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ORIGINAL ARTICLE

Some t-conorm-based distance measures and knowledge measures for Pythagorean fuzzy sets with their application in decision-making

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Abstract

The Pythagorean fuzzy sets are more robust than fuzzy sets and intuitionistic fuzzy sets in dealing with the problems involving uncertainty. To compare two Pythagorean fuzzy sets, distance measures play a crucial role. In this paper, we have proposed some novel distance measures for Pythagorean fuzzy sets using t-conorms. We have also discussed their various desirable properties. With the help of suggested distance measures, we have introduced some new knowledge measures for Pythagorean fuzzy sets. Through numerical comparison and linguistic hedges, we have established the effectiveness of the suggested distance measures and knowledge measures, respectively, over the existing measures in the Pythagorean fuzzy setting. At last, we have demonstrated the application of the suggested measures in pattern analysis and multi-attribute decision-making.

Keywords Pythagorean fuzzy set · t-conorm · Similarity measure · Entropy measure · Knowledge measure · Multi-attribute decision-making

Introduction

Zadeh [1] introduced the concept of fuzzy sets (FSs) for handling deterministic uncertainty. Each element in a fuzzy set (FS) possesses a grade of membership (μ) lying between 0 and 1. However, in an FS, the grade of non-membership (ϑ) is by default taken as $1 - \mu$. So to assign an independent non-membership grade to an element, a novel extension of FSs known as intuitionistic fuzzy set (IFS) was given by Atanassov [2]. In an IFS, the sum of membership grades is less or equal to one i.e., $\mu + \vartheta \leq 1$. Though there are numerous applications of intuitionistic fuzzy sets (IFSs) but their scope is limited due to the restriction on sum of membership grades. The IFSs fail to handle those uncertain problems where $\mu + \vartheta > 1$. So, the concept of Pythagorean fuzzy set (PFS) was proposed by Yager [3] as an extension of intuitionistic fuzzy sets [2] (IFSs) and fuzzy sets [1] (FSs) for solving the problems involving uncertainty more precisely. Each element of a PFS has a membership grade (μ) and a non-membership grade (ϑ) with their square sum at most one $\left(\mu^2 + \vartheta^2 \leq 1\right)$. The FSs and IFSs form a part of the space of PFSs and therefore the space of PFSs is wider than the space of FSs and IFSs.So, PFSs are more powerful than FSs and IFSs in handling uncertain problems. The technique for order of preference by similarity to ideal solution (TOPSIS) in the Pythagorean fuzzy (PF) setting and the concept of the Pythagorean fuzzy number were suggested by Zhang and Xu [4]. Various PF aggregation functions with their utility in decision-making were given by Yager [5]. Wei and Lu [6] introduced some power aggregation functions for PFSs. Using Einstein operations, Garg [7] proposed some new aggregation functions in the PF environment. Wei [8] suggested some PF interaction aggregation functions with their utility in multi-attribute decision-making (MADM). Many studies [9–12] concerning the PF aggregation functions with their various applications are available in the literature. The TODIM (an acronym in Portuguese for Interactive and Multicriteria Decision Making) method for PFSs was introduced by Ren et al. [13]. Peng et al. [14] proposed some information measures for PFSs. A novel PF distance measure was proposed by Peng and Dai [15]. Some PF measures of correlation with their utility were proposed by Singh and Ganie [16]. Various researchers [17–24] have studied PFSs and applied them in distinct uncertain situations. The current study is related to the development of some novel PF distance measures and knowledge measures.

The main contributions of this paper are as:

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- We suggest a new method of constructing the PF distance measures from t-conorms and introduce four new PF distance measures.
- We discuss various desirable properties of the suggested PF distance measures.
- We propose four weighted PF distance measures.
- We suggest a general method of constructing the PF knowledge measures from the proposed PF distance measures and propose four new knowledge measures based on PFSs.
- We compare the suggested PF measures of distance and knowledge with the available PF measures of compatibility.
- We demonstrate the applicability of the suggested measures in pattern recognition and MADM.

Related work

Distance measures are very powerful in comparing two objects based on their inequality content. For FSs and IFSs there are a lot of studies [25–42] concerning distance measures along with their various applications. Karmakar et al. [43] suggested a Minkowski distance measure for Type-2 IFSs (T2IFSs) using the Hausdorff metric. Some recent studies related to Type 2 FSs and T2IFSs are given in [44–46]. The application of some PF measures of distance and similarity in MADM was shown by Zeng et al. [47]. Hussain and Yang [48] proposed some Hausdorff metric-based PF measures of distance and similarity with their applicability in PF TOPSIS. Some generalized measures of distance and their continuous versions for PFSs were given by Li and Lu [49]. They also proposed set-theoretic-based, matching function-based, and complement-based PF similarity measures. Based on the membership grades, Ejegwa [19] proposed some distance and similarity measures for PFSs. Some cosine function-based PF similarity measures were suggested by Wei and Wei [50]. Twelve PF measures of distance and similarity with their applicability were given by Peng et al. [14]. For PFSs Zhang [51] introduced a measure of similarity and its utility in decision-making. Some novel measures of similarity and distance for PFSs based on L_p norm and level of uncertainty were given by Peng [52]. By combining the Euclidean distance measure and cosine similarity measures, Mohd and Abdullah [53] developed some novel PF similarity measures. Zhang et al. [54] proposed some exponential PF similarity measures and demonstrated their application in MADM, pattern analysis, and medical diagnosis. Some PF Dice similarity measures with application in decision-making were given by Wang et al. [55]. Verma and Merigo [56] developed some trigonometric function-based PF measures of similarity. The application of some multiparametric PF measures of similarity in classification problems was demonstrated by Peng and Garg [57]. Some novel PF similarity measures based on exponential function with their application in classification problems were given by Nguyen et al. [58]. Some recent studies related to PF distance measures along with their applications are given in [59–64].

The entropy of an FS is the ambiguous content present in it. Entropy measure is very essential for computing the weight of attributes in a MADM problem involving fuzzy data. The concept of fuzzy entropy was suggested by De Luca and Termini [65]. Some axiomatic requirements for a measure to be a fuzzy entropy measure were given by Ebanks [66]. Some more studies concerning fuzzy entropy measures and their utility are given in [67–72]. For IFSs Szmidt and Kacprzyk [73] suggested an entropy measure. Xue et al.[74] introduced the axiomatic definition of PF entropy measure and used the PF entropy measure in decision-making. Some probabilistic and non-probabilistic PF entropy measures were given by Yang and Hussain [75]. With the help of a new PF entropy measure, Thao and Smarandache [76] introduced the COR-PAS MADM method in the PF environment.

Knowledge of an FS is the amount of precision present in it. Knowledge measure (KM) plays a great role in determining the weight of attributes in a MADM problem involving fuzzy data. Singh et al. [77] introduced the axiomatic definition of a fuzzy knowledge measure (FKM) and used it in decision-making. They also proposed a fuzzy accuracy measure and utilized it in image processing. Later on, Singh et al. [78] also introduced a one-parametric generalization of the FKM and discussed its various applications. A two-parametric fuzzy knowledge measure and accuracy measure with their applicability in decision-making and classification problems were given by Singh and Ganie [79]. For IFSs, there are various studies [80–86] regarding the knowledge measures (KMs) with their practical applications. Lin et al. [87] proposed a knowledge measure (KM) for picture fuzzy sets with its utility in decision-making. Some PF KMs with their various applications were introduced by Singh et al. [88]. For hesitant fuzzy sets, Singh and Ganie [89] introduced a generalized KM.

The following are the primary motivating aspects for this research:

- There is no study concerning the generation of the PF distance measures and knowledge measures from t-conorms.
- Axiomatic conditions are not met by several existing PF distance metrics.
- When computing the distance between distinct PFSs, the majority of the existing PF distance metrics produce unacceptable results.
- From the perspective of linguistic hedging, all of the known PF entropy/knowledge measurements are insufficient.

So, keeping the above facts in mind, we suggest some new distance measures and knowledge measures based on PFSs.

The rest of the paper is organized as: Sect. 3 is preliminary. Some novel t-conorm-based PF distance measures along with desirable properties are given in Sect. 4. Section 5 is devoted to the introduction of some distance-based PF knowledge measures. The comparison of the suggested PF distance measures and knowledge measures with the available PF measures of compatibility is shown in Sect. 6. Section 7 demonstrates the applicability of the suggested measures in pattern analysis and MADM. At last, the conclusion and future study are given in Sect. 8.

Preliminaries

Let $W = \{m_1, m_2, \ldots, m_l\}$ be the universe of discourse and $PFS(W)$ be the set of all PFSs of W.

Definition 1 [3] A PFS M_1 in W is given by

$$M_1 = \{(m_j, \mu_{M_1}(m_j), \vartheta_{M_1}(m_j)) | m_j \in W\}$$

With $\mu_{M_1}(m_j)$ and $\vartheta_{M_1}(m_j)$ representing the membership and non-membership grades of the element m_j in M_1 such that $0 \leq \mu_{M_1}(m_j), \vartheta_{M_1}(m_j) \leq 1$ and $0 \leq \mu^2_{M_1}(m_j) + \vartheta^2_{M_1}(m_j) \leq 1$. Also, $\pi_{M_1}(m_j) = \sqrt{1 - \mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)}$ is the hesitancy grade of the element m_j in M_1.

Definition 2 [3] For two PFSs M_1 and M_2 in W, some operations are given as:

$$M_1 \cup M_2 = \left\{ \begin{pmatrix} m_j, max(\mu_{M_1}(m_j), \mu_{M_2}(m_j)), \\ min(\vartheta_{M_1}(m_j), \vartheta_{M_2}(m_j)) \end{pmatrix} \\ |m_j \in W \right\}.$$

$$M_1 \cap M_2 = \left\{ \begin{pmatrix} m_j, min(\mu_{M_1}(m_j), \mu_{M_2}(m_j)), \\ max(\vartheta_{M_1}(m_j), \vartheta_{M_2}(m_j)) \end{pmatrix} \\ |m_j \in W \right\}.$$

$M_1 \subseteq M_2$ iff $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \geq \vartheta_{M_2}(m_j) \forall m_j \in W$.

$$(M_1)^c = \left\{ (m_j, \vartheta_{M_1}(m_j), \mu_{M_1}(m_j)) \\ |m_j \in W \right\}.$$

Definition 4 [90] A function $g : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if $\forall x, y, z, t \in [0, 1]$

$g(x, y) = g(y, x)$;
$g(x, y) \leq g(z, t)$, whenever $x \leq z$ and $y \leq t$;
$g(x, 1) = x$;
$g(x, g(y, z)) = g(g(x, y), z)$.

Definition 5 [90] A function $g : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm if $\forall x, y, z, t \in [0, 1]$

$g(x, y) = g(y, x)$;
$g(x, y) \leq g(z, t)$, whenever $x \leq z$ and $y \leq t$;
$g(x, 0) = x$;
$g(x, g(y, z)) = g(g(x, y), z)$.

Definition 6 [14] A function $S : PFS(W) \times PFS(W) \to [0, 1]$ is called a PF similarity measure if $\forall M_1, M_2$ and $M_3 \in PFS(W)$, we have:

(S1) $0 \leq S(M_1, M_2) \leq 1$;
(S2) $S(M_1, M_2) = S(M_2, M_1)$;
(S3) $S(M_1, M_2) = 1$ iff $M_1 = M_2$;
(S4) $S(M_1, (M_1)^c) = 0$ iff M_1 is a crisp set;
(S5) If $M_1 \subseteq M_2 \subseteq M_3$, then $S(M_1, M_2) \geq S(M_1, M_3)$ and $S(M_2, M_3) \geq S(M_1, M_3)$.

Definition 7 [14] A function $D : PFS(W) \times PFS(W) \to [0, 1]$ is called a PF distance measure if $\forall M_1, M_2$ and $M_3 \in PFS(W)$, we have:

(D1) $0 \leq D(M_1, M_2) \leq 1$;
(D2) $D(M_1, M_2) = D(M_2, M_1)$;
(D3) $D(M_1, M_2) = 0$ iff $M_1 = M_2$;
(D4) $D(M_1, (M_1)^c) = 1$ iff M_1 is a crisp set;
(D5) If $M_1 \subseteq M_2 \subseteq M_3$, then $D(M_1, M_2) \leq D(M_1, M_3)$ and $D(M_2, M_3) \leq D(M_1, M_3)$.

Definition 8 [14] A function $E : PFS(W) \to [0, 1]$ is called a PF entropy measure if $\forall M_1$ and $M_2 \in PFS(W)$, we have:

(E1) $0 \leq E(M_1) \leq 1$;
(E2) $E(M_1) = 0$ iff M_1 is a crisp set;
(E3) $E(M_1) = 1$ iff $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j) \forall m_j \in W$;
(E4) $E(M_1) = E((M_1)^c)$;
(E5) $E(M_1) \leq E(M_2)$ if $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j) \leq \vartheta_{M_2}(m_j) \leq \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \geq \mu_{M_2}(m_j) \geq \vartheta_{M_2}(m_j) \geq \vartheta_{M_1}(m_j) \forall m_j \in W$.

Definition 9 [88] A function $K : PFS(W) \to [0, 1]$ is called a PF knowledge measure if $\forall M_1$ and $M_2 \in PFS(W)$, we have:

(K1) $0 \leq K(M_1) \leq 1$;
(K2) $K(M_1) = 1$ iff M_1 is a crisp set;
(K3) $K(M_1) = 0$ iff $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j) \forall m_j \in W$;
(K4) $K(M_1) = K((M_1)^c)$;
(K5) $K(M_1) \geq K(M_2)$ if $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j) \leq \vartheta_{M_2}(m_j) \leq \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \geq \mu_{M_2}(m_j) \geq \vartheta_{M_2}(m_j) \geq \vartheta_{M_1}(m_j) \forall m_j \in W$.

In the next section, we introduce some novel t-conorm-based distance measures for PFSs along with their properties.

New measures of distance for PFSs

Here, we propose some PF measures of distance based on t-conorms.

Definition 10 Let $M_1, M_2 \in PFS(W)$, then we define a function.

$$D_G : PFS(W) \times PFS(W) \to \mathbb{R}$$

given by

$$D_G(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \mu^2_{M_2}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_2}(m_j)\right|\end{array}\right), \quad (1)$$

where g is a t-conorm.

Theorem 1 The function D_G given in Eq. (1) is a valid PF distance measure.

Proof To prove that D_G is a PF distance measure, we show that it satisfies the properties given in Definition 7.

(D1) Clearly $0 \le D_G(M_1, M_2) \le 1$.
(D2) $D_G(M_1, M_2) = D_G(M_2, M_1)$ is obvious.
(D3) $D_G(M_1, M_2) = 0$

$$\iff g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \mu^2_{M_2}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_2}(m_j)\right|\end{array}\right) = 0\forall j,$$

$\iff \left|\mu^2_{M_1}(m_j) - \mu^2_{M_2}(m_j)\right| = 0$ and $\left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_2}(m_j)\right| = 0\forall j,$

$\iff \mu^2_{M_1}(m_j) = \mu^2_{M_2}(m_j)$ and $\vartheta^2_{M_1}(m_j) = \vartheta^2_{M_2}(m_j)\forall j,$

$$\iff M_1 = M_2.$$

(D4) $D_G\left(M_1, M_1^c\right) = 1$

$$\iff g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \mu^2_{M_1}(m_j)\right|\end{array}\right) = 1\forall j,$$

$\iff \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right| = 1$ and $\left|\vartheta^2_{M_1}(m_j) - \mu^2_{M_1}(m_j)\right| = 1\forall j,$

$$\iff \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right| = 1\forall j,$$

$$\iff \mu^2_{M_1}(m_j) = 1 \text{ and } \vartheta^2_{M_1}(m_j) = 0$$

$$\text{or}\mu^2_{M_1}(m_j) = 0 \text{ and } \vartheta^2_{M_1}(m_j) = 1\forall j,$$

$$\iff \mu_{M_1}(m_j) = 1 \text{ and } \vartheta_{M_1}(m_j) = 0$$

$$\text{or}\mu_{M_1}(m_j) = 0 \text{ and } \vartheta_{M_1}(m_j) = 1\forall j,$$

$\iff M_1$ is a crisp set.

(D5) Let $M_1 \subseteq M_2 \subseteq M_3$, then $\mu^2_{M_1}(m_j) \le \mu^2_{M_2}(m_j) \le \mu^2_{M_3}(m_j)$ and $\vartheta^2_{M_1}(m_j) \ge \vartheta^2_{M_2}(m_j) \ge \vartheta^2_{M_3}(m_j)\forall j$. Therefore, we get

$$\left|\mu^2_{M_1}(m_j) - \mu^2_{M_2}(m_j)\right| \le \left|\mu^2_{M_1}(m_j) - \mu^2_{M_3}(m_j)\right|,$$

$$\left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_2}(m_j)\right| \le \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_3}(m_j)\right|$$

and

$$\left|\mu^2_{M_2}(m_j) - \mu^2_{M_3}(m_j)\right| \le \left|\mu^2_{M_1}(m_j) - \mu^2_{M_3}(m_j)\right|,$$

$$\left|\vartheta^2_{M_2}(m_j) - \vartheta^2_{M_3}(m_j)\right| \le \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_3}(m_j)\right|.$$

So,

$$g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \mu^2_{M_2}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_2}(m_j)\right|\end{array}\right) \le g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \mu^2_{M_3}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_3}(m_j)\right|\end{array}\right)$$

and

$$g\left(\begin{array}{l}\left|\mu^2_{M_2}(m_j) - \mu^2_{M_3}(m_j)\right|,\\ \left|\vartheta^2_{M_2}(m_j) - \vartheta^2_{M_3}(m_j)\right|\end{array}\right) \le g\left(\begin{array}{l}\left|\mu^2_{M_1}(m_j) - \mu^2_{M_3}(m_j)\right|,\\ \left|\vartheta^2_{M_1}(m_j) - \vartheta^2_{M_3}(m_j)\right|\end{array}\right).$$

Thus, $D_G(M_1, M_2) \le D_G(M_1, M_3)$ and $D_G(M_2, M_3) \le D_G(M_1, M_3)$.

Hence, D_G is a valid PF measure of distance.

Theorem 2 The PF measure of distance D_G given in Eq. (1) has the following properties:

1. $D_G\left(M_1^c, M_2^c\right) = D_G(M_1, M_2)\forall M_1, M_2 \in PFS(W)$,
2. $D_G\left(M_1, M_2^c\right) = D_G\left(M_1^c, M_2\right)\forall M_1, M_2 \in PFS(W)$,
3. $D_G\left(M_1, M_1^c\right) = 0$ if and only if $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j)$, $\forall j$,
4. $D_G(M_1 \cap M_2, M_2) \le D_G(M_1, M_2)$ for every $M_1, M_2 \in PFS(W)$,
5. $D_G(M_1 \cup M_2, M_2) \le D_G(M_1, M_2)$ for every $M_1, M_2 \in PFS(W)$.

Proof 1. $D_G\left(M_1^c, M_2^c\right)$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|, \\ \left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|\end{array}\right)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right)$$

$$= D_G(M_1, M_2).$$

2. $D_G\left(M_1, M_2^c\right)$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|\end{array}\right)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\vartheta_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right)$$

$$= D_G\left(M_1^c, M_2\right).$$

3. $D_G\left(M_1, M_1^c\right) = 0$

$$\Longleftrightarrow \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_1}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \mu_{M_1}^2\left(m_j\right)\right|\end{array}\right) = 0,$$

$$\Longleftrightarrow g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_1}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \mu_{M_1}^2\left(m_j\right)\right|\end{array}\right) = 0, \forall j,$$

$$\Longleftrightarrow \left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_1}^2\left(m_j\right)\right| = 0$$

and $\left|\vartheta_{M_1}^2\left(m_j\right) - \mu_{M_1}^2\left(m_j\right)\right| = 0 \,\forall j,$

$$\Longleftrightarrow \left|\mu_{M_1}^2\left(m_j\right) - \vartheta_{M_1}^2\left(m_j\right)\right| = 0, \forall j,$$

$$\Longleftrightarrow \mu_{M_1}^2\left(m_j\right) = \vartheta_{M_1}^2\left(m_j\right), \forall j,$$

$$\Longleftrightarrow \mu_{M_1}\left(m_j\right) = \vartheta_{M_1}\left(m_j\right), \forall j.$$

4. $D_G(M_1 \cap M_2, M_2) =$

$$\frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|min\left(\begin{array}{c}\mu_{M_1}^2\left(m_j\right), \\ \mu_{M_2}^2\left(m_j\right)\end{array}\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|max\left(\begin{array}{c}\vartheta_{M_1}^2\left(m_j\right), \\ \vartheta_{M_2}^2\left(m_j\right)\end{array}\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right)$$

We have the following cases:

(a) When $\mu_{M_1}\left(m_j\right) \geq \mu_{M_2}\left(m_j\right)$ and $\vartheta_{M_1}\left(m_j\right) \geq \vartheta_{M_2}\left(m_j\right) \forall j$, then

$$D_G(M_1 \cap M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_2}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right),$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(0, \left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\right),$$

$$\leq \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right),$$

$$= D_G(M_1, M_2).$$

(b) When $\mu_{M_1}\left(m_j\right) \geq \mu_{M_2}\left(m_j\right)$ and $\vartheta_{M_1}\left(m_j\right) \leq \vartheta_{M_2}\left(m_j\right) \forall j$, then

$$D_G(M_1 \cap M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_2}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_2}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right),$$

$$= \frac{1}{l}\sum_{j=1}^{l} g(0, 0) = 0 \leq S_G(M_1, M_2).$$

(c) When $\mu_{M_1}\left(m_j\right) \leq \mu_{M_2}\left(m_j\right)$ and $\vartheta_{M_1}\left(m_j\right) \geq \vartheta_{M_2}\left(m_j\right) \forall j$, then

$$D_G(M_1 \cap M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_1}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right),$$

$$= D_G(M_1, M_2).$$

(d) When $\mu_{M_1}\left(m_j\right) \leq \mu_{M_2}\left(m_j\right)$ and $\vartheta_{M_1}\left(m_j\right) \leq \vartheta_{M_2}\left(m_j\right) \forall j$, then

$$D_G(M_1 \cap M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\begin{array}{c}\left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, \\ \left|\vartheta_{M_2}^2\left(m_j\right) - \vartheta_{M_2}^2\left(m_j\right)\right|\end{array}\right),$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^2\left(m_j\right) - \mu_{M_2}^2\left(m_j\right)\right|, 0\right),$$

$$\leq \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= D_G(M_1, M_2).$$

5. $D_G(M_1 \cup M_2, M_2)$

$$= \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|max\begin{pmatrix}\mu_{M_1}^2(m_j), \\ \mu_{M_2}^2(m_j)\end{pmatrix} - \mu_{M_2}^2(m_j)\right|, \\ \left|min\begin{pmatrix}\vartheta_{M_1}^2(m_j), \\ \vartheta_{M_2}^2(m_j)\end{pmatrix} - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix}.$$

We have the following cases:

(a) When $\mu_{M_1}(m_j) \geq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \geq \vartheta_{M_2}(m_j)\forall j$, then

$$D_G(M_1 \cup M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_2}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, 0\right),$$

$$\leq \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= D_G(M_1, M_2).$$

(b) When $\mu_{M_1}(m_j) \geq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \leq \vartheta_{M_2}(m_j)\forall j$, then

$$D_G(M_1 \cup M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= D_G(M_1, M_2).$$

(c) When $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \geq \vartheta_{M_2}(m_j)\forall j$, then

$$D_G(M_1 \cup M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_2}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_2}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= \frac{1}{l}\sum_{j=1}^{l} g(0, 0) = 0 \leq S_G(M_1, M_2).$$

(d) When $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \leq \vartheta_{M_2}(m_j)\forall j$, then

$$D_G(M_1 \cup M_2, M_2)$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_2}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= \frac{1}{l}\sum_{j=1}^{l} g\left(0, \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right|\right),$$

$$\leq \frac{1}{l}\sum_{j=1}^{l} g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix},$$

$$= D_G(M_1, M_2).$$

Example 1 Some examples of PF distance measures are given in Table 1.

In most decision-making problems, the weights w_j of the elements m_j, $j = 1, 2, \ldots, l$ are taken into consideration, so we introduce the weighted PF distance measures.

Definition 11 Let $M_1, M_2 \in PFS(W)$, then we define a function

$$D_G^W : PFS(W) \times PFS(W) \to \mathbb{R}$$

given by

$$D_G^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j g\begin{pmatrix} \left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right| \\ \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| \end{pmatrix}, \quad (2)$$

where g is a t-conorm.

Theorem 3 The function D_G^W given in Eq. (2) is a valid PF distance measure.

Proof Similar to Theorem 1.

Example 2 Some examples of weighted PF distance measures are given in Table 2.

Next, we propose some novel PF measures of knowledge based on the proposed PF distance measures.

Table 1 Examples of some t-conorm-based PF distance measures

t-conorms	Corresponding PF distance measures
$g(m_1, m_2) = \frac{m_1+m_2-2m_1m_2}{1-m_1m_2}$	$D_{G1}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\frac{\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\| - 2\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|}{1-\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}\right]$
$g(m_1, m_2) = m_1 + m_2 - m_1m_2$	$D_{G2}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\| - \left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|\right]$
$g(m_1, m_2) = min(1, m_1 + m_2)$	$D_{G3}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} min\left(1, \left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|\right)$
$g(m_1, m_2) = \frac{m_1+m_2}{1\mp m_2}$	$D_{G4}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\frac{\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}{1+\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}\right]$

Table 2 Weighted distance measures for PFSs

t-conorms	Corresponding weighted PF distance measures
$g(m_1, m_2) = \frac{m_1+m_2-2m_1m_2}{1-m_1m_2}$	$D_{G1}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\frac{\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\| - 2\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|}{1-\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}\right]$
$g(m_1, m_2) = m_1 + m_2 - m_1m_2$	$D_{G2}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\| - \left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|\right]$
$g(m_1, m_2) = min(1, m_1 + m_2)$	$D_{G3}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j min\left(1, \left\|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right\| + \left\|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right\|\right)$
$g(m_1, m_2) = \frac{m_1+m_2}{1\mp m_2}$	$D_{G4}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\frac{\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}{1+\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|}\right]$

PF distance-based knowledge measures

The entropy measures are used to compute the amount of ambiguity present in a PFS, whereas the knowledge measures acting as the soft duals of entropy measures are used to calculate the amount of precision in a PFS. Here, we introduce a method of constructing PF knowledge measures from the PF distance measures.

Definition 12 Let $M_1 \in PFS(W)$, then we define a function

$$K_G : PFS(W) \to [0,\ 1]$$

given by

$$K_G(M_1) = 1 - D_G\left(M_1,\ M_1^c\right) \tag{3}$$

where D_G is a PF distance measure.

Theorem 4 The function K_G defined in Eq. (3) is a valid PF knowledge measure.

Proof To show that the function K_G is a PF measure of knowledge, we show it has the properties of a PF measure of knowledge given in Definition 9.

(K1) Clearly $0 \le K_G(M_1) \le 1$ as $0 \le D_G\left(M_1,\ M_1^c\right)$

Table 3 Some suggested PF knowledge measures

Proposed PF distance measures	Corresponding PF knowledge measures
D_{G1}	$K_{G1}(M_1)=\frac{1}{l}\sum_{j=1}^{l}\frac{2\left(\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert-\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert^2\right)}{1-\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert^2}$
D_{G2}	$K_{G2}(M_1)=\frac{1}{l}\sum_{j=1}^{l}2\left(\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert-\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert^2\right)$
D_{G3}	$K_{G3}(M_1)=\frac{1}{l}\sum_{j=1}^{l}min\left(1,\ 2\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert\right)$
D_{G4}	$K_{G4}(M_1)=\frac{1}{l}\sum_{j=1}^{l}\frac{2\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert}{1+\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert^2}$

≤ 1.

(K2) $K_G(M_1)=1 \iff D_G(M_1, M_1^c)=0 \iff M_1$ is a crisp set.

(K3)

$$K_G(M_1)=0 \iff D_G(M_1, M_1^c)=1 \iff \mu_{M_1}(m_j) = \vartheta_{M_1}(m_j),\ \forall j.$$

(K4) $K_G(M_1^c)=K_G(M_1)$ is obvious.

(K5) Let M_1 be less fuzzy than M_2 i.e., $\mu_{M_1}(m_j)\leq\mu_{M_2}(m_j)\leq\vartheta_{M_2}(m_j)\leq\vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j)\geq\mu_{M_2}(m_j)\geq\vartheta_{M_2}(m_j)\geq\vartheta_{M_1}(m_j)$.

When $\mu_{M_1}(m_j)\leq\mu_{M_2}(m_j)\leq\vartheta_{M_2}(m_j)\leq\vartheta_{M_1}(m_j)$, then we get

$$\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert\geq\left\lvert\mu_{M_2}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\rvert.$$

So, $D_G(M_1, M_1^c)$

$$=\frac{1}{l}\sum_{j=1}^{l}g\left(\begin{array}{l}\left\lvert\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right\rvert,\\ \left\lvert\vartheta_{M_1}^2(m_j)-\mu_{M_1}^2(m_j)\right\rvert\end{array}\right),$$
$$\geq\frac{1}{l}\sum_{j=1}^{l}g\left(\begin{array}{l}\left\lvert\mu_{M_2}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\rvert,\\ \left\lvert\vartheta_{M_2}^2(m_j)-\mu_{M_2}^2(m_j)\right\rvert\end{array}\right),$$
$$=D_G(M_2, M_2^c).$$

Thus, $K_G(M_1)\geq K_G(M_2)$.

Similarly, when $\mu_{M_1}(m_j)\geq\mu_{M_2}(m_j)\geq\vartheta_{M_2}(m_j)\geq\vartheta_{M_1}(m_j)$, we get $K_G(M_1)\geq K_G(M_2)$. Hence the function K_G given in Eq. (3) is a valid PF knowledge measure.

With the help of Eq. (3) and based on the suggested PF measures of distance, some PF measures of knowledge are given in Table 3 below:

Now, we compare the suggested PF measures of distance and knowledge with some available PF measures of information.

Comparative analysis

In this section, we show that our suggested PF measures of distance and knowledge give better results than most of the available PF measures of information.

Comparison of the proposed PF distance measures with various available PF measures of similarity/distance

To contrast the performance of the suggested PF measures of distance, we first list the PF measures of similarity/distance available in the literature as shown in Table 4.

Now we compare the suggested measures with the existing ones through some numerical examples related to the computation of the distance/similarity between different PFSs.

Example 3 Consider three different cases of PFSs with each case consisting of two different PFSs as shown below.

$$\text{Case I: }\begin{cases}M_1=\{(m_1, 0.5, 0.5)\}\\ M_2=\{(m_1, 0.0, 0.0)\}\end{cases}$$
$$\text{Case II: }\begin{cases}M_1=\{(m_1, 0.4, 0.3)\}\\ M_2=\{(m_1, 0.5, 0.3)\}\end{cases}$$
$$\text{Case III: }\begin{cases}M_1=\{(m_1, 0.4, 0.3)\}\\ M_2=\{(m_1, 0.5, 0.2)\}\end{cases}$$

The computed distance/similarity values for these three cases using the available measures of distance/similarity along with the suggested ones are shown in Table 5.

From Table 5, we have

1. The PF distance measures D_{PYY1}, D_{PYY4}, D_{PYY5}, D_{PYY6}, D_{PYY9}, and D_{PYY10} gives the same distance for the two distinct cases (Case II and Case III).
2. The PF distance measure D_{PYY2} gives "0" as the distance between the two different PFSs (Case I) and thus fails to satisfy the axiom (D3) of the PF distance measure given in Definition 7.

Table 4 Existing PF distance/similarity measures due to Peng et al. [14]

Distance/similarity measure	Expression
$D_{PYY1}(M_1, M_2)$	$\frac{1}{2l}\sum_{j=1}^{l}\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|+\left\|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right\|\right)$
$D_{PYY2}(M_1, M_2)$	$\frac{1}{2l}\sum_{j=1}^{l}\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right\|\right)$
$D_{PYY3}(M_1, M_2)$	$\frac{1}{4l}\left\{\begin{array}{c}\sum_{j=1}^{l}\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|+\left\|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right\| \\ +\sum_{j=1}^{l}\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right\|\end{array}\right\}$
$D_{PYY4}(M_1, M_2)$	$\frac{1}{l}\sum_{j=1}^{l} max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\ \left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)$
$D_{PYY5}(M_1, M_2)$	$\frac{2}{l}\sum_{j=1}^{l}\frac{max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}{1+max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}$
$D_{PYY6}(M_1, M_2)$	$\frac{2\sum_{j=1}^{l} max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}{\sum_{j=1}^{l} 1+max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}$
$D_{PYY7}(M_1, M_2)$	$1-x\frac{\sum_{j=1}^{l} min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-y\frac{\sum_{j=1}^{l} min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}, x+y=1,\ x,\ y\in[0,\ 1]$
$D_{PYY8}(M_1, M_2)$	$1-\frac{x}{l}\frac{\sum_{j=1}^{l} min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-\frac{y}{l}\frac{\sum_{j=1}^{l} min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}, x+y=1,\ x,\ y\in[0,\ 1]$
$D_{PYY9}(M_1, M_2)$	$1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$
$D_{PYY10}(M_1, M_2)$	$1-\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$
$D_{PYY11}(M_1, M_2)$	$1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$
$D_{PYY12}(M_1, M_2)$	$1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$
$S_{PYY1}(M_1, M_2)$	$1-\frac{1}{2l}\sum_{j=1}^{l}\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|+\left\|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right\|\right)$
$S_{PYY2}(M_1, M_2)$	$1-\frac{1}{2l}\sum_{j=1}^{l}\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right\|\right)$
$S_{PYY3}(M_1, M_2)$	$1-\frac{1}{4l}\left\{\begin{array}{c}\sum_{j=1}^{l}\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|+\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|+\left\|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right\|\right) \\ +\sum_{j=1}^{l}\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right\|\end{array}\right\}$
$S_{PYY4}(M_1, M_2)$	$1-\frac{1}{l}\sum_{j=1}^{l} max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\ \left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)$
$S_{PYY5}(M_1, M_2)$	$\frac{1}{l}\sum_{j=1}^{l}\frac{1-max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}{1+max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}$
$S_{PYY6}(M_1, M_2)$	$\frac{\sum_{j=1}^{l} 1-max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}{\sum_{j=1}^{l} 1+max\left(\left\|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right\|,\left\|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right\|\right)}$
$S_{PYY7}(M_1, M_2)$	$x\frac{\sum_{j=1}^{l} min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}+y\frac{\sum_{j=1}^{l} min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}, x+y=1,\ x,\ y\in[0,\ 1]$
$S_{PYY8}(M_1, M_2)$	$\frac{x}{l}\frac{\sum_{j=1}^{l} min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-\frac{y}{l}\frac{\sum_{j=1}^{l} min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l} max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}, x+y=1,\ x,\ y\in[0,\ 1]$

Table 4 (continued)

Distance/similarity measure	Expression
$S_{PYY9}(M_1, M_2)$	$\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$
$S_{PYY10}(M_1, M_2)$	$\frac{\sum_{j=1}^{l}min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$
$S_{PYY11}(M_1, M_2)$	$\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$
$S_{PYY12}(M_1, M_2)$	$\frac{\sum_{j=1}^{l}min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{\sum_{j=1}^{l}max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$

Table 5 Computed values of various PF distance/similarity measures regarding Example 3

	Case I	Case II	Case III
D_{PYY1}	0.50000	**0.0900**	**0.0900**
D_{PYY2}	**0**	0.0450	0.0700
D_{PYY3}	0.2500	0.1350	0.1850
D_{PYY4}	0.2500	**0.0900**	**0.0900**
D_{PYY5}	0.4000	**0.1651**	**0.1651**
D_{PYY6}	0.4000	**0.1651**	**0.1651**
D_{PYY7}	**1.0000**	0.1080	0.4969
D_{PYY8}	**1.0000**	0.1080	0.4969
D_{PYY9}	**1.0000**	**0.3600**	**0.3600**
D_{PYY10}	**1.0000**	**0.3600**	**0.3600**
D_{PYY11}	0.4000	0.0776	0.1157
D_{PYY12}	0.4000	0.0776	0.1157
S_{PYY1}	0.5000	**0.9100**	**0.9100**
S_{PYY2}	**1.0000**	0.9550	0.9300
S_{PYY3}	0.7500	0.8650	0.8150
S_{PYY4}	0.7500	**0.9100**	**0.9100**
S_{PYY5}	0.6000	**0.8349**	**0.8349**
S_{PYY6}	0.6000	**0.8349**	**0.8349**
S_{PYY7}	**0**	**– 0.5080**	**– 0.1191**
S_{PYY8}	**0**	**– 0.5080**	**– 0.1191**
S_{PYY9}	**0**	**0.6400**	**0.6400**
S_{PYY10}	**0**	**0.6400**	**0.6400**
S_{PYY11}	0.6000	0.9224	0.8843
S_{PYY12}	0.6000	0.9224	0.8843
D_{G1}(Proposed)	0.4000	0.0900	0.1316
D_{G2}(Proposed)	0.4375	0.0900	0.1355
D_{G3}(Proposed)	0.5000	0.0900	0.1400
D_{G4}(Proposed)	0.4706	0.0900	0.1394

Bold values indicate unreasonable results. $x = 0.2$ and $y = 0.8$ in D_{PYY7}, D_{PYY8}, S_{PYY7}, and S_{PYY8}

3. The PF distance measures D_{PYY7}, D_{PYY8}, D_{PYY9}, and D_{PYY10} gives "1" as the distance between the two different PFSs (Case I) although they are not a complement of each other.
4. The PF similarity measures S_{PYY1}, S_{PYY4}, S_{PYY5}, S_{PYY6}, S_{PYY9}, andS_{PYY10} give the same degree of similarity for the two distinct cases (Case II and Case III).
5. The PF similarity measure S_{PYY2} gives "1" as a similarity degree for the two different PFSs (Case I) and thus fails to satisfy the axiom (S3) of the PF measure of similarity given in Definition 6.
6. The PF similarity measures S_{PYY7}, S_{PYY8}, S_{PYY9}, and S_{PYY10} gives "0" as the similarity degree for the two different PFSs (Case I) although they are not a complement of each other.
7. The similarity degree of the different PFSs (Case II and III) by the similarity measures S_{PYY7} and S_{PYY8} comes out to be negative, which is unreasonable.
8. The proposed PF distance measures D_{Gj}, $1 \leq j \leq 4$ outperforms the majority of the available PF measures of distance/similarity.

Example 4 Consider six different cases of PFSs with each case consisting of two different PFSs as shown below.

$$\text{Case I: }\begin{cases} M_1 = \{(m_1, 0.4, 0.2)\} \\ M_2 = \{(m_1, 0.5, 0.2)\} \end{cases}$$

$$\text{Case II: }\begin{cases} M_1 = \{(m_1, 0.4, 0.2)\} \\ M_2 = \{(m_1, 0.5, 0.1)\} \end{cases}$$

$$\text{Case III: }\begin{cases} M_1 = \{(m_1, 0.5, 0.5)\} \\ M_2 = \{(m_1, 0, 0)\} \end{cases}$$

$$\text{Case IV: }\begin{cases} M_1 = \{(m_1, 1, 0)\} \\ M_2 = \{(m_1, 0, 0)\} \end{cases}$$

$$\text{Case V: }\begin{cases} M_1 = \{(m_1, 0.3, 0.4)\} \\ M_2 = \{(m_1, 0.4, 0.3)\} \end{cases}$$

Table 6 Computed values of various PF distance/similarity measures regarding Example 4

	Case I	Case II	Case III	Case IV	Case V	Case VI
D_{PYY1}	**0.0900**	**0.0900**	0.5000	1	0.0700	0.8200
D_{PYY2}	0.0450	0.0600	**0**	0.5000	0.0700	0.4000
D_{PYY3}	0.1350	0.1650	0.2500	**1.5000**	0.1750	**1.2100**
D_{PYY4}	**0.0900**	**0.0900**	0.2500	1	0.0700	0.8100
D_{PYY5}	**0.1651**	**0.1651**	0.4000	1	0.1308	0.8950
D_{PYY6}	**0.1651**	**0.1651**	0.4000	1	0.1308	0.8950
D_{PYY7}	0.1080	0.6330	**1**	**NaN**	0.4375	**1**
D_{PYY8}	0.1080	0.6330	**1**	**NaN**	0.4375	**1**
D_{PYY9}	**0.3600**	**0.3600**	**1**	1	0.4375	**1**
D_{PYY10}	**0.3600**	**0.3600**	**1**	1	0.4375	**1**
D_{PYY11}	0.0744	0.0968	0.4000	0.5000	0.1308	0.8119
D_{PYY12}	0.0744	0.0968	0.4000	0.5000	0.1308	0.8119
S_{PYY1}	**0.9100**	**0.9100**	0.5000	0	0.9300	0.1800
S_{PYY2}	0.9550	0.9400	**1**	0.5000	0.9300	0.6000
S_{PYY3}	0.8650	0.8350	0.7500	**– 0.5000**	0.8250	**– 0.2100**
S_{PYY4}	**0.9100**	**0.9100**	0.7500	0	0.9300	0.1900
S_{PYY5}	**0.8349**	**0.8349**	0.6000	0	0.8692	0.1050
S_{PYY6}	**0.8349**	**0.8349**	0.6000	0	0.8692	0.1050
S_{PYY7}	**– 0.5080**	0.0170	**0**	**NaN**	**– 0.2250**	**0**
S_{PYY8}	**– 0.5080**	0.0170	**0**	**NaN**	**– 0.2250**	**0**
S_{PYY9}	**0.6400**	**0.6400**	**0**	0	0.5625	**0**
S_{PYY10}	**0.6400**	**0.6400**	**0**	0	0.5625	**0**
S_{PYY11}	0.9256	0.9032	0.6000	0.5000	0.8692	0.1881
S_{PYY12}	0.9256	0.9032	0.6000	0.5000	0.8692	0.1881
D_{G1}(Proposed)	0.0900	0.1149	0.4000	1	0.1308	0.8104
D_{G2}(Proposed)	0.0900	0.1173	0.4375	1	0.1351	0.8119
D_{G3}(Proposed)	0.0900	0.1200	0.5000	1	0.1400	0.1700
D_{G4}(Proposed)	0.0900	0.1197	0.4706	1	0.1393	0.8134

Bold values indicate unreasonable results. NaN means cannot be calculated. $x = 0.2$ and $y = 0.8$ in D_{PYY7}, D_{PYY8}, S_{PYY7}, and S_{PYY8}

Case VI: $\begin{cases} M_1 = \{(m_1, 0.1, 0.9)\} \\ M_2 = \{(m_1, 0, 0)\} \end{cases}$

The computed distance/similarity values for these six cases using the available measures of distance/similarity along with the suggested ones are shown in Table 6.

From Table 6, we have the following:

1. The PF distance measures D_{PYY1}, D_{PYY4},

D_{PYY5}, D_{PYY6}, D_{PYY9}, and D_{PYY10} give the same distance for two distinct cases (Case I and Case II).

2. The PF distance measure D_{PYY2} gives "0" as the distance between the two unequal PFSs (Case III).
3. The PF distance measures D_{PYY7}, D_{PYY8},

D_{PYY9}, and D_{PYY10} gives "1" as the distance between two PFSs M_1 and M_2 (Case III and Case VI) when neither M_1 is a crisp set nor $M_1 = M_2$. So, they fail to satisfy the axiom (D4) of Definition 7.

4. The PF distance measure D_{PYY3} indicates that the distance between the PFSs (Case IV and Case VI) is greater than "1" and therefore does not follow the axiom (D1) of definition 7.
5. The PF distance measures D_{PYY7} and D_{PYY8} fail to compute the distance between the two PFSs (Case IV).
6. The PF similarity measures S_{PYY1}, S_{PYY4},

S_{PYY5}, S_{PYY6}, S_{PYY9} and S_{PYY10} give the same similarity for two distinct cases (Case I and Case II).

7. The PF similarity measure S_{PYY2} gives "1" as the similarity between the two unequal PFSs (Case III).
8. The similarity between the different PFSs comes out to be negative (Case I, Case IV, Case V, and Case V) by the PF similarity measures S_{PYY3}, S_{PYY7}, and S_{PYY8}. So, these similarity measures fail to satisfy the axiom (S1) of Definition 6.
9. The PF similarity measures S_{PYY7} and S_{PYY8} fails to compute the similarity between the PFSs (Case IV).
10. The PF similarity measures S_{PYY7}, S_{PYY8},

S_{PYY9}, and S_{PYY10} gives "0" as the similarity between the PFSs M_1 and M_2 (Case III and Case VI), when neither $M_2 = {M_1}^c$ nor M_1 is a crisp set.

11. The suggested PF distance measures D_{Gj}, $1 \leq j \leq 4$ computes the distance of all the PFSs without any counterintuitive results.

Thus, from Examples 3 and 4, we conclude that the suggested distance measures are more robust and effective than most of the available distance/similarity measures in PF theory.

Next, we compare the suggested PF knowledge measures with the available PF measures of entropy/knowledge.

Comparison of the suggested PF measures of knowledge with the available PF measures of entropy/knowledge

To contrast the performance of the newly introduced PF measures of knowledge, we first list the PF entropy/knowledge measures available in the literature.

Entropy measures due to Peng et al. [14]

$$E_{PYY1}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\frac{\pi_{M_1}^2(m_j)+1-\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|}{\pi_{M_1}^2(m_j)+1+\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|};$$

$$E_{PYY2}(M_1) = \frac{1}{l}\frac{\sum_{j=1}^{l}\left(1-\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|\right)}{\sum_{j=1}^{l}\left(1+\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|\right)};$$

$$E_{PYY3}(M_1) = 1-\frac{1}{l}\sum_{j=1}^{l}\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|;$$

$$E_{PYY4}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\,\vartheta_{M_1}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\,\vartheta_{M_1}^2(m_j)\right)};$$

$$E_{PYY5}(M_1) = \frac{1}{\left(\sqrt{2}-1\right)l}\sum_{j=1}^{l}\left(\begin{array}{c} sin\frac{1+\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)}{4}\pi \\ +sin\frac{1-\mu_{M_1}^2(m_j)+\vartheta_{M_1}^2(m_j)}{4}\pi - 1\end{array}\right);$$

$$E_{PYY6}(M_1) = \frac{1}{\left(\sqrt{2}-1\right)l}\sum_{j=1}^{l}\left(\begin{array}{c} cos\frac{1+\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)}{4}\pi \\ +cos\frac{1-\mu_{M_1}^2(m_j)+\vartheta_{M_1}^2(m_j)}{4}\pi - 1\end{array}\right);$$

$$E_{PYY7}(M_1) = \frac{1}{l}\sum_{j=1}^{l}cot\left(\frac{\pi}{4}+\frac{\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|}{4\left(1+\pi_{M_1}^2(m_j)\right)}\pi\right)$$

$$E_{PYY8}(M_1) = \frac{1}{l}\sum_{j=1}^{l}tan\left(\frac{\pi}{4}-\frac{\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|}{4\left(1+\pi_{M_1}^2(m_j)\right)}\pi\right)$$

Entropy measure due to Xue et al. [74]

$$E_{XXZT}(M_1) = \frac{1}{l}\sum_{j=1}^{t}\left[\begin{array}{c}1-\left(\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right)\\ \times\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|\end{array}\right].$$

Entropy measure due to Thao and Smarandache [76]

$$E_{TS}(M_1) = \frac{1}{l}\sum_{j=1}^{t}\left[\begin{array}{c}1-\left|\mu_{M_1}^2(m_j)-\frac{1}{3}\right|\\ -\left|\vartheta_{M_1}^2(m_j)-\frac{1}{3}\right|\end{array}\right].$$

Entropy measure due to Yang and Hussain [75]

$$E_{YH}(M_1) = 1-\sqrt{\frac{1}{l}\sum_{j=1}^{t}\left(\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right)^2}.$$

Knowledge measures due to Singh et al. [88]

$$K_{SSG1}(M_1) = \sqrt{\frac{1}{l}\sum_{j=1}^{t}\left(\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right)^2};$$

$$K_{SSG2}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\left|\mu_{M_1}^2(m_j)-\vartheta_{M_1}^2(m_j)\right|;$$

$$K_{SSG3}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\frac{2\mu_{M_1}^2(m_j)\vartheta_{M_1}^2(m_j)}{\mu_{M_1}^4(m_j)+\vartheta_{M_1}^4(m_j)}.$$

Now, using linguistic hedges, we show the effectiveness of the suggested PF measures of knowledge.

Definition 13 [75] For any $M_1 \in PFS(W)$, its modifier $(M_1)^\lambda$, $\lambda > 0$ is defined as

$$(M_1)^\lambda = \left\{ \left(\begin{array}{c} m_j, \left(\mu_{M_1}(m_j)\right)^\lambda, \\ \left(1 - \left(1 - \vartheta_{M_1}^2(m_j)\right)^\lambda\right)^{\frac{1}{2}} \end{array} \right) \middle| m_j \in W \right\}.$$

Then, we have the following PFSs:

M_1: LARGE; $(M_1)^2$: very LARGE; $(M_1)^3$: quite very LARGE; $(M_1)^4$: very very LARGE; $(M_1)^{\frac{1}{2}}$: more or less LARGE.

Since a PF entropy measure, E computes the ambiguous content in a PFS, it has to satisfy the following requirement:

$$E\left((M_1)^{\frac{1}{2}}\right) > E(M_1) > E\left((M_1)^2\right) > E\left((M_1)^3\right) > E\left((M_1)^4\right). \quad (4)$$

Also, a PF knowledge measure K acts as a soft dual of a PF entropy measure and calculates the amount of precision in a PFS, so it has to satisfy the following requirement:

$$K\left((M_1)^{\frac{1}{2}}\right) < K(M_1) < K\left((M_1)^2\right) < K\left((M_1)^3\right) < K\left((M_1)^4\right). \quad (5)$$

We now consider an example related to the ambiguous computation of the above-mentioned PFSs.

Example 5 Let $M_1 \in PFS(W)$ be given as:

$$M_1 = \left\{ \begin{array}{c} (m_1, 0.35, 0.47), (m_2, 0.45, 0.72), \\ (m_3, 0.21, 0.60), (m_4, 0.80, 35), \\ (m_5, 0.48, 0.56) \end{array} \right\}.$$

With the help of Definition 13, we construct the following PFSs:

$$(M_1)^{\frac{1}{2}} = \left\{ \begin{array}{c} (m_1, 0.5916, 0.3425), (m_2, 0.6708, 0.5532), \\ (m_3, 0.4583, 0.4472), (m_4, 0.8944, 0.2515), \\ (m_5, 0.6928, 0.4141) \end{array} \right\}.$$

$$(M_1)^2 = \left\{ \begin{array}{c} (m_1, 0.1225, 0.6269), (m_2, 0.2025, 0.8764), \\ (m_3, 0.0441, 0.7684), (m_4, 0.6400, 0.4796), \\ (m_5, 0.2304, 0.7272) \end{array} \right\}.$$

$$(M_1)^3 = \left\{ \begin{array}{c} (m_1, 0.0429, 0.7260), (m_2, 0.0911, 0.9425), \\ (m_3, 0.0093, 0.8590), (m_4, 0.5120, 0.5695), \\ (m_5, 0.1106, 0.8226) \end{array} \right\}.$$

$$(M_1)^4 = \left\{ \begin{array}{c} (m_1, 0.0150, 0.7947), (m_2, 0.0410, 0.9727), \\ (m_3, 0.0019, 0.9123), (m_4, 0.4096, 0.6380), \\ (m_5, 0.0531, 0.8821) \end{array} \right\}.$$

The ambiguous content of these PFSs using the suggested PF knowledge measures and the existing ones is shown in Table 7.

From Table 7, we have the following:

$$E_{PYY1}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY1}(M_1) > E_{PYY1}\left((M_1)^2\right) > E_{PYY1}\left((M_1)^3\right) > E_{PYY1}\left((M_1)^4\right);$$

$$E_{PYY2}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY2}(M_1) > E_{PYY2}\left((M_1)^2\right) > E_{PYY2}\left((M_1)^3\right) > E_{PYY2}\left((M_1)^4\right);$$

$$E_{PYY3}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY3}(M_1) > E_{PYY3}\left((M_1)^2\right) > E_{PYY3}\left((M_1)^3\right) > E_{PYY3}\left((M_1)^4\right);$$

$$E_{PYY4}\left((M_1)^{\frac{1}{2}}\right) > E_{PYY4}(M_1) > E_{PYY4}\left((M_1)^2\right) < E_{PYY4}\left((M_1)^3\right) < E_{PYY4}\left((M_1)^4\right);$$

$$E_{PYY5}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY5}(M_1) > E_{PYY5}\left((M_1)^2\right) > E_{PYY5}\left((M_1)^3\right) > E_{PYY5}\left((M_1)^4\right);$$

$$E_{PYY6}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY6}(M_1) > E_{PYY6}\left((M_1)^2\right) > E_{PYY6}\left((M_1)^3\right) > E_{PYY6}\left((M_1)^4\right);$$

$$E_{PYY7}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY7}(M_1) > E_{PYY7}\left((M_1)^2\right) > E_{PYY7}\left((M_1)^3\right) > E_{PYY7}\left((M_1)^4\right);$$

$$E_{PYY8}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY8}(M_1) > E_{PYY8}\left((M_1)^2\right) > E_{PYY8}\left((M_1)^3\right) > E_{PYY8}\left((M_1)^4\right);$$

$$E_{XXZT}\left((M_1)^{\frac{1}{2}}\right) < E_{XXZT}(M_1) < E_{XXZT}\left((M_1)^2\right) < E_{XXZT}\left((M_1)^3\right) < E_{XXZT}\left((M_1)^4\right);$$

$$E_{TS}\left((M_1)^{\frac{1}{2}}\right) < E_{TS}(M_1) > E_{TS}\left((M_1)^2\right) > E_{TS}\left((M_1)^3\right) > E_{TS}\left((M_1)^4\right);$$

$$E_{YH}\left((M_1)^{\frac{1}{2}}\right) < E_{YH}(M_1) > E_{YH}\left((M_1)^2\right) > E_{YH}\left((M_1)^3\right) > E_{YH}\left((M_1)^4\right);$$

$$K_{SSG1}\left((M_1)^{\frac{1}{2}}\right) > K_{SSG1}(M_1) < K_{SSG1}\left((M_1)^2\right) < K_{SSG1}\left((M_1)^3\right) < K_{SSG1}\left((M_1)^4\right);$$

$$K_{SSG2}\left((M_1)^{\frac{1}{2}}\right) > K_{SSG2}(M_1) < K_{SSG2}\left((M_1)^2\right) < K_{SSG2}\left((M_1)^3\right) < K_{SSG2}\left((M_1)^4\right)$$

$$K_{SSG3}\left((M_1)^{\frac{1}{2}}\right) > K_{SSG3}(M_1) < K_{SSG3}\left((M_1)^2\right) < K_{SSG3}$$

Table 7 Values of the PF measures of entropy/knowledge regarding Example 5

	$(M_1)^{\frac{1}{2}}$	M_1	$(M_1)^2$	$(M_1)^3$	$(M_1)^4$
E_{PYY1}	0.6715	0.6927	0.5066	0.4183	0.3050
E_{PYY2}	0.5548	0.5796	0.3608	0.2707	0.1873
E_{PYY3}	0.7136	0.7338	0.5302	0.4261	0.3155
E_{PYY4}	0.4808	0.3988	0.1514	0.1679	0.0836
E_{PYY5}	1.3220	1.3273	1.3088	1.2909	1.2769
E_{PYY6}	12.2178	12.2388	12.1662	12.0964	12.0414
E_{PYY7}	0.7171	0.7434	0.5753	0.4786	0.3640
E_{PYY8}	0.7171	0.7434	0.5753	0.4786	0.3640
E_{XXZT}	0.8574	0.9897	1.2424	1.4080	1.5283
E_{TS}	0.6643	0.6810	0.5302	0.4225	0.3155
E_{YH}	0.6224	0.6889	0.4948	0.3613	0.2732
K_{SSG1}	0.3776	0.3111	0.5052	0.6387	0.7268
K_{SSG2}	0.2864	0.2662	0.4698	0.5739	0.6845
K_{SSG3}	0.3355	0.3818	0.7516	0.7920	0.8568
K_{G1} (Proposed)	0.3938	0.3950	0.6162	0.6779	0.7827
K_{G2} (Proposed)	0.4302	0.4356	0.6843	0.7399	0.8407
K_{G3} (Proposed)	0.4781	0.5254	0.8133	0.8249	0.8957
K_{G4}(Proposed)	0.4524	0.4651	0.7222	0.7633	0.8611

$$\left((M_1)^3\right) < K_{SSG3}\left((M_1)^4\right);$$

$$K_{G1}\left((M_1)^{\frac{1}{2}}\right) < K_{G1}(M_1) < K_{G1}\left((M_1)^2\right) < K_{G1}\left((M_1)^3\right) < K_{G1}\left((M_1)^4\right);$$

$$K_{G2}\left((M_1)^{\frac{1}{2}}\right) < K_{G2}(M_1) < K_{G2}\left((M_1)^2\right) < K_{G2}\left((M_1)^3\right) < K_{G2}\left((M_1)^4\right);$$

$$K_{G3}\left((M_1)^{\frac{1}{2}}\right) < K_{G3}(M_1) < K_{G3}\left((M_1)^2\right) < K_{G3}\left((M_1)^3\right) < K_{G3}\left((M_1)^4\right);$$

$$K_{G4}\left((M_1)^{\frac{1}{2}}\right) < K_{G4}(M_1) < K_{G4}\left((M_1)^2\right) < K_{G4}\left((M_1)^3\right) < K_{G4}\left((M_1)^4\right).$$

Thus, it follows that all the available PF measures of entropy E_{PLj}, $1 \le j \le 8$, E_{XXZT}, E_{TS}, E_{YH}, and the PF knowledge measures K_{SSGj}, $j = 1, 2, 3$ do not satisfy the requirements given in Eqs. (4) and (5) respectively. However, all our suggested PF knowledge measures K_{Gj}, $j = 1, 2, 3, 4$, follow the desired requirement given in Eq. (5). This shows that from a linguistic hedge perspective, the suggested measures of knowledge are more robust than the available ones.

Now, we demonstrate the utility of the proposed PF distance and knowledge measures in pattern identification and decision-making.

Application of the proposed measures

In this section, we show how the suggested metrics can be used in pattern analysis and MCDM.

Pattern analysis

We demonstrate how the suggested PF distance metrics can be employed to solve pattern classification problems. An unfamiliar pattern is classed into one of the known patterns using compatibility measurements such as similarity measures, distance measures, correlation measures, and so on in pattern analysis. We also compare our findings to the existing compatibility measures.

Now, we solve some problems related to pattern analysis in the examples given below.

Example 6 (Nanometer material classification) The current nanometer materials collection $M = \{M_1, M_2, M_3\}$ which stands for nanometer-ceramics, nanometer-film, and nanometer-fiber respectively. The following collection of parameters primarily describes the form features of the three-nanometer materials: $W = \{m_1(\text{odour}), m_2(\text{layer}), m_3(\text{color})\}$.

The following are the standard model data for the form properties of the three-nanometer materials:

$$M_1 = \{(m_1, 0.7, 0.2), (m_2, 0.1, 0.8), (m_3, 0.4, 0.4)\},$$

$$M_2 = \{(m_1, 0.5, 0.5), (m_2, 0.7, 0.3), (m_3, 0, 0.8)\},$$

$$M_3 = \{(m_1, 0.1, 0.1), (m_2, 0.5, 0.1), (m_3, 0.1, 0.9)\}.$$

There is a nanometer material N that needs to be recognized in the following way:

$$N = \{(m_1, 0.4, 0.4), (m_2, 0.6, 0.2), (m_3, 0, 0.8)\}.$$

We need to find out the nanometer material that N belongs to. The similarity/distance between N and M_j, $j = 1, 2, 3$ by various PF similarity/distance measures are shown in Table 8.

From Table 8, we see that the unknown nanometer is assigned to the pattern M_2 as shown by most of the PF distance/similarity measures. We also observe that the PF distance/similarity measures D_{PYY8}, D_{PYY9}, S_{PYY8}, and S_{PYY9} fail to recognize the unknown nanometer N.

Example 7 (Bacterial detection) $M = \{M_1, M_2, M_3\}$ represents Salmonella, Shigella, and *Escherichia coli*, respectively, in the existing bacterial collection. The following set of numbers best describes the shape features of the three gut bacteria:

$$W = \left\{ \begin{array}{c} m_1(\text{large belly small morphology}), \\ m_2(\text{double micromorphology}), \\ m_3(\text{single micromorphology}), \\ m_4(\text{round head shape}) \end{array} \right\}.$$

The following are the typical model data for the shape features of the three gut bacteria:

$$M_1 = \left\{ \begin{array}{l} (m_1, 0.5, 0.4), (m_2, 0.4, 0.5), \\ (m_3, 0.3, 0.3), (m_4, 0.2, 0.2) \end{array} \right\},$$

$$M_2 = \left\{ \begin{array}{l} (m_1, 0.5, 0.5), (m_2, 0.1, 0.1), \\ (m_3, 0.5, 0.5), (m_4, 0.1, 0.1) \end{array} \right\},$$

$$M_3 = \left\{ \begin{array}{l} (m_1, 0.3, 0.3), (m_2, 0.4, 0.4), \\ (m_3, 0.4, 0.4), (m_4, 0.4, 0.4) \end{array} \right\}.$$

The following is the description of an unknown microbe N found in the laboratory:

$$N = \left\{ \begin{array}{l} (m_1, 0.4, 0.4), (m_2, 0.5, 0.5), \\ (m_3, 0.2, 0.2), (m_4, 0.3, 0.3) \end{array} \right\}.$$

Our aim is to find the bacteria to which N belongs. The similarity/distance between N and M_j, $j = 1, 2, 3$ by various PF similarity/distance measures are shown in Table 9.

From Table 9, we observe that the unknown microbe N is assigned to M_1 as shown by most of the PF distance/similarity measures. We also observe that the PF distance/similarity measures D_{PYY2} and S_{PYY2} fail to recognize the unknown microbe N.

Thus from Examples 6 and 7, it is clear that results due to the suggested PF distance measures are consistent with the existing ones and therefore are applicable in classification problems.

Multi-criteria decision-making

Here, we show that the suggested PF measures of knowledge and distance are useful for solving MCDM problems involving uncertainty and ambiguity. The main hurdle in an MCDM problem is the computation of criteria weights and we use the suggested knowledge measures for this purpose. For determining the best alternative, we take the help of the suggested distance measures. First, we give the algorithm for solving an MCDM problem having n alternatives M_j, $j = 1, 2, \ldots, n$ and k criteria N_k, $k = 1, 2, \ldots, m$ with w_k, $k = 1, 2, \ldots, m$ as criteria weights where $0 \leq w_k \leq 1$ and $\sum_{k=1}^{m} w_k = 1$.

Algorithm Step 1: Formulate the decision matrix $D = \left[\left(\mu_{jk}, \vartheta_{jk}\right)\right]_{n\times m}$ expressing the information of the available alternatives with respect to the criteria.

Step 2: Formulate the normalized decision matrix $E = \left[\left(\mu'_{jk}, \vartheta'_{jk}\right)\right]_{n\times m}$ where,

$$\left(\mu'_{jk}, \vartheta'_{jk}\right) = \begin{cases} \left(\mu_{jk}, \vartheta_{jk}\right), & if\, N_k\, is\, a\, benefit\, criteria \\ \left(\vartheta_{jk}, \mu_{jk}\right), & if\, N_k\, is\, a\, cost\, criteria \end{cases}.$$

Step 3: Compute the criteria weights w_k, $k = 1, 2, \ldots, m$ as:

$$w_k = \frac{1 - K(N_k)}{m - \sum_{k=1}^{m} K(N_k)}, \; k = 1, 2, \ldots, m.$$

Here, K is a PF entropy measure.

Step 4: Determine the PF ideal solution $M^* = \left\{\left(\mu_1^*, \vartheta_1^*\right), \left(\mu_2^*, \vartheta_2^*\right), \ldots, \left(\mu_m^*, \vartheta_m^*\right)\right\}$ where $\mu_k^* = \max_j \mu_{jk}$ and $\vartheta_k^* = \min_j \vartheta_{jk}$, $k = 1, 2, \ldots, m$.

Step 5: Compute the distance of each alternative M_j, $j = 1, 2, \ldots, n$ from the PF ideal solution M^* using the suggested weighted PF distance measures.

Table 8 Computed values of distance/similarity between the unknown pattern and known patterns regarding Example 6

	(N, M_1)	(N, M_2)	(N, M_3)	Result
D_{PYY1}	0.4700	0.1200	0.2067	M_2
D_{PYY2}	0.3400	0.0133	0.0400	M_2
D_{PYY3}	0.4050	0.0667	0.1233	M_2
D_{PYY4}	0.4700	0.0733	0.1433	M_2
D_{PYY5}	0.6316	0.1317	0.2499	M_2
D_{PYY6}	0.6395	0.1366	0.2507	M_2
D_{PYY7}	0.8330	0.1737	0.3791	M_2
D_{PYY8}	0.8264	NaN	0.6555	NaN
D_{PYY9}	0.8819	NaN	0.7477	NaN
D_{PYY10}	0.8327	0.2039	0.4026	M_2
D_{PYY11}	0.5567	0.0964	0.2837	M_2
D_{PYY12}	0.5411	0.1241	0.2160	M_2
S_{PYY1}	0.5300	0.8800	0.7933	M_2
S_{PYY2}	0.6600	0.9867	0.9600	M_2
S_{PYY3}	0.5950	0.9333	0.8767	M_2
S_{PYY4}	0.5300	0.9267	0.8567	M_2
S_{PYY5}	0.3684	0.8683	0.7501	M_2
S_{PYY6}	0.3197	0.0683	0.1254	M_1
S_{PYY7}	0.1670	0.8263	0.6209	M_2
S_{PYY8}	0.1736	NaN	0.3445	NaN
S_{PYY9}	0.1181	NaN	0.2523	NaN
S_{PYY10}	0.1673	0.7907	0.5974	M_2
S_{PYY11}	0.4433	0.9036	0.7163	M_2
S_{PYY12}	0.4589	0.8759	0.7840	M_2
D_{G1}(proposed)	0.5279	0.1111	0.1905	M_2
D_{G2}(proposed)	0.5712	0.1151	0.1975	M_2
D_{G3}(proposed)	0.6800	0.1200	0.2067	M_2
D_{G4}(proposed)	0.6041	0.1191	0.2042	M_2

"NaN" indicates cannot be computed.$x = 0.2$ and $y = 0.8$ in D_{PYY7}, D_{PYY8}, S_{PYY7}, and S_{PYY8}

Step 6: Rank the alternatives as $M_j > M_t$ if $D\big(M_j, M^*\big) < D(M_t, M^*)$, where D is a PF distance measure and $1 \leq j, t \leq n$.

Now, we solve an MCDM problem in the example given below.

Example 8 [91] Consider the problem of purchasing a house out of the five houses M_j, $j = 1, 2, 3, 4, 5$ by considering the following criteria:

N_1 : ceiling height, N_2 : design, N_3 : location, N_4 : purchase price, N_5 : ventilation.

The information about the five houses with respect to the above-mentioned five criteria is expressed in the form of PFSs as shown by the decision matrix D below:

$$D = \begin{pmatrix} \langle 0.7, 0.5\rangle & \langle 0.6, 0.8\rangle & \langle 0.4, 0.7\rangle & \langle 0.8, 0.3\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.6, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.2, 0.7\rangle & \langle 0.4, 0.6\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.29, 0.8\rangle & \langle 0.21, 0.9\rangle & \langle 0.6, 0.8\rangle & \langle 0.71, 0.3\rangle & \langle 0.1, 0.3\rangle \\ \langle 0.2, 0.9\rangle & \langle 0.2, 0.8\rangle & \langle 0.1, 0.6\rangle & \langle 0.5, 0.6\rangle & \langle 0.4, 0.7\rangle \\ \langle 0.3, 0.9\rangle & \langle 0.32, 0.9\rangle & \langle 0.4, 0.8\rangle & \langle 0.6, 0.6\rangle & \langle 0.3, 0.4\rangle \end{pmatrix}$$

As the criteria N_4 is a cost attribute, so the normalized decision matrix E with the help of Step 2 is given below:

$$E = \begin{pmatrix} \langle 0.7, 0.5\rangle & \langle 0.6, 0.8\rangle & \langle 0.4, 0.7\rangle & \langle 0.3, 0.8\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.6, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.2, 0.7\rangle & \langle 0.6, 0.4\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.29, 0.8\rangle & \langle 0.21, 0.9\rangle & \langle 0.6, 0.8\rangle & \langle 0.3, 0.71\rangle & \langle 0.1, 0.3\rangle \\ \langle 0.2, 0.9\rangle & \langle 0.2, 0.8\rangle & \langle 0.1, 0.6\rangle & \langle 0.6, 0.5\rangle & \langle 0.4, 0.7\rangle \\ \langle 0.3, 0.9\rangle & \langle 0.32, 0.9\rangle & \langle 0.4, 0.8\rangle & \langle 0.6, 0.6\rangle & \langle 0.3, 0.4\rangle \end{pmatrix}$$

Table 9 Computed values of distance/similarity between the unknown pattern and known patterns regarding Example 7

	(N, M_1)	(N, M_2)	(N, M_3)	Result
D_{PYY1}	0.0950	0.3100	0.1750	M_1
D_{PYY2}	0.0225	**0**	**0**	Unable to classify
D_{PYY3}	0.0587	0.1550	0.0875	M_1
D_{PYY4}	0.0700	0.1550	0.0875	M_1
D_{PYY5}	0.1302	0.2619	0.1603	M_1
D_{PYY6}	0.1308	0.2684	0.1609	M_1
D_{PYY7}	0.2179	0.7381	0.4795	M_1
D_{PYY8}	0.3138	0.7622	0.4962	M_1
D_{PYY9}	0.4578	0.7622	0.4962	M_1
D_{PYY10}	0.2992	0.7381	0.4795	M_1
D_{PYY11}	0.0908	0.2619	0.1603	M_1
D_{PYY12}	0.0907	0.2684	0.1609	M_1
S_{PYY1}	0.9050	0.6900	0.8250	M_1
S_{PYY2}	0.9775	**1**	**1**	Unable to classify
S_{PYY3}	0.9413	0.8450	0.9125	M_1
S_{PYY4}	0.9300	0.8450	0.9125	M_1
S_{PYY5}	0.8698	0.7381	0.8397	M_1
S_{PYY6}	0.0654	0.1342	0.0805	M_2
S_{PYY7}	0.7821	0.2619	0.5205	M_1
S_{PYY8}	0.6862	0.2378	0.5038	M_1
S_{PYY9}	0.5422	0.2378	0.5038	M_1
S_{PYY10}	0.7008	0.2619	0.5205	M_1
S_{PYY11}	0.9092	0.7381	0.8397	M_1
S_{PYY12}	0.9093	0.7316	0.8391	M_1
D_{G1}(proposed)	0.0926	0.2619	0.1603	M_1
D_{G2}(proposed)	0.0937	0.2809	0.1669	M_1
D_{G3}(proposed)	0.0950	0.3100	0.1750	M_1
D_{G4}(proposed)	0.0949	0.2984	0.1734	M_1

Bold values indicate unreasonable results. $x = 0.2$ and $y = 0.8$ in D_{PYY7}, D_{PYY8}, S_{PYY7}, and S_{PYY8}

With the help of Step 3 and using the suggested entropy measure K_{G1} given in Table 3, we obtain the criteria weights as:

$w_1 = 0.1744,\ w_2 = 0.1229,\ w_3 = 0.1813,\ w_4 = 0.2525,$ and $w_5 = 0.2688.$

Next, using Step 4, the PF ideal solution M^* is given as:

$$M^* = \left\{ \begin{array}{c} \langle 0.7, 0.5\rangle, \langle 0.7, 0.3\rangle, \langle 0.6, 0.6\rangle, \\ \langle 0.6, 0.4\rangle, \langle 0.6, 0.3\rangle \end{array} \right\}.$$

The computed values of the distance of each alternative M_j, $j = 1, 2, 3, 4, 5$ from the PF ideal solution M^* using the suggested weighted distance measures D_{Gj}^w, $j = 1, 2, 3, 4$ given in Table 2 are shown in Table 10.

The final ranking of alternatives with the help of Step 6 is shown in Table 11.

From Table 11, we conclude that M_2 is the most feasible alternative as all the suggested PF distance measures and the existing PF distance measures D_{PYY1} and D_{PYY4} indicate the same. Further, some existing q-rung orthopair correlation coefficients also indicate the same. This shows that the suggested distance measures are consistent with the existing distance measures.

Conclusion

With the use of t-conorms, this work offered a novel way of building some distance and knowledge metrics for PFSs. First, four new distance measures were presented using t-conorms, and then four new knowledge measures were developed using the proposed distance measures. In terms of the distance/similarity degree between distinct PFSs, the

Table 10 Computed values of the distance of each alternative from the PF ideal solution

	(M_1, M^*)	(M_2, M^*)	(M_3, M^*)	(M_4, M^*)	(M_5, M^*)
D_{G1}	0.0617	0.0507	0.0916	0.0830	0.0826
D_{G2}	0.0659	0.0555	0.0983	0.0901	0.0888
D_{G3}	0.0752	0.0650	0.1123	0.1090	0.1039
D_{G4}	0.0694	0.0593	0.1030	0.0950	0.0932

Table 11 Ranking of alternatives

	Ranking
D_{G1}(proposed)	$M_2 > M_1 > M_5 > M_4 > M_3$
D_{G2}(proposed)	$M_2 > M_1 > M_5 > M_4 > M_3$
D_{G3}(proposed)	$M_2 > M_1 > M_5 > M_4 > M_3$
D_{G4}(proposed)	$M_2 > M_1 > M_5 > M_4 > M_3$
D_{PYY1}	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{PYY4}	$M_2 > M_1 > M_4 > M_5 > M_3$
C_{SG1}[91]	$M_2 > M_1 > M_5 > M_3 > M_4$
C_{SG2}[91]	$M_2 > M_5 > M_3 > M_1 > M_4$
C_{SG3}[91]	$M_2 > M_1 > M_3 > M_5 > M_4$
C_{SG4}[91]	$M_2 > M_1 > M_3 > M_5 > M_4$
C_D[92]	$M_2 > M_1 > M_5 > M_3 > M_4$

suggested distance measures are more successful than most of the known PF distance/similarity measures. The majority of the existing PF distance/similarity measures produce the same distance/similarity between distinct PFSs, and some of them fail to satisfy all of the axiomatic conditions. The suggested PF distance metrics, on the other hand, are devoid of these flaws. Furthermore, from the linguistic hedging perspective, the suggested measures of knowledge for PFSs are more resilient than the known PF entropy/knowledge measures. The proposed PF distance metrics have shown to be effective in pattern recognition challenges. Finally, in a multi-criteria decision-making situation, the recommended knowledge measures are used to compute the weight of attributes, and the distance measures are utilized to rank the alternatives. The results of the recommended measures are compatible with the available measures in pattern recognition and decision-making situations.

The advantages of this study are:

1. The suggested method of constructing the distance measures from t-conorms can be utilized for obtaining the new distance measures for some recent generalizations of fuzzy sets.
2. The distance-based knowledge measures can be used in computing the ambiguity content of Pythagorean fuzzy sets where the existing entropy measures lead to unreasonable results.
3. The suggested distance measures can be applied to bidirectional approximate reasoning.

Our future studies include:

- To demonstrate the applicability of the suggested distance measures in clustering and medical diagnosis.
- To demonstrate the applicability of the suggested measures in real decision-making problems.
- To introduce the t-conorm-based distance measures and knowledge measures for picture fuzzy sets [93], spherical fuzzy sets [94], T-spherical fuzzy sets [94], etc.
- To introduce the parametric generalizations of the suggested measures along with their various applications.

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Declarations

Conflict of interest Authors declare that there is no conflict of interest.

Ethical approval The present article does not contain any studies with human participants or animals performed by any of the authors.

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