omega_1 \cup \Omega_2) \otimes \Omega_3 = (\Omega_1 \otimes \Omega_3) \cup (\Omega_2 \otimes \Omega_3)$.

Proof 1.

$$(\Omega_1 \cap \Omega_2) \oplus \Omega_3 = (\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \oplus (\delta_{\Omega_3}, \lambda_{\Omega_3})$$
$$= \left(\sqrt{\min\{\delta_{\Omega_1}^2, \delta_{\Omega_2}^2\} + \delta_{\Omega_3}^2 - \delta_{\Omega_3}^2\min\{\delta_{\Omega_1}^2, \delta_{\Omega_2}^2\}}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}\lambda_{\Omega_3}\right)$$
$$= \left(\sqrt{(1-\delta_{\Omega_3}^2)\min\{\delta_{\Omega_1}^2, \delta_{\Omega_2}^2\} + \delta_{\Omega_3}^2}, \max\{\lambda_{\Omega_1}\lambda_{\Omega_3}, \lambda_{\Omega_2}\lambda_{\Omega_3}\}\right).$$

And $(\Omega_1 \oplus \Omega_3) \cap (\Omega_2 \oplus \Omega_3)$

$$= \left(\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_3}^2 - \delta_{\Omega_1}^2\delta_{\Omega_3}^2}, \lambda_{\Omega_1}\lambda_{\Omega_3}\right) \cap \left(\sqrt{\delta_{\Omega_2}^2 + \delta_{\Omega_3}^2 - \delta_{\Omega_2}^2\delta_{\Omega_3}^2}, \lambda_{\Omega_2}\lambda_{\Omega_3}\right)$$
$$= \left(\min\left\{\sqrt{\delta_{\Omega_1}^2 + \delta_{\Omega_3}^2 - \delta_{\Omega_1}^2\delta_{\Omega_3}^2}, \sqrt{\delta_{\Omega_2}^2 + \delta_{\Omega_3}^2 - \delta_{\Omega_2}^2\delta_{\Omega_3}^2}\right\}, \max\{\lambda_{\Omega_1}\lambda_{\Omega_3}, \lambda_{\Omega_2}\lambda_{\Omega_3}\}\right)$$

$$= \Big(\min\Big\{\sqrt{(1-\delta^2_{\Omega_3})\delta^2_{\Omega_1}+\delta^2_{\Omega_3}}, \sqrt{(1-\delta^2_{\Omega_3})\delta^2_{\Omega_2}+\delta^2_{\Omega_3}}\Big\}, \max\{\lambda_{\Omega_1}\lambda_{\Omega_3}, \lambda_{\Omega_2}\lambda_{\Omega_3}\}\Big)$$
$$= \Big(\sqrt{(1-\delta^2_{\Omega_3})\min\{\delta^2_{\Omega_1}, \delta^2_{\Omega_2}\}+\delta^2_{\Omega_3}}, \max\{\lambda_{\Omega_1}\lambda_{\Omega_3}, \lambda_{\Omega_2}\lambda_{\Omega_3}\}\Big).$$

Hence, $(\Omega_1 \cap \Omega_2) \oplus \Omega_3 = (\Omega_1 \oplus \Omega_3) \cap (\Omega_2 \oplus \Omega_3)$.

2. Similar to 1.

3.

$$(\Omega_1 \cap \Omega_2) \otimes \Omega_3 = (\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}) \otimes \Omega_3$$
$$= \big(\min\{\delta_{\Omega_1}, \delta_{\Omega_2}\}\delta_{\Omega_3}, \max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\} + \lambda_{\Omega_3} - \lambda_{\Omega_3}\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\}\big)$$
$$= \big(\min\{\delta_{\Omega_1}\delta_{\Omega_3}, \delta_{\Omega_2}\delta_{\Omega_3}\}, (1-\lambda_{\Omega_3})\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\} + \lambda_{\Omega_3}\big).$$

And $(\Omega_1 \otimes \Omega_3) \cap (\Omega_2 \otimes \Omega_3)$

$$= \big(\delta_{\Omega_1}\delta_{\Omega_3}, \lambda_{\Omega_1} + \lambda_{\Omega_3} - \lambda_{\Omega_1}\lambda_{\Omega_3}\big) \cap \big(\delta_{\Omega_2}\delta_{\Omega_3}, \lambda_{\Omega_2} + \lambda_{\Omega_3} - \lambda_{\Omega_2}\lambda_{\Omega_3}\big)$$
$$= \big(\delta_{\Omega_1}\delta_{\Omega_3}, (1-\lambda_{\Omega_3})\lambda_{\Omega_1} + \lambda_{\Omega_3}\big) \cap \big(\delta_{\Omega_2}\delta_{\Omega_3}, (1-\lambda_{\Omega_3})\lambda_{\Omega_2} + \lambda_{\Omega_3}\big)$$
$$= \big(\min\{\delta_{\Omega_1}\delta_{\Omega_3}, \delta_{\Omega_2}\delta_{\Omega_3}\}, \max\big\{(1-\lambda_{\Omega_3})\lambda_{\Omega_1} + \lambda_{\Omega_3}, (1-\lambda_{\Omega_3})\lambda_{\Omega_2} + \lambda_{\Omega_3}\big\}\big)$$
$$= \big(\min\{\delta_{\Omega_1}\delta_{\Omega_3}, \delta_{\Omega_2}\delta_{\Omega_3}\}, (1-\lambda_{\Omega_3})\max\{\lambda_{\Omega_1}, \lambda_{\Omega_2}\} + \lambda_{\Omega_3}\big).$$

Hence, $(\Omega_1 \cap \Omega_2) \otimes \Omega_3 = (\Omega_1 \otimes \Omega_3) \cap (\Omega_2 \otimes \Omega_3)$.

4. Similar to 3. □

Theorem 10 *Let $\Omega_1 = (\delta_{\Omega_1}, \lambda_{\Omega_1})$, $\Omega_2 = (\delta_{\Omega_2}, \lambda_{\Omega_2})$ and $\Omega_3 = (\delta_{\Omega_3}, \lambda_{\Omega_3})$ be (2,1)-FSs on B. Then*

1.

$$\Omega_1 \oplus \Omega_2 \oplus \Omega_3 = \Omega_1 \oplus \Omega_3 \oplus \Omega_2.$$

2.

$$\Omega_1 \otimes \Omega_2 \otimes \Omega_3 = \Omega_1 \otimes \Omega_3 \otimes \Omega_2.$$

Proof

$$\Omega_1 \oplus \Omega_2 \oplus \Omega_3$$
$$= (\delta_{\Omega_1}, \lambda_{\Omega_1}) \oplus (\delta_{\Omega_2}, \lambda_{\Omega_2}) \oplus (\delta_{\Omega_3}, \lambda_{\Omega_3})$$
$$= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2}}, \lambda_{\Omega_1}\lambda_{\Omega_2}\right) \oplus (\delta_{\Omega_3}, \lambda_{\Omega_3})$$
$$= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2} + \delta^2_{\Omega_3} - \delta^2_{\Omega_3}(\delta^2_{\Omega_1} + \delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_2})}, \lambda_{\Omega_1}\lambda_{\Omega_2}\lambda_{\Omega_3}\right)$$
$$= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_2} + \delta^2_{\Omega_3} - \delta^2_{\Omega_1}\delta^2_{\Omega_2} - \delta^2_{\Omega_1}\delta^2_{\Omega_3} - \delta^2_{\Omega_2}\delta^2_{\Omega_3} + \delta^2_{\Omega_1}\delta^2_{\Omega_2}\delta^2_{\Omega_3}}, \lambda_{\Omega_1}\lambda_{\Omega_2}\lambda_{\Omega_3}\right)$$
$$= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_3} - \delta^2_{\Omega_1}\delta^2_{\Omega_3} + \delta^2_{\Omega_2} - \delta^2_{\Omega_2}(\delta^2_{\Omega_1} + \delta^2_{\Omega_3} - \delta^2_{\Omega_1}\delta^2_{\Omega_3})}, \lambda_{\Omega_1}\lambda_{\Omega_2}\lambda_{\Omega_3}\right)$$
$$= \left(\sqrt{\delta^2_{\Omega_1} + \delta^2_{\Omega_3} - \delta^2_{\Omega_1}\delta^2_{\Omega_3}}, \lambda_{\Omega_1}\lambda_{\Omega_3}\right) \oplus (\delta_{\Omega_2}, \lambda_{\Omega_2})$$
$$= \Omega_1 \oplus \Omega_3 \oplus \Omega_2.$$

2. Similar to 1. □

Aggregation of (2,1)-fuzzy sets

Herein, we generalize some aggregation operators to the environment of (2,1)-Fuzzy sets, and display some formulas which show the relationships between them.

Definition 13 Let $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})$ $(j = 1, 2, \ldots, m)$ be a family of (2,1)-FNs on B, and $w = (w_1, w_2, \ldots, w_m)^T$ be a weight vector of Ω_j with $w_j > 0$ and $\sum_{j=1}^{m} w_j = 1$. Then

1. a (2,1)-Fuzzy weighted average ((2,1)-FWA) operator is given by

$$(2,1)\text{-}FWA(\Omega_1, \Omega_2, \ldots, \Omega_m) = \left(\sum_{j=1}^{m} w_j\delta_{\Omega_j}, \sum_{j=1}^{m} w_j\lambda_{\Omega_j}\right).$$

2. a (2,1)-Fuzzy weighted geometric ((2,1)-FWG) operator is given by

$$(2,1)\text{-}FWG(\Omega_1, \Omega_2, \ldots, \Omega_m) = \left(\prod_{j=1}^{m} \delta^{w_j}_{\Omega_j}, \prod_{j=1}^{m} \lambda^{w_j}_{\Omega_j}\right).$$

3. a (2,1)-Fuzzy weighted power average ((2,1)-FWPA) operator is given by

$$(2,1)\text{-}FWPA(\Omega_1,\Omega_2,\ldots,\Omega_m) = \left(\left(\sum_{j=1}^{m} w_j \delta^2_{\Omega_j}\right)^{\frac{1}{2}}, \sum_{j=1}^{m} w_j \lambda_{\Omega_j}\right).$$

4. a (2,1)-Fuzzy weighted power geometric ((2,1)-FWPG) operator is given by

$$(2,1)\text{-}FWPG(\Omega_1,\Omega_2,\ldots,\Omega_m) = \left(\left(1-\prod_{j=1}^{m}(1-\delta^2_{\Omega_j})^{w_j}\right)^{\frac{1}{2}}, 1-\prod_{j=1}^{m}(1-\lambda_{\Omega_j})^{w_j}\right).$$

The next example presents the way of calculating the aggregations operators given above.

Example 9 For the following five (2,1)-FNs $\Omega_1=(0.52,0.7)$, $\Omega_2=(0.2,0.9)$, $\Omega_3=(0.8,0.3)$, $\Omega_4=(0.6,0.6)$ and $\Omega_5=(0.7,0.4)$ on $B=\{\nu\}$, let $w=(0.3,0.15,0.25,0.2,0.1)^T$ be a weight vector of Ω_j $(j=1,2,\ldots,5)$. Then

1. $$\begin{aligned}&(2,1)\text{-}FWA(\Omega_1,\Omega_2,\ldots,\Omega_5)\\&=(0.52\times0.3+0.2\times0.15+0.8\times0.25\\&\quad+0.6\times0.2+0.7\times0.1,\ 0.7\times0.3+0.9\times0.15\\&\quad+0.3\times0.25+0.6\times0.2+0.4\times0.1)\\&=(0.576,0.58).\end{aligned}$$

2. $$\begin{aligned}&(2,1)\text{-}FWG(\Omega_1,\Omega_2,\ldots,\Omega_5)\\&=(0.52^{0.3}\times0.2^{0.15}\times0.8^{0.25}\times0.6^{0.2}\\&\quad\times0.7^{0.1},\ 0.7^{0.3}\times0.9^{0.15}\times0.3^{0.25}\\&\quad\times0.6^{0.2}\times0.4^{0.1})\approx(053195,0.53924).\end{aligned}$$

3. $$\begin{aligned}&(2,1)\text{-}FWPA(\Omega_1,\Omega_2,\ldots,\Omega_5)\\&=((0.52^2\times0.3+0.2^2\times0.15+0.8^2\times0.25+0.6^2\\&\quad\times0.2+0.7^2\times0.1)^{\frac{1}{2}},\ 0.7\times0.3+0.9\times\\&\quad0.15+0.3\times0.25+0.6\times0.2+0.4\times0.1)\\&\approx(0.60673,0.58).\end{aligned}$$

4. $$\begin{aligned}&(2,1)\text{-}FWPG(\Omega_1,\Omega_2,\ldots,\Omega_5)\\&=((1-(1-0.52^2)^{0.3}\times(1-0.2^2)^{0.15}\\&\quad\times(1-0.8^2)^{0.25}\times(1-0.6^2)^{0.2}\\&\quad\times(1-0.7^2)^{0.1})^{\frac{1}{2}},\ 1-(1-0.7)^{0.3}\times(1-0.9)^{0.15}\\&\quad\times(1-0.3)^{0.25}\times(1-0.6)^{0.2}\times(1-0.4)^{0.1})\\&\approx(0.633346,0.643025).\end{aligned}$$

Remark 3 Note that the values obtained from the operators presented in the above definition need not be a (2,1)-FS. To illustrate that, take the ordered values (0.633346, 0.643025) given in 4 of the above example. By calculating, we find that $(0.633346)^2+0.643025=1.044>1$ which means that $(2,1)\text{-}FWPG(\Omega_1,\Omega_2,\ldots,\Omega_5)$ is not a (2,1)-FS.

Theorem 11 *Let $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})(i=1,2,\ldots,m)$ be a family of (2,1)-FNs on B, $\Omega=(\delta_\Omega,\lambda_\Omega)$ be a (2,1)-FN and $w=(w_1,w_2,\ldots,w_m)^T$ be a weight vector of Ω_j with $\sum_{j=1}^{m}w_j=1$. Then*

1. $(2,1)\text{-}FWA(\Omega_1\oplus\Omega,\Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)\geq(2,1)-FWA(\Omega_1\otimes\Omega,\Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$.

2. $(2,1)\text{-}FWG(\Omega_1\oplus\Omega,\Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)\geq(2,1)-FWG(\Omega_1\otimes\Omega,\Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$.

3. $(2,1)\text{-}FWPA(\Omega_1\oplus\Omega,\Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)\geq(2,1)-FWPA(\Omega_1\otimes\Omega,\Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$.

4. $(2,1)\text{-}FWPG(\Omega_1\oplus\Omega,\Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)\geq(2,1)-FWPG(\Omega_1\otimes\Omega,\Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$.

Proof We shall give the proofs of 1 and 4. Following similar technique, one can prove the other affirmations.
(1) For any $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})$ $(j=1,2,\ldots,m)$ and $\Omega=(\delta_\Omega,\lambda_\Omega)$, we obtain

$\sqrt{\delta^2_{\Omega_j}+\delta^2_\Omega-\delta^2_{\Omega_j}\delta^2_\Omega}\geq\sqrt{2\delta^2_{\Omega_j}\delta^2_\Omega-\delta^2_{\Omega_j}\delta^2_\Omega}=\delta_{\Omega_j}\delta_\Omega$, and $\lambda_{\Omega_j}+\lambda_\Omega-\lambda_{\Omega_j}\lambda_\Omega\geq2\lambda_{\Omega_j}\lambda_\Omega-\lambda_{\Omega_j}\lambda_\Omega=\lambda_{\Omega_j}\lambda_\Omega$.

That is,

$$\sum_{j=1}^{m} w_j\sqrt{\delta^2_{\Omega_j}+\delta^2_\Omega-\delta^2_{\Omega_j}\delta^2_\Omega}\geq\sum_{j=1}^{m} w_j\delta_{\Omega_j}\delta_\Omega \tag{1}$$

and

$$\sum_{j=1}^{m} w_j(\lambda_{\Omega_j}+\lambda_\Omega-\lambda_{\Omega_j}\lambda_\Omega)\geq\sum_{j=1}^{m} w_j\lambda_{\Omega_j}\lambda_\Omega. \tag{2}$$

According to item 1 of Definition 13 and items 1 and 2 of Definition 12, we have

$$(2,1)\text{-}FWA(\Omega_1\oplus\Omega,\Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)$$

$$= \left(\sum_{j=1}^{m} w_j\sqrt{\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}}, \sum_{j=1}^{m} w_j\lambda_{\Omega_j}\lambda_{\Omega}\right)$$

and

$$(2,1)\text{-}FWA(\Omega_1\otimes\Omega, \Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$$
$$= \left(\sum_{j=1}^{m} w_j\delta_{\Omega_j}\delta_{\Omega}, \sum_{j=1}^{m} w_j(\lambda_{\Omega_j}+\lambda_{\Omega}-\lambda_{\Omega_j}\lambda_{\Omega})\right).$$

Hence, from (1) and (2), we complete the proof.
(4) For any $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})(j = 1, 2, \ldots, m)$ and $\Omega = (\delta_{\Omega}, \lambda_{\Omega})$, we obtain

$$\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega} \geq 2\delta^2_{\Omega_j}\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega} = \delta^2_{\Omega_j}\delta^2_{\Omega}$$
$$\Rightarrow 1-(\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}) \leq 1-\delta^2_{\Omega_j}\delta^2_{\Omega}$$
$$\Rightarrow (1-(\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}))^{w_j} \leq (1-\delta^2_{\Omega_j}\delta^2_{\Omega})^{w_j}$$
$$\Rightarrow \prod_{j=1}^{m}(1-(\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}))^{w_j}$$
$$\leq \prod_{j=1}^{m}(1-\delta^2_{\Omega_j}\delta^2_{\Omega})^{w_j}$$
$$\Rightarrow 1-\prod_{j=1}^{m}(1-(\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}))^{w_j}$$
$$\geq 1-\prod_{j=1}^{m}(1-\delta^2_{\Omega_j}\delta^2_{\Omega})^{w_j}.$$

Similarly,

$$\Rightarrow 1-\prod_{j=1}^{m}(1-(\lambda_{\Omega_j}+\lambda_{\Omega}-\lambda_{\Omega_j}\lambda_{\Omega}))^{w_j}$$
$$\geq 1-\prod_{j=1}^{m}(1-\lambda_{\Omega_j}\lambda_{\Omega})^{w_j}.$$

According to items 1 and 2 of Definition 12, we have

$$(2,1)\text{-}FWPG(\Omega_1\oplus\Omega, \Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega)$$
$$= \left(\left(1-\prod_{j=1}^{m}(1-(\delta^2_{\Omega_j}+\delta^2_{\Omega}-\delta^2_{\Omega_j}\delta^2_{\Omega}))^{w_j}\right)^{\frac{1}{2}},\right.$$
$$\left.1-\prod_{j=1}^{m}(1-\lambda_{\Omega_j}\lambda_{\Omega})^{w_j}\right), \text{ and}$$
$$(2,1)\text{-}FWPG(\Omega_1\otimes\Omega, \Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega) =$$
$$\left(\left(1-\prod_{j=1}^{m}(1-\delta^2_{\Omega_j}\delta^2_{\Omega})^{w_j}\right)^{\frac{1}{2}},\right.$$
$$\left.1-\prod_{j=1}^{m}(1-(\lambda_{\Omega_j}+\lambda_{\Omega}-\lambda_{\Omega_j}\lambda_{\Omega}))^{w_j}\right).$$

Hence, $(2,1)\text{-}FWPG(\Omega_1\oplus\Omega, \Omega_2\oplus\Omega,\ldots,\Omega_m\oplus\Omega) \geq (2,1)\text{-}FWPG(\Omega_1\otimes\Omega, \Omega_2\otimes\Omega,\ldots,\Omega_m\otimes\Omega)$. □

Theorem 12 *Let $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})$ and $\Gamma_j = (\delta_{\Gamma_j}, \lambda_{\Gamma_j})(j = 1, 2, \ldots, m)$ be two families of (2,1)-FSs on B, and $w = (w_1, w_2, \ldots, w_m)^T$ be a weight vector of them with $\sum_{j=1}^{m} w_j = 1$. Then*

1. $(2,1)\text{-}FWA(\Omega_1\oplus\Gamma_1, \Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m) \geq (2,1)-FWA(\Omega_1\otimes\Gamma_1, \Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m).$
2. $(2,1)\text{-}FWG(\Omega_1\oplus\Gamma_1, \Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m) \geq (2,1)-FWG(\Omega_1\otimes\Gamma_1, \Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m).$
3. $(2,1)\text{-}FWPA(\Omega_1\oplus\Gamma_1, \Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m) \geq (2,1)-FWPA(\Omega_1\otimes\Gamma_1, \Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m).$
4. $(2,1)\text{-}xFWPG(\Omega_1\oplus\Gamma_1, \Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m) \geq (2,1)-FWPG(\Omega_1\otimes\Gamma_1, \Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m).$

Proof We shall give the proof for 1. Following similar technique, one can prove the other affirmations.
(1) For any $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})$ and $\Gamma_j = (\delta_{\Gamma_j}, \lambda_{\Gamma_j})$ $(j = 1, 2, \ldots, m)$, we can get

$$\sqrt{\delta^2_{\Omega_j}+\delta^2_{\Gamma_j}-\delta^2_{\Omega_j}\delta^2_{\Gamma_j}} \geq \sqrt{2\delta^2_{\Omega_j}\delta^2_{\Gamma_j}-\delta^2_{\Omega_j}\delta^2_{\Gamma_j}} = \delta_{\Omega_j}\delta_{\Gamma_j}.$$

That is,

$$\sum_{j=1}^{m} w_j\sqrt{\delta^2_{\Omega_j}+\delta^2_{\Gamma_j}-\delta^2_{\Omega_j}\delta^2_{\Gamma_j}} \geq \sum_{j=1}^{m} w_j\delta_{\Omega_j}\delta_{\Gamma_j}.$$

Similarly,

$$\sum_{j=1}^{m} w_j(\lambda_{\Omega_j}+\lambda_{\Gamma_j}-\lambda_{\Omega_j}\lambda_{\Gamma_j}) \geq \sum_{j=1}^{m} w_j\lambda_{\Omega_j}\lambda_{\Gamma_j}.$$

By items 1 and 2 of Definition 12, we have

$$(2,1)\text{-}FWA(\Omega_1\oplus\Gamma_1, \Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m)$$
$$= \left(\sum_{j=1}^{m} w_j\sqrt{\delta^2_{\Omega_j}+\delta^2_{\Gamma_j}-\delta^2_{\Omega_j}\delta^2_{\Gamma_j}}, \sum_{j=1}^{m} w_j\lambda_{\Omega_j}\lambda_{\Gamma_j}\right)$$

and

$$(2,1)\text{-}FWA(\Omega_1\otimes\Gamma_1,\Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m) = \left(\sum_{j=1}^{m} w_j\delta_{\Omega_j}\delta_{\Gamma_j}, \sum_{j=1}^{m} w_j(\lambda_{\Omega_j}+\lambda_{\Gamma_j}-\lambda_{\Omega_j}\lambda_{\Gamma_j})\right).$$

Hence, $(2,1)\text{-}FWA(\Omega_1\oplus\Gamma_1,\Omega_2\oplus\Gamma_2,\ldots,\Omega_m\oplus\Gamma_m)\geq (2,1)\text{-}FWA(\Omega_1\otimes\Gamma_1,\Omega_2\otimes\Gamma_2,\ldots,\Omega_m\otimes\Gamma_m)$. □

Theorem 13 *Let $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})(j=1,2,\ldots,m)$ be a family of (2,1)-FNs on B, and $w=(w_1,w_2,\ldots,w_m)^T$ be a weight vector of Ω_j with $\sum_{j=1}^{m} w_j=1$ and $\xi\geq 1$. Then*

1. $(2,1)\text{-}FWA(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m)\geq (2,1)-FWA(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi})$.
2. $(2,1)\text{-}FWG(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m)\geq (2,1)-FWG(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi})$.
3. $(2,1)\text{-}FWPA(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m)\geq (2,1)\text{-}FWPA(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi})$.
4. $(2,1)\text{-}FWPG(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m)\geq (2,1)\text{-}FWPG(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi})$.

Proof We shall give the proof for 1. Following similar technique, one can prove the other affirmations.
(1) For any $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})$ $(j=1,2,\ldots,m)$, we have

$$(2,1)\text{-}FWA(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m) = \left(\sum_{j=1}^{m} w_j\sqrt{1-(1-\delta_{\Omega_j}^2)^{\xi}}, \sum_{j=1}^{m} w_j\lambda_{\Omega_j}^{\xi}\right), \text{ and}$$

$$(2,1)\text{-}FWA(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi}) = \left(\sum_{j=1}^{m} w_j\delta_{\Omega_j}^{\xi}, \sum_{j=1}^{m} w_j(1-(1-\lambda_{\Omega_j})^{\xi})\right).$$

Let $f(\delta_{\Omega_j})=1-(1-\delta_{\Omega_j}^2)^{\xi}-(\delta_{\Omega_j}^2)^{\xi}$. We demonstrate that $f(\delta_{\Omega_j})\geq 0$. It follows from the Newton generalized binomial theorem that

$$(1-\delta_{\Omega_j}^2)^{\xi}+(\delta_{\Omega_j}^2)^{\xi}\leq (1-\delta_{\Omega_j}^2+\delta_{\Omega_j}^2)^{\xi}=1.$$

This means that $f(\delta_{\Omega_j})\geq 0$. Now,

$$1-(1-\delta_{\Omega_j}^2)^{\xi}-(\delta_{\Omega_j}^2)^{\xi}\geq 0$$
$$\Rightarrow 1-(1-\delta_{\Omega_j}^2)^{\xi}\geq (\delta_{\Omega_j}^2)^{\xi}$$
$$\Rightarrow \sqrt{1-(1-\delta_{\Omega_j}^2)^{\xi}}\geq \delta_{\Omega_j}^{\xi}$$
$$\Rightarrow \sum_{j=1}^{m} w_j\sqrt{1-(1-\delta_{\Omega_j}^2)^{\xi}}\geq \sum_{j=1}^{m} w_j\delta_{\Omega_j}^{\xi}.$$

Similarly,

$$\sum_{j=1}^{m} w_j(1-(1-\lambda_{\Omega_j})^{\xi})\geq \sum_{j=1}^{m} w_j\lambda_{\Omega_j}^{\xi}.$$

Hence, $(2,1)\text{-}FWA(\xi\Omega_1,\xi\Omega_2,\ldots,\xi\Omega_m)\geq (2,1)\text{-}FWA(\Omega_1^{\xi},\Omega_2^{\xi},\ldots,\Omega_m^{\xi})$.

□

Theorem 14 *Let $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})$ $(j=1,2,\ldots,m)$ be a family of (2,1)-FNs on B, $\Omega=(\delta_{\Omega},\lambda_{\Omega})$ be a (2,1)-FN on B and $w=(w_1,w_2,\ldots,w_m)^T$ be a weight vector of Ω_j with $\sum_{j=1}^{m} w_j=1$ and $\xi\geq 1$. Then*

1. $(2,1)\text{-}FWA(\xi\Omega_1\oplus\Omega,\xi\Omega_2\oplus\Omega,\ldots,\xi\Omega_m\oplus\Omega) \geq (2,1)\text{-}FWA(\Omega_1^{\xi}\otimes\Omega,\Omega_2^{\xi}\otimes\Omega,\ldots,\Omega_m^{\xi}\otimes\Omega)$.
2. $(2,1)\text{-}FWG(\xi\Omega_1\oplus\Omega,\xi\Omega_2\oplus\Omega,\ldots,\xi\Omega_m\oplus\Omega) \geq (2,1)\text{-}FWG(\Omega_1^{\xi}\otimes\Omega,\Omega_2^{\xi}\otimes\Omega,\ldots,\Omega_m^{\xi}\otimes\Omega)$.
3. $(2,1)\text{-}FWPA(\xi\Omega_1\oplus\Omega,\xi\Omega_2\oplus\Omega,\ldots,\xi\Omega_m\oplus\Omega) \geq (2,1)\text{-}FWPA(\Omega_1^{\xi}\otimes\Omega,\Omega_2^{\xi}\otimes\Omega,\ldots,\Omega_m^{\xi}\otimes\Omega)$.
4. $(2,1)\text{-}FWPG(\xi\Omega_1\oplus\Omega,\xi\Omega_2\oplus\Omega,\ldots,\xi\Omega_m\oplus\Omega) \geq (2,1)\text{-}FWPG(\Omega_1^{\xi}\otimes\Omega,\Omega_2^{\xi}\otimes\Omega,\ldots,\Omega_m^{\xi}\otimes\Omega)$.

Proof We shall give the proof for 1. Following similar technique, one can prove the other affirmations.
(1) For any $\Omega_j=(\delta_{\Omega_j},\lambda_{\Omega_j})$ $(j=1,2,\ldots,m)$ and $\Omega=(\delta_{\Omega},\lambda_{\Omega})$, we have

$$(2,1)\text{-}FWA(\xi\Omega_1\oplus\Omega,\xi\Omega_2\oplus\Omega,\ldots,\xi\Omega_m\oplus\Omega) = \left(\sum_{j=1}^{m} w_j\sqrt{1-(1-\delta_{\Omega_j}^2)^{\xi}(1-\delta_{\Omega}^2)}, \sum_{j=1}^{m} w_j\lambda_{\Omega_j}^{\xi}\lambda_{\Omega}\right),$$

$$\text{and}(2,1)\text{-}FWA(\Omega_1^{\xi}\otimes\Omega,\Omega_2^{\xi}\otimes\Omega,\ldots,\Omega_m^{\xi}\otimes\Omega) = \left(\sum_{j=1}^{m} w_j\delta_{\Omega_j}^{\xi}\delta_{\Omega}, \sum_{j=1}^{m} w_j(1-(1-\lambda_{\Omega_j})^{\xi}(1-\lambda_{\Omega}))\right).$$

Let $f(\delta_{\Omega_j})=1-(1-\delta_{\Omega_j}^2)^{\xi}(1-\delta_{\Omega}^2)-(\delta_{\Omega_j}^2)^{\xi}\delta_{\Omega}^2$. We demonstrate that $f(\delta_{\Omega_j})\geq 0$. To do this, let $g(\delta_{\Omega_j})=(1-$

$\delta^2_{\Omega_j})^{\xi} + (\delta^2_{\Omega_j})^{\xi}$. Then

$$\begin{aligned} g^{'}(\delta_{\Omega_j}) &= -2\xi\delta_{\Omega_j}(1-\delta^2_{\Omega_j})^{\xi-1} + 2\xi\delta_{\Omega_j}(\delta^2_{\Omega_j})^{\xi-1} \\ &= 2\xi\delta_{\Omega_j}((\delta^2_{\Omega_j})^{\xi-1} - (1-\delta^2_{\Omega_j})^{\xi-1}). \end{aligned}$$

Now, if $\delta_{\Omega_j} > \frac{1}{\sqrt{2}}$, then $g(\delta_{\Omega_j})$ is monotonic increasing and if $\delta_{\Omega_j} < \frac{1}{\sqrt{2}}$, then $g(\delta_{\Omega_j})$ is monotonic decreasing. Therefore, $g(\delta_{\Omega_j}) \le g(\delta_{\Omega_j})_{max} = max\{g(0), g(1)\} = 1$. Note that $(1-\delta^2_{\Omega_j})^{\xi}(1-\delta^2_{\Omega}) + (\delta^2_{\Omega_j})^{\xi}\delta^2_{\Omega} \le 1$. This automatically means that

$$\begin{aligned} & f(\delta_{\Omega_j}) = 1 - (1-\delta^2_{\Omega_j})^{\xi}(1-\delta^2_{\Omega}) - (\delta^2_{\Omega_j})^{\xi}\delta^2_{\Omega} \ge 0 \\ & \Rightarrow \sum_{j=1}^{m} w_j\sqrt{1-(1-\delta^2_{\Omega_j})^{\xi}(1-\delta^2_{\Omega})} \ge \sum_{j=1}^{m} w_j\delta^{\xi}_{\Omega_j}\delta_{\Omega}. \end{aligned}$$

Similarly,

$$\sum_{j=1}^{m} w_j(1-(1-\lambda_{\Omega_j})^{\xi}(1-\lambda_{\Omega})) \ge \sum_{j=1}^{m} w_j\lambda^{\xi}_{\Omega_j}\lambda_{\Omega}.$$

Hence, (2, 1)-$FWA(\xi\Omega_1 \oplus \Omega, \xi\Omega_2 \oplus \Omega, \ldots, \xi\Omega_m \oplus \Omega) \ge$ (2, 1)-$FWA(\Omega_1^{\xi} \otimes \Omega, \Omega_2^{\xi} \otimes \Omega, \ldots, \Omega_m^{\xi} \otimes \Omega)$. □

According to Remark 3, we need to impose a further condition to prove the following three results; this condition is that the values obtained from the operators presented in Definition 13 is a (2,1)-FS.

Theorem 15 *Let $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})(j = 1, 2, \ldots, m)$ be a family of (2,1)-FNs on B, $\Omega = (\delta_{\Omega}, \lambda_{\Omega})$ be a (2,1)-FN on B and $w = (w_1, w_2, \ldots, w_m)^T$ be a weight vector of Ω_j with $\sum_{j=1}^{m} w_j = 1$. Then*

1. (2, 1)-
$FWA(\Omega_1 \oplus \Omega, \Omega_2 \oplus \Omega, \ldots, \Omega_m \oplus \Omega)$
$\ge (2, 1) - FWA(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

2. (2, 1)-
$FWA(\Omega_1, \Omega_2, \ldots, \Omega_m) \oplus \Omega$
$\ge (2, 1) - FWA(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

3. (2, 1)-
$FWG(\Omega_1 \oplus \Omega, \Omega_2 \oplus \Omega, \ldots, \Omega_m \oplus \Omega)$
$\ge (2, 1) - FWG(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

4. (2, 1)-
$FWG(\Omega_1, \Omega_2, \ldots, \Omega_m) \oplus \Omega$
$\ge (2, 1) - FWG(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

5. (2, 1)-
$FWPA(\Omega_1 \oplus \Omega, \Omega_2 \oplus \Omega, \ldots, \Omega_m \oplus \Omega)$
$\ge (2, 1) - FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

6. (2, 1)-
$FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m) \oplus \Omega$
$\ge (2, 1) - FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

7. (2, 1)-
$FWPG(\Omega_1 \oplus \Omega, \Omega_2 \oplus \Omega, \ldots, \Omega_m \oplus \Omega)$
$\ge (2, 1) - FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

8. (2, 1)-
$FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \oplus \Omega$
$\ge (2, 1) - FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes \Omega.$

Proof Similar to the proof of Theorem 11. □

Theorem 16 *Let $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})$ and $\Gamma_j = (\delta_{\Gamma_j}, \lambda_{\Gamma_j})(j = 1, 2, \ldots, m)$ be two families of (2,1)-FSs on B, and $w = (w_1, w_2, \ldots, w_m)^T$ be a weight vector of them with $\sum_{j=1}^{m} w_j = 1$. Then*

1. (2, 1)-$FWA(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\oplus(2, 1) - FWA(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$
$\ge (2, 1) - FWA(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\otimes(2, 1) - FWA(\Gamma_1, \Gamma_2, \ldots, \Gamma_m).$

2. (2, 1)-$FWG(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\oplus(2, 1) - FWG(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$
$\ge (2, 1) - FWG(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\otimes(2, 1) - FWG(\Gamma_1, \Gamma_2, \ldots, \Gamma_m).$

3. (2, 1)-$FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\oplus(2, 1) - FWPA(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$
$\ge (2, 1) - FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m)$
$\otimes(2, 1) - FWPA(\Gamma_1, \Gamma_2, \ldots, \Gamma_m).$

4. (2, 1)-$FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \oplus (2, 1)$
$-FWPG(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) \ge (2, 1)$
$-FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \otimes (2, 1)$
$-FWPG(\Gamma_1, \Gamma_2, \ldots, \Gamma_m).$

Proof Similar to the proof of Theorem 12. □

Theorem 17 *Let* $\Omega_j = (\delta_{\Omega_j}, \lambda_{\Omega_j})(j = 1, 2, \ldots, m)$ *be a family of (2,1)-FNs on B, and* $w = (w_1, w_2, \ldots, w_m)^T$ *be a weight vector of* Ω_j *with* $\sum_{j=1}^{m} w_j = 1$ *and* $\xi \geq 1$. *Then*

1. $\xi(2, 1)\text{-}FWA(\Omega_1, \Omega_2, \ldots, \Omega_m) \geq ((2, 1) - FWA(\Omega_1, \Omega_2, \ldots, \Omega_m))^{\xi}.$
2. $\xi(2, 1)\text{-}FWG(\Omega_1, \Omega_2, \ldots, \Omega_m) \geq ((2, 1) - FWG(\Omega_1, \Omega_2, \ldots, \Omega_m))^{\xi}.$
3. $\xi(2, 1)\text{-}FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m) \geq ((2, 1) - FWPA(\Omega_1, \Omega_2, \ldots, \Omega_m))^{\xi}.$
4. $\xi(2, 1)\text{-}FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m) \geq ((2, 1) - FWPG(\Omega_1, \Omega_2, \ldots, \Omega_m))^{\xi}.$

Proof Similar to the proof of Theorem 17. □

Application of (2,1)-FSs to MCDM problems

We dedicated this section to investigating a MCDM problem using the four types of aggregations operators given in the foregoing section. We propose some algorithms that show how this type of problem is handled, and provide an illustrative example.

Representation of MCDM problems and their algorithms under the environment of (2,1)-FSs

MCDM problems are one of the challenging and fast techniques for all decision makers for getting the best alternative(s) among the set of possible ones according to multiple criteria. To illustrate that, assume $B = \{b_i : i = 1, 2, \ldots, n\}$ as a set of n different alternatives that have been evaluated (by the decision maker) under a set of m different criteria $C = \{c_j : j = 1, 2, \ldots, m\}$. Presume that the decision maker estimates the preferences in terms of (2,1)-FNs: $\theta_{ij} = \langle \delta_{ij}, \lambda_{ij} \rangle_{i \times j}$, where $0 \leq \delta_{ij}^2 + \lambda_{ij} \leq 1$ and $\delta_{ij}, \lambda_{ij} \in [0, 1]$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that δ_{ij} and λ_{ij} respectively represent the degree that the alternative b_i fulfills and doesn't fulfill the attribute c_j provided by the decision maker. Thus, MCDM problems can be concisely expressed in a (2,1)-Fuzzy decision matrix $\mathcal{A} = (\theta_{ij})_{n \times m} = \langle \delta_{ij}, \lambda_{ij} \rangle_{n \times m}$.

In what follows , we explain the steps used in the proposed methodology for MCDM:

Step 1 : formulate the (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n \times m}$ for a MCDM problem under study.

Step 2 : Convert (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n \times m}$ into the normalized (2,1)-Fuzzy decision matrix $\tau = (\tau_{ij})_{n \times m}$. In this step, if there are different kinds of criteria, namely benefit X and cost Y then the rating values of X and Y can be transformed using the below normalization formula: $\tau_{ij} = \begin{cases} \theta_{ij} & j \in X \\ (\theta_{ij})^c & j \in Y \end{cases}$

Step 3 : Assessment of the alternatives' aggregations based on the normalized (2,1)-Fuzzy decision matrix given in Step 2. That is, for each alternative b_i $(i = 1, 2, \ldots, n)$, compute all types of (2,1)-Fuzzy weighted operators given in Definition 13 (i.e., (2,1)-FWA, (2,1)-FWG, (2,1)-FWPA and (2,1)-FWPG operators).

Step 4 : Compute the scores and accuracy functions for each (2,1)-FNs provided in Step 3. According Remark 3 the ordered values obtained from these operators need not be a (2,1)-FS; however, we extend the formulas of scores and accuracy functions given in Definition 10 for those ordered values.

Step 5 : Compare the given alternatives based on the scores and accuracy.

Step 6 : Determine the optimal ranking order of the alternatives and recognize the optimal alternative(s) using Definition 10

Herein, we provide an algorithm for each aggregation operator: Algorithm 1 for (2,1)-FWA operator, Algorithm 2 for (2,1)-FWG operator, Algorithm 3 for (2,1)-FWPA operator, and Algorithm 4 for (2,1)-FWPG operator.

```
Input  : The set of alternatives B and the set of multi criteria C.
Output: select the most desirable alternative(s).
Initiate (2,1)-Fuzzy decision matrix θ = (θ_ij)_{n×m} for a MCDM
  problem under study;
Convert (2,1)-Fuzzy decision matrix θ = (θ_ij)_{n×m} into the
  normalized (2,1)-Fuzzy decision matrix τ = (τ_ij)_{n×m};
Compute (2,1)-FWA operator (using the formula given in
  Definition 13) for each alternative b_i (i = 1, 2, ..., n);
foreach i ≤ n do
  | Compute score function induced from (2,1)-FWA operator
  | for b_i.
end
Let D = {b_i : score(b_i) = max{score(b_i) : i = 1, 2, ..., n}};
if D is a singleton set, say, b_k then
  | return b_k is the desirable (optimal) alternative.
else
  | Compute accuracy function induced from (2,1)-FWA
  | operator for b_i ∈ D;
  | Let E = {b_i : acc(b_i) = max{acc(b_i) : b_i ∈ D}};
  | return each b_k ∈ E represents a desirable (an optimal)
  | alternative;
end
```

Algorithm 1: The algorithm of selection the optimal alternative(s) with respect to (2,1)-FWA operator

Input : The set of alternatives B, and the set of multi criteria C.
Output: select the most desirable (optimal) alternative.

Initiate (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ for a MCDM problem under study;
Convert (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ into the normalized (2,1)-Fuzzy decision matrix $\tau = (\tau_{ij})_{n\times m}$;
Compute (2,1)-FWG operator (using the formula given in Definition 13) for each alternative b_i $(i = 1, 2, \ldots, n)$;
foreach $i \leq n$ **do**
  Compute score function induced from (2,1)-FWG operator for b_i.
end
Let $D = \{b_i : score(b_i) = max\{score(b_i) : i = 1, 2, \ldots, n\}\}$;
if *D is a singleton set, say, b_k* **then**
  return b_k is the desirable (optimal) alternative.
else
  Compute accuracy function induced from (2,1)-FWG operator for $b_i \in D$;
  Let $E = \{b_i : acc(b_i) = max\{acc(b_i) : b_i \in D\}\}$;
  return each $b_k \in E$ represents a desirable (an optimal) alternative;
end

Algorithm 2: The algorithm of selection the optimal alternative(s) with respect to (2,1)-FWG operator

Input : The set of alternatives B, and the set of multi criteria C.
Output: select the most desirable alternative(s).

Initiate (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ for a MCDM problem under study;
Convert (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ into the normalized (2,1)-Fuzzy decision matrix $\tau = (\tau_{ij})_{n\times m}$;
Compute (2,1)-FWPA operator (using the formula given in Definition 13) for each alternative b_i $(i = 1, 2, \ldots, n)$;
foreach $i \leq n$ **do**
  Compute score function induced from (2,1)-FWPA operator for b_i.
end
Let $D = \{b_i : score(b_i) = max\{score(b_i) : i = 1, 2, \ldots, n\}\}$;
if *D is a singleton set, say, b_k* **then**
  return b_k is the desirable (optimal) alternative.
else
  Compute accuracy function induced from (2,1)-FWPA operator for $b_i \in D$;
  Let $E = \{b_i : acc(b_i) = max\{acc(b_i) : b_i \in D\}\}$;
  return each $b_k \in E$ represents a desirable (an optimal) alternative;
end

Algorithm 3: The algorithm of selection the optimal alternative(s) with respect to (2,1)-FWPA operator

In Fig. 3, we display the flow chart of selection the optimal alternative(s) with respect to (2,1)-FWA operator. Similarly, the flow charts induced from the other operators are displayed.

Input : The set of alternatives B, and the set of multi criteria C.
Output: select the most desirable alternative(s).

Initiate (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ for a MCDM problem under study;
Convert (2,1)-Fuzzy decision matrix $\theta = (\theta_{ij})_{n\times m}$ into the normalized (2,1)-Fuzzy decision matrix $\tau = (\tau_{ij})_{n\times m}$;
Compute (2,1)-FWPG operator (using the formula given in Definition 13) for each alternative b_i $(i = 1, 2, \ldots, n)$;
foreach $i \leq n$ **do**
  Compute score function induced from (2,1)-FWPG operator for b_i
end
Let $D = \{b_i : score(b_i) = max\{score(b_i) : i = 1, 2, \ldots, n\}\}$;
if *D is a singleton set, say, b_k* **then**
  return b_k is the desirable (optimal) alternative.
else
  Compute accuracy function induced from (2,1)-FWPG operators for $b_i \in D$;
  Let $E = \{b_i : acc(b_i) = max\{acc(b_i) : b_i \in D\}\}$;
  return each $b_k \in E$ represents a desirable (an optimal) alternative;
end

Algorithm 4: The algorithm of selection the optimal alternative(s) with respect to (2,1)-FWPG operator

Illustrative examples

In this subsection, we explain the above-mentioned approaches by the following example which investigated a multiple criteria decision-making problem.

Example 10 Assume that a certain university wants to assign a permanent faculty member from the set of candidates $U = \{$Redhwan, Al-Harith, Mustafa, Bushra, Sarah$\}$. For this, the university authorities consider the following five criteria $C = \{c_i : i = 1, 2, 3, 4, 5\}$, where:

- c_1 represents the number of research publications,
- c_2 represents the teaching experience,
- c_3 represents the regularity and punctuality,
- c_4 represents the number of conferences participated, and
- c_5 represents the behavior with students through the class.

After a deep discussion, a committee (forms by the university authorities) proposed a weight vector corresponding to every criteria $\omega = (0.25, 0.35, 0.1, 0.1, 0.2)^T$. A committee assesses the performance of these candidates under the (2,1)-FSs environment as given in Table 1. Every ordered pair (δ, λ) given in Table 1 represents the membership and non-membership degrees of a candidate to fulfill and dissatisfy the corresponding criteria (or attribute) such that $0 \leq (\delta)^2 + \lambda \leq 1$ and δ, λ lie in $[0, 1]$.

Assume that the proposed approach for accessing the best candidate with appreciation to every criterion provided using the committee is furnished according to the different types of (2,1)-FS operators introduced in Definition 13. Then, we

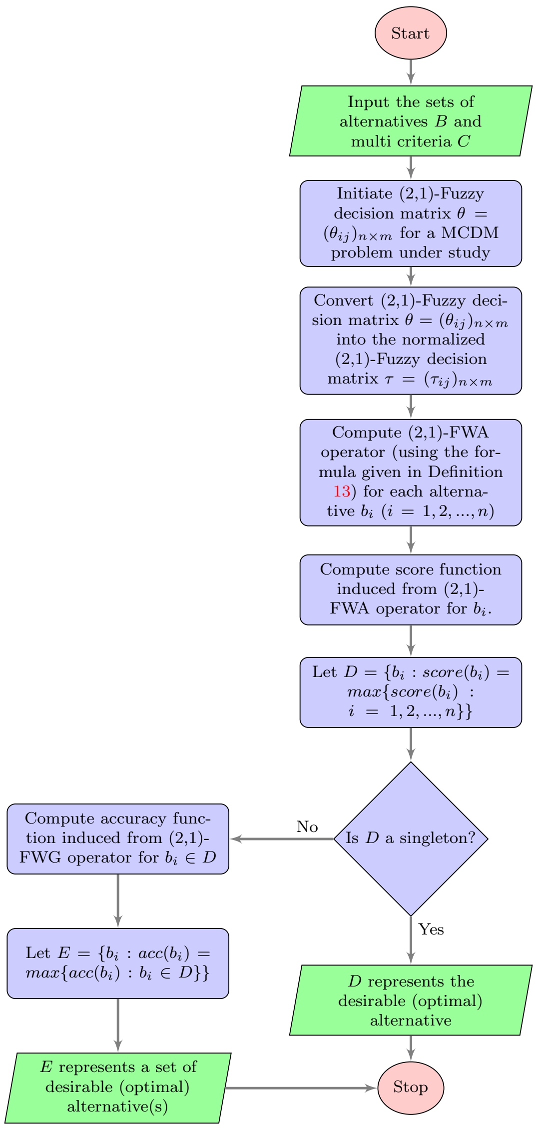

Fig. 3 Flow chart of selection the optimal alternative(s) with respect to (2,1)-FWA operator

compute the score function for each candidate. If there are some candidates who have the same score function, then we compute their accuracy function to decide who is the optimal candidate(s); see, Table 2.

According to the computations induced from the four operators of aggregation, we find that the optimal ranking order of the five candidates induced from a $(2, 1)$-FWA operator is Sarah. It should be noted that the candidates Redhwan and Sarah are equal with respect to the score function; so that, we complete comparison by computing their accuracy functions which show that Sarah is the best candidates to get this job. The rank of the candidates induced from a $(2, 1)$-FWA operator is

Sarah $\succ$ Redhwan $\succ$ Bushra $\succ$ Mustafa $\succ$ Al-Harith.

On the other hand, note that the values of score functions induced from the other aggregation operators are distinct for all candidates, so there is no need to compute the accuracy function. Thus, the rank of the five candidates respectively induced from $(2, 1)$-FWG, $(2, 1)$-$FWPA$ and $(2, 1)$-$FWPG$ operators are

Redhwan $\succ$ Sarah $\succ$ Mustafa $\succ$ Bushra $\succ$ Al-Harith.

Sarah $\succ$ Bushra $\succ$ Redhwan $\succ$ Mustafa $\succ$ Al-Harith.

Bushra $\succ$ Redhwan $\succ$ Sarah $\succ$ Mustafa $\succ$ Al-Harith.

It can be noted from the above discussion that the selection of the optimal permanent faculty member is based on two factors, first one is the type of generalizations of IFSs, which is herein a (2,1)-FS. The second one is the aggregation operator provided by the committee to evaluate the performance of the candidates.

Conclusions

In this paper, we have established a new class of orthopair fuzzy sets, namely (2,1)-Fuzzy sets. Two of the merits of this class are to, first, enlarge the space of membership and non-membership more than IFSs, which means overcoming some limitations of IFS in handling some situations that have the sum of membership and nonmembership grades exceed one. Second, to offer a convenient frame to model some real-life problems that are evaluated with different importance of their membership and non-membership grades. On the other hand, the limitation of the proposed class is that its grades space is smaller than the grades space of q-rung orthopair fuzzy sets.

Our contributions through the manuscript are as follows. We have defined some operations for (2,1)-Fuzzy sets and presented main characterizations. In addition, we have introduced four types of aggregation operators in the environment of (2,1)-Fuzzy sets and reveled the relationships among them. Ultimately, we have exploited the proposed aggregation operators to address the decision-making issues and provided the algorithms used in the evaluation with a flow chart. A numerical example has been given to show how the followed method assisted us with being effective in decision problems.

In future works, we intend to display a novel class of orthopair fuzzy sets that forms an umbral for all the generalizations of IFSs. Theoretically, we shall benefit from (2,1)-Fuzzy sets to construct a new type of fuzzy topologies.

Table 1 (2,1)-Fuzzy numbers

Candidates	c_1	c_2	c_3	c_4	c_5
Redhwan	(0.7, 0.4)	(0.5, 0.35)	(0.6, 0.45)	(0.4, 0.4)	(0.7, 0.3)
Al−Harith	(0.3, 0.8)	(0.7, 0.5)	(0.75, 0.35)	(0.25, 0.85)	(0.7, 0.35)
Mustafa	(0.75, 0.2)	(0.6, 0.6)	(0.2, 0.8)	(0.5, 0.4)	(0.8, 0.35)
Bushra	(0.8, 0.35)	(0.75, 0.42)	(0.55, 0.65)	(0.1, 0.75)	(0.6, 0.45)
Sarah	(0.7, 0.476625)	(0.65, 0.55)	(0.15, 0.9)	(0.85, 0.2)	(0.85, 0.25)

Table 2 Evaluation of scores with (2,1)-Fuzzy aggregation operators

	Redhwan	Al-Harith	Mustafa	Bushra	Sarah
$(2,1)\text{-}FWA$	(0.59, 0.3675)	(0.56, 0.565)	(0.6275, 0.45)	(0.6475, 0.4645)	(0.6725, 0.47165625)
$Score$	−0.0194	−0.2514	−0.05624375	−0.04524375	−0.0194
$Accuracy$	0.7156	0.8786	0.83376	0.88376	0.92391
$(2,1)\text{-}FWG$	(0.579368, 0.364662)	(0.514499, 0.532812)	(0.591202, 0.404519)	(0.577711, 0.450396)	(0.619768, 0.430315)
$Score$	−0.028995	−0.268103	−0.054999	−0.116646	−0.046203
$(2,1)\text{-}FWPA$	$(\sqrt{0.36}, 0.3675)$	$(\sqrt{0.3545}, 0.565)$	$(\sqrt{0.423625}, 0.45)$	$(\sqrt{0.460125}, 0.4645)$	$(\sqrt{0.489375}, 0.471656)$
$Score$	−0.0075	−0.2105	−0.026375	−0.004375	0.017719
$(2,1)\text{-}FWPG$	$(\sqrt{0.372341}, 0.369147)$	$(\sqrt{0.383094}, 0.6186)$	$(\sqrt{0.451216}, 0.490676)$	$(\sqrt{0.488822}, 0.483938)$	$(\sqrt{0.526384}, 0.528318)$
$Score$	0.003194	−0.235506	−0.03946	0.004884	−0.001934

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Declarations

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