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Topological approach to generate new rough set models

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Abstract

In this paper, we introduce a topological method to produce new rough set models. This method is based on the idea of "somewhat open sets" which is one of the celebrated generalizations of open sets. We first generate some topologies from the different types of N_ρ-neighborhoods. Then, we define new types of rough approximations and accuracy measures with respect to somewhat open and somewhat closed sets. We study their main properties and prove that the accuracy and roughness measures preserve the monotonic property. One of the unique properties of these approximations is the possibility of comparing between them. We also compare our approach with the previous ones, and show that it is more accurate than those induced from open, α-open, and semi-open sets. Moreover, we examine the effectiveness of the followed method in a problem of Dengue fever. Finally, we discuss the strengths and limitations of our approach and propose some future work.

Keywords Somewhat open set · Upper and lower approximations · Accuracy and roughness measures · Closure and interior operators · Rough set · Topology

Introduction

Rough set theory, proposed by Pawlak [31], is a non-statistical tool to address uncertain knowledge. Every subset in rough set theory is described by two ways are classifications (upper and lower approximations) and accuracy measure. We determine whether the subset is exact or inexact by the boundary region which is known as the difference between the upper and lower approximations. The set's approximations give some insights into the boundary region structure without information of its size. Whereas, the set's accuracy measure shows the boundary region size without saying anything of its structure; it answers the question: To what extent our knowledge is complete?

As we know, rough set theory starts from an equivalence relation which seems a stringent condition that limits the rough set's applications. In an attempt to solve such unreasonableness, some extensions under various relations were proposed such as [46,47]. To different purposes including improving the set's accuracy values, new types of neighborhoods were introduced such as minimal right (left) neighborhoods [4,5], intersection (union) neighborhoods [1], maximal neighborhoods [18], remote neighborhood [42], P_j-neighborhoods [29], E_j-neighborhoods [12], C_j-neighborhoods [7], and recently S_j-neighborhoods [10].

Through rough sets, the concepts are defined according to the information that we know about them. For instance, we say that the sets with different elements are roughly equal if they have identical upper and/or lower approximations. These thoughts refer to the topological spaces when we contrast the sets in terms of their closure and interior points, instead of their elements. In this direction, Skowron [41] and Wiweger [44] discussed rough set theory in view of topological concepts. From binary relations, Lashin et al. [27] generated a topology that is applied to generalize the essential concepts in rough set theory. Abu-Donia [3] made use of rough approximations and topology to introduce multi knowledge bases. Salama [38] applied topological notions to solve the missing attribute values problem. Kondo [24] discussed some methods of generating topologies from coverings of approximation spaces. In [9], the authors explored separation axioms via topological spaces induced from the system of N_j-neighborhoods. El-Bably and Al-shami [16] illustrated some techniques to constitute a topology from different types of neighborhoods. They also discussed a medical application using the concept of generalized nanotopology. Studying the interaction between topology and rough set theory was the main target for many articles such as [2,19,25,26,28,39,40,48]. This path of study also

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included some topology's extensions such as minimal structure [15,17] and bitopology [36]. Hybridization of rough sets with some uncertainty tools such as soft and fuzzy sets was investigated in [32,34].

Near open sets are one of the major areas of research in topology. They are applied to redefine the original topological concepts such as compactness, connectedness, and separation axioms. Abd El-Monsef et al. [1] initiated new kinds of topological approximations in cases of fore-set and after-set using some near open sets. Amer et al. [14] applied five types of near open sets to set up new kinds of topological approximations. Hosny [20] defined new topological approximations using $\delta\beta$-open sets and $\bigwedge_\beta$-sets and proved that her methods produced a higher accuracy than Amer et al.'s methods. Salama [35] made much iterations of closure and interior operators to define higher order sets as a novel family of near closed and open sets. Recently, Al-shami [8] has capitalized from one of the generalizations of open sets called somewhere dense sets to improve the approximations and accuracy measures of rough subsets.

This manuscript contributes to this direction; it exploits a topological concept called "somewhat open sets" to initiate new rough set models. It is natural to ask what are the motivations to introduce these models? In fact, there are four main motivations to study these models are, first, to improve the approximations and increase their accuracy measures displayed in the published literature. This matter was illustrated with the help of some comparisons that validate that our approach is better than those given in [1,14,37]. Second, to keep most properties of Pawlak's approximations that are evaporated by the previous approximations as illustrated in Proposition 3 and Proposition 4. Third, to preserve the monotonic property for the accuracy and roughness measures without further conditions as shown in Proposition 6 and Corollary 2. Finally, we can compare between the different types of ρso-approximations and ρso-accuracy measures (as investigated in Proposition 10 and Corollary 4); this preferred property is not guaranteed for the types of approximations and accuracy measures induced from the other generalizations, because they are defined using interior and closure operators which are working against each other with respect to the size of a set.

The layout of this manuscript is as follows. The concepts and some properties of topological spaces and rough sets that help to understand this work are mentioned in Sect. 2. We divide Sect. 3, the main section, into three subsections. In the first subsection, we utilize somewhat open and somewhat closed sets to present and study new types of approximations and accuracy measures. In the second subsection, we compare the followed technique with the previous ones in terms of the approximations and accuracy measures. In the third subsection, we apply our technique to a medical issue. In Sect. 4, we investigate the advantages of our method and show its limitations compared with the previous methods. Finally, we give some conclusions and suggest some future work in Sect. 5.

Preliminaries

In the current section, we recall the main definitions and results of topology and rough set theory that we need through this article.

Definition 1 [31] Let $\mathcal{E}$ be an equivalence relation in a finite set $U \neq \emptyset$. We associate each $\Omega \subseteq U$ with two subsets

$$\underline{\mathcal{E}}(\Omega) = \bigcup\{G \in U/\mathcal{E} : G \text{ is a subset of } \Omega\}, \text{ and}$$
$$\overline{\mathcal{E}}(\Omega) = \bigcup\{G \in U/\mathcal{E} : G \text{ and } \Omega \text{ has a non-empty intersection. }\}$$

We respectively call $\overline{\mathcal{E}}(\Omega)$ and $\underline{\mathcal{E}}(\Omega)$ upper and lower approximations of Ω.
From now onwards, we consider U to be a non-empty finite set, if not otherwise specified.
The major properties of these approximations are described in the next result.

Proposition 1 [31] *Let $\mathcal{E}$ be an equivalence relation in U and $\Omega, \Sigma \subseteq U$. The next properties are satisfied.*

$(L1)$ $\underline{\mathcal{E}}(\Omega) \subseteq \Omega$
$(U1)$ $\Omega \subseteq \overline{\mathcal{E}}(\Omega)$
$(L2)$ $\underline{\mathcal{E}}(\emptyset) = \emptyset$
$(U2)$ $\overline{\mathcal{E}}(\emptyset) = \emptyset$
$(L3)$ $\underline{\mathcal{E}}(U) = U$
$(U3)$ $\overline{\mathcal{E}}(U) = U$
$(L4)$ $If\, \Omega \subseteq \Sigma,\ then\ \underline{\mathcal{E}}(\Omega) \subseteq \underline{\mathcal{E}}(\Sigma)$
$(U4)$ $If\, \Omega \subseteq \Sigma,\ then\ \overline{\mathcal{E}}(\Omega) \subseteq \overline{\mathcal{E}}(\Sigma)$
$(L5)$ $\underline{\mathcal{E}}(\Omega \cap \Sigma) = \underline{\mathcal{E}}(\Omega) \cap \underline{\mathcal{E}}(\Sigma)$
$(U5)$ $\overline{\mathcal{E}}(\Omega \cap \Sigma) \subseteq \overline{\mathcal{E}}(\Omega) \cap \overline{\mathcal{E}}(\Sigma)$
$(L6)$ $\underline{\mathcal{E}}(\Omega) \cup \underline{\mathcal{E}}(\Sigma) \subseteq \underline{\mathcal{E}}(\Omega \cup \Sigma)$
$(U6)$ $\overline{\mathcal{E}}(\Omega \cup \Sigma) = \overline{\mathcal{E}}(\Omega) \cup \overline{\mathcal{E}}(\Sigma)$
$(L7)$ $\underline{\mathcal{E}}(\Omega^c) = (\overline{\mathcal{E}}(\Omega))^c$
$(U7)$ $\overline{\mathcal{E}}(\Omega^c) = (\underline{\mathcal{E}}(\Omega))^c$
$(L8)$ $\underline{\mathcal{E}}(\underline{\mathcal{E}}(\Omega)) = \underline{\mathcal{E}}(\Omega)$
$(U8)$ $\overline{\mathcal{E}}(\overline{\mathcal{E}}(\Omega)) = \overline{\mathcal{E}}(\Omega)$
$(L9)$ $\underline{\mathcal{E}}((\underline{\mathcal{E}}(\Omega))^c) = (\overline{\mathcal{E}}(\Omega))^c$
$(U9)$ $\overline{\mathcal{E}}((\overline{\mathcal{E}}(\Omega))^c) = (\overline{\mathcal{E}}(\Omega))^c$
$(L10)$ $\forall K \in U/\mathcal{E} \Rightarrow \underline{\mathcal{E}}(K) = K$
$(U10)$ $\forall K \in U/\mathcal{E} \Rightarrow \overline{\mathcal{E}}(K) = K.$

Definition 2 [1,4,5,46,47] Let $\mathcal{E}$ be an arbitrary relation in U. The ρ-neighborhoods of $v \in U$ (denoted by $N_\rho(v)$) are defined for each $\rho \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ as follows:

(i) $N_r(v) = \{w \in U : v\mathcal{E}w\}$.
(ii) $N_l(v) = \{w \in U : w\mathcal{E}v\}$.
(iii)

$$N_{\langle r\rangle}(v) = \begin{cases} \bigcap\limits_{v\in N_r(w)} N_r(w) : & \text{there exists } N_r(w) \text{ including } v \\ \emptyset & : \text{Otherwise.} \end{cases}$$

(iv)

$$N_{\langle l\rangle}(v) = \begin{cases} \bigcap\limits_{v\in N_l(w)} N_l(w) : & \text{there exists } N_l(w) \text{ including } v \\ \emptyset & : \text{Otherwise.} \end{cases}$$

(v) $N_i(v)$ equals the intersection of $N_r(v)$ and $N_l(v)$.
(vi) $N_u(v)$ equals the union of $N_r(v)$ and $N_l(v)$.
(vii) $N_{\langle i\rangle}(v)$ equals the intersection of $N_{\langle r\rangle}(v)$ and $N_{\langle l\rangle}(v)$.
(viii) $N_{\langle u\rangle}(v)$ equals the union of $N_{\langle r\rangle}(v)$ and $N_{\langle l\rangle}(v)$.

From now onwards, we deem $\rho \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$, if not otherwise specified.

Definition 3 [1] Consider $\mathcal{E}$ is an arbitrary relation in U and $\phi_\rho : U \longrightarrow 2^U$ is a map which associates each $v \in U$ with its ρ-neighborhood in 2^U. We call $(U, \mathcal{E}, \phi_\rho)$ a ρ-neighborhood space (briefly, ρ-NS)

A class of subsets of $U \neq \emptyset$ which is closed under finite intersection and arbitrary union is called a topology. A topology is called a quasi-discrete topology (or locally indiscrete topology) if all open subsets are also closed. A topology is called hyperconnected if the closure of any non-empty open set is U. We called a topology a strongly hyperconnected if a set is dense $\Longleftrightarrow$ it is a non-empty open set.

The next theorem provides one of the interesting and significant methods of generating topological spaces using the concept of neighborhoods system. It also opens a door for more interaction between the notions of topological space and rough set theory

Theorem 1 [1] *If $(U, \mathcal{E}, \phi_\rho)$ is a ρ-NS, then a class $\vartheta_\rho = \{G \subseteq U : N_\rho(v) \subseteq G \text{ for each } v \in G\}$ constitutes a topology on U for every ρ.*

Definition 4 [1] A subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ is called ρ-open if $\Omega \in \vartheta_\rho$. The complement of Ω is called ρ-closed.

The class of all ρ-closed sets is denoted by Γ_ρ.

The rough approximations were defined with a topological flavor as follows.

Definition 5 [1] The ρ-lower and ρ-upper approximations of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are, respectively, formulated as follows:

$$\underline{\mathcal{E}}_\rho(\Omega) = \bigcup\{G \in \vartheta_\rho : G \subseteq \Omega\}, \text{ and}$$
$$\overline{\mathcal{E}}_\rho(\Omega) = \bigcap\{H \in \Gamma_\rho : \Omega \subseteq H\}.$$

Obviously, $\underline{\mathcal{E}}_\rho(\Omega)$ and $\overline{\mathcal{E}}_\rho(\Omega)$ are, respectively, the interior and closure of Ω in a topological structure (U, ϑ_ρ). Therefore, we can write $\underline{\mathcal{E}}_\rho(\Omega) = int_\rho(\Omega)$ and $\overline{\mathcal{E}}_\rho(\Omega) = cl_\rho(\Omega)$.

Definition 6 [1] The ρ-boundary, ρ-positive and ρ-negative regions, and ρ-accuracy and ρ-roughness measures of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are, respectively, formulated as follows:

$$B_\rho(\Omega) = \overline{\mathcal{E}}_\rho(\Omega) \backslash \underline{\mathcal{E}}_\rho(\Omega),$$
$$POS_\rho(\Omega) = \underline{\mathcal{E}}_\rho(\Omega),$$
$$NEG_\rho(\Omega) = U \backslash \overline{\mathcal{E}}_\rho(\Omega), \text{ and}$$
$$M_\rho(\Omega) = \frac{|\underline{\mathcal{E}}_\rho(\Omega)|}{|\overline{\mathcal{E}}_\rho(\Omega)|} \text{ provided that } \overline{\mathcal{E}}_\rho(\Omega) \neq \emptyset.$$
$$R_\rho(\Omega) = 1 - M_\rho(\Omega).$$

It is clear that $M_\rho(\Omega) \in [0, 1]$ for every $\Omega \subseteq U$.

Definition 7 (see, [6,13]) A set Ω in a topological structure (U, ϑ) is said to be:

(i) α-open if $\Omega \subseteq int(cl(int(\Omega)))$.
(ii) semi-open if $\Omega \subseteq cl(int(\Omega))$.
(iii) somewhat open if $int(\Omega) \neq \emptyset$.
(iv) somewhere dense if $int(cl(\Omega)) \neq \emptyset$.

Their complements are respectively called α-closed, semi-closed, somewhat closed, and cs-dense sets.

These near open sets were familiarized in a ρ-NS in a similar way.

Definition 8 [14,37] A subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ is said to be $\rho\alpha$-open (resp. ρ-semi-open) if $\Omega \subseteq int_\rho(cl_\rho(int_\rho(\Omega)))$ $(resp.\ \Omega \subseteq cl_\rho(int_\rho(\Omega)))$.

We call Ω^c (complement of Ω) a $\rho\alpha$-closed (resp. ρ-semiclosed) set.

Remark 1 The classes of $\rho\alpha$-open, ρ-semi-open, $\rho\alpha$-closed, and ρ-semiclosed sets are, respectively, symbolized by $\alpha O(\vartheta_\rho)$, $semi O(\vartheta_\rho)$, $\alpha C(\Gamma_\rho)$, and $semi C(\Gamma_\rho)$.

Definition 9 [14,37] For every $k \in \{semi, \alpha\}$, the ρk-lower and ρk-upper approximations of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are defined, respectively, by

$$\underline{\mathcal{E}}^k_\rho(\Omega) = \bigcup\{G \in kO(\vartheta_\rho) : G \subseteq \Omega\} = kint_\rho(\Omega), \text{ and}$$
$$\overline{\mathcal{E}}^k_\rho(\Omega) = \bigcap\{H \in kC(\Gamma_\rho) : \Omega \subseteq H\} = kcl_\rho(\Omega).$$

From now onwards, we consider $k \in \{\alpha, semi\}$, if not otherwise specified.

Definition 10 [14,37] The ρk-boundary, ρk-positive and ρk-negative regions and ρk-accuracy and ρk-roughness measures of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are, respectively, defined by

$$B^k_\rho(\Omega) = \overline{\mathcal{E}}^k_\rho(\Omega)\backslash\underline{\mathcal{E}}^k_\rho(\Omega),$$
$$POS^k_\rho(\Omega) = \underline{\mathcal{E}}^k_\rho(\Omega)$$
$$NEG^k_\rho(\Omega) = U\backslash\overline{\mathcal{E}}^k_\rho(\Omega),$$
$$M^k_\rho(\Omega) = \frac{|\ \underline{\mathcal{E}}^k_\rho(\Omega)\ |}{|\ \overline{\mathcal{E}}^k_\rho(\Omega)\ |} \text{ provided that } \overline{\mathcal{E}}^k_\rho(\Omega) \neq \emptyset, \text{ and}$$
$$R^k_\rho(\Omega) = 1 - M^k_\rho(\Omega).$$

It is clear that $M^k_\rho(\Omega) \in [0, 1]$ for every $\Omega \subseteq U$.

Definition 11 (see, [13]) For a subset Ω of (U, ϑ):

(i) the sw-interior of Ω (briefly, $swint(\Omega)$) is the union of all subsets of Ω that are somewhat open.
(ii) the sw-closure of Ω (briefly, $swcl(\Omega)$) is the intersection of all supersets of Ω that are somewhat closed.

From now on, if we want to compute $N_\rho(v)$, $\underline{\mathcal{E}}^k_\rho(\Omega)$, $\overline{\mathcal{E}}^k_\rho(\Omega)$, $B^k_\rho(\Omega)$, $POS^k_\rho(\Omega)$, $NEG^k_\rho(\Omega)$, and $M^k_\rho(\Omega)$ from two different ρ-NSs $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$, we write $(N_{1\rho}(v), N_{2\rho}(v), \underline{\mathcal{E}}^k_{1\rho}(\Omega), \overline{\mathcal{E}}^k_{1\rho}(\Omega), B^k_{1\rho}(\Omega), POS^k_{1\rho}(\Omega), NEG^k_{1\rho}(\Omega), M^k_{1\rho}(\Omega))$ and $(\underline{\mathcal{E}}^k_{2\rho}(\Omega), \overline{\mathcal{E}}^k_{2\rho}(\Omega), B^k_{2\rho}(\Omega), POS^k_{2\rho}(\Omega), NEG^k_{2\rho}(\Omega), M^k_{2\rho}(\Omega))$.

Definition 12 [18] Consider $\mathcal{E}_1$ and $\mathcal{E}_2$ are two binary relations in U. We say that $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ have the monotonicity-accuracy (resp., monotonicity-roughness) property provided that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ implies that $M_{\mathcal{E}_1}(\Omega) \geq M_{\mathcal{E}_2}(\Omega)$ (resp., $R_{\mathcal{E}_1}(\Omega) \leq R_{\mathcal{E}_2}(\Omega)$).

Proposition 2 *[10] Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2,)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Then, $N_{1\rho}(v) \subseteq N_{2\rho}(v)$ for each $v \in U$ and $\rho \in \{r, l, i, u\}$.*

Approximations using somewhat open sets

In this section, we define new rough approximations and accuracy measures using the concepts of somewhat open and somewhat closed sets which are one of the open sets generalizations. We establish their main properties and prove that our approach offers accuracy measures and approximations better than those displayed by open, α-open, and semi-open sets [1,14,37]. Also, we compare between the approximations induced from our approach and show that the accuracy measures given in cases of $\rho \in \{i, \langle i\rangle\}$ are the best. Finally, we provide a medical example illustrating that how the somewhat open sets are applied to improve the approximations and accuracy measures.

ρso-Lower and ρso-upper approximations

Definition 13 A subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ is said to be ρ-somewhat open if $int_\rho(\Omega) \neq \emptyset$. The complement of Ω is called ρ-somewhat closed.

The classes of ρ-somewhat open and ρ-somewhat closed sets are, respectively, denoted by $so(\vartheta_\rho)$ and $sc(\vartheta_\rho)$.

Definition 14 We define ρso-lower approximation $\underline{\mathcal{E}}^{so}_\rho$ and ρso-upper approximation $\overline{\mathcal{E}}^{so}_\rho$ of a subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ as follows:

$$\underline{\mathcal{E}}^{so}_\rho(\Omega) = \bigcup\{G \in so(\vartheta_\rho) : G \subseteq \Omega\} = swint_\rho(\Omega), \text{ and}$$
$$\overline{\mathcal{E}}^{so}_\rho(\Omega) = \bigcap\{H \in sc(\vartheta_\rho) : \Omega \subseteq H\} = swcl_\rho(\Omega).$$

We elucidate the main properties of ρso-lower and ρso-upper approximations in the following two results.

Proposition 3 *Let Ω and Σ be subsets of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$. Then, the next properties are satisfied.*

(i) $\underline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \Omega$.
(ii) $\underline{\mathcal{E}}^{so}_\rho(\emptyset) = \emptyset$.
(iii) $\underline{\mathcal{E}}^{so}_\rho(U) = U$.
(iv) *If* $\Omega \subseteq \Sigma$, *then* $\underline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Sigma)$.
(v) $\underline{\mathcal{E}}^{so}_\rho(\Omega \cap \Sigma) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega) \cap \underline{\mathcal{E}}^{so}_\rho(\Sigma)$.
(vi) $\underline{\mathcal{E}}^{so}_\rho(\Omega) \cup \underline{\mathcal{E}}^{so}_\rho(\Sigma) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega \cup \Sigma)$.
(vii) $\underline{\mathcal{E}}^{so}_\rho(\Omega^c) = (\overline{\mathcal{E}}^{so}_\rho(\Omega))^c$.
(viii) $\underline{\mathcal{E}}^{so}_\rho(\underline{\mathcal{E}}^{so}_\rho(\Omega)) = \underline{\mathcal{E}}^{so}_\rho(\Omega)$.

Proof The proof comes from the properties of an sw-interior operator which is a counterpart of ρso-near lower approximation $\underline{\mathcal{E}}^{so}_\rho$. □

Proposition 4 *Let Ω and Σ be subsets of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$. Then, the next properties are satisfied.*

(i) $\Omega \subseteq \overline{\mathcal{E}}^{so}_\rho(\Omega)$.
(ii) $\overline{\mathcal{E}}^{so}_\rho(\emptyset) = \emptyset$.
(iii) $\overline{\mathcal{E}}^{so}_\rho(U) = U$.
(iv) *If* $\Omega \subseteq \Sigma$, *then* $\overline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \overline{\mathcal{E}}^{so}_\rho(\Sigma)$.
(v) $\overline{\mathcal{E}}^{so}_\rho(\Omega \cap \Sigma) \subseteq \overline{\mathcal{E}}^{so}_\rho(\Omega) \cap \overline{\mathcal{E}}^{so}_\rho(\Sigma)$.
(vi) $\overline{\mathcal{E}}^{so}_\rho(\Omega) \cup \overline{\mathcal{E}}^{so}_\rho(\Sigma) \subseteq \overline{\mathcal{E}}^{so}_\rho(\Omega \cup \Sigma)$.

(vii) $\overline{\mathcal{E}}_{\rho}^{so}(\Omega^c) = (\underline{\mathcal{E}}_{\rho}^{so}(\Omega))^c$.

(viii) $\overline{\mathcal{E}}_{\rho}^{so}(\overline{\mathcal{E}}_{\rho}^{so}(\Omega)) = \overline{\mathcal{E}}_{\rho}^{so}(\Omega)$.

Proof The proof comes from the properties of an sw-closure operator which is a counterpart of ρso-near upper approximation $\overline{\mathcal{E}}_{\rho}^{so}$. □

The inclusion relations of (i) and (iv-vi) of Proposition 3 and Proposition 4 are proper as the next example validates this matter in case of $\rho = r$.

Example 1 Let $(U, \mathcal{E}, \phi_\rho)$ be a ρ-NS, such that $\mathcal{E} = \{(tx, tx), (ty, ty), (tv, tw), (tv, tx), (tx, tw)\}$ is a relation in the universal set $U = \{tv, tw, tx, ty\}$. Then, $N_r(tv) = N_r(tx) = \{tw, tx\}$, $N_r(tw) = \emptyset$, and $N_r(ty) = \{ty\}$. According to Theorem 1, a topology generated from r-neighborhoods on U is $\vartheta_r = \{\emptyset, U, \{tw\}, \{ty\}, \{tw, ty\}, \{tw, tx\}, \{tw, tx, ty\}, \{tv, tw, tx\}\}$. Let $V = \{tx\}$, $W = \{tv, tw\}$, $\Omega = \{tv, ty\}$, $\Sigma = \{tv, tx\}$, and $Z = \{tw, ty\}$. By calculation, we obtain $\underline{\mathcal{E}}_{r}^{so}(V) = \emptyset$, $\overline{\mathcal{E}}_{r}^{so}(V) = \{tv, tx\}$, $\underline{\mathcal{E}}_{r}^{so}(W) = \overline{\mathcal{E}}_{r}^{so}(W) = W$, $\underline{\mathcal{E}}_{r}^{so}(\Omega) = \overline{\mathcal{E}}_{r}^{so}(\Omega) = \Omega$, $\underline{\mathcal{E}}_{r}^{so}(\Sigma) = \emptyset$, $\overline{\mathcal{E}}_{r}^{so}(\Sigma) = \Sigma$, $\underline{\mathcal{E}}_{r}^{so}(Z) = Z$, and $\overline{\mathcal{E}}_{r}^{so}(Z) = U$. Now, we note the following:

(i) $V \nsubseteq \underline{\mathcal{E}}_{r}^{so}(V)$ and $\overline{\mathcal{E}}_{r}^{so}(V) \nsubseteq V$.

(ii) $\underline{\mathcal{E}}_{r}^{so}(V) \subseteq \underline{\mathcal{E}}_{r}^{so}(W)$, but $V \nsubseteq W$. Also, $\overline{\mathcal{E}}_{r}^{so}(W) \subseteq \overline{\mathcal{E}}_{r}^{so}(Z)$, but $W \nsubseteq Z$

(iii) $\underline{\mathcal{E}}_{r}^{so}(W) \cap \underline{\mathcal{E}}_{r}^{so}(\Omega) = \{tv\} \nsubseteq \underline{\mathcal{E}}_{r}^{so}(W \cap \Omega) = \emptyset$. Also, $\overline{\mathcal{E}}_{r}^{so}(\Sigma) \cap \overline{\mathcal{E}}_{r}^{so}(Z) = \Sigma \nsubseteq \overline{\mathcal{E}}_{r}^{so}(\Sigma \cap Z) = \emptyset$.

(iv) $\underline{\mathcal{E}}_{r}^{so}(V \cup W) = V \cup W = \{tv, tw, tx\} \nsubseteq \underline{\mathcal{E}}_{r}^{so}(V) \cup \underline{\mathcal{E}}_{r}^{so}(W) = \{tv, tw\}$. Also, $\overline{\mathcal{E}}_{r}^{so}(\{tw\} \cup \{ty\}) = U \nsubseteq \overline{\mathcal{E}}_{r}^{so}(\{tw\}) \cup \overline{\mathcal{E}}_{r}^{so}(\{ty\}) = \{tw, ty\}$.

Remark 2 If (U, ϑ_ρ) is a hyperconnected space, then the class of somewhat open sets is closed under finite intersection which means it forms a topology; so that, the equality relations presented in (v) of Proposition 3 and (vi) of Proposition 4 are satisfied. These properties are kept for the approximations defined using somewhere dense sets [8] under a strongly hyperconnected spaces. This implies that our approach preserves all Pawlak properties under a weaker condition.

Proposition 5 *Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\rho \in \{r, l, i, u\}$. Then, $\vartheta_{2\rho} \subseteq \vartheta_{1\rho}$.*

Proof Let a subset G of U be a member in $\vartheta_{2\rho}$, where $\rho \in \{r, l, i, u\}$. Then, $N_{2\rho}(\mu) \subseteq G$ for each $\mu \in G$. Since $\mathcal{E}_1 \subseteq \mathcal{E}_2$, it follows from Proposition 2 that $N_{1\rho}(\mu) \subseteq N_{2\rho}(\mu)$. This implies that G is a member in $\vartheta_{1\rho}$. Hence, $\vartheta_{2\rho} \subseteq \vartheta_{1\rho}$, as required. □

Corollary 1 *Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\rho \in \{r, l, i, u\}$. Then, the class of somewhat open sets in $(U, \vartheta_{2\rho})$ is a subset of the class of somewhat open sets in $(U, \vartheta_{1\rho})$.*

Definition 15 The ρso-accuracy measure and ρso-roughness measure of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are defined, respectively, by

$$M_{\rho}^{so}(\Omega) = \frac{|\underline{\mathcal{E}}_{\rho}^{so}(\Omega)|}{|\overline{\mathcal{E}}_{\rho}^{so}(\Omega)|} \text{ provided that } \overline{\mathcal{E}}_{\rho}^{so}(\Omega) \neq \emptyset.$$

$$H_{\rho}^{so}(\Omega) = 1 - M_{\rho}^{so}(\Omega).$$

Obviously, $M_{\rho}^{so}(\Omega), H_{\rho}^{so}(\Omega) \in [0, 1]$ for every subset Ω of U.

In the following two results, we show the monotonicity of M_{ρ}^{so}-accuracy and M_{ρ}^{so}-roughness measures.

Proposition 6 *Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\rho \in \{r, l, i, u\}$. Then, $M_{1\rho}^{so}(\Omega) \geq M_{2\rho}^{so}(\Omega)$ for every set Ω.*

Proof Since $\underline{\mathcal{E}}_{\rho}^{so}(\Omega) = swint_\rho(\Omega)$ and $\overline{\mathcal{E}}_{\rho}^{so}(\Omega) = swcl_\rho(\Omega)$, it follows from Corollary 1 that $|\underline{\mathcal{E}}_{2\rho}^{so}(\Omega)| \leq |\underline{\mathcal{E}}_{1\rho}^{so}(\Omega)|$ and $\frac{1}{|\overline{\mathcal{E}}_{2\rho}^{so}(\Omega)|} \leq \frac{1}{|\overline{\mathcal{E}}_{1\rho}^{so}(\Omega)|}$. Therefore, $\frac{|\underline{\mathcal{E}}_{2\rho}^{so}(\Omega)|}{|\overline{\mathcal{E}}_{2\rho}^{so}(\Omega)|} \leq \frac{|\underline{\mathcal{E}}_{1\rho}^{so}(\Omega)|}{|\overline{\mathcal{E}}_{1\rho}^{so}(\Omega)|}$ which means that $M_{1\rho}^{so}(\Omega) \geq M_{2\rho}^{so}(\Omega)$. Hence, the desired result is obtained. □

Corollary 2 *Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\rho \in \{r, l, i, u\}$. Then, $H_{1\rho}^{so}(\Omega) \leq H_{2\rho}^{so}(\Omega)$ for every set Ω.*

Definition 16 A subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ is called ρso-exact if $\underline{\mathcal{E}}_{\rho}^{so}(\Omega) = \overline{\mathcal{E}}_{\rho}^{so}(\Omega) = \Omega$. Otherwise, it is called a ρso-rough set.

From the well-known relationships between α-open ($semi$-open) and so-open sets, we easily note that $\rho\alpha$-exact ($\rho semi$-exact) set is ρso-exact, but the converses need not be true as the next example elucidates.

Example 2 Let $\Omega = \{tv, ty\}$ be a set in a r-NS $(U, \mathcal{E}, \phi_r)$ displayed in Example 1. As we showed that $\underline{\mathcal{E}}_{r}^{so}(\Omega) = \overline{\mathcal{E}}_{r}^{so}(\Omega) = \Omega$. Then, Ω is a rso-exact set. However, $\underline{\mathcal{E}}_{r}^{semi}(\Omega) = \underline{\mathcal{E}}_{r}^{\alpha}(\Omega) = \{ty\} \neq \overline{\mathcal{E}}_{r}^{semi}(\Omega) = \overline{\mathcal{E}}_{r}^{\alpha}(\Omega) = \Omega$; so that, Ω is neither a $rsemi$-exact set nor a $r\alpha$-exact set.

Proposition 7 *A set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ is ρso-exact iff $B_{\rho}^{so}(\Omega) = \emptyset$.*

Proof Let Ω be a ρso-exact set. Then, $B_{\rho}^{so}(\Omega) = \overline{\mathcal{E}}_{\rho}^{so}(\Omega) \setminus \underline{\mathcal{E}}_{\rho}^{so}(\Omega) = \overline{\mathcal{E}}_{\rho}^{so}(\Omega) \setminus \overline{\mathcal{E}}_{\rho}^{so}(\Omega) = \emptyset$. Conversely, let $B_{\rho}^{so}(\Omega) = \emptyset$. Then, $\overline{\mathcal{E}}_{\rho}^{so}(\Omega) \setminus \underline{\mathcal{E}}_{\rho}^{so}(\Omega) = \emptyset$ which means that $\overline{\mathcal{E}}_{\rho}^{so}(\Omega) \subseteq \underline{\mathcal{E}}_{\rho}^{so}(\Omega)$. However, $\underline{\mathcal{E}}_{\rho}^{so}(\Omega) \subseteq \overline{\mathcal{E}}_{\rho}^{so}(\Omega)$, so that $\overline{\mathcal{E}}_{\rho}^{so}(\Omega) = \underline{\mathcal{E}}_{\rho}^{so}(\Omega)$. Hence, Ω is ρso-exact. □

Definition 17 The ρso-boundary, ρso-positive, and ρso-negative regions of a set Ω in a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ are defined, respectively, by

$$B^{so}_\rho(\Omega) = \overline{\mathcal{E}}^{so}_\rho(\Omega) \backslash \underline{\mathcal{E}}^{so}_\rho(\Omega),$$
$$POS^{so}_\rho(\Omega) = \underline{\mathcal{E}}^{so}_\rho(\Omega),$$
$$NEG^{so}_\rho(\Omega) = U \backslash \overline{\mathcal{E}}^{so}_\rho(\Omega).$$

The proof of the following proposition comes from Proposition 6.

Proposition 8 *Let $(U, \mathcal{E}_1, \phi_\rho)$ and $(U, \mathcal{E}_2, \phi_\rho)$ be two ρ-NSs, such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\rho \in \{r, l, i, u\}$. Then, we have the following results for every non-empty set Ω and $\rho \in \{r, l, i, u\}$.*

(i) $B^{so}_{1\rho}(\Omega) \subseteq B^{so}_{2\rho}(\Omega)$.
(ii) $NEG^{so}_{2\rho}(\Omega) \subseteq NEG^{so}_{1\rho}(\Omega)$.

Proposition 9 *Let ϑ_1 and ϑ_2 be two topologies on U, such that $\vartheta_1 \subseteq \vartheta_2$. Then, $so(\vartheta_1) \subseteq so(\vartheta_2)$ and $sc(\vartheta_1) \subseteq sc(\vartheta_2)$,*

Proof Let $G \subseteq U$ be a set in $so(\vartheta_1)$. Then, $int_{\vartheta_1}(G) \neq \emptyset$. By hypothesis $\vartheta_1 \subseteq \vartheta_2$, we obtain $int_{\vartheta_2}(G) \neq \emptyset$. Therefore, $G \in so(\vartheta_2)$. Thus, $so(\vartheta_1) \subseteq so(\vartheta_2)$. Similarly, it can be proved that $sc(\vartheta_1) \subseteq sc(\vartheta_2)$. □

Corollary 3 *Let ϑ_1 and ϑ_2 be two topologies on U such that $\vartheta_1 \subseteq \vartheta_2$. Then, $swint_{\vartheta_1}(\Omega) \subseteq swint_{\vartheta_2}(\Omega)$ and $swcl_{\vartheta_2}(\Omega) \subseteq swcl_{\vartheta_1}(\Omega)$ for every $\Omega \subseteq U$.*

Now, we are in a position to prove the following two results which are a unique characteristic of the accuracy measures and approximations obtained from somewhat open sets. They mainly show that the larger the given topologies are, the better the accuracy measures are.

Proposition 10 *Let $(U, \mathcal{E}, \phi_\rho)$ be a ρ-NS and $\Omega \subseteq U$. Then*

(i) $\underline{\mathcal{E}}^{so}_u(\Omega) \subseteq \underline{\mathcal{E}}^{so}_r(\Omega) \subseteq \underline{\mathcal{E}}^{so}_i(\Omega)$.
(ii) $\underline{\mathcal{E}}^{so}_u(\Omega) \subseteq \underline{\mathcal{E}}^{so}_l(\Omega) \subseteq \underline{\mathcal{E}}^{so}_i(\Omega)$.
(iii) $\underline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega) \subseteq \underline{\mathcal{E}}^{so}_{\langle r\rangle}(\Omega) \subseteq \underline{\mathcal{E}}^{so}_{\langle i\rangle}(\Omega)$.
(iv) $\underline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega) \subseteq \underline{\mathcal{E}}^{so}_{\langle l\rangle}(\Omega) \subseteq \underline{\mathcal{E}}^{so}_{\langle i\rangle}(\Omega)$.
(v) $\overline{\mathcal{E}}^{so}_i(\Omega) \subseteq \overline{\mathcal{E}}^{so}_r(\Omega) \subseteq \overline{\mathcal{E}}^{so}_u(\Omega)$.
(vi) $\overline{\mathcal{E}}^{so}_i(\Omega) \subseteq \overline{\mathcal{E}}^{so}_l(\Omega) \subseteq \overline{\mathcal{E}}^{so}_u(\Omega)$.
(vii) $\overline{\mathcal{E}}^{so}_{\langle i\rangle}(\Omega) \subseteq \overline{\mathcal{E}}^{so}_{\langle r\rangle}(\Omega) \subseteq \overline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega)$.
(viii) $\overline{\mathcal{E}}^{so}_{\langle i\rangle}(\Omega) \subseteq \overline{\mathcal{E}}^{so}_{\langle l\rangle}(\Omega) \subseteq \overline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega)$.

Proof To prove (i), let $\mu \in \underline{\mathcal{E}}^{so}_u(\Omega)$. Then there exists $G \in so(\vartheta_u)$, such that $\mu \in G \subseteq \Omega$. Since $\vartheta_u \subseteq \vartheta_r$, it follows from Proposition 9 that $so(\vartheta_u) \subseteq so(\vartheta_r)$. Therefore, $\mu \in \underline{\mathcal{E}}^{so}_r(\Omega)$. Thus, $\underline{\mathcal{E}}^{so}_u(\Omega) \subseteq \underline{\mathcal{E}}^{so}_r(\Omega)$. Similarly, we prove that $\underline{\mathcal{E}}^{so}_r(\Omega) \subseteq \underline{\mathcal{E}}^{so}_i(\Omega)$.

Table 1 N_ρ-neighborhood of every member in U

	tv	tw	tx	ty
$\mathcal{N}_r$	$\{tw, tx\}$	$\emptyset$	$\{tw, tx\}$	$\{ty\}$
$\mathcal{N}_l$	$\emptyset$	$\{tv, tx\}$	$\{tv, tx\}$	$\{ty\}$
$\mathcal{N}_i$	$\emptyset$	$\emptyset$	$\{tx\}$	$\{ty\}$
$\mathcal{N}_u$	$\{tw, tx\}$	$\{tv, tx\}$	$\{tv, tw, tx\}$	$\{ty\}$
$\mathcal{N}_{\langle r\rangle}$	$\emptyset$	$\{tw, tx\}$	$\{tw, tx\}$	$\{ty\}$
$\mathcal{N}_{\langle l\rangle}$	$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$	$\{ty\}$
$\mathcal{N}_{\langle i\rangle}$	$\emptyset$	$\emptyset$	$\{tx\}$	$\{ty\}$
$\mathcal{N}_{\langle u\rangle}$	$\{tv, tx\}$	$\{tw, tx\}$	$\{tv, tw, tx\}$	$\{ty\}$

To prove (v), let $\mu \in \overline{\mathcal{E}}^{so}_i(\Omega)$. Then every somewhat closed set in ϑ_i containing μ has a non-empty intersection with Ω. Since $sc(\vartheta_r) \subseteq sc(\vartheta_i)$, every somewhat closed set in ϑ_r containing μ has a non-empty intersection with Ω. So that, $\mu \in \overline{\mathcal{E}}^{so}_r(\Omega)$. Thus, $\overline{\mathcal{E}}^{so}_i(\Omega) \subseteq \overline{\mathcal{E}}^{so}_r(\Omega)$. Similarly, we prove that $\overline{\mathcal{E}}^{so}_r(\Omega) \subseteq \overline{\mathcal{E}}^{so}_u(\Omega)$.

Following similar arguments, the other cases are proved. □

Corollary 4 *Let Ω be a subset of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$. Then*

(i) $M^{so}_u(\Omega) \leq M^{so}_r(\Omega) \leq M^{so}_i(\Omega)$.
(ii) $M^{so}_u(\Omega) \leq M^{so}_l(\Omega) \leq M^{so}_i(\Omega)$.
(iii) $M^{so}_{\langle u\rangle}(\Omega) \leq M^{so}_{\langle r\rangle}(\Omega) \leq M^{so}_{\langle i\rangle}(\Omega)$.
(iv) $M^{so}_{\langle u\rangle}(\Omega) \leq M^{so}_{\langle l\rangle}(\Omega) \leq M^{so}_{\langle i\rangle}(\Omega)$.

Proof We give a proof for (i). The other cases are proved similarly.
Since $\underline{\mathcal{E}}^{so}_u(\Omega) \subseteq \underline{\mathcal{E}}^{so}_r(\Omega) \subseteq \underline{\mathcal{E}}^{so}_i(\Omega)$, we obtain

$$| \underline{\mathcal{E}}^{so}_u(\Omega) | \leq | \underline{\mathcal{E}}^{so}_r(\Omega) | \leq | \underline{\mathcal{E}}^{so}_i(\Omega) | . \tag{1}$$

Since $\overline{\mathcal{E}}^{so}_i(\Omega) \subseteq \overline{\mathcal{E}}^{so}_r(\Omega) \subseteq \overline{\mathcal{E}}^{so}_u(\Omega)$, we obtain $| \overline{\mathcal{E}}^{so}_i(\Omega) | \leq | \overline{\mathcal{E}}^{so}_r(\Omega) | \leq | \overline{\mathcal{E}}^{so}_u(\Omega) |$. Therefore

$$\frac{1}{| \overline{\mathcal{E}}^{so}_u(\Omega) |} \leq \frac{1}{| \overline{\mathcal{E}}^{so}_r(\Omega) |} \leq \frac{1}{| \overline{\mathcal{E}}^{so}_i(\Omega)(\Omega) |}. \tag{2}$$

By (1) and (2), we find

$$\frac{| \underline{\mathcal{E}}^{so}_u(\Omega) |}{| \overline{\mathcal{E}}^{so}_u(\Omega) |} \leq \frac{| \underline{\mathcal{E}}^{so}_r(\Omega) |}{| \overline{\mathcal{E}}^{so}_r(\Omega) |} \leq \frac{| \underline{\mathcal{E}}^{so}_i(\Omega) |}{| \overline{\mathcal{E}}^{so}_i(\Omega) |}.$$

Hence, the proof is complete. □

To confirm the results obtained in the above proposition and corollary, we consider a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ presented in Example 1. First, we compute the different types of N_ρ-neighborhoods in Table 1.

Table 2 The approximations and their accuracy measures when $\rho \in \{u, r, l, i\}$

Ω	$\underline{\mathcal{E}}_u^{so}(\Omega)$	$\overline{\mathcal{E}}_u^{so}(\Omega)$	$M_u^{so}(\Omega)$	$\underline{\mathcal{E}}_r^{so}(\Omega)$	$\overline{\mathcal{E}}_r^{so}(\Omega)$	$M_r^{so}(\Omega)$	$\underline{\mathcal{E}}_l^{so}(\Omega)$	$\overline{\mathcal{E}}_l^{so}(\Omega)$	$M_l^{so}(\Omega)$	$\underline{\mathcal{E}}_i^{so}(\Omega)$	$\overline{\mathcal{E}}_i^{so}$	$M_i^{so}(\Omega)$
$\{tv\}$	$\emptyset$	$\{tv\}$	0	$\emptyset$	$\{tv\}$	0	$\{tv\}$	$\{tv\}$	1	$\{tv\}$	$\{tv\}$	1
$\{tw\}$	$\emptyset$	$\{tw\}$	0	$\{tw\}$	$\{tw\}$	1	$\emptyset$	$\{tw\}$	0	$\{tw\}$	$\{tw\}$	1
$\{tx\}$	$\emptyset$	$\{tx\}$	0	$\emptyset$	$\{tx\}$	0	$\emptyset$	$\{tx\}$	0	$\{tx\}$	$\{tx\}$	1
$\{ty\}$	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1
$\{tv, tw\}$	$\emptyset$	$\{tv, tw\}$	0	$\{tv, tw\}$	$\{tv, tw\}$	1	$\{tv, tw\}$	$\{tv, tw\}$	1	$\{tv, tw\}$	$\{tv, tw\}$	1
$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$	0	$\emptyset$	$\{tv, tx\}$	0	$\{tv, tx\}$	$\{tv, tx\}$	1	$\{tv, tx\}$	$\{tv, tx\}$	1
$\{tv, ty\}$	$\{tv, ty\}$	U	$\frac{1}{2}$	$\{tv, ty\}$	$\{tv, ty\}$	1	$\{tv, ty\}$	U	$\frac{1}{2}$	$\{tv, ty\}$	$\{tv, ty\}$	1
$\{tw, tx\}$	$\emptyset$	$\{tw, tx\}$	0	$\{tw, tx\}$	$\{tw, tx\}$	1	$\emptyset$	$\{tw, tx\}$	0	$\{tw, tx\}$	$\{tw, tx\}$	1
$\{tw, ty\}$	$\{tw, ty\}$	U	$\frac{1}{2}$	$\{tw, ty\}$	U	$\frac{1}{2}$	$\{tw, ty\}$	$\{tw, ty\}$	1	$\{tw, ty\}$	$\{tw, ty\}$	1
$\{tx, ty\}$	$\{tx, ty\}$	U	$\frac{1}{2}$	$\{tx, ty\}$	$\{tx, ty\}$	1	$\{tx, ty\}$	$\{tx, ty\}$	1	$\{tx, ty\}$	$\{tx, ty\}$	1
$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1
$\{tv, tw, ty\}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	$\{tv, tw, ty\}$	1
$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	U	$\frac{3}{4}$	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	1	$\{tv, tx, ty\}$	U	$\frac{3}{4}$	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	1
$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	U	$\frac{3}{4}$	$\{tw, tx, ty\}$	U	$\frac{3}{4}$	$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	1	$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	1
U	U	U	1	U	U	1	U	U	1	U	U	1

Second, we apply Theorem 1 to determine the topologies ϑ_ρ generated from these neighborhoods as follows:

$$\begin{cases} \vartheta_r = & \{\emptyset, U, \{tw\}, \{ty\}, \{tw, ty\}, \{tw, tx\}, \{tv, tw, tx\}, \{tw, tx, ty\}\}; \\ \vartheta_l = & \{\emptyset, U, \{ty\}, \{tv\}, \{tv, ty\}, \{tv, tx\}, \{tv, tx, ty\}, \{tv, tw, tx\}\}; \\ \vartheta_i = & P(U); \\ \vartheta_u = & \{\emptyset, U, \{ty\}, \{tv, tw, tx\}\}; \\ \vartheta_{\langle r\rangle} = & \{\emptyset, U, \{tv\}, \{ty\}, \{tv, ty\}, \{tw, tx\}, \{tv, tw, tx\}, \{tw, tx, ty\}\}; \\ \vartheta_{\langle l\rangle} = & \{\emptyset, U, \{tw\}, \{ty\}, \{tw, ty\}, \{tv, tx\}, \{tv, tw, tx\}, \{tv, tx, ty\}\}; \\ \vartheta_{\langle i\rangle} = & P(U); \\ \vartheta_{\langle u\rangle} = & \{\emptyset, U, \{ty\}, \{tv, tw, tx\}\}. \end{cases} \tag{3}$$

Finally, we compute the approximations and their accuracy measures for $\rho \in \{u, r, l, i\}$ in Table 2, and for $\rho \in \{\langle u\rangle, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$ in Table 3.

It can be seen from Tables 2 and 3 that the approximations and their accuracy measures in case of $\rho = i$ are better than those given in cases of $\rho = r, l, u$, and the approximations and their accuracy measures in case of $\rho = \langle i\rangle$ are better than those given in cases of $\rho = \langle r\rangle, \langle l\rangle, \langle u\rangle$. This is due to that the topology generated by N_i-neighborhoods contains the topologies generated by N_r-neighborhoods, N_l-neighborhoods and N_u-neighborhoods, and the topology generated by $N_{\langle i\rangle}$-neighborhoods contains the topologies generated by $N_{\langle r\rangle}$-neighborhoods, $N_{\langle l\rangle}$-neighborhoods, and $N_{\langle u\rangle}$-neighborhoods.

In fact, these comparisons are a unique characteristic of the approximations and accuracy measures induced from somewhat open sets, because somewhat open sets are only based on a factor of interior operator which is proportional to the size of a given topology; and we know that $\vartheta_u \subseteq (\vartheta_r \bigcup \vartheta_l) \subseteq \vartheta_i$ and $\vartheta_{\langle u\rangle} \subseteq (\vartheta_{\langle r\rangle} \bigcup \vartheta_{\langle l\rangle}) \subseteq \vartheta_{\langle i\rangle}$. On the other hand, the approach of α-open (semi-open, pre-open, b-open, β-open) sets is based on two factors, interior, and closure operators which are working against each other with respect to the size of a given topology. Therefore, the approximations and accuracy measures induced from these approaches are incomparable.

In Algorithm 1 and Flowchart (in Fig. 1), we show how the accuracy measures induced from the family of somewhat open and somewhat closed sets are calculated.

Table 3 The approximations and their accuracy measure when $\rho \in \{\langle u\rangle, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$

Ω	$\underline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega)$	$\overline{\mathcal{E}}^{so}_{\langle u\rangle}(\Omega)$	$M^{so}_{\langle u\rangle}(\Omega)$	$\underline{\mathcal{E}}^{so}_{\langle r\rangle}(\Omega)$	$\overline{\mathcal{E}}^{so}_{\langle r\rangle}(\Omega)$	$M^{so}_{\langle r\rangle}(\Omega)$	$\underline{\mathcal{E}}^{so}_{\langle l\rangle}(\Omega)$	$\overline{\mathcal{E}}^{so}_{\langle l\rangle}(\Omega)$	$M^{so}_{\langle l\rangle}(\Omega)$	$\underline{\mathcal{E}}^{so}_{\langle i\rangle}(M)$	$\overline{\mathcal{E}}^{so}_{\langle i\rangle}$	$M^{so}_{\langle i\rangle}(M)$
$\{tv\}$	$\emptyset$	$\{tv\}$	0	$\{tv\}$	$\{tv\}$	1	$\emptyset$	$\{tv\}$	0	$\{tv\}$	$\{tv\}$	1
$\{tw\}$	$\emptyset$	$\{tw\}$	0	$\emptyset$	$\{tw\}$	0	$\{tw\}$	$\{tw\}$	1	$\{tw\}$	$\{tw\}$	1
$\{tx\}$	$\emptyset$	$\{tx\}$	0	$\emptyset$	$\{tx\}$	0	$\emptyset$	$\{tx\}$	0	$\{tx\}$	$\{tx\}$	1
$\{ty\}$	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1	$\{ty\}$	$\{ty\}$	1
$\{tv, tw\}$	$\emptyset$	$\{tv, tw\}$	0	$\{tv, tw\}$	$\{tv, tw\}$	1	$\{tv, tw\}$	$\{tv, tw\}$	1	$\{tv, tw\}$	$\{tv, tw\}$	1
$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$	0	$\{tv, tx\}$	$\{tv, tx\}$	1	$\{tv, tx\}$	$\{tv, tx\}$	1	$\{tv, tx\}$	$\{tv, tx\}$	1
$\{tv, ty\}$	$\{tv, ty\}$	U	$\frac{1}{2}$	$\{tv, ty\}$	$\{tv, ty\}$	1	$\{tv, ty\}$	$\{tv, ty\}$	1	$\{tv, ty\}$	$\{tv, ty\}$	1
$\{tw, tx\}$	$\emptyset$	$\{tw, tx\}$	0	$\{tw, tx\}$	$\{tw, tx\}$	1	$\{tw, tx\}$	$\{tw, tx\}$	1	$\{tw, tx\}$	$\{tw, tx\}$	1
$\{tw, ty\}$	$\{tw, ty\}$	U	$\frac{1}{2}$	$\{tw, ty\}$	$\{tw, ty\}$	1	$\{tw, ty\}$	$\{tw, ty\}$	1	$\{tw, ty\}$	$\{tw, ty\}$	1
$\{tx, ty\}$	$\{tx, ty\}$	U	$\frac{1}{2}$	$\{tx, ty\}$	$\{tx, ty\}$	1	$\{tx, ty\}$	$\{tx, ty\}$	1	$\{tx, ty\}$	$\{tx, ty\}$	1
$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	1
$\{tv, tw, ty\}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	U	$\frac{3}{4}$	$\{tv, tw, ty\}$	$\{tv, tw, ty\}$	1
$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	U	$\frac{3}{4}$	$\{tv, tx, ty\}$	U	$\frac{3}{4}$	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	1	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	1
$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	U	$\frac{3}{4}$	$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	1	$\{tw, tx, ty\}$	U	$\frac{3}{4}$	$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	1
U	U	U	1	U	U	1	U	U	1	U	U	1

Input : A binary relation $\mathcal{E}$ that associated the elements of the universal set U under study.
Output: An accuracy measure $\mathcal{E}^{so}_{\rho}$ of each subset Ω of U.

Input the binary relation $\mathcal{E}$ that associated the elements of the universal set U;
for *for each* ρ **do**
  Compute N_ρ-neighborhoods induced from $\mathcal{E}$;
  Generate a topology θ_ρ using Theorem 1;
  Let $so(\theta_\rho) = \{H \subseteq U : int_\rho(H) \neq \emptyset\}$;
  if $\Omega \in so(\theta_\rho)$ **then**
    Put $\underline{\mathcal{E}}^{so}_{\rho}(\Omega) = \Omega$;
    Compute $swcl_\rho(\Omega)$;
    Put $\overline{\mathcal{E}}^{so}_{\rho}(\Omega) = swcl_\rho(\Omega)$;
    Calculate $M^{so}_{\rho}(\Omega) = \frac{|\underline{\mathcal{E}}^{so}_{\rho}(\Omega)|}{|\overline{\mathcal{E}}^{so}_{\rho}(\Omega)|}$
  else
    $M^{so}_{\rho}(\Omega) = 0$
  end
end

Algorithm 1: The algorithm of accuracy measures induced from the family of somewhat open and somewhat closed sets.

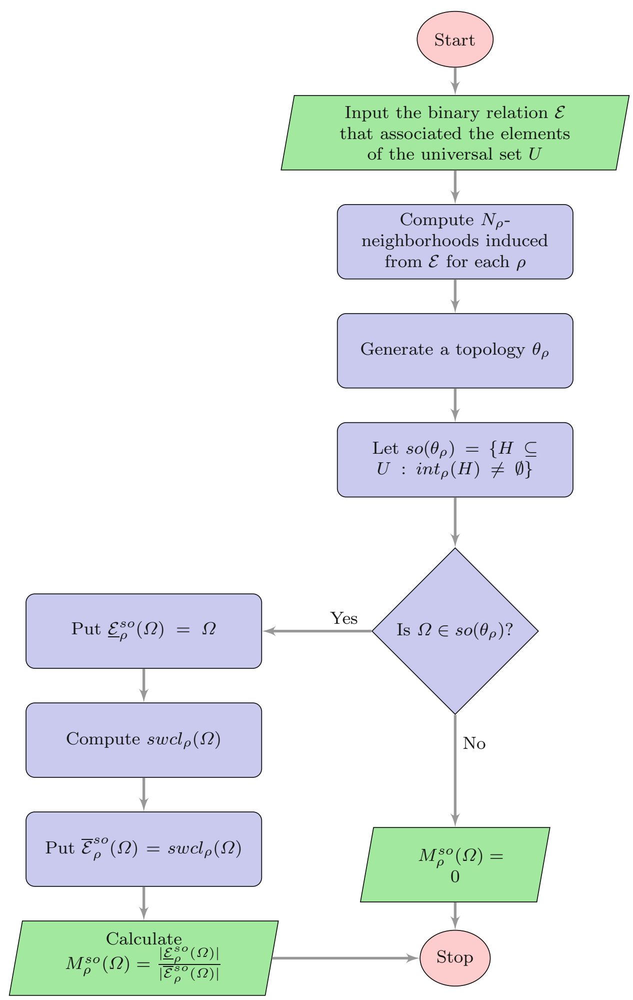

Fig. 1 Flowchart of the accuracy measures induced from the family of somewhat open and somewhat closed sets

Comparison of our approach with the previous ones

In this subsection, we compare our approach with the previous approaches introduced in [1,14,37]. In [1], the authors approximated a subset using interior and closure topological operators, whereas the authors of [14,37] approximated a subset using some generalizations of interior and closure topological operators, such as α-interior and α-closure and semi-interior and semi-closure topological operators. Through this subsection, we show that our approach improves the approximations and accuracy measures more than the approaches induced from open sets as given in [1] and the approaches induced from α-open and semi-open sets as given in [14,37].

We begin with the following two results which show the grade of approximations and accuracy values according to some generalizations of open sets.

Theorem 2 *Let $(U, \mathcal{E}, \phi_\rho)$ be a ρ-NS and $\Omega \subseteq U$. Then*

$$\underline{\mathcal{E}}_\rho(\Omega) \subseteq \underline{\mathcal{E}}^k_\rho(\Omega) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \Omega \subseteq \overline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \overline{\mathcal{E}}^k_\rho(\Omega) \subseteq \overline{\mathcal{E}}_\rho(\Omega), where k \in \{\alpha o, semio\}.$$

Proof As we know that the class of α-open (semi-open) subsets of (U, ϑ_ρ) contains a topology ϑ_ρ. Then, for each $\Omega \subseteq U$, we have $\underline{\mathcal{E}}_\rho(\Omega) \subseteq \underline{\mathcal{E}}^k_\rho(\Omega)$. Also, the class of somewhat open subsets of (Ω, ϑ_ρ) contains the classes of α-open and semi-open subsets. Then, $\underline{\mathcal{E}}^k_\rho(\Omega) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega)$. It comes from Proposition 3 that $\underline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \Omega$. Hence, $\underline{\mathcal{E}}_\rho(\Omega) \subseteq \underline{\mathcal{E}}^k_\rho(\Omega) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \Omega$. Similarly, we prove that $\Omega \subseteq \overline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \overline{\mathcal{E}}^k_\rho(\Omega) \subseteq \overline{\mathcal{E}}_\rho(\Omega)$. □

Proposition 11 *The next two results are satisfied for every subset Ω of a ρ-NS $(U, \mathcal{E}, \phi_\rho)$ and $k \in \{\alpha, semi\}$.*

(i) $B^{so}_\rho(\Omega) \subseteq B^k_\rho(\Omega) \subseteq B_\rho(\Omega)$.
(ii) $M_\rho(\Omega) \leq M^k_\rho(\Omega) \leq M^{so}_\rho(\Omega)$.

Proof (i): The proof comes from Theorem 2.
(ii): According to Theorem 2, we obtain $\underline{\mathcal{E}}^k_\rho(\Omega) \subseteq \underline{\mathcal{E}}^{so}_\rho(\Omega)$ and $\overline{\mathcal{E}}^{so}_\rho(\Omega) \subseteq \overline{\mathcal{E}}^k_\rho(\Omega)$. This means that $\mid \underline{\mathcal{E}}^k_\rho(\Omega) \mid \leq \mid \underline{\mathcal{E}}^{so}_\rho(\Omega) \mid$ and $\mid \overline{\mathcal{E}}^{so}_\rho(\Omega) \mid \leq \mid \overline{\mathcal{E}}^k_\rho(\Omega) \mid$. Therefore, $\mid \underline{\mathcal{E}}^k_\rho(\Omega) \mid \times \mid \overline{\mathcal{E}}^{so}_\rho(\Omega) \mid \leq \mid \underline{\mathcal{E}}^{so}_\rho(\Omega) \mid \times \mid \overline{\mathcal{E}}^k_\rho(\Omega) \mid$. Thus, we get the next inequality

$$\frac{\mid \underline{\mathcal{E}}^k_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}^k_\rho(\Omega) \mid} \leq \frac{\mid \underline{\mathcal{E}}^{so}_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}^{so}_\rho(\Omega) \mid}. \quad (4)$$

Similarly, we get the next inequality

$$\frac{\mid \underline{\mathcal{E}}_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}_\rho(\Omega) \mid} \leq \frac{\mid \underline{\mathcal{E}}^k_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}^k_\rho(\Omega) \mid}. \quad (5)$$

It follows from the two equalities (4) and (5) that:

$$\frac{\mid \underline{\mathcal{E}}_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}_\rho(\Omega) \mid} \leq \frac{\mid \underline{\mathcal{E}}^k_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}^k_\rho(\Omega) \mid} \leq \frac{\mid \underline{\mathcal{E}}^{so}_\rho(\Omega) \mid}{\mid \overline{\mathcal{E}}^{so}_\rho(\Omega) \mid}.$$

Hence, the proof is complete. □

We give the following example to confirm that our approach gives accuracy measures and approximations better than the methods introduced in [1] and the methods introduced in [14,37] in cases of α-open and semi-open sets. For the sake of economy, we only illustrate case $\rho = r$.

Example 3 Let $(U, \mathcal{E}, \phi_r)$ be a ρ-NS given in Example 1. Then, $\vartheta_r = \{\emptyset, U, \{ty\}, \{tw\}, \{ty, tw\}, \{tw, tx\}, \{tv, tw, tx\}, \{tw, tx, ty\}\}$. the family of semi-open sets contains the family of α-open sets, so that we will suffice by the class of semi-open sets.

$semio(\vartheta_r) = \{\emptyset, U, \{ty\}, \{tw\}, \{tw, ty\}, \{tw, tx\}, \{tv, tw\}, \{tv, ty\}, \{tw, tx, ty\}, \{tv, tw, tx\}, \{tv, tw, ty\}\}$, and $so(\vartheta_r) = \{\emptyset, U, \{ty\}, \{tw\}, \{tw, ty\}, \{tw, tx\}, \{tv, tw\}, \{tv, ty\}, \{tx, ty\}, \{tw, tx, ty\}, \{tv, tw, tx\}, \{tv, tw, ty\}, \{tv, tx, ty\}\}$.

Table 4 presents the r-approximations, $rsemi$-approximations, and our approximations for all subsets of U.

Now, we initiate Table 5 to compare between the r-accuracy, $rsemi$-accuracy, and rso-accuracy for all subsets of U.

Tables 4 and 5 display some approximations and accuracy measures that are generated from three different methods are (1) open and closed subsets of r-neighborhood topology, (2) semi-open and semi-closed subsets of r-neighborhood topology, and (3) somewhat open and somewhat closed subsets of r-neighborhood topology. It is clear that our approach reduces the size of boundary regions and increases the accuracy measures of subsets more than the other two methods. This is due to that the class of somewhat open sets is wider than the classes of open and semi-open sets which leads to maximizing the ρso-lower approximation and minimizing the ρso-upper approximation. Hence, the accuracy measures are increasing. Finally, it should be noted that the two classes of somewhat open and semi-open sets coincide if the generated topology is hyperconnected which means that our approach and semi-open approach produce identical approximations and accuracy measures. To elucidate this matter, consider $\mathcal{E} = \{(tv, tx), (tx, tv), (tv, tw), (tx, tw)\}$ to be a relation in $U = \{tv, tw, tx\}$. Then, $N_r(tv) = N_r(tx) = \{tw, tx\}$ and $N_r(tw) = \emptyset$. Therefore, $\vartheta_r = \{\emptyset, U, \{tw\}\}$. It is clear that (U, ϑ_r) is a hyperconnected space which means that the classes of semi-open and somewhat open sets are identical.

Medical example: Dengue fever

In this subsection, we analyze a problem of Dengue fever disease. The virus-carrying Dengue mosquitoes is responsible for transmitting this disease to humans [45]. The symptoms of this disease start from 3 to 4 days of infection. Usually, recovery requires two days to a week [33]. It is a common disease

Table 4 Comparison among the approximations in cases of $r, rsemi, rso$

ϑ_ρ $P(U)$	ϑ_r		$semi\,O(\vartheta_r)$		$so(\vartheta_r)$	
	$\underline{\mathcal{E}}_r$	$\overline{\mathcal{E}}_r$	$\underline{\mathcal{E}}_r^{semi}$	$\overline{\mathcal{E}}_r^{semi}$	$\underline{\mathcal{E}}_r^{so}$	$\overline{\mathcal{E}}_r^{so}$
$\{tv\}$	$\emptyset$	$\{tv\}$	$\emptyset$	$\{tv\}$	$\emptyset$	$\{tv\}$
$\{tw\}$	$\{tw\}$	$\{tv, tx, ty\}$	$\{tw\}$	$\{tw, tx\}$	$\{tw\}$	$\{tw\}$
$\{tx\}$	$\emptyset$	$\{tv, tx\}$	$\emptyset$	$\{tx\}$	$\emptyset$	$\{tx\}$
$\{ty\}$	$\{ty\}$	$\{ty\}$	$\{ty\}$	$\{ty\}$	$\{ty\}$	$\{ty\}$
$\{tv, tw\}$	$\{tw\}$	$\{tv, tw, tx\}$	$\{tv, tw\}$	$\{tv, tw, tx\}$	$\{tv, tw\}$	$\{tv, tw\}$
$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$	$\emptyset$	$\{tv, tx\}$
$\{tv, ty\}$	$\{ty\}$	$\{tv, ty\}$	$\{tv, ty\}$	$\{tv, ty\}$	$\{tv, ty\}$	$\{tv, ty\}$
$\{tw, tx\}$	$\{tw, tx\}$	$\{tv, tw, tx\}$	$\{tw, tx\}$	$\{tv, tw, tx\}$	$\{tw, tx\}$	$\{tw, tx\}$
$\{tw, ty\}$	$\{tw, ty\}$	U	$\{tw, ty\}$	U	$\{tw, ty\}$	U
$\{tx, ty\}$	$\{ty\}$	$\{tv, tx, ty\}$	$\{ty\}$	$\{tx, ty\}$	$\{tx, ty\}$	$\{tx, ty\}$
$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$	$\{tv, tw, tx\}$
$\{tv, tw, ty\}$	$\{tw, ty\}$	U	$\{tv, tw, ty\}$	U	$\{tv, tw, ty\}$	U
$\{tv, tx, ty\}$	$\{ty\}$	$\{tv, tx, ty\}$	$\{tv, ty\}$	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$	$\{tv, tx, ty\}$
$\{tw, tx, ty\}$	$\{tw, tx, ty\}$	U	$\{tw, tx, ty\}$	U	$\{tw, tx, ty\}$	U

Table 5 Comparison among the accuracy measures in cases of $r, rsemi, rso$

$Accuracy$ $P(U)$	M_r	M_r^{semi}	M_r^{so}
$\{tv\}$	0	0	0
$\{tw\}$	$\frac{1}{3}$	$\frac{1}{2}$	1
$\{tx\}$	0	0	0
$\{ty\}$	1	1	1
$\{tv, tw\}$	$\frac{1}{3}$	$\frac{2}{3}$	1
$\{tv, tx\}$	0	0	0
$\{tv, ty\}$	$\frac{1}{2}$	1	1
$\{tw, tx\}$	$\frac{2}{3}$	$\frac{2}{3}$	1
$\{tw, ty\}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{tx, ty\}$	$\frac{1}{3}$	$\frac{1}{2}$	1
$\{tv, tw, tx\}$	1	1	1
$\{tv, tw, ty\}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{tv, tx, ty\}$	$\frac{1}{3}$	$\frac{2}{3}$	1
$\{tw, tx, ty\}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$

Table 6 Original Dengue fever information system

U	J	H	S	T	Dengue fever
μ_1	✓	✓	✓	h	✓
μ_2	✓	×	×	h	×
μ_3	✓	×	×	h	✓
μ_4	×	×	×	vh	×
μ_5	×	✓	✓	h	×
μ_6	✓	✓	×	vh	✓
μ_7	✓	✓	×	n	×
μ_8	✓	✓	×	vh	✓

in more than 120 countries around the world, mainly South America and Asia [45]. It causes about 13600 status deaths as well as 60 million symptomatic infections worldwide. Therefore, we are concerned with this disease and will analyze using our approach. The data examine the Dengue fever problem as given in Table 6, where the columns represent the symptoms of Dengue fever (attributes): muscle and joint pains J, headache with vomiting H, characteristic skin rash S, and T is a temperature [very high (vh), high (h), normal (n)] as given in [45]. Attribute D is the decision of disease and the rows of attributes $U = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$ are the patients. All attributes except for T have two values: '✓' and '×', respectively, denote the patient has a symptom and the patient has no symptom.

In Table 7, we transmit the variables descriptions of attributes $\{A_1 = J, A_2 = H, A_3 = S, A_4 = T\}$ into quantity values that clarify the similarities among the symptoms patients. Note that the degree of similarity $\alpha(v, w)$ between any two patients v, w is calculated by

$$\alpha(v, w) = \frac{\sum_{j=1}^{n} (A_j(v) = A_j(w))}{m}, \quad (6)$$

where m denotes the number of conditions attributes.

Now, we initiate a relation in each case based on the requirements of experts in charge of the system. For example, let $v\mathcal{E}w \iff \alpha(v, w) > 0.65$, where $\alpha(v, w)$ given in equation (6). It is worthy to note that the proposed relation $>$ and number 0.65 are changed according to the viewpoint

Table 7 Similarities between symptoms of eight of patients

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
μ_1	1	0.5	0.5	0	0.75	0.5	0.5	0.5
μ_2	0.5	1	1	0.5	0.25	0.5	0.5	0.5
μ_3	0.5	1	1	0.5	0.25	0.5	0.5	0.5
μ_4	0	0.5	0.5	1	0.25	0.5	0.25	0.5
μ_5	0.75	0.25	0.25	0.25	1	0.25	0.25	0.25
μ_6	0.5	0.5	0.5	0.5	0.25	1	0.75	1
μ_7	0.5	0.5	0.5	0.25	0.25	0.75	1	0.75
μ_8	0.5	0.5	0.5	0.5	0.25	1	0.75	1

Table 8 N_ρ-neighborhoods for each patient μ_i

	$\mathcal{N}_\rho$
μ_1	$\{\mu_1, \mu_5\}$
μ_2	$\{\mu_2, \mu_3\}$
μ_3	$\{\mu_2, \mu_3\}$
μ_4	$\{\mu_4\}$
μ_5	$\{\mu_1, \mu_5\}$
μ_6	$\{\mu_6, \mu_7, \mu_8\}$
μ_7	$\{\mu_6, \mu_7, \mu_8\}$
μ_8	$\{\mu_6, \mu_7, \mu_8\}$

of system's experts. Since the given relation $\mathcal{E}$ is an equivalence relation, we have only one type of N_ρ-neighborhoods. It should be noted that relation $\mathcal{E}$ needs not be an equivalence in general; for example, if we replace the number 0.65 by 0.4, then $\mathcal{E}$ is not transitive, because (μ_4, μ_3) and $(\mu_3, \mu_1) \in \mathcal{E}$, but $(\mu_4, \mu_1) \notin \mathcal{E}$.

In Table 8, we compute the N_r-neighborhoods for each patient μ_i.

The topology ϑ_ρ generated from N_ρ-neighborhoods is the topology induced from the basis $\{N_\rho(\mu) : \mu \in U\}$. To validate the advantages of the followed technique in improving the approximations and accuracy measures compared with the techniques given in [14,37], we consider $\Omega = \{\mu_2, \mu_4, \mu_5, \mu_7\}$ which is the set of patients who do not have Dengue fever. We calculate the approximations and accuracy measures in the following:

1. $\underline{\mathcal{E}}^{\alpha}_{\rho}(\Omega) = \underline{\mathcal{E}}^{semi}_{\rho}(\Omega) = \{\mu_4\}$ and $\overline{\mathcal{E}}^{\alpha}_{\rho}(\Omega) = \overline{\mathcal{E}}^{semi}_{\rho}(\Omega) = U$. Then, $M^{\alpha}_{\rho}(\Omega) = M^{s}_{\rho}(\Omega) = \frac{1}{8}$.
2. $\underline{\mathcal{E}}^{so}_{\rho}(\Omega) = \Omega$ and $\overline{\mathcal{E}}^{so}_{\rho}(\Omega) = U$. Then, $M^{so}_{\rho}(\Omega) = \frac{1}{2}$.

It follows from 1 and 2 above that the approximations and accuracy measures induced from our method are better than the those defined in [14,37].

As we see that ϑ_ρ is a quasi-discrete topology which leads to the equality between the classes of open, α-open and semi-open sets. This means that the three types of accuracy measures M_ρ, $M^{\alpha o}_\rho$ and M^{so}_ρ are equal for each subset.

It is natural to ask about the values of accuracy measures induced from the class of pre-open subsets of a quasi-discrete topology. Since we deal with a finite space, every subset of a finite quasi-discrete topology is pre-open. So that, the accuracy measures induced from this class are one for any subset. This means that approximations and accuracy measures induced from the class of pre-open sets are the best under this circumstance. This matter is applied also in the classes of b-open, β-open, and somewhere dense sets, because they are wider than the class of pre-open sets.

Discussion: strengths and limitations

- Strengths

  1. Our approach preserves the monotonic property for the accuracy and roughness measures (see, Proposition 6 and Corollary 2); whereas, this property is losing in the previous topological approaches given in [14,37]. This is due to that our approach is only based on the interior operator which is proportional to the size of a given topology. However, the other approaches are based on two factors, interior and closure operators, which are working against each other with respect to the size of a given topology. That is, when the size of a given topology enlarges, the interior points of a subset is increasing and the closure points of a subset are decreasing which means that we cannot anticipate the behaviours of the approximations in cases of α-open, semi-open, pre-open, b-open, and β-open and somewhere dense sets.
  2. All Pawlak properties are preserved by ρso-lower and ρso-upper approximations except for (L5) and (U6) given in Proposition 1 (see their counterparts:(v) and (vi) given, respectively, in Proposition 3 and Proposition 4). These two properties are kept by ρso-approximations under a hyperconnectedness condition, whereas we need a strong hyperconnectedness condition to keep them by the approximations generated from somewhere dense sets. That is, the properties (L5) and (U6) are preserved by ρso-approximations under relaxed conditions than the other approximations.
  3. Comparisons between the different types of ρso-approximations and ρso-accuracy measures are investigated in Proposition 10 and Corollary 4. Whereas, we cannot compare between the different types of approximations and accuracy measures induced from α-open and α-closed sets, because

they are defined using interior and closure operators which are working against each other. This matter does not guarantee standard behaviour between $\rho\alpha$-approximations and $\rho\alpha$-accuracy measures. For the same reason, this matter applied to the other approximations and accuracy measures induced from semi-open, pre-open, b-open, β-open sets, and somewhere dense sets.
4. The approximations and accuracy measures induced from our approach are better than those given in [1] and those given in [14,37] in the cases of α-open and semi-open sets.

- limitations

1. Our approach is incomparable with those given in [14,37] in cases of pre-open, b-open, and β-open sets. To validate this matter, consider the collections given in (3), and let $\Omega = \{tx, ty\}$ and $\Sigma = \{tv\}$ be subsets of (U, ϑ_r) and (U, ϑ_u), respectively. By calculation, we find that $cl(int(cl(\Omega))) = \{ty\}$ and $int(\Omega) = \{ty\}$ which means that Ω is somewhat open, but not pre-open (b-open, β-open). Also, $int(cl(\Sigma)) = \{tv, tw, tx\}$ and $int(\Sigma) = \emptyset$ which means that Σ is pre-open (b-open, β-open), but not somewhat open. However, the accuracy measures and approximations generated by the class of pre-open subsets are better than our approach under a finite quasi-discrete topology, because all subsets of a finite quasi-discrete topology are pre-open; hence, the accuracy measures induced from this class are equal to one for any subset; this matter is also applied to all classes that are wider than the class of pre-open sets such as b-open, β-open, and somewhere dense sets.
2. One can note that every somewhat open set is somewhere dense; so that, $\underline{\mathcal{E}}_\rho^{so}(\Omega) \subseteq \underline{\mathcal{E}}_\rho^{SD}(\Omega) \subseteq \Omega \subseteq \overline{\mathcal{E}}_\rho^{SD}(\Omega) \subseteq \overline{\mathcal{E}}_\rho^{so}(\Omega)$. Consequently, $M_\rho^{so}(\Omega) \leq M_\rho^{SD}(\Omega)$. Hence, the approximations and accuracy measures generated from the method of somewhere dense sets given in [8] are better than their counterparts given in this manuscript.

Conclusion

It is well known that the topological concepts provide a vital tool to study rough set theory. In this manuscript, we have applied a topological approach called "somewhat open and somewhat closed sets" to investigate new types of rough set models. We have studied the main properties of the given models and discussed their unique characteristics. We have made some comparisons between the different kinds of our models as well as compared our model with the previous ones. Also, we have provide a medical example to examine the performance of our approach. We complete this article by discussing the strengths and limitations of our approach.

In the upcoming works, we are going to study the following.

(i) Explore the concepts introduced herein using a topology generated from different systems of neighborhoods like E_ρ- (C_ρ-, S_ρ-)neighborhoods.
(ii) Familiarize the concepts displayed herein in the frame of soft rough set.
(iii) Improve the given results by adding the ideals to the topological structures such those presented in [11,21,22,30].

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