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A decomposition-based many-objective evolutionary algorithm with optional performance indicators

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Abstract

Evolutionary algorithms (EAs) have shown excellent performance for solving optimization problems with multiple objectives as they can get a set of compromising solutions on a single run. However, when the number of objectives increases, an efficient selection is significant to find a good set of solutions. In this paper, a decomposition-based many-objective evolutionary algorithm with optional performance indicators is proposed, in which the decomposition strategy is utilized to convert a many-objective optimization problem into a set of single-objective optimization problems, and the criterion to select a solution for the next generation along each reference is randomly set to convergence or diversity performance. The performance of the proposed method is evaluated on two sets of benchmark problems, and the experimental results showed the efficiency of the proposed method compared with seven state-of-the-art MaOEAs.

Keywords Decomposition-based evolutionary optimization · Performance indicators · Many-objective optimization problems

Introduction

In real-world applications, such as parallel machine scheduling [1], hybrid electric vehicle optimization [2], workflow scheduling in clouds [3], and chemical refinery process optimization [4], many problems are required to simultaneously optimize a number of conflicting objectives, which are called multi-objective optimization problems (MOPs) when the number of objectives is not more than three and many-objective optimization problems (MaOPs) when that

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is not less than four. Generally, the mathematical model of a multi-/many-objective optimization problem can be given as follows:

$$\min \ \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$$
$$\text{s.t.} \quad \mathbf{x} \in \mathbb{R}^D \tag{1}$$

where m is the number of objectives, $\mathbb{R}^D$ represents D-dimensional decision space, and $\mathbf{x} = (x_1, x_2, \dots, x_D)$ is the decision vector.

Due to the conflicts between the objectives, it is impossible to find a solution that has the best performance on all objectives [5]. Instead, a set of optimal solutions will be found for a multi-/many-objective optimization problem, which is called the Pareto set (PS) in the decision space and the Pareto front (PF) in the objective space. The evolutionary algorithms have been widely applied to solve multi-objective optimization problems because they can provide a number of optimal solutions in a single run. However, it is difficult to find a good set of solutions for the problem when the number of objectives increases due to the loss of selection pressure. A number of evolutionary algorithms have been proposed for solving many-objective optimization problems [6,7], denoted as MaOEAs, which can mainly be classified into three categories: (1) Pareto dominance-based MaOEAs,

(2) indicator-based MaOEAs, and (3) decomposition-based MaOEAs [6].

The Pareto dominance-based MaOEAs mainly focus on convergence performance. However, the selection pressure of most evolutionary algorithms will vanish when the number of objectives increases. Many approaches have been proposed to modify the dominance relationship to push the population toward the true Pareto front [8]. In [9], the fuzzification of the Pareto-dominance relation is used to support a new ranking method to select individuals in the environmental selection. ϵ-dominance [10] employed a relaxed factor ϵ to control the dominance relation between solutions. Yang et al. [11] proposed a Pareto dominance based on grid-dominance to select the solution having the higher priority of dominance. In α-dominance [12], an α-domination strategy that relaxes the domination introduced a weak trade-off among optimization objectives.

The indicator-based MaOEAs adopt the performance indicators for environmental selection. Normally, the number of performance indicators is much less than the number of objectives, thus increasing the selection pressure. Villalobos and Coello [13] proposed a ΔP indicator in conjunction with a differential evolution [14] (called ΔP-MOEA) for many-objective optimization problems. Ishibuchi et al. [15] proposed an inverse generational distance plus (IGD$^+$) indicator-based evolutionary algorithm (IGD$^+$-EMOA) for MaOPs with no more than eight objectives. Tian et al. [16] proposed a new evolutionary algorithm based on an improved inverted generational distance indicator for solving many-objective optimization problems with different shapes of the Pareto front. Recently, Li et al. proposed to apply the stochastic ranking technique in SRA [17] to balance the search biases on different indicators. In [18], Wang et al. suggested sorting on convergence and diversity performances for environmental selection.

The decomposition-based MaOEAs normally solve a many-objective optimization problem by converting it into a set of single-objective optimization problems or dividing the Pareto front into a group of segments to be a number of many-objective optimization sub-problems [19]. Yuan et al. [19] proposed to preserve the diversity of solutions in high-dimensional objective space by exploiting the perpendicular distance from the solution to the reference vector in the objective space. In [20], Cheng et al. presented an adaptive strategy to adjust the distribution of the reference vectors dynamically and adopted the angle-penalized distance to balance the convergence and diversity of solutions in the high-dimensional objective space. Based on MOEA/D-M2M, a MOEA/D variant, Liu et al. [21] tried to detect the importance of different objectives for adaptively adjusting the subregions of its sub-problems. In [22], Chen et al. proposed to adaptively partition an objective space based on the contributions of subspaces to the population convergence to improve the population convergence while keeping the population diversity.

Note that some MaOEAs are not able to fall into the above three categories as multiple strategies are simultaneously utilized. For example, in MOEA/DD [23] and NSGA-III [24], both the dominance and decomposition strategies are utilized. In [25], the indicator and dominance strategies are used.

The literature review shows that the balance between the convergence and uniformity of the optimal solutions found for MaOPs is significantly important for the optimization algorithms. The decomposition-based MaOEAs decompose a many-objective optimization problem into a set of sub-problems, which are optimized simultaneously by using scalar functions [20,26]. Thus, the reference vectors and the aggregation functions play significant roles in finding a good set of solutions uniformly distributed on the front and approaching the true Pareto front as much as possible. Although the reference vectors are able to assist in maintaining the uniform of a solution set, the performance of the method for finding optimal solutions mainly relies on the search direction and the diversity of the population. Correct search directions are used to ensure the quick convergence to the PF, and the diversity of the population can assist in preventing falling into local optima.

Generally, the same performance indicator will be used for all solutions in the current population for environmental selection. The performance indicator used focuses on the convergence only, such as the Tchebycheff function, or the convergence and diversity simultaneously in a criterion, such as PBI and APD. When only the convergence performance is used for environmental selection, the diversity will not be considered, resulting in poor exploration and incapable of covering the Pareto front. On the other hand, the indicator simultaneously considering the convergence and diversity performance of the solution in a criterion for environmental selection may slow down the speed to find the optimal solutions.

Thus, in our method, convergence and diversity performances of solutions are used simultaneously at each generation but are applied for different reference vectors for selecting solutions for the next generation. Some solutions in the population are selected based on the convergence performance, which is expected to guide the population to approach the Pareto front quickly. In contrast, others are selected based on the diversity performance, which is expected to explore different regions for finding optimal solutions as much as possible. Thus, the trade of balance between exploitation and exploration is met.

The main contributions of this paper are summarized as follows:

1. Each solution plays a role in exploitation or exploration. Thus, the balance between convergence and diversity can be ensured for the population.
2. A mating pool is used for mating selection. The solutions in the mating pool are selected based on the closeness to each reference vector.
3. The computational complexity of the proposed method is $O(N \times |A|)$ if $|A| > N$, where $|A|$ is the number of solutions in the archive A and N represents the population size. However, when $|A| < N$, the computational complexity of MaOEA/D-OPI is $O(N^2)$.

The remainder of this paper is organized as follows: Section "Reference vectors" briefly reviews the reference vector used in this work. Section "A decomposition-based many-objective evolutionary algorithm with optional performance indicators (MaOEA/D-OPI)" gives a detailed description of the proposed algorithm MaOEA/D-OPI. An empirical analysis of the experimental results is provided in Section "Experimental studies". Finally, the conclusions and future work are summarized in Section "Conclusion".

Reference vectors

The reference vectors are unit vectors starting from the origin and distributing uniformly in the objective space. In our proposed method, the two-layer weight vector generation method proposed by Li et al. [27] is adopted to generate a set of reference vectors. Two sets of uniformly distributed points, denoted as $BD = \{\mathbf{bd}_1, \mathbf{bd}_2, \ldots, \mathbf{bd}_{N_1}\}$ and $IN = \{\mathbf{in}_1, \mathbf{in}_2, \ldots, \mathbf{in}_{N_2}\}$, respectively, are firstly generated on a unit hyperplane using the canonical simplex-lattice design method, in which

$$
\begin{cases}
\mathbf{bd}_i = (bd_i^1, bd_i^2, \ldots, bd_i^m), \\
\mathbf{in}_j = (in_j^1, in_j^2, \ldots, in_j^m), \\
db_i^k \in \{\frac{0}{H}, \frac{1}{H}, \ldots, \frac{H}{H}\}, \sum_{k=1}^m bd_i^k = 1, \\
in_j^k \in \{\frac{0}{H}, \frac{1}{H}, \ldots, \frac{H}{H}\}, \sum_{k=1}^m in_j^k = 1, \\
i = 1, 2, \ldots, N_1, \\
j = 1, 2, \ldots, N_2
\end{cases}
\tag{2}
$$

where N_1 and N_2 are the number of uniformly distributed points, $N_1 + N_2 = N$ and H is a positive integer for the simplex-lattice design [24]. Then the coordinates of solutions in IN are shrunk by a coordinate transformation, which will be changed to

$$
in_j^k = \frac{1 - \tau}{m} + \tau \times in_j^k
\tag{3}
$$

where $\tau \in [0, 1]$ is called a shrinkage factor, which is set to 0.5 in our method as that in [27]. After that, the corresponding unit reference vectors $\mathbf{w}$ can be obtained by mapping the reference points, including those in both $\mathbf{bd}$ and $\mathbf{in}$, from a hyperplane to a hypersphere.

A decomposition-based many-objective evolutionary algorithm with optional performance indicators (MaOEA/D-OPI)

In one category of the decomposition-based many-objective evolutionary algorithms, a multi-/many-objective optimization problem will be converted to a number of single-objective optimization problems using a scalarization method, such as weighted sum, weighted Tchebycheff [28], penalty-based boundary intersection (PBI) [26], and angle penalized distance (APD) [20]. Normally, the same performance indicator is used for the environmental selection of the many-objective evolutionary algorithms from beginning to end. Thus, when only the convergence performance is used for environmental selection in the decomposition-based algorithms, the exploration of the method will be poor, and the non-dominated solutions will be incapable of covering the Pareto front due to the lack of diversity. On the other hand, when both convergence and diversity are considered for environmental selection, the speed of finding the optimal solutions will be slow down. Thus, in this paper, we propose to use convergence and diversity performance simultaneously at each generation, which is applied to different reference vectors to select the solutions for the next generation. Some solutions in the population are chosen based on the convergence performance, which is expected to guide the population to approach the Pareto front quickly. The remainders are selected based on the diversity performance, which is expected to explore different regions to find optimal solutions as much as possible. The details of the proposed method are given in this section.

The general framework of MaOEA/D-OPI

Algorithm 1 presents the pseudocode of the proposed MaOEA/D-OPI algorithm. An initial population P with N solutions will be randomly generated in the decision space and evaluated using the objective functions. The non-dominated solutions in P will be selected by the fast sorting technique proposed by Deb et al. [29] and saved to an archive Arc. Then a set of uniformly distributed reference vectors $\mathbf{W}$ will be generated using the method given in [27]. The following procedure will be repeated when the stopping criterion is not met. Two solutions will be selected from the current parent population P for each reference vector. The angle between the reference vector and the selected two solutions

will be calculated, respectively, and the solution closest to this reference vector will be kept for the reproduction of the offspring. The solutions in the offspring population will be used to update the archive Arc. Simultaneously, each solution in the parent and offspring populations will be associated with its closest reference vector, and the environmental selection strategy proposed in this paper is adopted to choose the solutions that survive to the next generation. N solutions in the archive Arc closest to each reference vector will be output.

Algorithm 1 The pseudocode of the proposed MaOEA/D-OPI

Input: The maximum number of objective evaluations $MaxFEs$, a set of unit reference vectors $\mathbf{W}$, the population size N, an archive for saving the non-dominated solutions Arc;
Output: a number of optimal solutions in the archive Arc;
1: Generate an initial population P in the decision space and evaluate the objective values of each solution;
2: Find the non-dominated solutions in P using the fast sorting method [29] and save to the archive Arc;
3: Define a set of reference vectors $\mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N\}$ with the same size of the population P;
4: **while** the stopping criterion is not met **do**
5:     Mating selection (**Details refer to section 3.2**);
6:     Generate an offspring population O;
7:     Evaluate the objective values of each solution in the population O;
8:     Update the archive Arc;
9:     Associate each solution in the P and O populations to its closest reference vector;
10:     Environmental selection;(**Details refer to section 3.3**)
11: **end while**
12: Output N solutions of the archive Arc;

Mating selection

The simulated binary crossover (SBX) [30] and the polynomial mutation [31] are used for the reproduction of the population. In order that the solutions participating in the crossover are uniformly distributed in the objective space, in our proposed method, a pair of solutions in the parent population will be randomly selected for each reference vector, and the solution with the smaller angle to this reference vector will be kept for reproduction. The method to calculate the angle between the objective vector of the solution $\mathbf{x}_i$, denoted as $\mathbf{f}(\mathbf{x}_i)$, and the reference vector $\mathbf{w}_j$ is given in the following:

$$\theta_{ij} = \arccos \frac{\mathbf{f}(\mathbf{x}_i) \cdot \mathbf{w}_j}{\|\mathbf{f}(\mathbf{x}_i)\| \|\mathbf{w}_j\|} \tag{4}$$

where θ_{ij} represents the angle between the objective vector $\mathbf{f}(\mathbf{x}_i)$ and the reference vector $\mathbf{w}_j$.

Algorithm 2 gives the pseudocode of the mating selection.

Algorithm 2 Mating selection

Input: the current parent population P;
Output: the mating pool P';
1: $P' = \emptyset$;
2: **for** $k = 1$ to $|\mathbf{W}|$ **do**
3:     Randomly select two solutions, $\mathbf{x}_i$ and $\mathbf{x}_j$ from the current population P;
4:     Calculate the angles θ_{ik} and θ_{jk}, respectively, using Eq. (4);
5:     **if** $\theta_{ik} < \theta_{jk}$ **then**
6:         $P' = P' \bigcup \mathbf{x}_i$;
7:     **else**
8:         $P' = P' \bigcup \mathbf{x}_j$;
9:     **end if**
10: **end for**
11: Output P';

Environmental selection

Both convergence and diversity of a population play a significantly important role in finding a set of good solutions that approach close to the true Pareto front and are distributed uniformly as much as possible on the Pareto front. Thus, either the exploitation or the exploration capability of a solution is critical in the optimization. In most existing evolutionary algorithms, a unified criterion, normally focusing on convergence only or convergence and diversity simultaneously, is applied to measure the performance of each solution in the population. Then a new population will be selected. However, in the case that only the convergence performance is considered in environmental selection, poor exploration may happen due to the lack of diversity of the population. Thus, it will result in the incapability of covering the Pareto front. On the other hand, when the convergence and diversity performances are ensembled for the environmental selection to measure the performance of a solution, the moment for finding the optimal solutions may be put off. Thus, to find a set of solutions with good convergence and uniformly distribution, we propose to provide two scalarization functions for each reference vector, and one of them will be used for choosing one of all solutions associated with this reference vector to survive to next generation. Thus, some solutions with good convergence will lead the population to exploit the region with good objective values, and others will take on the duty to explore the search space to find more solutions as much as possible. Equations (5) and (6) give the scalarization functions focusing on exploitation and exploration, respectively, of the objective vector of solution $\mathbf{x}_i$ associated for each reference vector $\mathbf{w}_k$.

$$C_{ik} = \|\mathbf{f}(\mathbf{x}_i)\| \times \cos \theta_{ik} \tag{5}$$

$$D_{ik} = \|\mathbf{f}(\mathbf{x}_i)\| \times \sin \theta_{ik} \tag{6}$$

C_{ik} and D_{ik} represent two criteria for environmental selection. Note that only one of Eqs. (5) and (6) will be used for

each reference vector to select a solution to be kept to the next generation.

Algorithm 2 gives the pseudocode of the mating selection used in our method. In Algorithm 3, the angle of each solution $\mathbf{x}_i$ of the populations P and O to each reference vector $\mathbf{w}_k$, denoted as θ_{ik}, will be calculated using Eq. (4), and $\mathbf{x}_i$ will be associated with the reference vector $\mathbf{w}_k$ that satisfies $k = \{k|argmin_{k=1,2,...,N}\theta_{ik}\}$. After that, for each reference vector $\mathbf{w}_k$ that has solutions associated with, a scalarization function will be selected randomly, and the solution with the best value of the scalarization function will be kept to the next generation.

Algorithm 3 Environmental selection

Input: the parent and offspring population P and O, the reference vectors $\mathbf{W}$;
Output: the next parent population P;
1: $P_{next} = \emptyset$;
2: Associate each solution in P and O to its closest reference vector based on the angles between this solution and all reference vectors;
3: **for** $k = 1$ to N **do**
4:    **if** the reference vector $\mathbf{w}_k$ has been associated at least one solution **then**
5:       **if** $rand() \leq 0.5$ **then**
6:          Evaluate the convergence performance of each solution associated with $\mathbf{w}_k$ using
            Eq. (5);
7:          Save the solution with minimum C value to P_{next};
8:       **else**
9:          Evaluate the diversity performance of each solution associated with $\mathbf{w}_k$ using
            Eq. (6);
10:          Save the solution with minimum D value to P_{next};
11:       **end if**
12:    **end if**
13: **end for**
14: $P = P_{next}$.

Figure 1 gives a simple example to show the environmental selection strategy used in our method. Suppose there are three solutions associated with the reference vector $\mathbf{w}_k$. Clearly, solution $\mathbf{x}_1$ will be kept to the next generation if Eq. (5) is adopted as the scalarization function because $\mathbf{x}_1$ has the smallest distance to the ideal point along this reference vector. Otherwise, if Eq. (6) is utilized as the scalarization function, solution $\mathbf{x}_3$ will be selected to be survived to the next generation. It is because solution $\mathbf{x}_3$ is closest to the reference vector, which is expected to contribute to the exploration of good solutions.

Experimental studies

In order to evaluate the performance of the proposed method, we conduct a number of experiments on DTLZ [32], WFG [33] and MaF [34] benchmark problems, and compare the
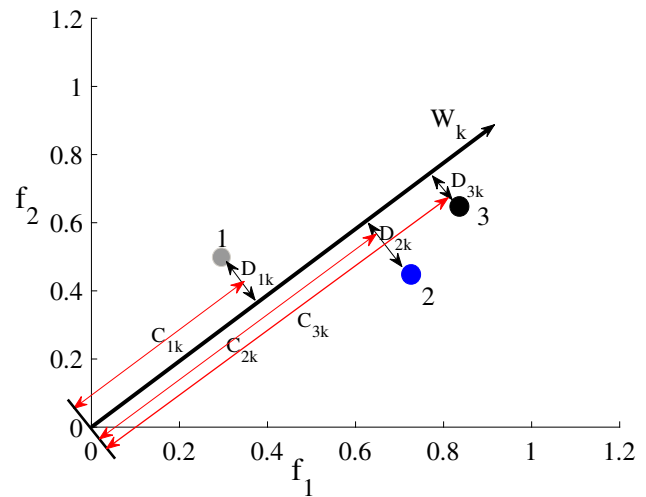

Fig. 1 An example of environmental selection

results to some state-of-the-art algorithms for many-objective optimization problems.

Test problems and performance metrics

Three widely used test suits, DTLZ [32], WFG [33] and MaF [34], are adopted for the empirical studies in this paper. The number of objectives is set to 3, 5, 8, 10, 15 and 20 for DTLZ and WFG, and is set to 10, 15 and 20 for MaF test problems, respectively. The characteristics of DTLZ, WFG and MaF are summarized in Table 1. The number of decision variables is set to $D = m + L - 1$ as recommended in [35], where m is the number of objectives, and $L = 5$ for DTLZ1, $L = 10$ for DTLZ2-6, WFG1-9, MaF1-7 and MaF10-12, while $L = 20$ for DTLZ7. For other MaF test problems, the number of decision variables is set to 2 for MaF8 and MaF9, and $m \times 20$ for MaF14 and MaF15, respectively.

Two performance criteria, including the inverted generational distance (IGD) [36] and hypervolume (HV) [37], are used to evaluate the performance of different algorithms for DTLZ, MaF and WFG, respectively, in this paper. Both of them can simultaneously measure the convergence and diversity of the optimized solutions.

Let P^* represents a set of solutions sampled uniformly on the true Pareto front, and Q is the set of optimal solutions found by an algorithm. Then the IGD value of the set Q can be calculated using the following equation:

$$IGD(Q, P^*) = \frac{\sum_{\mathbf{x}^* \in P^*} dist(\mathbf{x}^*, Q)}{|P^*|} \tag{7}$$

where $dist(\mathbf{x}^*, S)$ represents the Euclidean distance between a point $\mathbf{x}^*$ in P and its closest neighbor point in S. $|P^*|$ denotes the number of solutions in P^*. From Eq. (7), we can see that if the number of solutions in P^* is large enough,

Table 1 The characteristics of three benchmark test suits

Problem	Characteristics
DTLZ1	Linear, multi-modal
DTLZ2	Concave
DTLZ3	Concave, multi-modal
DTLZ4	Concave, biased
DTLZ5	Concave, degenerate
DTLZ6	Concave, degenerate, biased
DTLZ7	Mixed, disconnected, multi-modal, Scaled
WFG1	Mixed, biased, Scaled
WFG2	Convex, disconnected, multi-modal, non-separable, scaled
WFG3	Linear, degenerate, non-separable, scaled
WFG4	Concave, multi-modal, scaled
WFG5	Concave, biased, scaled
WFG6	Concave, non-separable, scaled
WFG7	Concave, biased, scaled
WFG8	Concave, biased, non-separable, scaled
WFG9	Concave, biased, multi-modal, deceptive, non-separable, scaled
MaF1	Linear
MaF2	Concave
MaF3	Convex, multi-modal
MaF4	Concave, multi-modal
MaF5	Convex, biased
MaF6	Concave, degenerate
MaF7	Mixed, disconnected, multi-modal
MaF8	Linear, degenerate
MaF9	Linear, degenerate
MaF10	Mixed, biased
MaF11	Convex, disconnected, non-separable
MaF12	Concave, non-separable, biased, deceptive
MaF13	Concave, unimodal, non-separable, degenerate
MaF14	Linear, partiallyseparable, largescale
MaF15	Convex, partiallyseparable, largescale

the solutions sampled will uniformly cover the true Pareto front. Thus, the smaller the IGD value is, the better the performance of the algorithm is. In our experiments, the number of solutions in P^* is set to 10000.

The hypercube column (HV) metric [37] can provide a comprehensive assessment of the convergence and diversity of the optimized solutions by calculating the volume of the objective space enclosed by the optimized solution set and a reference point. Let $Z^r = (z_1^r, z_1^r, ..., z_m^r)^T$ be the reference point in the objective space that is dominated by all solutions in the current Pareto set. The HV value of the optimization solutions can be calculated using the following equation:

$$HV(Q) = VOL(\bigcup_{x \in Q} [f_1(\mathbf{x}), z_1^r] \times ...[f_m(\mathbf{x}), z_m^r]) \qquad (8)$$

where $VOL(\cdot)$ represents the Lebesgue measure. The larger the HV value is, the better the quality of Q approaching the true PF.

The parameter settings

All algorithms taking part in the comparisons are run on the PlatEMO2.0 [38] platform. The crossover probability is set to $P_c = 1.0$ with distribution index $\eta_c = 20$, and the mutation probability is set to $P_m = 1.0$ with distribution $\eta_m = 20$.

For making a fair comparison, the population size is set to be the same number based on the number of objectives for all algorithms, which are listed in Table 2. p_1 and p_2 represent the number of reference points on the boundary and inside layers, respectively. All algorithms perform 20 independent runs for each problem and will be terminated when the maximum number of objective evaluations reaches 50, 000. The Wilcoxon rank-sum test [39] with a significance level of 0.05 is applied to assess whether the performance of a solution obtained by one of the two methods is expected to be better than the other [40]. The symbols '+', '≈' and '−' represent the compared algorithms that are significantly better, equivalent to, and worse than MaOEA/D-OPI, respectively, in terms of the median values.

Comparison with MaOEA/D-OPI variants

Unlike the existing methods, two optional criteria are provided for any reference vector to select the solution that survived to the next generation along this reference vector. In order to evaluate the performance of the proposed strategy, we substitute the environmental selection criterion used in this paper with the existing environmental

Table 2 The setting of the population size N

m	p_1	p_2	N
3	16	0	153
5	6	0	210
8	3	2	156
10	3	2	275
15	2	1	135
20	2	1	230

selection criteria, including APD, Tchebycheff and PBI. These variants are denoted as MaOEA/D-APD, MaOEA/D-Tchebycheff and MaOEA/D-PBI, respectively. Tables 3 and 4 give the statistical results on DTLZ1-DTLZ7 with 3, 5, 8, 10, 15, 20 objectives and WFG1-WFG9 with 10, 15, and 20 objectives, respectively. From Table 3, we can see that MaOEA/D-OPI can obtain 18/42, 15/42 and 26/42 better results than MaOEA/D-APD, MaOEA/D-Tchebycheff and MaOEA/D-PBI, respectively, and from Table 4, we can see that MaOEA/D-OPI can obtain 23, 27 and 27 better results out of 27 problems, showing that our proposed environmental selection strategy is much better than existing approaches. In order to see the efficiency of providing two performance indicators for environmental selection, we further modify our proposed MaOEA/D-OPI method by using only one of the performance indicators, denoted MaOEA/D-convergence and MaOEA/D-diversity, respectively. The statistical results obtained by MaOEA/D-convergence and MaOEA/D-diversity on DTLZ and WFG are also given in Tables 3 and 4, respectively. From Table 3, we can see that MaOEA/D-OPI can obtain 16/42 and 20/42 better results than MaOEA/D-convergence and MaOEA/D-diversity, respectively. However, we can find that the MaOEA/D-diversity method can obtain better results on DTLZ2, DTLZ4 and DTLZ7. It can be understood that the DTLZ2 and DTLZ4 test problems are normally used to validate the capability of an approach for finding the uniform distributed solutions on the Pareto front, and the DTLZ7 problem is used for testing the ability of a method to find solutions on the disconnected PF as much as possible. Thus, A good population diversity of a method is prefered for optimization compared to a good convergence performance. From Table 4, we can see that MaOEA/D-OPI loses on none WFG problems than MaOEA/D-convergence, but is defeated by MaOEA/D-diversity. The reason is that the optimization population at each generation is close to the true Pareto front. Thus, the solutions focusing on convergence do not play an important role in speeding up the search. Hence, the MaOEA/D-diversity method can obtain better results than MaOEA/D-OPI. However, from the results of 69 test problems in total, we can find that our proposed MaOEA/D-OPI obtains 25/69 better results than MaOEA/D-diversity and loses on 19 out of 69 problems, showing the efficiency of our proposed method.

Comparison with seven state-of-the-art algorithms for many-objective optimization problems

To evaluate the efficiency of MaOEA/D-OPI, seven state-of-the-art MaOEAs are adopted for performance comparison, including PREA [41], MaOEAIGD [21], MaOEAIT [42], NMPSO [43], GrEA[11], KnEA [44] and Two_Arch2 [25] on DTLZ and MaF benchmark problems. In PREA [41],

there are two stages to choose the next population. A promising region will be constructed by a ratio based indicator in the first stage to abandon some solutions with poor quality, then a strategy according to the parallel distance is introduced in the second stage to choose solutions for surviving to the next generation. The MaOEAIGD [45] utilized the IGD indicator in each generation to select solutions with good convergence and diversity. Furthermore, an efficient decomposition-based nadir point estimation method for constructing the utopian Pareto front was proposed, which is regarded as the best approximated Pareto front during the iteration. In MaOEAIT [42], the convergence and diversity performances are addressed in two independent and sequential stages. More specifically, a non-dominated dynamic weight aggregation method based on the genetic algorithm is utilized to locate the Pareto-optimal solutions for MaOPs with different PF shapes, which are then employed to learn the Pareto-optimal subspace for convergence. Afterwards, the diversity of the population is addressed by a set of single-objective optimization problems with reference lines within the learned Pareto-optimal subspace. In NMPSO [43], a balanceable fitness estimation method (BFE) was proposed to overcome the limitations of both Pareto dominance and decomposition approaches. GrEA [11] proposed a grid-based approach to improve the selection pressure toward the optimal direction. KnEA [44] is an excellent algorithm for solving many-objective optimization problems in which a new strategy was proposed to select knee solutions to accelerate the convergence. In Two_Arch2 [25], solutions with good convergence are selected by Pareto dominance and saved in archive CA, and solutions with excellent diversity are selected by I_ϵ [46] and saved in archive DA. The selection of offspring to CA and DA are independent of each other by different methodologies.

Table 5 presents the results of MaOEA/D-OPI and seven state-of-the-art algorithms in terms of the median and median absolute deviation on the 42 DTLZ instances. The best median result and the statistically comparable results are shown in the gray background. From Table 5, we can see that MaOEA/D-OPI obtained the best or competitive results on 24 problems out of the 42 instances than the other seven approaches. The proportions of the test instances on which MaOEA/D-OPI significantly outperforms PREA, MaOEAIGD, MaOEAIT, NMPSO, GrEA, KnEA and Two_Arch2 are 29/42, 32/42, 41/42, 28/42, 32/42, 29/42 and 29/42, respectively. However, from Table 5, MaOEA/D-OPI is not good for solving problems with degenerate or disconnected Pareto fronts, such as DTLZ5, DTLZ6 and DTLZ7. The reason is probably that the reference vectors have not been updated in the optimization, resulting in misguiding the direction of the search. From Table 5, we can further notice that MaOEA/D-OPI can obtain better results than other algorithms on problems with mul-

Table 3 Performance comparison with three variants on seven DTLZ benchmark problems

Problem	m	MaOEA/D-APD	MaOEA/D-Tchebycheff	MaOEA/D-PBI	MaOEA/D-convergence	MaOEA/D-diversity	MaOEA/D-OPI
DTLZ1	3	**1.5477e-2 (5.72e-5)** ≈	2.2417e-2 (3.04e-3) −	1.5575e-2 (5.51e-2) −	1.5572e-2 (8.59e-5) −	**1.5455e-2 (3.74e-5)** ≈	**1.5470e-2 (3.14e-5)**
	5	5.3062e-2 (2.38e-4) −	1.1126e-1 (2.61e-2) −	2.5812e-1 (3.04e-1) −	**5.2067e-2 (8.08e-4)** ≈	5.2743e-2 (1.61e-4) −	**5.2207e-2 (2.52e-4)**
	8	**9.6352e-2 (4.44e-4)** ≈	**1.5538e-1 (3.00e-2)** ≈	9.3453e-1 (8.69e-1) −	9.8314e-2 (1.82e-3) −	9.8805e-2 (5.58e-4) −	**9.6178e-2 (7.23e-4)**
	10	1.0913e-1 (1.49e-3) ≈	1.6803e-1 (3.69e-2) ≈	7.8774e-1 (9.54e-1) −	**1.0861e-1 (1.82e-3)** +	1.1296e-1 (2.99e-3) −	1.1036e-1 (4.44e-3)
	15	1.4929e-1 (5.89e-3) −	**2.4098e-1 (5.44e-2)** ≈	5.8728e-1 (6.47e-1) −	1.5569e-1 (1.24e-2) −	1.6892e-1 (9.37e-3) −	**1.4438e-1 (3.93e-3)**
	20	**1.6389e-1 (1.52e-2)** +	2.9118e-1 (5.29e-2) −	2.0570e+0 (7.02e-1) −	1.9576e-1 (8.42e-3) −	1.7258e-1 (1.00e-2) +	1.8096e-1 (1.21e-2)
DTLZ2	3	4.0924e-2 (7.79e-5) +	5.5428e-2 (6.51e-3) ≈	**4.0910e-2 (3.68e-6)** +	4.1204e-2 (4.24e-5) −	4.1008e-2 (2.60e-5) +	4.1091e-2 (2.82e-5)
	5	1.6523e-1 (4.44e-5) +	3.1772e-1 (6.81e-2) ≈	**1.6521e-1 (4.31e-5)** +	1.6646e-1 (4.06e-4) −	1.6539e-1 (1.96e-4) +	1.6579e-1 (3.06e-4)
	8	3.1621e-1 (2.33e-4) +	5.6735e-1 (1.17e-1) ≈	**3.1603e-1 (2.87e-4)** +	3.4065e-1 (1.18e-3) −	3.3971e-1 (1.46e-3) −	3.3932e-1 (9.91e-4)
	10	4.2263e-1 (1.18e-3) +	6.4440e-1 (1.10e-1) ≈	**4.1604e-1 (6.79e-3)** +	4.4389e-1 (2.22e-3) −	4.3220e-1 (1.97e-3) −	4.2928e-1 (2.64e-3)
	15	**6.2600e-1 (9.79e-3)** ≈	9.3444e-1 (1.57e-1) −	6.9187e-1 (3.84e-2) −	6.4626e-1 (1.74e-2) −	6.2695e-1 (1.70e-3) −	**6.2609e-1 (5.27e-3)**
	20	6.2241e-1 (5.01e-3) −	1.1079e+0 (2.03e-1) −	8.3831e-1 (7.56e-2) −	6.5338e-1 (1.02e-2) −	**6.2334e-1 (2.36e-3)** +	6.2698e-1 (3.71e-2)
DTLZ3	3	9.3577e-2 (1.47e-1) −	5.6224e-2 (3.52e-3) −	8.3744e+0 (5.75e+0) −	5.2156e-2 (7.82e-3) +	1.6125e-1 (2.71e-1) −	**5.1361e-1 (1.08e-3)**
	5	7.8900e-1 (1.00e+0) −	**3.2262e-1 (1.64e-1)** ≈	2.7927e+1 (1.20e+1) −	**5.4103e-1 (5.79e-1)** ≈	7.3555e-1 (6.72e-1) −	**5.6548e-1 (5.78e-1)**
	8	**4.1598e-1 (7.52e-1)** ≈	**6.7157e-1 (1.86e-1)** ≈	3.2495e+1 (1.59e+1) −	**5.1204e-1 (2.74e-1)** ≈	1.0939e+0 (1.65e+0) −	**7.6130e-1 (6.37e-1)**
	10	3.0312e+0 (1.22e+0) −	**8.4160e-1 (1.53e-1)** +	2.9647e+1 (1.21e+1) −	1.8897e+0 (1.32e+0) ≈	2.6127e+0 (1.33e+0) ≈	1.8990e+0 (1.25e+0)
	15	9.8493e-1 (8.04e-1) ≈	1.0411e+0 (1.74e-1) ≈	4.4634e+1 (1.47e+1) −	**9.4608e-1 (3.79e-1)** +	1.8538e+0 (1.15e+0) −	1.0028e-1 (3.86e-1)
	20	2.7254e+0 (1.38e+0) ≈	1.1900e+0 (1.76e-1) +	5.0822e+1 (1.27e+1) −	**1.1466e+0 (8.82e-1)** +	3.0038e+0 (2.00e+0) −	2.2626e+0 (1.38e+0)
DTLZ4	3	**4.0917e-2 (2.22e-5)** +	5.5455e-2 (2.33e-1) −	5.4109e-1 (2.49e-1) −	4.1359e-2 (7.42e-5) ≈	4.1155e-2 (7.47e-5) +	6.6273e-2 (1.12e-2)
	5	**1.6514e-1 (4.47e-5)** +	4.9249e-1 (1.17e-1) −	6.4134e-1 (1.70e-1) −	1.7949e-1 (5.18e-2) −	1.6631e-1 (3.54e-4) +	1.6662e-1 (3.23e-4)
	8	**3.1818e-1 (1.22e-2)** +	7.3880e-1 (6.44e-2) −	7.7503e-1 (1.32e-1) −	3.6761e-1 (1.32e-3) −	3.5605e-1 (2.78e-2) −	3.5045e-1 (2.05e-2)
	10	**4.3291e-1 (2.30e-3)** +	8.1118e-1 (7.04e-2) −	8.8162e-1 (9.39e-2) −	4.3735e-1 (1.99e-3) +	4.4348e-1 (2.40e-3) −	4.3971e-1 (2.13e-3)
	15	**6.2858e-1 (1.84e-3)** +	9.7496e-1 (9.59e-2) −	1.0460e+0 (9.14e-2) −	6.5115e-1 (6.37e-3) −	6.2554e-1 (7.07e-3) +	6.3381e-1 (7.26e-3)
	20	**6.2295e-1 (2.46e-3)** +	1.1266e+0 (1.52e-1) −	1.1001e+0 (7.65e-2) −	6.2535e-1 (3.32e-3) ≈	6.1875e-1 (2.65e-3) +	6.2546e-1 (2.78e-4)

Table 3 continued

Problem	m	MaOEA/D-APD	MaOEA/D-Tchebycheff	MaOEA/D-PBI	MaOEA/D-convergence	MaOEA/D-diversity	MaOEA/D-OPI
DTLZ5	3	4.9651e-2 (8.44e-6) −	**8.9190e-3 (7.31e-3)** ≈	2.5410e-2 (3.21e-4) −	1.9360e-2 (1.07e-3) −	**1.8124e-2 (1.29e-3)** ≈	**1.8261e-2 (1.07e-3)**
	5	1.8166e-1 (8.05e-3) −	3.8253e-2 (7.01e-3) +	**2.2594e-2 (3.67e-4)** +	4.9667e-2 (9.50e-3) ≈	5.6489e-2 (9.56e-3) −	4.9252e-2 (8.32e-3)
	8	2.6966e-1 (4.72e-2) −	4.1548e-2 (8.61e-3) +	**2.4397e-2 (5.12e-4)** +	1.4126e-1 (5.07e-2) ≈	1.6831e-1 (4.61e-2) ≈	1.4675e-1 (4.61e-2)
	10	2.4048e-1 (4.24e-2) −	4.4772e-2 (1.14e-2) +	**1.8750e-2 (6.67e-5)** +	1.5535e-1 (4.10e-2) ≈	2.0029e-1 (5.16e-2) ≈	1.7932e-1 (5.80e-2)
	15	6.8675e-1 (1.72e-1) −	**9.0081e-2 (8.55e-3)** +	9.5728e-2 (1.54e-4) +	2.7162e-1 (8.43e-2) +	3.1909e-1 (8.71e-2) ≈	3.2239e-1 (5.78e-2)
	20	1.9449e-1 (4.92e-2) +	**8.5519e-2 (1.25e-2)** +	1.0203e-1 (5.03e-4) +	3.9650e-1 (6.16e-2) ≈	3.9998e-1 (5.82e-2) ≈	3.8555e-1 (6.04e-2)
DTLZ6	3	4.9640e-2 (1.63e-4) −	**8.9274e-3 (7.34e-3)** +	2.5363e-2 (1.59e-3) ≈	2.5189e-2 (6.46e-4) ≈	2.5179e-2 (6.62e-4) ≈	2.5048e-2 (6.26e-4)
	5	1.5268e-1 (1.65e-2) −	3.5825e-2 (6.98e-3) +	**2.2508e-2 (7.21e-4)** +	5.1848e-2 (9.15e-3) ≈	6.9721e-2 (1.65e-2) −	5.1772e-2 (1.16e-3)
	8	2.7673e-1 (5.11e-2) ≈	3.1706e-1 (6.35e-3) +	**2.6360e-2 (5.65e-1)** +	2.3907e-1 (3.91e-2) ≈	2.3954e-1 (5.13e-2) ≈	2.4084e-1 (3.49e-2)
	10	2.3760e-1 (1.60e-1) ≈	**3.3316e-2 (8.00e-3)** +	2.4644e-2 (1.02e+0) ≈	2.8508e-1 (4.64e-2) ≈	4.5355e-1 (1.81e-1) −	2.8865e-1 (5.36e-2)
	15	1.9042e-1 (1.09e-2) +	**7.3846e-2 (1.06e-2)** +	9.5766e-2 (8.11e-2) +	4.4626e-1 (1.10e-1) ≈	6.5293e-1 (2.47e-1) −	4.6966e-1 (6.40e-2)
	20	1.8594e-1 (1.48e-2) +	**7.0945e-2 (1.45e-2)** +	1.0226e-1 (2.53e-1) +	4.9329e-1 (8.61e-2) ≈	9.5410e-1 (2.98e-1) −	4.9167e-1 (7.09e-2)
DTLZ7	3	1.0484e-1 (2.05e-4) −	1.8300e-1 (1.02e-1) −	1.0501e-1 (7.27e-2) −	9.7307e-2 (4.62e-3) ≈	**9.1064e-2 (2.72e-3)** +	9.9037e-2 (6.19e-3)
	5	**5.5067e-1 (2.93e-3)** +	6.0748e-1 (5.51e-2) +	1.1114e+0 (2.50e-1) −	7.9509e-1 (7.22e-2) ≈	1.1207e+0 (3.58e-1) −	8.3597e-1 (1.78e-1)
	8	1.7626e+0 (2.20e-1) −	**1.3467e+0 (3.49e-1)** ≈	1.5271e+0 (2.21e-1) −	**1.3403e+0 (1.35e-1)** ≈	1.3629e+0 (1.33e-1) ≈	**1.3720e+0 (1.39e-1)**
	10	2.2186e+0 (2.82e-1) −	**1.8243e+0 (3.62e-1)** ≈	2.5184e+0 (2.43e-1) −	**1.6696e+0 (1.76e-1)** ≈	1.6053e+0 (7.95e-2) ≈	**1.6181e+0 (8.75e-2)**
	15	2.6217e+0 (3.08e-1) −	3.1997e+0 (8.28e-1) −	3.1301e+0 (3.21e-1) −	2.4665e+0 (2.04e-1) −	**2.3642e+0 (1.36e-1)** +	2.4031e+0 (1.53e-1)
	20	3.2127e+0 (2.89e-1) −	5.5990e+0 (1.65e+0) −	3.9366e+0 (5.86e-1) −	2.9810e+0 (2.29e-1) −	**2.8526e+0 (2.15e-1)** +	2.8797e+0 (2.24e-1)
+/−/≈		15/18/9	14/15/13	13/26/3	7/16/19	11/20/11	

Table 4 Performance comparison with three variants on nine WFG benchmark problems

Problem	m	MaOEA/D-APD	MaOEA/D-Tchebycheff	MaOEA/D-PBI	MaOEA/D-convergence	MaOEA/D-diversity	MaOEA/D-OPI
WFG1	10	4.8669e-1 (5.11e-2) −	4.8252e-1 (7.22e-2) −	3.8537e-1 (2.93e-2) −	9.2284e-1 (5.59e-2) ≈	**9.4392e-1 (6.53e-2)** +	9.0153e-1 (5.04e-2)
	15	**9.1847e-1 (7.64e-2)** +	3.1964e-1 (3.58e-2) −	2.2366e-1 (1.62e-2) −	8.2478e-1 (1.10e-1) ≈	7.9861e-1 (1.17e-1) ≈	8.1788e-1 (1.07e-1)
	20	**7.1947e-1 (5.98e-2)** ≈	2.3795e-1 (2.54e-2) −	1.7779e-1 (9.93e-3) −	**8.3008e-1 (1.27e-1)**	7.9387e-1 (1.30e-1) ≈	**7.1878e-1 (1.07e-1)**
WFG2	10	9.5940e-1 (7.12e-3) −	9.1204e-1 (1.68e-2) −	6.5315e-1 (4.09e-2) −	**9.6812e-1 (1.15e-2)**	**9.6872e-1 (5.01e-3)**	**9.6885e-1 (5.07e-3)**
	15	9.4670e-1 (1.15e-2) −	8.9432e-1 (2.45e-2) −	6.1054e-1 (4.40e-2) −	9.4175e-1 (2.71e-2) −	9.4840e-1 (2.92e-2) −	**9.6212e-1 (1.10e-2)**
	20	**9.6291e-1 (2.26e-2)** ≈	8.8937e-1 (4.04e-2) −	5.8123e-1 (4.64e-2) −	9.4652e-1 (3.96e-2) ≈	9.6393e-1 (1.36e-2) ≈	**9.6011e-1 (1.26e-2)**
WFG3	10	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
	15	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
	20	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
WFG4	10	8.0158e-1 (1.66e-2) −	3.3709e-1 (3.79e-2) −	3.0232e-1 (7.26e-2) −	8.2557e-1 (1.08e-2) −	**8.8254e-1 (8.63e-3)** +	8.6620e-1 (8.74e-3)
	15	8.1613e-1 (1.66e-2) −	2.7009e-1 (3.37e-2) −	3.3161e-1 (3.13e-1) −	8.7860e-1 (4.68e-2) −	**9.3431e-1 (2.93e-2)** ≈	**9.3333e-1 (1.58e-2)**
	20	9.0870e-1 (3.50e-2) −	2.3037e-1 (2.94e-2) −	7.6918e-1 (3.58e-1) −	9.0636e-1 (3.77e-2) −	**9.3312e-1 (2.58e-2)** ≈	**9.3691e-1 (1.54e-2)**
WFG5	10	8.0399e-1 (9.37e-3) −	4.3964e-1 (2.54e-2) −	4.1413e-1 (4.19e-2) −	7.9401e-1 (9.64e-3) −	7.9966e-1 (8.43e-3) −	**8.1991e-1 (5.97e-3)**
	15	8.2166e-1 (6.50e-3) −	2.5212e-1 (3.19e-2) −	1.7385e-1 (4.32e-2) −	8.3457e-1 (8.48e-3) −	**8.9853e-1 (6.19e-3)** +	8.6806e-1 (6.14e-3)
	20	8.0222e-1 (9.86e-3) −	2.4298e-1 (3.76e-2) −	9.3697e-2 (5.02e-2) −	7.9134e-1 (1.82e-2) −	**8.4656e-1 (2.03e-2)** +	8.2086e-1 (1.10e-2)
WFG6	10	6.6855e-1 (2.23e-2) −	2.2353e-1 (4.60e-2) −	1.4708e-1 (5.45e-2) −	7.4338e-1 (1.99e-2) −	7.6021e-1 (2.70e-2) −	**7.7384e-1 (1.59e-2)**
	15	4.8299e-1 (8.14e-2) −	9.2597e-2 (3.40e-2) −	9.6408e-2 (8.00e-2) −	6.4974e-1 (7.85e-2) −	7.9881e-1 (3.62e-2) −	**8.0504e-1 (3.29e-2)**
	20	5.7997e-1 (4.31e-2) −	9.2656e-2 (4.15e-2) −	8.9259e-2 (1.93e-1) −	6.9158e-1 (7.93e-2) −	**8.3948e-1 (2.96e-2)** +	8.0197e-1 (3.79e-2)
WFG7	10	8.5325e-1 (1.07e-2) −	3.0069e-1 (3.83e-2) −	3.6165e-1 (4.94e-2) −	8.3106e-1 (1.29e-2) −	**8.9969e-1 (8.25e-3)** +	8.7860e-1 (1.06e-2)
	15	5.3745e-1 (2.15e-1) −	1.4509e-1 (1.48e-2) −	2.0963e-1 (1.64e-1) −	7.9749e-1 (8.57e-2) −	**9.5549e-1 (7.94e-3)** ≈	**9.3998e-1 (1.36e-2)**
	20	8.0140e-1 (1.61e-1) −	1.5251e-1 (2.88e-2) −	3.8419e-1 (2.58e-1) −	**8.2525e-1 (1.33e-1)** ≈	**9.4750e-1 (1.34e-2)** ≈	**9.1744e-1 (2.69e-2)**
WFG8	10	6.6149e-1 (4.24e-2) −	3.1475e-2 (3.10e-2) −	2.2524e-1 (2.45e-2) −	6.7992e-1 (3.06e-2) −	**7.5324e-1 (3.60e-2)** +	7.1404e-1 (4.06e-2)
	15	3.8587e-1 (2.13e-1) −	1.7898e-2 (3.18e-1) −	4.7098e-1 (3.18e-1) −	6.2996e-1 (1.34e-1) −	**8.5028e-1 (3.48e-2)** +	8.4393e-1 (4.88e-2)
	20	5.9129e-1 (1.42e-1) −	2.7743e-2 (2.55e-1) −	6.0744e-1 (2.24e-1) −	6.3184e-1 (1.31e-1) −	**8.6875e-1 (5.05e-2)** ≈	**8.5901e-1 (9.68e-2)**
WFG9	10	7.4649e-1 (1.98e-2) −	3.3504e-1 (7.78e-2) −	2.7558e-1 (8.17e-2) −	**7.5878e-1 (4.08e-2)** ≈	7.5093e-1 (3.81e-2) −	**7.7142e-1 (2.03e-2)**
	15	6.7289e-1 (3.57e-2) −	1.3448e-1 (1.75e-1) −	1.1185e-1 (1.61e-1) −	**7.5082e-1 (7.43e-2)** ≈	7.2931e-1 (7.47e-2) ≈	**7.5445e-1 (3.29e-2)**
	20	7.0279e-1 (5.74e-2) −	2.7443e-1 (1.94e-1) −	3.6119e-1 (1.81e-1) −	**7.3282e-1 (4.45e-2)** ≈	7.2105e-1 (5.96e-2) ≈	**7.3003e-1 (4.47e-2)**
+/−/≈		1/21/5	0/24/3	0/24/3	0/15/12	8/5/14	

Table 5 Median and median absolute deviation of the IGD values obtained by PREA, MaOEAIGD, MaOEAIT, NMPSO, GrEA, KnEA, Two_Arch2 and MaOEA/D-OPI

Problem	M	RPEA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
DTLZ1	3	1.9084e-2 (2.42e-3) −	3.3339e-1 (2.37e-1) −	5.7353e-2 (7.10e-2) −	1.9016e-2 (1.10e-3) −	6.3007e-2 (5.03e-2) −	4.8418e-2 (3.68e-2) −	1.9701e-2 (2.05e-3) −	**1.5470e-2 (3.14e-5)**
	5	6.3026e-2 (8.92e-3) −	1.9300e-1 (3.99e-1) −	1.8778e+0 (5.24e+0) −	5.9882e-2 (6.77e-3) −	1.7172e-1 (6.05e-2) −	1.9077e-1 (6.23e-2) −	5.9869e-2 (3.71e-3) −	**5.2207e-2 (2.52e-4)**
	8	1.2020e-1 (7.63e-3) −	2.9304e-1 (2.48e-1) −	1.0076e+1 (1.01e+1) −	1.7816e-1 (9.54e-2) −	3.1385e-1 (8.87e-2) −	6.8463e-1 (6.36e-1) −	1.0658e-1 (2.69e-3) −	**9.6178e-2 (7.23e-4)**
	10	4.1262e-1 (4.29e-1) −	1.5099e-1 (6.89e-2) −	7.8136e+0 (7.30e+0) −	2.2453e-1 (1.07e-1) −	4.9778e-1 (2.81e-1) −	2.4887e+0 (1.07e+0) −	1.3658e-1 (1.54e-2) −	**1.1036e-1 (4.44e-3)**
	15	2.0850e+0 (1.56e+0) −	3.7884e-1 (2.24e-1) −	1.3700e+1 (1.08e+1) −	2.8650e+1 (1.56e+1) −	7.8629e+0 (1.20e+1) −	6.6712e+0 (5.15e+0) −	1.7930e-1 (2.92e-2) −	**1.4438e-1 (3.93e-3)**
	20	2.6513e+0 (1.79e+0) −	2.5920e-1 (9.40e-2) −	1.4936e+1 (1.15e+1) −	2.9878e+1 (7.70e+0) −	4.3013e+0 (7.34e+0) −	7.4328e+0 (6.96e+0) −	2.9021e-1 (9.37e-2) −	**1.8096e-1 (1.21e-2)**
DTLZ2	3	5.0733e-2 (5.50e-3) −	1.7067e-1 (2.30e-4) −	2.3371e-1 (1.00e-1) −	6.4089e-2 (2.33e-3) −	5.5687e-2 (1.00e-3) −	5.1732e-2 (2.00e-3) −	4.7195e-2 (8.84e-4) −	**4.1091e-2 (2.82e-5)**
	5	1.9409e-1 (2.01e-2) −	1.7031e-1 (7.05e-4) −	4.8814e-1 (7.00e-2) −	1.8873e-1 (2.01e-2) −	1.7175e-1 (1.05e-3) −	1.8011e-1 (3.18e-3) −	1.7350e-1 (2.08e-3) −	**1.6579e-1 (3.06e-4)**
	8	3.8487e-1 (9.00e-3) −	3.4318e-1 (9.59e-3) ≈	8.1341e-1 (7.21e-2) −	3.5798e-1 (1.68e-3) −	3.5022e-1 (1.35e-3) −	3.8382e-1 (6.85e-3) −	3.7974e-1 (4.32e-3) −	**3.3932e-1 (9.91e-4)**
	10	4.9992e-1 (1.50e-2) −	4.6598e-1 (8.08e-2) ≈	8.6009e-1 (6.22e-2) −	4.1036e-1 (2.11e-3) +	**4.0605e-1 (1.16e-3)** +	4.3370e-1 (7.53e-3) −	4.2238e-1 (3.55e-3) +	4.2928e-1 (2.64e-3)
	15	6.4831e-1 (3.89e-2) ≈	8.0033e-1 (4.63e-2) −	1.1510e+0 (8.39e-2) −	7.7622e-1 (5.63e-2) −	**5.9252e-1 (2.07e-2)** +	6.1302e-1 (6.27e-3) +	6.4060e-1 (1.07e-2) −	6.2609e-1 (5.27e-3)
	20	7.9459e-1 (8.74e-2) −	7.9607e-1 (5.79e-2) −	1.2216e+0 (4.43e-2) −	8.3732e-1 (3.78e-2) −	**6.1124e-1 (2.53e-3)** +	6.1335e-1 (2.14e-2) +	8.3203e-1 (1.31e-2) −	6.2698e-1 (3.71e-2)
DTLZ3	3	6.9804e-2 (3.67e-2) −	1.0627e+1 (4.57e+0) −	3.3164e+0 (2.63e+0) −	8.1840e+0 (7.16e+0) −	9.5930e-2 (8.44e-2) −	1.2415e-1 (2.14e-1) −	3.2308e-1 (3.80e-1) −	**5.1361e-2 (1.08e-2)**
	5	4.4495e-1 (3.22e-1) ≈	1.2907e+1 (6.10e+0) −	2.0238e+1 (3.37e+1) −	1.7636e+1 (7.16e+0) −	8.2721e-1 (4.58e-1) −	**4.3844e-1 (1.72e-1)** ≈	7.3972e-1 (3.91e-1) −	**5.6548e-1 (5.78e-1)**
	8	3.0230e+0 (2.23e+0) −	9.9027e+0 (5.56e+0) −	2.9535e+2 (1.42e+2) −	1.6747e+1 (1.72e+1) −	8.3805e+0 (3.87e+0) −	7.5334e+1 (1.91e+1) −	2.0683e+0 (1.32e+0) −	**7.6130e-1 (6.37e-1)**

Table 5 continued

Problem	M	RPEA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
	10	1.4620e+1 (9.69e+0) −	6.0909e+0 (3.13e+0) −	2.6876e+2 (1.02e+2) −	2.2254e+1 (1.33e+1) −	1.7606e+1 (9.13e+0) −	2.3864e+2 (1.15e+2) −	1.0787e+1 (2.92e+0) −	**1.8990e+0 (1.25e+0)**
	15	4.5384e+1 (1.90e+1) −	6.9639e+0 (7.08e+0) −	2.8935e+2 (1.24e+2) −	1.2824e+2 (6.81e+1) −	1.4731e+2 (4.11e+1) −	4.9038e+2 (1.36e+2) −	1.7696e+1 (7.01e+0) −	**1.0028e+0 (3.86e-1)**
	20	6.2421e+1 (2.82e+1) −	3.9098e+0 (2.13e+0) −	3.1123e+2 (7.17e+1) −	1.4526e+2 (8.32e+1) −	2.6749e+2 (9.95e+1) −	4.8926e+2 (1.89e+2) −	4.5085e+1 (1.63e+1) −	**2.2626e+0 (1.38e+0)**
DTLZ4	3	2.9050e-1 (2.88e-1) −	2.7890e-1 (1.70e-1) −	8.4170e-1 (2.20e-1) −	8.7857e-2 (1.07e-1) −	8.1301e-2 (1.09e-1) −	5.2197e-2 (1.77e-3) +	**4.6008e-2 (7.82e-4)** +	6.6273e-2 (1.12e-1)
	5	3.2897e-1 (2.01e-1) −	2.6312e-1 (1.18e-1) −	7.8436e-1 (1.32e-1) −	1.8986e-1 (1.49e-1) −	1.7353e-1 (1.14e-3) −	1.7927e-1 (3.46e-3) −	1.7420e-1 (2.31e-3) −	**1.6662e-1 (3.23e-4)**
	8	4.4582e-1 (8.33e-2) −	3.9411e-1 (8.58e-2) ≈	1.0516e+0 (8.21e-2) −	4.1388e-1 (8.06e-2) −	3.5143e-1 (1.04e-3) −	3.7539e-1 (4.78e-3) −	3.8371e-1 (4.67e-3) −	**3.5045e-1 (2.05e-2)**
	10	4.6516e-1 (7.53e-2) ≈	4.5564e-1 (3.12e-2) ≈	1.0560e+0 (7.07e-2) −	4.1701e-1 (1.89e-3) +	**4.0825e-1 (7.44e-4)** +	4.3683e-1 (3.83e-3) +	4.2060e-1 (3.25e-3) +	4.3971e-1 (2.13e-3)
	15	6.1519e-1 (2.06e-2) +	6.5179e-1 (1.65e-2) −	1.1101e+0 (6.51e-2) −	7.1620e-1 (1.29e-1) ≈	**5.7940e-1 (1.15e-3)** +	6.1634e-1 (6.89e-3) +	6.1291e-1 (6.66e-3) +	6.3381e-1 (7.26e-3)
	20	6.2884e-1 (3.78e-2) ≈	6.2915e-1 (8.81e-3) ≈	1.1765e+0 (7.25e-2) −	6.1940e-1 (1.41e-2) ≈	6.2601e-1 (6.09e-3) ≈	**6.1716e-1 (4.17e-3)** +	7.0949e-1 (8.08e-3) −	6.2546e-1 (2.78e-3)
DTLZ5	3	3.6845e-3 (7.57e-4) +	5.4197e-1 (2.12e-1) −	2.9948e-2 (9.76e-2) −	1.3527e-2 (2.48e-3) −	2.0903e-2 (5.08e-4) −	5.5178e-3 (5.26e-4) +	**3.4131e-3 (7.59e-5)** +	1.8261e-2 (1.07e-3)
	5	1.1603e-1 (2.78e-2) −	4.3359e-1 (1.78e-1) −	2.9268e-1 (1.74e-1) −	**3.8028e-2 (4.25e-3)** +	1.4960e-1 (3.08e-2) −	2.2414e-1 (7.09e-2) −	6.1948e-2 (6.85e-3) −	4.9252e-2 (8.32e-3)
	8	3.2415e-1 (5.87e-2) −	5.5975e-1 (1.85e-1) −	2.9744e-1 (1.09e-1) −	6.9826e-1 (1.30e-1) −	2.8429e-1 (6.93e-2) −	3.4750e-1 (6.57e-2) −	**1.4643e-1 (2.59e-2)** ≈	**1.4675e-1 (4.61e-2)**
	10	4.0986e-1 (9.74e-2) −	5.2704e-1 (2.14e-1) −	3.6574e-1 (1.96e-1) −	7.5534e-1 (1.60e-1) −	3.0350e-1 (4.64e-2) −	3.2360e-1 (7.15e-2) −	**1.5773e-1 (3.31e-2)** ≈	**1.7932e-1 (5.80e-2)**
	15	3.7694e-1 (9.13e-2) ≈	6.1485e-1 (1.93e-1) −	4.3284e-1 (9.23e-2) −	7.5560e-1 (3.14e-2) −	6.1811e-1 (1.15e-1) −	7.1276e-1 (1.95e-1) −	4.4041e-1 (1.04e-1) −	**3.2239e-1 (5.78e-2)**
	20	4.3223e-1 (1.26e-1) ≈	**2.9677e-1 (9.79e-2)** +	3.1290e-1 (1.64e-1) +	8.2053e-1 (1.98e-1) −	7.7737e-1 (1.19e-1) −	6.3892e-1 (2.17e-1) −	5.7957e-1 (4.89e-2) −	3.8555e-1 (6.04e-2)

Table 5 continued

Problem	M	RPEA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
DTLZ6	3	3.6758e-3 (8.12e-4) +	6.5639e-1 (1.19e-1) −	2.6975e-2 (9.81e-2) −	1.3798e-2 (2.56e-3) +	2.2247e-2 (2.55e-4) +	4.7698e-3 (3.09e-3) +	**3.4614e-3 (1.99e-5)** +	2.5048e-2 (6.26e-4)
	5	2.2262e-1 (5.70e-2) −	6.0654e-1 (1.10e-1) −	8.5254e+0 (1.11e-1) −	**3.6843e-2 (3.84e-3)** +	3.2371e-1 (4.57e-2) −	4.0972e-1 (1.64e-1) −	9.1566e-2 (1.90e-2) −	5.1772e-2 (1.16e-2)
	8	4.3100e-1 (1.62e-1) −	7.3575e-1 (2.09e-1) −	8.5449e+0 (1.29e-1) −	7.0897e-1 (1.42e-1) −	7.1079e-1 (3.71e-1) −	1.0866e+0 (5.20e-1) −	**2.8977e-1 (1.10e-1)** ≈	**2.4084e-1 (3.49e-2)**
	10	1.0492e+0 (5.08e-1) −	6.8709e-1 (8.76e-2) −	8.5445e+0 (1.42e-1) −	7.4083e-1 (5.64e-3) −	1.4064e+0 (4.00e-1) −	2.6093e+0 (4.72e-1) −	4.0642e-1 (1.45e-1) −	**2.8865e-1 (5.36e-2)**
	15	2.1885e+0 (8.37e-1) −	7.4603e-1 (2.17e-1) −	8.3874e+0 (2.77e-1) −	7.4209e-1 (2.28e-16) −	2.4778e+0 (4.40e-1) −	2.3986e+0 (6.52e-1) −	1.2194e+0 (6.05e-1) −	**4.6966e-1 (6.40e-2)**
	20	2.8759e+0 (7.65e-1) −	**4.0969e-1 (5.73e-2)** +	8.5712e+0 (1.50e-1) −	7.6045e-1 (4.49e-2) −	3.9693e+0 (3.73e-1) −	2.6395e+0 (5.32e-1) −	4.1579e+0 (8.00e-1) −	4.9167e-1 (7.09e-2)
DTLZ7	3	1.0901e-1 (1.24e-1) −	6.4298e-1 (4.33e-2) −	1.6017e-1 (8.78e-2) −	5.3885e-2 (2.59e-3) +	7.5056e-2 (3.12e-3) +	**5.2653e-2 (2.12e-3)** +	5.9535e-2 (6.62e-2) +	9.9037e-2 (6.19e-3)
	5	2.8212e-1 (8.16e-2) +	6.9266e-1 (4.24e-2) +	3.2071e+0 (2.10e+0) −	**2.2606e-1 (4.47e-3)** +	2.3570e-1 (7.46e-3) +	2.4044e-1 (5.85e-3) +	2.3608e-1 (3.44e-2) +	8.3597e-1 (1.78e-1)
	8	6.6115e-1 (3.96e-2) +	1.2253e+0 (4.40e-2) +	2.0955e+1 (3.31e+0) −	8.0663e-1 (1.25e-1) +	8.0520e-1 (5.39e-2) +	6.5111e-1 (3.12e-2) +	**6.1523e-1 (4.35e-3)** +	1.3720e+0 (1.39e-1)
	10	9.0938e-1 (8.94e-2) +	1.5268e+0 (4.04e-2) +	3.0657e+1 (2.07e+0) −	8.7275e-1 (4.67e-2) +	2.6237e+0 (1.18e-1) −	**8.5086e-1 (7.23e-3)** +	9.8514e-1 (3.67e-2) +	1.6181e+0 (8.75e-2)
	15	**2.5579e+0 (3.59e-1)** ≈	2.6852e+0 (1.40e-1) −	5.9473e+1 (2.74e+0) −	**2.5157e+0 (3.47e-1)** ≈	9.3058e+0 (1.04e+0) −	3.2618e+0 (6.82e-1) −	2.7879e+0 (5.80e-1) −	**2.4031e+0 (1.53e-1)**
	20	1.1884e+1 (4.37e+0) −	3.4463e+0 (2.37e-1) −	6.9988e+1 (3.35e+0) −	**2.7313e+0 (2.89e-1)** +	1.4057e+1 (1.11e+0) −	1.4247e+1 (3.66e+0) −	1.8660e+1 (2.34e+0) −	2.8797e+0 (2.24e-1)
+/−/≈		6/29/7	5/32/5	1/41/0	11/28/3	9/32/1	12/29/1	10/29/3	

The best median result and its comparatives without significant difference in each row are shown in gray background

(a) PREA on 20 objectives of DTLZ3

(b) MaOEAIGD on 20 objectives of DTLZ3

(c) MaOEAIT on 20 objectives of DTLZ3

(d) NMPSO on 20 objectives of DTLZ3

(e) GrEA on 20 objectives of DTLZ3

(f) KnEA on 20 objectives of DTLZ3

(g) Two_Arch2 on 20 objectives of DTLZ3
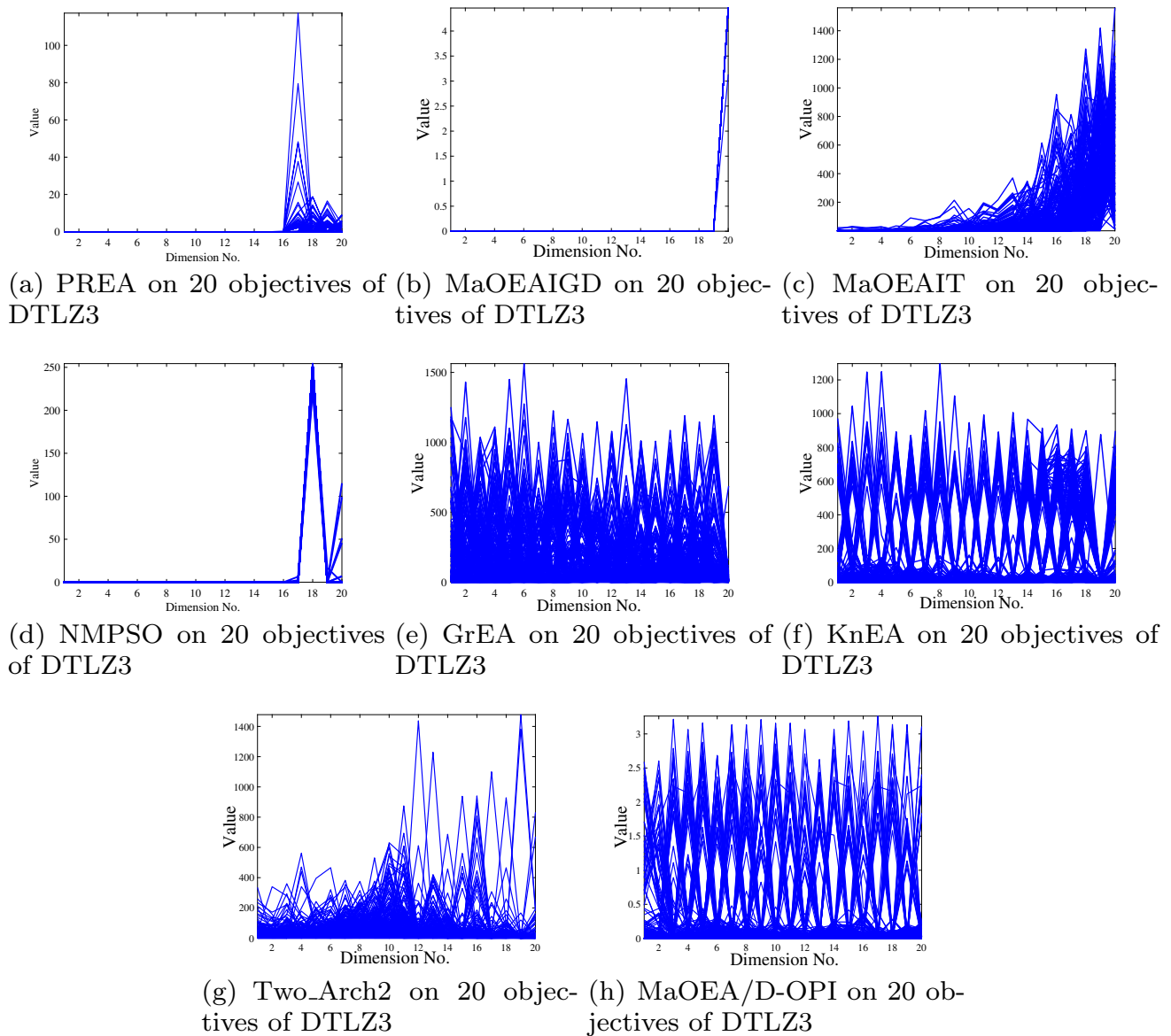
(h) MaOEA/D-OPI on 20 objectives of DTLZ3

Fig. 2 Parallel coordinates of the final solutions obtained by seven compared algorithms for the 20-objective DTLZ3 instance. **a** PREA. (b) MaOEAIGD. **c** MaOEAIT. **d** NMPSO. **e** GrEA. **f** KnEA. **g** Two_Arch2. **h** MaOEA/D-OPI

timodal PF landscapes such as DTLZ1 and DTLZ3. For a more intuitive understanding of the behaviour of MaOEA/D-OPI on the multimodal problems, we graphically plot the parallel coordinates of the final solutions obtained by the compared algorithms for 20-objective DTLZ3 in Fig. 2. The horizontal and vertical axes represent each objective and the final optimal value of each individual on the corresponding objective, respectively. From Fig. 2, we can see that MaOEA/D-OPI has much better convergence and diversity performances than other seven methods for solving high-dimensional many-objective optimization problems with multimodal PF landscapes. From Fig. 2, we can see that MaOEA/D-OPI are much better at convergence and

diversity performances than the other seven methods to solve high-dimensional many-objective optimization problems with multimodal PF landscapes.

In order to further evaluate the performance of MaOEA/D-OPI on many-objective optimization problems, a number of experiments are conducted on the MaF test suit with 10, 15 and 20 objectives. Table 6 shows the statistical results on MaF test problems. From Table 6, the proportion of problems on which MaOEA/D-OPI performs better than PREA, MaOEAIGD, MaOEAIT, NMPSO, GrEA, KnEA and Two_Arch2 are 18/45, 32/45, 39/45, 20/45, 23/45, 22/45 and 19/45. The method loses to win than RPEA, NMPSO, and Two-Arch2 on MaF test problems. However,

Table 6 Median and median absolute deviation of the IGD values obtained by PREA MaOEAIGD, MaOEAIT, NMPSO, GrEA, KnEA, Two_Arch2 and MaOEA/D-OPI on MaF1-15

Problem	M	PREA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
MaF1	10	2.4025e-1 (2.63e-3) +	3.4563e-1 (5.38e-3) +	4.9872e-1 (1.02e-1) ≈	2.8457e-1 (1.06e-2) +	2.3894e-1 (3.86e-3) +	**2.2618e-1** (**4.03e-3**) +	2.3781e-1 (3.89e-3) +	4.7195e-1 (2.37e-2)
	15	3.7433e-1 (8.46e-3) +	4.1329e-1 (1.69e-2) +	7.3227e-1 (1.51e-1) ≈	4.1275e-1 (2.47e-2) +	3.6413e-1 (1.89e-2) +	3.4407e-1 (3.16e-2) +	**3.2044e-1** (**6.78e-3**) +	6.7060e-1 (6.17e-2)
	20	**3.8694e-1** (**6.70e-3**) +	5.1881e-1 (1.00e-2) +	7.9322e-1 (2.03e-1) ≈	4.5496e-1 (2.49e-2) +	4.8687e-1 (2.22e-2) +	4.0009e-1 (2.86e-2) +	4.3945e-1 (1.33e-2) +	7.6001e-1 (5.75e-2)
MaF2	10	**1.6003e-1** (**3.67e-3**) +	4.6593e-1 (1.92e-2) −	3.6107e-1 (5.22e-2) −	1.7469e-1 (3.95e-3) +	3.7691e-1 (1.47e-2) −	1.6756e-1 (8.78e-3) +	1.6339e-1 (2.51e-3) +	2.1891e-1 (3.08e-2)
	15	1.9244e-1 (6.00e-3) +	4.6095e-1 (2.84e-2) −	5.3600e-1 (3.75e-2) −	3.2301e-1 (2.27e-2) +	5.2214e-1 (2.31e-2) −	2.1360e-1 (5.28e-3) +	**1.8864e-1** (**4.64e-3**) +	3.6563e-1 (4.31e-2)
	20	2.0621e-1 (2.19e-2) +	5.1168e-1 (2.98e-2) −	4.9422e-1 (7.38e-2) −	3.0512e-1 (3.12e-2) +	3.9581e-1 (1.33e-2) ≈	**1.8211e-1** (**3.15e-3**) +	1.8353e-1 (3.08e-3) +	3.8205e-1 (3.67e-2)
MaF3	10	1.5188e+3 (1.87e+3) −	**8.3582e+0** (**9.26e+0**) +	4.8731e+8 (1.49e+9) −	4.7743e+5 (9.43e+5) −	7.6915e+4 (9.87e+4) −	1.2187e+7 (4.50e+7) −	6.4085e+7 (9.09e+7) −	3.6479e+2 (4.91e+2)
	15	3.6225e+3 (2.84e+3) −	**2.3962e+1** (**4.32e+1**) +	1.6560e+9 (4.30e+9) −	1.6917e+8 (5.91e+8) −	1.8610e+4 (3.13e+4) −	2.9898e+9 (7.18e+9) −	1.4229e+5 (5.35e+5) −	2.9427e+2 (3.11e+2)
	20	3.7120e+3 (2.41e+3) −	**3.3995e+0** (**2.99e+0**) +	1.9090e+8 (5.96e+8) −	7.8814e+8 (1.32e+9) −	6.4732e+4 (1.60e+5) −	3.5856e+10 (4.61e+10) −	3.6023e+7 (1.18e+8) −	6.0749e+2 (5.91e+2)
MaF4	10	3.7304e+2 (7.10e+2) +	1.0993e+3 (4.24e+2) −	4.8676e+3 (7.28e+3) −	5.0837e+4 (1.68e+4) −	**1.7237e+2** (**1.83e+2**) +	1.2277e+3 (1.04e+3) −	7.3363e+2 (7.78e+2) ≈	4.0331e+2 (1.37e+2)
	15	**3.7880e+3** (**6.73e+3**) +	2.9588e+4 (1.75e+4) −	2.7408e+5 (3.44e+5) −	7.2945e+5 (4.32e+5) −	5.1334e+3 (4.03e+2) +	1.8543e+4 (1.99e+4) ≈	1.3030e+4 (1.59e+4) ≈	1.4805e+4 (3.20e+3)
	20	**9.2677e+4** (**9.68e+4**) +	9.3959e+5 (1.37e+5) −	6.1544e+6 (1.24e+7) −	5.8609e+7 (1.98e+7) −	2.6853e+5 (2.78e+5) +	1.4572e+6 (1.65e+6) ≈	6.0921e+5 (4.84e+5) ≈	4.8888e+5 (1.85e+5)
MaF5	10	5.2728e+1 (1.57e+1) +	2.9362e+2 (1.04e+1) −	2.0550e+2 (4.99e+1) −	5.1943e+1 (1.55e+1) +	**4.6745e+1** (**1.42e+0**) +	6.3647e+1 (3.54e+0) +	4.7975e+1 (1.36e+0) +	1.3743e+2 (1.05e+1)
	15	1.8275e+3 (2.92e+2) +	7.3257e+3 (5.30e-1) −	5.1862e+3 (1.07e+3) +	2.9561e+3 (1.19e+3) +	1.6433e+3 (1.50e+2) +	2.1509e+3 (1.73e+2) +	**1.5432e+3** (**1.07e+2**) +	5.5133e+3 (6.29e+2)
	20	1.9298e+4 (2.14e+3) +	1.7095e+5 (8.27e-1) −	1.3326e+5 (5.18e+4) +	2.9120e+4 (1.93e+4) +	**1.7842e+4** (**1.05e+3**) +	2.1744e+4 (1.62e+3) +	3.6068e+4 (3.20e+3) +	1.4313e+5 (8.40e+3)

Table 6 continued

Problem	M	PREA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
MaF6	10	**1.7076e-3 (2.60e-5)** +	6.4605e-1 (1.72e-1) −	1.1084e-1 (1.92e-1) −	2.9055e+0 (2.17e+0) −	2.2584e+0 (1.46e+0) −	6.1185e+0 (4.92e+0) −	3.4572e-1 (2.11e-3) −	5.9201e-2 (1.56e-2)
	15	2.9214e-1 (2.01e-1) −	7.0387e-1 (1.19e-1) −	1.3478e+1 (8.91e+0) −	3.8727e+0 (2.43e+0) −	9.7919e+0 (3.71e+0) −	1.6659e+1 (1.19e+1) −	7.4244e-1 (1.75e-4) −	**1.1790e-1 (2.22e-2)**
	20	1.0113e+0 (5.73e-1) −	4.1454e-1 (1.98e-1) −	1.8062e+1 (7.83e+0) −	1.1839e+1 (7.52e+0) −	1.0987e+1 (5.47e+0) −	9.6384e+0 (1.40e+1) −	7.4642e-1 (3.16e-3) −	**1.1532e-1 (1.66e-2)**
MaF7	10	**8.3722e-1 (5.95e-3)** +	1.5148e+0 (5.51e-2) +	3.1150e+1 (1.68e+0) −	9.2722e-1 (1.25e-1) +	2.5883e+0 (1.42e-1) −	8.5022e-1 (8.85e-3) +	9.9335e-1 (5.19e-2) +	1.6605e+0 (1.09e-1)
	15	**2.7851e+0 (4.64e-1)** ≈	2.6784e+0 (1.53e-1) −	5.9818e+1 (2.61e+0) −	2.6163e+0 (3.45e-1) −	1.0058e+1 (1.04e+0) −	3.0947e+0 (5.58e-1) −	**2.8244e+0 (8.88e-1)** ≈	**2.5586e+0 (2.98e-1)**
	20	1.3668e+1 (4.08e+0) −	3.5324e+0 (1.41e-1) −	6.7741e+1 (4.03e+0) −	**2.7555e+0 (3.51e-1)** +	1.3848e+1 (8.54e-1) −	1.3168e+1 (4.06e+0) −	1.9508e+1 (2.86e+0) −	3.1467e-1 (3.78e-1)
MaF8	10	**1.2747e-1 (2.83e-2)** +	1.3395e+0 (1.73e-1) −	7.8152e+1 (1.12e+2) −	2.1361e-1 (1.94e-2) +	1.4609e-1 (4.80e-3) +	2.1024e-1 (1.47e-1) +	1.6869e-1 (6.25e-2) +	2.5402e-1 (5.78e-2)
	15	2.0794e-1 (2.87e-2) +	1.9031e+0 (2.99e-1) −	2.7000e+2 (4.86e+2) −	5.3268e-1 (6.22e-2) −	2.1062e-1 (6.85e-3) +	2.9139e-1 (6.86e-2) +	**1.9238e-1 (6.75e-3)** +	4.9034e-1 (5.74e-2)
	20	**1.7430e-1 (1.18e-2)** +	2.0764e+0 (1.55e-1) −	7.4459e+1 (1.09e+2) −	5.2939e-1 (1.12e-1) ≈	2.0520e-1 (5.63e-3) +	7.1710e-1 (3.49e-1) −	2.0091e-1 (6.46e-2) +	4.9467e-1 (5.72e-2)
MaF9	10	9.1917e+0 (1.00e+1) −	1.2095e+0 (2.81e-1) −	1.5619e+2 (1.12e+2) −	**1.3196e-1 (6.50e-3)** +	1.4431e+0 (3.84e-2) −	2.5236e+1 (2.34e+1) −	1.0437e+0 (1.07e-1) −	3.1077e-1 (7.94e-3)
	15	6.6536e+0 (5.88e+0) ≈	6.8014e+0 (5.61e+0) −	3.4186e+2 (3.86e+2) −	6.3917e-1 (4.13e-1) +	4.3880e+0 (4.56e+0) −	2.8264e+0 (4.76e+0) ≈	**1.8845e-1 (9.33e-3)** +	1.3693e+0 (2.41e+0)
	20	3.6906e+1 (1.06e+1) −	1.7232e+1 (4.04e+0) −	1.7931e+2 (1.08e+2) −	**1.1943e+0 (8.26e-1)** ≈	8.9694e+0 (7.95e+0) −	3.0323e+1 (9.20e+0) −	1.9734e+0 (6.56e-1) −	1.7071e+0 (3.28e+0)
MaF10	10	1.6710e+0 (7.14e-2) −	2.1959e+0 (4.24e-1) −	3.3686e+0 (1.05e-1) −	2.3033e+0 (1.86e-1) −	1.3899e+0 (1.06e-1) −	1.5321e+0 (1.12e-1) −	2.2301e+0 (1.20e-1) −	**1.2004e+0 (5.66e-2)**
	15	2.2822e+0 (8.56e-2) −	8.4135e+0 (6.87e+0) −	4.4220e+0 (2.54e-2) −	2.4914e+0 (2.28e-1) −	2.1705e+0 (9.48e-2) −	2.1511e+0 (4.93e-1) ≈	2.3274e+0 (1.62e-1) −	**1.9395e+0 (8.54e-2)**
	20	**4.0932e+0 (1.40e-1)** ≈	5.3597e+0 (1.13e+0) −	5.7902e+0 (3.39e-2) −	4.5088e+0 (2.57e-1) −	4.0934e+0 (1.90e-1) ≈	4.1893e+0 (3.14e-1) ≈	4.4787e+0 (1.67e-1) −	**4.0527e+0 (2.02e-1)**

Table 6 continued

Problem	M	PREA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2	MaOEA/D-OPI
MaF11	10	1.4260e+0 (4.83e-2) −	2.2300e+0 (5.86e-1) −	4.2718e+0 (2.38e+0) −	1.9039e+0 (2.19e-1) −	1.1629e+0 (3.17e-2) −	1.1899e+0 (3.42e-2) −	**9.7099e-1 (1.41e-2)** −	1.0415e+0 (2.18e-2)
	15	2.4166e+0 (1.50e-1) −	3.4422e+0 (3.10e+0) −	6.3491e+0 (1.20e+0) −	3.1363e+0 (2.33e-1) −	1.9263e+0 (9.93e-2) ≈	1.8739e+0 (2.14e-1) −	**1.7050e+0 (3.93e-2)** +	1.8799e+0 (9.51e-2)
	20	4.0043e+0 (7.56e-2) −	4.3903e+0 (9.01e-1) −	9.6022e+0 (3.93e+0) −	4.3677e+0 (3.23e-1) −	4.3295e+0 (1.14e-1) −	3.7809e+0 (1.86e-1) ≈	**3.2973e+0 (6.30e-2)** +	3.8569e+0 (1.57e-1)
MaF12	10	**4.0000e+0 (3.75e-2)** +	5.4160e+0 (3.55e-1) −	7.9121e+0 (4.07e-1) −	4.0549e+0 (3.64e-2) +	4.1661e+0 (3.15e-2) −	4.2314e+0 (5.22e-2) −	4.2918e+0 (3.72e-2) −	4.1945e+0 (3.22e-2)
	15	**8.0955e+0 (1.35e-1)** +	1.1997e+1 (6.88e+0) ≈	1.7573e+1 (1.74e+0) −	8.4937e+0 (2.00e-1) ≈	9.4843e+0 (4.48e-1) −	8.7187e+0 (1.33e-1) −	8.9306e+0 (1.30e-1) −	8.7119e+0 (9.77e-1)
	20	1.1670e+1 (2.21e-1) −	1.1792e+1 (7.98e-1) ≈	2.6537e+1 (1.23e+0) −	1.3600e+1 (6.25e-1) −	1.2522e+1 (1.97e-1) −	**1.1075e+1 (2.02e-1)** +	1.4885e+1 (2.13e-1) −	1.1266e+1 (2.41e-1)
MaF13	10	**8.9657e-2 (1.20e-2)** +	1.3582e+0 (6.89e-2) −	1.0048e+0 (4.83e-1) −	1.9183e-1 (5.89e-2) +	4.0028e-1 (1.68e-1) −	1.9910e-1 (5.53e-2) +	1.5521e-1 (1.85e-2) +	4.2090e-1 (4.10e-2)
	15	**1.3403e-1 (1.39e-2)** +	1.6177e+0 (2.35e-1) −	3.3822e+7 (3.05e+7) −	3.2926e-1 (9.21e-2) +	5.4837e-1 (2.38e-1) −	1.9532e-1 (1.37e-2) +	2.3843e-1 (2.30e-2) +	8.7507e-1 (2.79e-1)
	20	**1.1773e-1 (1.22e-2)** +	2.0043e+0 (4.98e-2) −	1.1198e+0 (4.23e-1) ≈	4.3464e-1 (1.13e-1) +	7.0625e-1 (3.07e-1) +	2.7558e-1 (9.16e-2) +	2.1245e-1 (1.78e-2) +	9.9613e-1 (2.30e-1)
MaF14	10	5.4074e+0 (1.86e+0) −	**1.4743e+0 (3.73e-1)** +	3.2653e+3 (2.10e+3) −	1.8297e+1 (2.81e+1) −	6.6119e+2 (8.86e+2) −	2.6642e+1 (1.86e+1) −	4.1741e+0 (4.77e+0) +	4.2287e+0 (1.52e+0)
	15	8.0785e+0 (4.96e+0) ≈	1.6767e+1 (7.33e+0) −	2.0567e+2 (3.72e+2) −	4.4924e+1 (9.62e+0) −	1.7461e+2 (3.41e+2) ≈	1.3984e+2 (1.17e+2) −	**2.8562e+0 (5.83e+0)** +	7.6362e+0 (4.18e+0)
	20	5.0746e+0 (2.78e+0) ≈	**1.4058e+0 (3.52e-1)** +	3.6393e+2 (5.11e+2) −	2.8767e+1 (1.89e+1) −	1.6024e+3 (1.63e+3) −	4.9483e+2 (4.42e+2) −	1.5438e+1 (9.47e+0) −	5.3317e+0 (3.37e+0)
MaF15	10	1.0114e+1 (2.32e+0) −	1.8265e+0 (9.13e-2) −	5.7284e+1 (3.47e+0) −	2.5949e+0 (4.75e-0) ≈	3.3725e+0 (8.50e-1) −	1.8054e+1 (1.06e-1) −	1.1812e+1 (2.38e+0) −	**1.4997e+0 (4.04e-1)**
	15	2.1392e+1 (5.39e+0) −	**1.4215e+0 (2.51e-1)** +	7.6102e+1 (6.16e+0) −	1.4850e+0 (6.37e-2) +	1.5918e+0 (1.18e-1) +	5.5128e+1 (1.85e+1) −	2.0882e+1 (6.13e+0) −	3.9626e+0 (1.87e+0)
	20	4.9227e+1 (1.41e+1) −	4.3599e+0 (5.43e-1) +	9.2787e+1 (4.96e+0) −	4.0540e+0 (5.69e-1) +	**2.1616e+0 (1.04e+0)** +	1.5288e+2 (3.43e+1) −	4.3253e+1 (4.84e+0) −	7.7647e+0 (2.47e+0)
+/−/≈		22/18/5	11/32/2	1/39/5	20/20/5	17/23/5	16/22/7	22/19/4	

The best median result and its comparatives without significant difference in each row are shown in gray background

Table 7 Summary of the Results on DTLZ and MaF test problems, where seven compared algorithms is better than (+), worst than (−), and comparable (≈) to than MaOEA/D-OPI according to the Wilcoxon rank sum test with Bonferroni correction

	PREA	MaOEAIGD	MaOEAIT	NMPSO	GrEA	KnEA	Two_Arch2
−	47	64	80	48	55	51	48
+	28	16	2	31	26	28	32
≈	12	7	5	8	6	8	7

from Table 7, we can see that when all results obtained on DTLZ and MaF test problems are considered, our proposed MaOEA/D-OPI can achieve better results on 47, 48, and 48 out of 87 test problems than PREA, NMPSO, and Two_Arch2 respectively. The overall results show that MOEA/D-OPI has better performance than RPEA, NMPSO and Two-Arch2 for solving many-objective problems. From Table 7, we can further find that our proposed MaOEA/D-OPI method achieves better results on 64, 80, 55, and 51 out of 87 test problems than MaOEAIGD, MaOEAIT, GrEA, and KnEA, respectively. Therefore, the overall performance of MOEA/D-OPI is efficient for many-objective problems.

Computational complexity analysis

This subsection gives an upper bound of the computation complexity within one generation of MOEA/D-OPI. There are four main parts in one generation:

1. The angle between each solution and reference vector will be calculated before the solution is associated with its closest reference vector. Thus, its computational complexity is $O(N^2)$, where N is the size of the population. On the other hand, the computational complexity of convergence evaluation is $O(N^2)$. Thus, The computational complexity to calculate two performance indicators is $O(N^2)$.
2. N solutions will be kept to the mating pool, each of which is selected from two solutions according to the closeness to a reference vector. Thus, the computational complexity will be $O(N)$ for mating selection.
3. In the environmental selection, the population is selected according to sort the diversity or convergence. Thus, the time complexity of the environmental selection is $O(N^2)$.
4. The computational complexity of archive updating costs $O(m \times |A|)$ for each solution of the population because the non-dominated sorting is applied. m is the number of objectives, and $|A|$ represents the number of solutions in the archive. Thus, the total computational complexity to update the archive is $O(m \times N \times |A|)$ for all solutions in the population to update the archive.

As $m << N$, the overall computational complexity of MaOEA/D-OPI for one generation is $O(N \times |A|)$ if $|A| > N$.

Otherwise, the computational complexity of MaOEA/D-OPI will be $O(N^2)$.

Conclusion

Optional performance indicators were proposed to be utilized in this paper to balance the convergence and diversity of a population. A many-objective optimization problem is converted to a number of single problems based on reference vectors. The criterion for selecting a solution to survive to the next generation along each reference vector is randomly set to convergence or diversity. The experimental results on DTLZ, WFG and MaF test problems showed good performance of our proposed method for many-objective optimization problems. Nevertheless, when MaOEA/D-OPI is used for handling problems with irregular Pareto fronts, the performance of MaOEA/D-OPI is poor. The main reason is that the reference vectors are fixed in the optimization, which is not suitable for problems with irregular fronts. Thus, in future work, an effective strategy will be studied for enhancing the performance of MaOEA/D-OPI for solving MaOPs with irregular Pareto fronts.

Declarations

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