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ORIGINAL ARTICLE

Intuitionistic fuzzy-valued neutrosophic multi-sets and numerical applications to classification

Mehmet Ünver[1] · Ezgi Türkarslan[2] · Nuri Çelik[3] · Murat Olgun[1] · Jun Ye[4]

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Abstract

A single-valued neutrosophic multi-set is characterized by a sequence of truth membership degrees, a sequence of indeterminacy membership degrees and a sequence of falsity membership degrees. Nature of a single-valued neutrosophic multi-set allows us to consider multiple information in the truth, indeterminacy and falsity memberships which is pretty useful in multi-criteria group decision making. In this paper, we consider sequences of intuitionistic fuzzy values instead of numbers to define the concept of intuitionistic fuzzy-valued neutrosophic multi-set. In this manner, such a set gives more powerful information. We also present some set theoretic operations and a partial order for intuitionistic fuzzy-valued neutrosophic sets and provide some algebraic operations between intuitionistic fuzzy-valued neutrosophic values. Then, we develop two types of weighted aggregation operators with the help of intuitionistic fuzzy t-norms and t-conorms. By considering some well-known additive generators of ordinary t-norms, we give the Algebraic weighted arithmetic and geometric aggregation operators and the Einstein weighted arithmetic and geometric aggregation operators that are the particular cases of the weighted aggregation operators defined via general t-norms and t-conorms. We also define a simplified neutrosophic valued similarity measure and we use a score function for simplified neutrosophic values to rank similarities of intuitionistic fuzzy-valued neutrosophic multi-values. Finally, we give an algorithm to solve classification problems using intuitionistic fuzzy-valued neutrosophic multi-values and proposed aggregation operators and we apply the theoretical part of the paper to a real classification problem.

Keywords Intuitionistic fuzzy-valued neutrosophic multi-set · Weighted arithmetic aggregation operator · Weighted geometric aggregation operator · Classification

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✉ Murat Olgun
olgun@ankara.edu.tr

Mehmet Ünver
munver@ankara.edu.tr

Ezgi Türkarslan
ezgi.turkarslan@tedu.edu.tr

Nuri Çelik
nuricelik@gtu.edu.tr

Jun Ye
yejun1@nbu.edu.cn

1 Department of Mathematics, Faculty of Science, Ankara University, 06100 Ankara, Turkey

2 Department of Mathematics, Faculty of Arts and Science, TED University, 06420 Ankara, Turkey

3 Department of Mathematics, Faculty of Science, Gebze Technical University, 41400 Gebze, Kocaeli, Turkey

4 School of Civil and Environmental Engineering, Ningbo University, 818 Fenghua Road, Jiangbei, Ningbo, Zhejiang Province, People's Republic of China

Introduction

Fuzzy sets (FSs) [1] are characterized by membership functions and a fuzzy set is a successful tool to handle uncertainties arising from partial belongingness of an element to a set. Atanassov [2] extended the concept of fuzzy set to the concept of intuitionistic fuzzy set (IFS) via a membership function μ_A and a non-membership function ν_A such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. For a fixed $x \in X$, the pair $(\mu_A(x), \nu_A(x))$ is called an intuitionistic fuzzy value (IFV) or an intuitionistic fuzzy number [3]. Moreover, Atanassov [2] described the hesitant function of an IFS that is given by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. This function indicates exactly the level of indecision of a decision maker and it is not included in the classical intuitionistic fuzzy set notion. Theory of IFS has been extensively studied by many authors and it has been applied to vary fields such as decision making, image fusion and segmentation systems, classification and clustering (see e.g., [4–9]). Recently, many methods and approaches also have been developed to solve the problems in many areas. For example, Krawczak and Szkatula [10] have studied on IFSs's perturbation and they have applied this concept to classification problems. He et al. [11] have proposed some distance measures between IFSs via dissimilarity function and have given a pattern recognition application. Kumar [12] have formulated the crisp, fuzzy and intuitionistic fuzzy optimization problems. Lohani et al. [13] have carried out experimental study of intuitionistic fuzzy c-mean algorithm over machine learning dataset.

Smarandache [14] has proposed the concept of neutrosophic set (NS) from a philosophical point of view as a generalization of the concept of FS and IFS. A NS is characterized by a truth membership function, an indeterminacy membership function and a falsity membership function and each membership degree is a real standard or a non-standard subset of the non-standard unit interval $]^{-}0, 1^{+}[$. Unlike IFSs, there is no restriction on the membership functions in a NS, and the degree of hesitancy is included in the NS. Nonetheless, NSs are hard to apply in practical problems since the values of the truth, indeterminacy and falsity membership functions lie in $]^{-}0, 1^{+}[$. Therefore, this concept has been extended to various NSs whose truth, falsity and indeterminacy membership functions take only one value from the closed interval [0, 1] such as single-valued neutrosophic sets (SVNSs) [15], simplified neutrosophic sets (SNSs) [16], neutrosophic soft sets (NSSs) [17] and they have been applied to various multi-criteria decision making problems. However, decision makers get hard to determine the truth, indeterminacy and falsity membership degree of an element to a set in some real-life situation and so they give rise of giving a few different values due to doubt such as fuzzy multi-sets (FMs) and intuitionistic fuzzy multi-sets (IFMSs) (see e.g., [18–20]). In such a situation, it may be useful to use the concept of single-valued neutrosophic multi-set (SVNMS) which was proposed by Ye and Ye [21] in 2014. A SVNMS is characterized by sequences of truth, indeterminacy and falsity membership degrees coming from [0, 1].

The process of combining several numerical values into a single representative one is called aggregation, and a numerical function performing this process is called an aggregation operator. This concept has various application areas such as artificial intelligence, operations research, economics and finance, pattern recognition and image processing, data fusion, multi-criteria decision making, classification and clustering, automated reasoning, etc. (see, e.g., [35]). The concepts of t-norm and t-conorm are often used to define algebraic operations and aggregation operators for fuzzy sets. Most researchers have identified various aggregation operators for an IFS and its generalizations using several types of $t-$norm and $t-$ conorm. For example, Beliakov et al. [22] constructed some operations for IFSs based on Archimedean t-norm and t-conorm, from which an aggregation principle is proposed for intuitionistic fuzzy information. Liu et al. [23] proposed Heronian aggregation operators of IFVs based on the Archimedean $t-$norm and $t-$conorm and Liu and You [24] developed Heronian mean operators based on Einstein $t-$norm and $t-$conorm for linguistic IFSs. Garg and Rani [25] have proposed Bonferroni mean aggregation operators based on Archimedean $t-$norm and $t-$conorm for complex intuitionistic fuzzy information and Garg and Arora [26] introduced Maclaurin symmetric mean aggregation operator based on Archimedean $t-$norm for intuitionistic fuzzy soft sets.

Classification is a pattern identification method in the field of data science and statistics. This method is used for rearranging the data into predefined classes according to the some specific algorithms. These algorithms have been commonly used in machine learning as a supervised rule learning method. On the other hand, cluster analysis is an unsupervised learning method which is used for classification without labeled responses. Once data have been classified or clustered, the correct output can be checked using some statistical features like accuracy. There are plenty of classification and clustering algorithms in machine learning studies. These algorithms are commonly based on some statistical features or analysis. The concept is preferred by researchers in various fields such as pattern recognition, information retrieval, microbiology analysis, data mining, etc. In the meantime, recently many researchers have been using fuzzy methods as well as fuzzy set theory in classification analysis (see, e.g., [36–38]) and clustering analysis (see, e.g., [39,40]).

In this paper, we expand the idea of SVNMS to the concepts of intuitionistic fuzzy-valued neutrosophic multi-set (IFVNMS) and intuitionistic fuzzy-valued neutrosophic multi-value (IFVNMV), which combine NS theory with IFS theory with the help of IFVs instead of numbers in membership sequences. Actually, an IFVNMS lets us model membership and non-membership degrees of an element to truth, indeterminacy and falsity sequences in the neutrosophic environment. Therefore, more detailed information can be carried when transforming the data to fuzzy information and so decision makers can assign less strict fuzzy values. Thus, an IFVNMS or IFVNMV prevents the loss of the information and relaxes the decision process. We also present some set theoretic operations and a partial order for IFVNMSs. Then, we give some fundamental algebraic operations with the help of intuitionistic fuzzy t-norms and t-conorms and we provide some weighted arithmetic and geometric aggregation operators for IFVNMVs. A compar-

Table 1 Literature survey based on aggregation operators for IFS, NS

References	Type of the set	Aggregation operator	t−norm/t-conorm	Case study
Beliakov et al. [22]	IFS	Averaging	Archimedean, Łukasiewicz	–
Liu et al. [23]	IFS	Heronian	Archimedean	–
Liu and You [24]	Linguistic IFS	Heronian	Einstein	Supplier selection problem
Garg and Rani [25]	Complex IFS	Bonferroni	Archimedean	Software selection problem
Garg and Arora [26]	IF Soft S	Maclaurin symmetric	Archimedean	Investment selection problem
Zhang et al. [27]	IFS	Frank power	Frank	Investment selection problem
Zhu et al. [28]	Linguistic Interval-valued IFS	Hamacher weighted	Hamacher	Supplier selection problem
Chen et al. [29]	SVNS	Dombi weighted	Dombi	Investment selection problem
Peng et al. [30]	SNS	Arithmetic/geometric	Archimedean	Investment selection problem
Peng et al. [31]	Multi-valued NS	Power	Einstein	Investment selection problem
Ye et al. [32]	Neutrosophic enthalpy	Arithmetic/geometric	Algebraic, Einstein	Car selection problem
Jamil et al. [33]	Bipolar NS	Bipolar N Hamacher	Hamacher	Investment selection problem
Wang et al. [34]	Bipolar NS	Frank Choquet Bonferroni	Frank	Plant location selection
Present study	IFVNMS	Arithmetic/geometric	Algebraic, Einstein	Classification

Table 2 Comparison of IFVNMS with some existing sets

Sets	Truth	Indeterminacy	Falsity	Restriction on Membership function	Repetitive information	Representable as sequence of two dimensional information in multi-valued set setting
FS	✓	×	×	×	×	×
IFS	✓	×	✓	✓	×	×
FMS	✓	×	×	×	✓	×
IFMS	✓	×	✓	✓	✓	×
NS	✓	✓	✓	×	×	×
SVNMS	✓	✓	✓	×	✓	×
IFVNMS	✓	✓	✓	✓	✓	✓

ison of aggregation operators from the literature and those of the present study is provided in Table 1 and the advantages of the concept of IFVNMS is emphasized in Table 2. Next, we aim to solve a real classification problem with the help of the proposed aggregation operators. For this purpose, we first define a simplified neutrosophic valued cosine similarity measure between IFVNMVs and we rank similarity results via a score function for SNVs. As an example, we consider a real-life example and construct a new classification method based on IFVNMVs, the proposed aggregation operators and the cosine similarity measure for an applied classification problem.

Preliminaries

The concepts of triangular norm (t-norm) and triangular conorm (t-conorm) have a significant importance in the definition of algebraic operations and aggregation operators for fuzzy sets. A t-norm and a t-conorm are functions that map pairs of numbers from [0, 1] to [0, 1]. Deschrijver et al. [41] extended the notions of t-norm and t-conorm to the intuitionistic fuzzy case by defining these functions from the domain of $I^* \times I^*$ to I^* where $I^* := \{(x_1, x_2) : x_1, x_2 \in [0, 1] \text{ and } x_1 + x_2 \leq 1\}$. Before recalling these concepts, we recall a partial order for IFVs.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two IFVs. According to the partial order introduced by Atanassov [2] $x \leq_{(int)} y$ if and only if $x_1 \leq y_1$ and $x_2 \geq y_2$.

A function $\mathcal{T} : I^* \times I^* \to I^*$ is called an intuitionistic fuzzy t-norm if

(1) For any $x \in I^*$, $\mathcal{T}(x, (1, 0)) = x$ (border condition),
(2) For any $x, y \in I^*$, $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ (commutativity),
(3) For any $x, y, z \in I^*$, $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(y, \mathcal{T}(x, z))$ (associativity),
(4) For any $x = (x_1, x_2)$, $x' = (x_1', x_2')$, $y = (y_1, y_2)$, $y' = (y_1', y_2') \in I^*$, $\mathcal{T}(x, y) \leq_{(int)} \mathcal{T}(x', y')$ whenever

$(x_1, x_2) \leq_{(int)} (x_1', x_2')$ and $(y_1, y_2) \leq_{(int)} (y_1', y_2')$ (monotonicity) [41].

A function $\mathcal{S} : I^* \times I^* \to I^*$ is called an intuitionistic fuzzy t-conorm if

(1) For any $x \in I^*$, $\mathcal{S}(x, (0, 1)) = x$ (border condition),
(2) For any $x, y \in I^*$, $\mathcal{S}(x, y) = \mathcal{S}(y, x)$ (commutativity),
(3) For any $x, y, z \in I^*$, $\mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(y, \mathcal{S}(x, z))$ (associativity),
(4) For any $x = (x_1, x_2), x' = (x_1', x_2'), y = (y_1, y_2), y' = (y_1', y_2') \in I^*$, $\mathcal{S}(x, y) \leq_{(int)} \mathcal{S}(x', y')$ whenever $(x_1, x_2) \leq_{(int)} (x_1', x_2')$ and $(y_1, y_2) \leq_{(int)} (y_1', y_2')$ (monotonicity) [41].

Definitions of t-norm and t-conorm in the ordinary sense can be found in [42,43].

A function $\mathcal{N} : I^* \to I^*$ is called a fuzzy negator if

(1) For any $x = (x_1, x_2), y = (y_1, y_2) \in I^*$, $\mathcal{N}(x) \leq_{(int)} \mathcal{N}(y)$ whenever $x \geq_{(int)} y$, i.e., $\mathcal{N}$ is decreasing,
(2) $\mathcal{N}((0, 1)) = (1, 0)$ and $\mathcal{N}((1, 0)) = (0, 1)$ [41].

Remark 1 (i) The mapping $\mathcal{N}_s$ defined by $\mathcal{N}_s((x_1, x_2)) = (x_2, x_1)$ is a fuzzy negator and it is called the standard negator (see e.g., [41]).
(ii) Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and let $\mathcal{N}$ be a fuzzy negator. Then, the function $\mathcal{T}^* : I^* \times I^* \to I^*$ defined by $\mathcal{T}^*(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$ is a t-conorm which is called the dual intuitionistic fuzzy t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}$ [41].
(iii) Let T be a t-norm and let S be a t-conorm in the ordinary sense. If

$$T(a, b) \leq 1 - S(1 - a, 1 - b) \text{ for any } a, b \in [0, 1], \quad (2.1)$$

then the mapping $\mathcal{T} : I^* \times I^* \to I^*$ defined by $\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$ is an intuitionistic fuzzy t-norm and the mapping $\mathcal{T}^* : I^* \times I^* \to I^*$ defined by $\mathcal{T}^*(x, y) = (S(x_1, y_1), T(x_2, y_2))$ is the dual intuitionistic fuzzy t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}_s$ [41].

Remark 2 Additive generators of t-norms and t-conorms of ordinary sense play an important role while defining the algebraic operators. Now, we generate an intuitionistic fuzzy t-norm and an intuitionistic fuzzy t-conorm using generators in the ordinary sense. Let $g : [0, 1] \to [0, \infty]$ be the additive generator of a t-norm T and let S be the dual t-conorm of T. In this case, we know that $h(t) = g(1 - t)$ (see, e.g., [44]). Then, (2.1) is satisfied. Thus, using (iii) of Remark 1 we obtain an intuitionistic fuzzy t-norm $\mathcal{T}$ defined by

$$\mathcal{T}(x, y) = \left(g^{-1}(g(x_1) + g(y_1)), h^{-1}(h(x_2) + h(y_2))\right)$$

and its dual t-conorm

$$\mathcal{T}^*(x, y) = \left(h^{-1}(h(x_1) + h(y_1)), g^{-1}(g(x_2) + g(y_2))\right)$$

with respect to $\mathcal{N}_s$. In this case, we say that $\mathcal{T}$ is the intuitionistic fuzzy t-norm generated by g.

Example 1 Let $g, h : [0, 1] \to [0, \infty]$ defined by $g(t) = -\log t$ and $h(t) = -\log(1 - t)$. Then, we obtain the algebraic intuitionistic fuzzy t-norm

$$\mathcal{T}(x, y) = (x_1 y_1, x_2 + y_2 - x_2 y_2)$$

given in [41] (see pg.48) and its dual intuitionistic fuzzy dual t-conorm

$$\mathcal{T}^*(x, y) = (x_1 + y_1 - x_1 y_1, x_2 y_2).$$

Let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. Then, the pair (α, β) is called an IFV [2]. Now, we are ready to introduce the concepts of IFVNMS and IFVNMV.

Definition 1 Let $X = \{x_1, ..., x_n\}$ be a finite set. An IFVNMS defined on X is given with

$$A = \left\{\left\langle x_i, \left((\mathbb{T}_A^{i,j})_{j=1}^{p_i}, (\mathbb{I}_A^{i,j})_{j=1}^{p_i}, (\mathbb{F}_A^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, ...n\right\} \quad (2.2)$$

where $(\mathbb{T}_A^{i,j})_{j=1}^{p_i}$, $(\mathbb{I}_A^{i,j})_{j=1}^{p_i}$ and $(\mathbb{F}_A^{i,j})_{j=1}^{p_i}$ are the truth, the indeterminacy and the falsity membership sequences of IFVs, respectively, i.e., $i = 1, ..., n$, $j = 1, ..., p_i$

$$\mathbb{T}_A^{i,j} = (T_A^{j,t_i}, T_A^{j,f_i}), \text{ with } T_A^{j,t_i}, T_A^{j,f_i} \in [0, 1] \text{ such that } 0 \leq T_A^{j,t_i} + T_A^{j,f_i} \leq 1 \quad (2.3)$$

$$\mathbb{I}_A^{i,j} = (I_A^{j,t_i}, I_A^{j,f_i}), \text{ with } I_A^{j,t_i}, I_A^{j,f_i} \in [0, 1] \text{ such that } 0 \leq I_A^{j,t_i} + I_A^{j,f_i} \leq 1 \quad (2.4)$$

and

$$\mathbb{F}_A^{i,j} = (F_A^{j,t_i}, F_A^{j,f_i}), \text{ with } F_A^{j,t_i}, F_A^{j,f_i} \in [0, 1] \text{ such that } 0 \leq F_A^{j,t_i} + F_A^{j,f_i} \leq 1. \quad (2.5)$$

For a fixed $i = 1, ..., n$ the expression

$$\alpha = \left\langle (\mathbb{T}_\alpha^j)_{j=1}^p, (\mathbb{I}_\alpha^j)_{j=1}^p, (\mathbb{F}_\alpha^j)_{j=1}^p \right\rangle := \left\langle x_i, \left((\mathbb{T}_A^{i,j})_{j=1}^{p_i}, (\mathbb{I}_A^{i,j})_{j=1}^{p_i}, (\mathbb{F}_A^{i,j})_{j=1}^{p_i}\right)\right\rangle$$

denotes an IFVNMV.

Example 2 Let $X = \{x_1, x_2\}$. The following is an IFVNMS:

$$A = \left\{ \begin{array}{l} \langle x_1, \ ((0.5, 0.4), (0.7, 0.1)), ((0.3, 0.2), (0.1, 0.1)), ((0.9, 0.1), (0.2, 0.6)) \ \rangle, \\ \left\langle x_2, \begin{array}{l} ((0.3, 0.5), (0.4, 0.2), (0.1, 0.2)), ((0.1, 0.9), (0.5, 0.5), (0.7, 0.2)), \\ ((0.0, 0.8), (0.3, 0.6), (0.5, 0.2)) \end{array} \right\rangle \end{array} \right\}.$$

Now, we introduce some set theoretic operations among IFVNMSs. Throughout this manuscript "$\subset_{(int)}, \cap_{(int)}, \cup_{(int)}, (\cdot)^{c(int)}$"denote the set operations for IFSs [2].

Definition 2 Let $X = \{x_1, ..., x_n\}$ be a finite set and let A and B be two IFVNMSs in X. Some set operations among IFVNMSs can be defined as follows:

a) $A \subset B$ if and only if

(i) $\mathbb{T}_A^{i,j} \subset_{(int)} \mathbb{T}_B^{i,j}$ i.e., $T_A^{j,t_i} \leq T_B^{j,t_i}$ and $T_A^{j,f_i} \geq T_B^{j,f_i}$
(ii) $\mathbb{I}_A^{i,j} \supset_{(int)} \mathbb{I}_B^{i,j}$ i.e., $I_A^{j,t_i} \geq I_B^{j,t_i}$ and $I_A^{j,f_i} \leq I_B^{j,f_i}$
(iii) $\mathbb{F}_A^{i,j} \supset_{(int)} \mathbb{F}_B^{i,j}$ i.e., $F_A^{j,t_i} \geq F_B^{j,t_i}$ and $F_A^{j,f_i} \leq F_B^{j,f_i}$

for $j = 1, 2, ...p_i$ and $i = 1, 2, ..., n$.

(b) $A = B$ if and only if $A \subset B$ and $A \supset B$.

(c) $A^c = \left\{\left\langle x_i, \left((\mathbb{F}_A^{i,j})_{j=1}^{p_i}, ([\mathbb{I}_A^{i,j}]^{c(int)})_{j=1}^{p_i}, (\mathbb{T}_A^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}$

(d) $A \cup B = \left\{\left\langle x_i, \left((\mathbb{T}_A^{i,j} \cup_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}, (\mathbb{I}_A^{i,j} \cap_{(int)} \mathbb{I}_B^{i,j})_{j=1}^{p_i}, (\mathbb{F}_A^{i,j} \cap_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}$ where

$$(\mathbb{T}_A^{i,j} \cup_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i} = (\max_j(T_A^{j,t_i}, T_B^{j,t_i}), \min_j(T_A^{j,f_i}, T_B^{j,f_i}))_j^{p_i},$$

$$(\mathbb{I}_A^{i,j} \cap_{(int)} \mathbb{I}_B^{i,j})_{j=1}^{p_i} = (\min_j(I_A^{j,t_i}, I_B^{j,t_i}), \max_j(I_A^{j,f_i}, I_B^{j,f_i}))_j^{p_i},$$

and

$$(\mathbb{F}_A^{i,j} \cap_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i} = (\min_j(F_A^{j,t_i}, F_B^{j,t_i}), \max_j(F_A^{j,f_i}, F_B^{j,f_i}))_j^{p_i}.$$

(e) $A \cap B = \left\{\left\langle x_i, \left((\mathbb{T}_A^{i,j} \cap_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}, (\mathbb{I}_A^{i,j} \cup_{(int)} \mathbb{I}_B^{i,j})_{j=1}^{p_i}, (\mathbb{F}_A^{i,j} \cup_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}$ where

$$(\mathbb{T}_A^{i,j} \cap_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i} = (\min_j(T_A^{j,t_i}, T_B^{j,t_i}), \max_j(T_A^{j,f_i}, T_B^{j,f_i}))_j^{p_i},$$

$$(\mathbb{I}_A^{i,j} \cup_{(int)} \mathbb{I}_B^{i,j})_{j=1}^{p_i} = (\max_j(I_A^{j,t_i}, I_B^{j,t_i}), \min_j(I_A^{j,f_i}, I_B^{j,f_i}))_j^{p_i},$$

and

$$(\mathbb{F}_A^{i,j} \cup_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i} = (\max_j(F_A^{j,t_i}, F_B^{j,t_i}), \min_j(F_A^{j,f_i}, F_B^{j,f_i}))_j^{p_i}.$$

Example 3 Let $X = \{x_1, x_2\}$ and let A and B be two IFVNMSs given with:

$$A = \left\{ \begin{array}{l} \langle x_1, ((0.5, 0.4), (0.7, 0.1)), ((0.3, 0.2), (0.1, 0.0)), ((0.9, 0.1), (0.2, 0.4))\rangle, \\ \left\langle x_2, \begin{array}{l} ((0.3, 0.5), (0.1, 0.2), (0.1, 0.1)), ((0.4, 0.2), (0.5, 0.5), (0.7, 0.3)), \\ ((0.7, 0.2), (0.3, 0.5), (0.2, 0.3)) \end{array} \right\rangle \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{l} \langle x_1, ((0.6, 0.2), (0.9, 0.1)), ((0.1, 0.5), (0.1, 0.3)), ((0.6, 0.3), (0.2, 0.6))\rangle, \\ \left\langle x_2, \begin{array}{l} ((0.5, 0.4), (0.4, 0.1), (0.1, 0.0)), ((0.1, 0.3), (0.2, 0.6), (0.4, 0.6)), \\ ((0.0, 0.8), (0.3, 0.6), (0.0, 0.9)) \end{array} \right\rangle \end{array} \right\}$$

then $A \subset B$. On the other hand, it is easy to check that

$$A^c = \left\{ \begin{array}{l} \langle x_1, ((0.9, 0.1), (0.2, 0.4)), ((0.2, 0.3), (0.0, 0.1)), ((0.5, 0.4), (0.7, 0.1))\rangle, \\ \left\langle x_2, \begin{array}{l} ((0.7, 0.2), (0.3, 0.5), (0.2, 0.3)), ((0.2, 0.4), (0.5, 0.5), (0.3, 0.7)), \\ ((0.3, 0.5), (0.1, 0.2), (0.1, 0.1)) \end{array} \right\rangle \end{array} \right\},$$

$$A \cup B = \left\{ \begin{array}{l} \langle x_1, ((0.6, 0.2), (0.9, 0.1)), ((0.1, 0.5), (0.1, 0.3)), ((0.6, 0.3), (0.2, 0.6))\rangle, \\ \left\langle x_2, \begin{array}{l} ((0.5, 0.4), (0.4, 0.1), (0.1, 0.0)), ((0.1, 0.3), (0.2, 0.5), (0.4, 0.6)), \\ ((0.0, 0.8), (0.3, 0.6), (0.0, 0.9)) \end{array} \right\rangle \end{array} \right\},$$

and

$$A \cap B = \left\{ \begin{array}{l} \langle x_1, ((0.5, 0.4), (0.7, 0.1)), ((0.3, 0.2), (0.1, 0.0)), ((0.9, 0.1), (0.2, 0.4))\rangle, \\ \left\langle x_2, \begin{array}{l} ((0.3, 0.5), (0.1, 0.2), (0.1, 0.1)), ((0.4, 0.2), (0.5, 0.5), (0.7, 0.3)), \\ ((0.7, 0.2), (0.3, 0.5), (0.2, 0.3)) \end{array} \right\rangle \end{array} \right\}.$$

The following shows that De Morgan's rules are valid for set operations defined in Definition 2.

Theorem 1 *Let $X = \{x_1, ..., x_n\}$ be a finite set and let*

$$A = \left\{\left\langle x_i, \left((\mathbb{T}_A^{i,j})_{j=1}^{p_i}, (\mathbb{I}_A^{i,j})_{j=1}^{p_i}, (\mathbb{F}_A^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}$$

and

$$B = \left\{\left\langle x_i, \left((\mathbb{T}_B^{i,j})_{j=1}^{p_i}, (\mathbb{I}_B^{i,j})_{j=1}^{p_i}, (\mathbb{F}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}$$

be two IFVNMSs in X. The following De Morgan's rules are satisfied:

(1) $(A \cup B)^c = A^c \cap B^c$,
(2) $(A \cap B)^c = A^c \cup B^c$.

Proof We obtain

$$\begin{aligned}(A \cup B)^c = &\left\{\left\langle x_i, \left((\mathbb{F}_A^{i,j} \cap_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i},\right.\right.\right.\\ &((\mathbb{I}_A^{i,j} \cap_{(int)} \mathbb{I}_B^{i,j})^{c(int)})_{j=1}^{p_i},\\ &\left.\left.\left.(\mathbb{T}_A^{i,j} \cup_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}\\ = &\left\{\left\langle x_i, \left((\mathbb{F}_A^{i,j} \cap_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i},\right.\right.\right.\\ &((\mathbb{I}_A^{i,j})^{c(int)} \cup_{(int)} (\mathbb{I}_B^{i,j})^{c(int)})_{j=1}^{p_i},\\ &\left.\left.\left.(\mathbb{T}_A^{i,j} \cup_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}\\ = &A^c \cap B^c\end{aligned}$$

$$\begin{aligned}(A \cap B)^c = &\left\{\left\langle x_i, \left((\mathbb{F}_A^{i,j} \cup_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i},\right.\right.\right.\\ &((\mathbb{I}_A^{i,j} \cup_{(int)} \mathbb{I}_B^{i,j})^{c(int)})_{j=1}^{p_i},\\ &\left.\left.\left.(\mathbb{T}_A^{i,j} \cap_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}\\ = &\left\{\left\langle x_i, \left((\mathbb{F}_A^{i,j} \cup_{(int)} \mathbb{F}_B^{i,j})_{j=1}^{p_i},\right.\right.\right.\\ &((\mathbb{I}_A^{i,j})^{c(int)} \cap_{(int)} (\mathbb{I}_B^{i,j})^{c(int)})_{j=1}^{p_i},\\ &\left.\left.\left.(\mathbb{T}_A^{i,j} \cap_{(int)} \mathbb{T}_B^{i,j})_{j=1}^{p_i}\right)\right\rangle : i = 1, 2, ...n\right\}\\ = &A^c \cup B^c\end{aligned}$$

which finish the proof. □

Now we define a partial order over the set of all IFVNMSs on a given set X using"$\leq_{(int)}$".

Proposition 1 *The relation defined on the set of all IFVNMSs on a set X by $\alpha \preceq \beta$ if and only if $\mathbb{T}_\alpha^j \leq_{(int)} \mathbb{T}_\beta^j$, $\mathbb{I}_\alpha^j \geq_{(int)} \mathbb{I}_\beta^j$ and $\mathbb{F}_\alpha^j \geq_{(int)} \mathbb{F}_\beta^j$ for any $j = 1, ..., p$ where $\alpha = \left\langle(\mathbb{T}_\alpha^j)_{j=1}^p, (\mathbb{I}_\alpha^j)_{j=1}^p, (\mathbb{F}_\alpha^j)_{j=1}^p\right\rangle$ and $\beta = \left\langle(\mathbb{T}_\beta^j)_{j=1}^p, (\mathbb{I}_\beta^j)_{j=1}^p, (\mathbb{F}_\beta^j)_{j=1}^p\right\rangle$ is a partial order.*

Proof Reflexivity and antisymmetry are trivial. To prove transitivity let $\alpha \preceq \beta$ and $\beta \preceq \gamma$ where $\gamma = \left\langle(\mathbb{T}_\gamma^j)_{j=1}^p, (\mathbb{I}_\gamma^j)_{j=1}^p, (\mathbb{F}_\gamma^j)_{j=1}^p\right\rangle$. Therefore, for any $j = 1, ..., p$, we have

$$\mathbb{T}_\alpha^j \leq_{(int)} \mathbb{T}_\beta^j, \mathbb{I}_\alpha^j \geq_{(int)} \mathbb{I}_\beta^j, \mathbb{F}_\alpha^j \geq_{(int)} \mathbb{F}_\beta^j \tag{2.6}$$

and

$$\mathbb{T}_\beta^j \leq_{(int)} \mathbb{T}_\gamma^j, \mathbb{I}_\beta^j \geq_{(int)} \mathbb{I}_\gamma^j, \mathbb{F}_\beta^j \geq_{(int)} \mathbb{F}_\gamma^j. \tag{2.7}$$

Since,"$\leq_{(int)}$" is a partial order we have from (2.6) and (2.7)

$$\mathbb{T}^j_\alpha \leq_{(int)} \mathbb{T}^j_\gamma, \mathbb{I}^j_\alpha \geq_{(int)} \mathbb{I}^j_\gamma, \mathbb{F}^j_\alpha \geq_{(int)} \mathbb{F}^j_\gamma$$

which yields that $\alpha \preceq \gamma$. □

Note that

$$\overline{\mathbf{0}} = (((0,1), \dots (0,1)), ((1,0), \dots (1,0)), ((1,0), \dots (1,0)))$$

and

$$\overline{\mathbf{1}} = (((1,0), \dots (1,0)), ((0,1), \dots (0,1)), ((0,1), \dots (0,1)))$$

are the minimal and the maximal elements of the IFVNMVs with respect to"$\preceq$".

Algebraic operations for IFVNMVs

In this section, using intuitionistic fuzzy t-norms and intuitionistic fuzzy t-conorms, we define some algebraic operations for IFVNMVs.

Definition 3 Let $\alpha = \left\langle (\mathbb{T}^j_\alpha)^p_{j=1}, (\mathbb{I}^j_\alpha)^p_{j=1}, (\mathbb{F}^j_\alpha)^p_{j=1} \right\rangle$ and $\beta = \left\langle (\mathbb{T}^j_\beta)^p_{j=1}, (\mathbb{I}^j_\beta)^p_{j=1}, (\mathbb{F}^j_\beta)^p_{j=1} \right\rangle$ be two IFVNMVs with same sequence lengths $p \in \mathbb{Z}^+$, let $\mathcal{T}$ be an intuitionistic t-norm and let $\mathcal{N}$ be a fuzzy negator. Then,

(i) $\alpha \oplus \beta := \left\langle \left(\mathcal{T}^*\left(\mathbb{T}^j_\alpha, \mathbb{T}^j_\beta\right)\right)^p_{j=1}, \left(\mathcal{T}\left(\mathbb{I}^j_\alpha, \mathbb{I}^j_\beta\right)\right)^p_{j=1}, \left(\mathcal{T}\left(\mathbb{F}^j_\alpha, \mathbb{F}^j_\beta\right)\right)^p_{j=1} \right\rangle$,

(ii) $\alpha \otimes \beta := \left\langle \left(\mathcal{T}\left(\mathbb{T}^j_\alpha, \mathbb{T}^j_\beta\right)\right)^p_{j=1}, \left(\mathcal{T}^*\left(\mathbb{I}^j_\alpha, \mathbb{I}^j_\beta\right)\right)^p_{j=1}, \left(\mathcal{T}^*\left(\mathbb{F}^j_\alpha, \mathbb{F}^j_\beta\right)\right)^p_{j=1} \right\rangle$

where $\mathcal{T}^*$ is the dual intuitionistic fuzzy t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}$.

The following proposition confirms that the sum and the product of two IFVNMVs are also IFVNMVs.

Proposition 2 *Let α and β be two IFVNMVs, let $\mathcal{T}$ be an intuitionistic t-norm and let $\mathcal{T}^*$ be the dual intuitionistic t-conorm of $\mathcal{T}$ with respect to a fuzzy negator. Then, $\alpha \oplus \beta$ and $\alpha \otimes \beta$ are IFVNMVs.*

Proof Since $\mathcal{T}$ and $\mathcal{T}^*$ have range I^* the proof is trivial. □

Now, we define multiplication by a positive constant and a positive power of IFVNMVs using additive generators of ordinary t-norms.

Definition 4 Let $\alpha = \left\langle (\mathbb{T}^j_\alpha)^p_{j=1}, (\mathbb{I}^j_\alpha)^p_{j=1}, (\mathbb{F}^j_\alpha)^p_{j=1} \right\rangle$ be an IFVNMV with sequence length $p \in \mathbb{Z}^+$ and let g be the additive generator of a t-norm and let $h(t) = g(1-t)$. Then,

i) For $\lambda > 0$, $\lambda\alpha = \left\{\left\langle (\mathbb{T}^j_{\lambda\alpha})^p_{j=1}, (\mathbb{I}^j_{\lambda\alpha})^p_{j=1}, (\mathbb{F}^{i,j}_{\lambda\alpha})^p_{j=1} \right\rangle : i = 1, 2, \dots n\right\}$ where $\mathbb{T}^j_{\lambda\alpha} = (T^{j,t}_{\lambda\alpha}, T^{j,f}_{\lambda\alpha})$, $\mathbb{I}^j_{\lambda\alpha} = (I^{j,t}_{\lambda\alpha}, I^{j,f}_{\lambda\alpha})$, $\mathbb{F}^j_{\lambda\alpha} = (F^{j,t}_{\lambda\alpha}, F^{j,f}_{\lambda\alpha})$ and

$$T^{j,t}_{\lambda\alpha} := h^{-1}\left(\lambda h\left(T^{j,t}_\alpha\right)\right)$$
$$T^{j,f}_{\lambda\alpha} := g^{-1}\left(\lambda g\left(T^{j,f}_\alpha\right)\right)$$
$$I^{j,t}_{\lambda\alpha} := g^{-1}\left(\lambda g\left(I^{j,t}_\alpha\right)\right)$$
$$I^{j,f}_{\lambda\alpha} := h^{-1}\left(\lambda h\left(I^{j,f}_\alpha\right)\right)$$

and

$$F^{j,t}_{\lambda\alpha} := g^{-1}\left(\lambda g\left(F^{j,t}_\alpha\right)\right)$$
$$F^{j,f}_{\lambda\alpha} := h^{-1}\left(\lambda h\left(F^{j,f}_\alpha\right)\right)$$

for any $j = 1, \dots, p$.

(ii) For $\lambda > 0$, $\alpha^\lambda := \left\{\left\langle (\mathbb{T}^j_{\alpha^\lambda})^p_{j=1}, (\mathbb{I}^j_{\alpha^\lambda})^p_{j=1}, (\mathbb{F}^{i,j}_{\alpha^\lambda})^p_{j=1} \right\rangle : i = 1, 2, \dots n\right\}$ where $\mathbb{T}^j_{\alpha^\lambda} := (T^{j,t}_{\alpha^\lambda}, T^{j,f}_{\alpha^\lambda})$, $\mathbb{I}^j_{\alpha^\lambda} := (I^{j,t}_{\alpha^\lambda}, I^{j,f}_{\alpha^\lambda})$, $\mathbb{F}^j_{\alpha^\lambda} := (F^{j,t}_{\alpha^\lambda}, F^{j,f}_{\alpha^\lambda})$ and

$$T^{j,t}_{\alpha^\lambda} := g^{-1}\left(\lambda g\left(T^{j,t}_\alpha\right)\right)$$
$$T^{j,f}_{\alpha^\lambda} := h^{-1}\left(\lambda h\left(T^{j,f}_\alpha\right)\right)$$
$$I^{j,t}_{\alpha^\lambda} := h^{-1}\left(\lambda h\left(I^{j,t}_\alpha\right)\right)$$
$$I^{j,f}_{\alpha^\lambda} := g^{-1}\left(\lambda g\left(I^{j,f}_\alpha\right)\right)$$

and

$$F^{j,t}_{\alpha^\lambda} := h^{-1}\left(\lambda h\left(F^{j,t}_\alpha\right)\right)$$
$$F^{j,f}_{\alpha^\lambda} := g^{-1}\left(\lambda g\left(F^{j,f}_\alpha\right)\right)$$

for any $j = 1, \dots, p$.

The following theorem validates that $\lambda\alpha$ and α^λ are also an IFVNMVs.

Theorem 2 *Let α and β be two IFVNMVs and $\lambda > 0$. Then, $\lambda\alpha$ and α^{λ} are IFVNMVs on X.*

Proof We prove that $\lambda\alpha$ is an IFVNMV. Since $T_{\alpha}^{j,t} \leq 1 - T_{\alpha}^{j,f}$ and h, h^{-1} are increasing we obtain for any $j = 1, ..., p$ that

$$\begin{aligned} 0 &\leq T_{\lambda\alpha}^{j,t} + T_{\lambda\alpha}^{j,f} \\ &= h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)\right) + g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right) \\ &\leq h^{-1}\left(\lambda h\left(1 - T_{\alpha}^{j,f}\right)\right) + g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right) \\ &= 1 - g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right) + g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right) \\ &= 1 \end{aligned}$$

which proves that $\mathbb{T}_{\lambda\alpha}^{j}$ is an IFV. Similarly, we can prove that $\mathbb{I}_{\lambda\alpha}^{j}, \mathbb{F}_{\lambda\alpha}^{j}, \mathbb{T}_{\alpha^{\lambda}}^{j}, \mathbb{I}_{\alpha^{\lambda}}^{j}$ and $\mathbb{F}_{\alpha^{\lambda}}^{j}$ are IFVs. □

Some properties of the operational laws are given in the following theorem.

Theorem 3 *Let α, β and ν be IFVNMVs on X. Assume that t-norm T, t-conorm S and intuitionistic fuzzy t-norm $\mathcal{T}$ are generated by an algebraic generator g and let $h(t) = g(1 - t)$. Then,*

(i) $\alpha \oplus \beta = \beta \oplus \alpha$,
(ii) $\alpha \otimes \beta = \beta \otimes \alpha$,
(iii) $(\alpha \oplus \beta) \oplus \nu = \alpha \oplus (\beta \oplus \nu)$,
(iv) $(\alpha \otimes \beta) \otimes \nu = \alpha \otimes (\beta \otimes \nu)$,
(v) $\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$, $\lambda > 0$,
(vi) $(\lambda + \gamma)\alpha = \lambda\alpha \oplus \gamma\alpha$, $\lambda, \gamma > 0$,
(vii) $(\alpha \otimes \beta)^{\lambda} = \alpha^{\lambda} \otimes \beta^{\lambda}$, $\lambda > 0$,
(viii) $\alpha^{\lambda} \otimes \alpha^{\gamma} = \alpha^{\lambda+\gamma}$, $\lambda, \gamma > 0$.
(ix) *If $\alpha \preceq \beta$, then $\alpha \oplus \varphi \preceq \beta \oplus \varphi$.*
(x) *If $\alpha \preceq \beta$, then $\alpha \otimes \varphi \preceq \beta \otimes \varphi$.*
(xi) $\alpha \oplus \overline{0} = \alpha$.
(xii) $\alpha \otimes \overline{1} = 1$.

Proof (i) and (ii) are trivial.
(iii) We know that

$$\begin{aligned} &\mathbb{T}_{\alpha\oplus\beta}^{j} \\ &= \left(h^{-1}\left(h\left(T_{\alpha}^{;j,t}\right) + h\left(T_{\beta}^{;j,t}\right)\right), g^{-1}\left(g\left(T_{\alpha}^{;j,f}\right)\right.\right. \\ &\left.\left.+g\left(T_{\beta}^{;j,f}\right)\right)\right) \end{aligned}$$

for any $j = 1, ..., p$ which yields that

$$\begin{aligned} \mathbb{T}_{(\alpha\oplus\beta)\oplus\nu}^{j} &= \mathcal{T}^{*}\left(\mathbb{T}_{\alpha\oplus\beta}^{j}, \mathbb{T}_{\nu}^{j}\right) \\ &= \left(h^{-1}\left(h\left(h^{-1}\left(h\left(T_{\alpha}^{;j,t}\right)\right.\right.\right.\right. \\ &\quad \left.\left.\left.+h\left(T_{\beta}^{;j,t}\right)\right)\right) + h\left(T_{\nu}^{;j,t}\right)\right), \\ &\quad g^{-1}\left(g\left(g^{-1}\left(g\left(T_{\alpha}^{;j,f}\right) + g\left(T_{\beta}^{;j,f}\right)\right)\right)\right. \\ &\quad \left.\left.+g\left(T_{\nu}^{;j,f}\right)\right)\right) \\ &= \left(h^{-1}\left(h\left(T_{\alpha}^{;j,t}\right) + h\left(T_{\beta}^{;j,t}\right)\right.\right. \\ &\quad \left.+h\left(T_{\nu}^{;j,t}\right)\right), \\ &\quad g^{-1}\left(g\left(T_{\alpha}^{;j,f}\right) + g\left(T_{\beta}^{;j,f}\right) + g\left(T_{\nu}^{;j,f}\right)\right)\Big) \\ &= \left(h^{-1}\left(h\left(T_{\alpha}^{;j,t}\right) + h\left(h^{-1}\left(h\left(T_{\beta}^{;j,t}\right)\right.\right.\right.\right. \\ &\quad \left.\left.\left.+h\left(T_{\nu}^{;j,t}\right)\right)\right)\right), \\ &\quad g^{-1}\left(g\left(T_{\alpha}^{;j,f}\right) + g\left(g^{-1}\left(g\left(T_{\beta}^{;j,f}\right)\right.\right.\right. \\ &\quad \left.\left.\left.\left.+g\left(T_{\nu}^{;j,f}\right)\right)\right)\right)\right) \\ &= \mathcal{T}^{*}\left(\mathbb{T}_{\alpha}^{j}, \mathbb{T}_{\beta\oplus\nu}^{j}\right) \\ &= \mathbb{T}_{\alpha\oplus(\beta\oplus\nu)}^{j}. \end{aligned}$$

Similarly, we get $\mathbb{I}_{(\alpha\oplus\beta)\oplus\nu}^{j} = \mathbb{I}_{\alpha\oplus(\beta\oplus\nu)}^{j}$ and $\mathbb{F}_{(\alpha\oplus\beta)\oplus\nu}^{j} = \mathbb{F}_{\alpha\oplus(\beta\oplus\nu)}^{j}$ for any $j = 1, ..., p$.
(iv) It is clear that

$$\begin{aligned} &\mathbb{T}_{\alpha\otimes\beta}^{j} \\ &= \left(g^{-1}\left(g\left(T_{\alpha}^{;j,t}\right) + g\left(T_{\beta}^{;j,t}\right)\right), h^{-1}\left(h\left(T_{\alpha}^{;j,f}\right)\right.\right. \\ &\left.\left.+h\left(T_{\beta}^{;j,f}\right)\right)\right) \end{aligned}$$

for any $j = 1, ..., p$. Therefore, we obtain

$$\begin{aligned} \mathbb{T}_{(\alpha\otimes\beta)\otimes\nu}^{j} &= \mathcal{T}\left(\mathbb{T}_{\alpha\otimes\beta}^{j}, \mathbb{T}_{\nu}^{j}\right) \\ &= \left(g^{-1}\left(g\left(g^{-1}\left(g\left(T_{\alpha}^{;j,t}\right) + g\left(T_{\beta}^{;j,t}\right)\right)\right)\right.\right. \\ &\quad \left.+g\left(T_{\nu}^{;j,t}\right)\right), \\ &\quad h^{-1}\left(h\left(h^{-1}\left(h\left(T_{\alpha}^{;j,f}\right) + h\left(T_{\beta}^{;j,f}\right)\right)\right)\right. \\ &\quad \left.\left.+h\left(T_{\nu}^{;j,f}\right)\right)\right) \\ &= \left(g^{-1}\left(g\left(T_{\alpha}^{;j,t}\right) + g\left(T_{\beta}^{;j,t}\right) + g\left(T_{\nu}^{;j,t}\right)\right),\right. \\ &\quad h^{-1}\left(h\left(T_{\alpha}^{;j,f}\right) + h\left(T_{\beta}^{;j,f}\right) + h\left(T_{\nu}^{;j,f}\right)\right)\Big) \\ &= \left(g^{-1}\left(g\left(T_{\alpha}^{;j,t}\right) + g\left(g^{-1}\left(g\left(T_{\beta}^{;j,t}\right)\right.\right.\right.\right. \end{aligned}$$

$$+g\left(T_{\nu}^{j,t}\right)\right)\right)\right),$$
$$h^{-1}\left(h\left(T_{\alpha}^{j,f}\right)+h\left(h^{-1}\left(h\left(T_{\beta}^{j,f}\right)\right.\right.\right.$$
$$\left.\left.\left.+h\left(T_{\nu}^{j,f}\right)\right)\right)\right)\right)$$
$$=\mathcal{T}\left(\mathbb{T}_{\alpha}^{j},\mathbb{T}_{\beta\otimes\nu}^{j}\right)$$
$$=\mathbb{T}_{\alpha\otimes(\beta\otimes\nu)}^{j}.$$

Similarly, we get $\mathbb{I}_{(\alpha\otimes\beta)\otimes\nu}^{j}=\mathbb{I}_{\alpha\otimes(\beta\otimes\nu)}^{j}$ and $\mathbb{F}_{(\alpha\otimes\beta)\otimes\nu}^{j}=\mathbb{F}_{\alpha\otimes(\beta\otimes\nu)}^{j}$ for any $j=1,...,p$.

(v) We have for any $j=1,...,p$ that

$$\begin{aligned}\mathbb{T}_{\lambda(\alpha\oplus\beta)}^{j}&=\left(h^{-1}\left(\lambda h\left(T_{\alpha\oplus\beta}^{j,t}\right)\right),g^{-1}\left(\lambda g\left(T_{\alpha\oplus\beta}^{j,f}\right)\right)\right)\\&=\left(h^{-1}\left(\lambda h\left(S\left(T_{\alpha}^{j,t},T_{\beta}^{j,t}\right)\right)\right),\right.\\&\quad\left.g^{-1}\left(\lambda g\left(T\left(T_{\alpha}^{j,f},T_{\beta}^{j,f}\right)\right)\right)\right)\\&=\left(h^{-1}\left(\lambda h\left(h^{-1}\left(h\left(T_{\alpha}^{j,t}\right)+h\left(T_{\beta}^{j,t}\right)\right)\right)\right)\right)\\&\quad g^{-1}\left(\lambda g\left(g^{-1}\left(g\left(T_{\alpha}^{j,f}\right)+g\left(T_{\alpha}^{j,f}\right)\right)\right)\right)\right)\\&=\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)+\lambda h\left(T_{\beta}^{j,t}\right)\right)\right)\\&\quad g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)+\lambda g\left(T_{\alpha}^{j,f}\right)\right)\right).\end{aligned}\tag{3.1}$$

On the other hand, we obtain

$$\begin{aligned}\mathbb{T}_{\lambda\alpha\oplus\lambda\beta}^{j}&=\mathcal{T}^{*}\left(\mathbb{T}_{\lambda\alpha}^{j},\mathbb{T}_{\lambda\beta}^{j}\right)\\&=\left(S\left(T_{\lambda\alpha}^{j,t},T_{\lambda\beta}^{j,t}\right),T\left(T_{\lambda\alpha}^{j,f},T_{\lambda\beta}^{j,f}\right)\right)\\&\quad\left(h^{-1}\left(h\left(T_{\lambda\alpha}^{j,t}\right)+h\left(T_{\lambda\beta}^{j,t}\right)\right),g^{-1}\left(g\left(T_{\lambda\alpha}^{j,f}\right)\right.\right.\\&\quad\left.\left.+g\left(T_{\lambda\beta}^{j,f}\right)\right)\right)\\&=\left(h^{-1}\left(h\left(h^{-1}\left(\lambda h\left(T_{\lambda\alpha}^{j,t}\right)\right)\right)\right.\right.\\&\quad\left.\left.+h\left(h^{-1}\left(\lambda h\left(T_{\lambda\beta}^{j,t}\right)\right)\right)\right),\right.\\&\quad g^{-1}\left(g\left(g^{-1}\left(\lambda g\left(T_{\lambda\alpha}^{j,f}\right)\right)\right)\right.\\&\quad\left.\left.+g\left(h^{g-1}\left(\lambda g\left(T_{\lambda\beta}^{j,f}\right)\right)\right)\right)\right)\\&=\left(h^{-1}\left(\lambda h\left(T_{\lambda\alpha}^{j,t}\right)+\lambda h\left(T_{\lambda\beta}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(\lambda g\left(T_{\lambda\alpha}^{j,f}\right)+\lambda g\left(T_{\lambda\beta}^{j,f}\right)\right)\right).\end{aligned}\tag{3.2}$$

From (3.1) and (3.2) we have $\mathbb{T}_{\lambda(\alpha\oplus\beta)}^{j}=\mathbb{T}_{\lambda\alpha\oplus\lambda\beta}^{j}$. Similarly, we obtain $\mathbb{I}_{\lambda\alpha\oplus\lambda\beta}^{j}=\mathbb{I}_{\lambda\alpha\oplus\lambda\beta}^{j}$ and $\mathbb{F}_{\lambda\alpha\oplus\lambda\beta}^{j}=\mathbb{F}_{\lambda\alpha\oplus\lambda\beta}^{j}$ for any $j=1,...,p$ which proves the claim.

vi) We get for any $j=1,...,p$ that

$$\begin{aligned}\mathbb{T}_{\lambda\alpha\oplus\gamma\alpha}^{j}&=\mathcal{T}^{*}\left(\mathbb{T}_{\lambda\alpha}^{j},\mathbb{T}_{\gamma\alpha}^{j}\right)\\&=\mathcal{T}^{*}\left(\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)\right),g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right)\right),\right.\\&\quad\left.\left(h^{-1}\left(\gamma h\left(T_{\alpha}^{j,t}\right)\right),g^{-1}\left(\gamma g\left(T_{\alpha}^{j,f}\right)\right)\right)\right)\\&=\left(S\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)\right),h^{-1}\left(\gamma h\left(T_{\alpha}^{j,t}\right)\right)\right),\right.\\&\quad\left.T\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right),g^{-1}\left(\gamma g\left(T_{\alpha}^{j,f}\right)\right)\right)\right)\\&=\left(h^{-1}\left(h\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)\right)\right)\right.\right.\\&\quad\left.\left.+h\left(h^{-1}\left(\gamma h\left(T_{\alpha}^{j,t}\right)\right)\right)\right),\right.\\&\quad g^{-1}\left(g\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)\right)\right)\right.\\&\quad\left.\left.+g\left(g^{-1}\left(\gamma g\left(T_{\alpha}^{j,f}\right)\right)\right)\right)\right)\\&=\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,t}\right)+\gamma h\left(T_{\alpha}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(\lambda g\left(T_{\alpha}^{j,f}\right)+\gamma g\left(T_{\alpha}^{j,f}\right)\right)\right)\\&=\left(h^{-1}\left((\lambda+\gamma)h\left(T_{\alpha}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left((\lambda+\gamma)g\left(T_{\alpha}^{j,f}\right)\right)\right)\\&=\mathbb{T}_{(\lambda+\gamma)\alpha}^{j}.\end{aligned}$$

Similarly, we obtain $\mathbb{I}_{\lambda\alpha\oplus\gamma\alpha}^{j}=\mathbb{I}_{(\lambda+\gamma)\alpha}^{j}$ and $\mathbb{F}_{\lambda\alpha\oplus\gamma\alpha}^{j}=\mathbb{F}_{(\lambda+\gamma)\alpha}^{j}$ for any $j=1,...,p$ which proves the claim.

vii) We obtain for any $j=1,...,p$ that

$$\begin{aligned}\mathbb{T}_{(\alpha\otimes\beta)^{\lambda}}^{j}&=\left(g^{-1}\left(\lambda g\left(T_{\alpha\otimes\beta}^{j,t}\right)\right),h^{-1}\left(\lambda h\left(T_{\alpha\otimes\beta}^{j,f}\right)\right)\right)\\&=\left(g^{-1}\left(\lambda g\left(T\left(T_{\alpha}^{j,t},T_{\beta}^{j,t}\right)\right)\right),\right.\\&\quad\left.h^{-1}\left(\lambda h\left(S\left(T_{\alpha}^{j,f},T_{\beta}^{j,f}\right)\right)\right)\right)\\&=\left(g^{-1}\left(\lambda g\left(g^{-1}\left(g\left(T_{\alpha}^{j,t}\right)+g\left(T_{\beta}^{j,t}\right)\right)\right)\right),\right.\\&\quad\left.h^{-1}\left(\lambda h\left(h^{-1}\left(h\left(T_{\alpha}^{j,f}\right)+h\left(T_{\beta}^{j,f}\right)\right)\right)\right)\right)\\&=\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)+\lambda g\left(T_{\beta}^{j,t}\right)\right),\right.\\&\quad\left.h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)+\lambda h\left(T_{\beta}^{j,f}\right)\right)\right)\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\mathbb{T}_{\alpha^{\lambda}\otimes\beta^{\lambda}}^{j}&=\left(T\left(T_{\alpha^{\lambda}}^{j,t},T_{\beta^{\lambda}}^{j,t}\right),S\left(T_{\alpha^{\lambda}}^{j,f},T_{\beta^{\lambda}}^{j,f}\right)\right)\\&=\left(T\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)\right),g^{-1}\left(\lambda g\left(T_{\beta}^{j,t}\right)\right)\right),\right.\\&\quad\left.S\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)\right),h^{-1}\left(\lambda h\left(T_{\beta}^{j,f}\right)\right)\right)\right)\\&=\left(g^{-1}\left(g\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)\right)\right)\right.\right.\\&\quad\left.\left.+g\left(g^{-1}\left(\lambda g\left(T_{\beta}^{j,t}\right)\right)\right)\right),\right.\\&\quad h^{-1}\left(h\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)\right)\right)\right.\end{aligned}$$

$$+h\left(h^{-1}\left(\lambda h\left(T_{\beta}^{j,f}\right)\right)\right)\Big)\Big)$$
$$=\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)+\lambda g\left(T_{\beta}^{j,t}\right)\right),\right.$$
$$\left.h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)+\lambda h\left(T_{\beta}^{j,f}\right)\right)\right). \tag{3.4}$$

Now, from (3.3) and (3.4) we get $\mathbb{T}^{j}_{(\alpha\otimes\beta)^{\lambda}}=\mathbb{T}^{j}_{\alpha^{\lambda}\otimes\beta^{\lambda}}$. Similarly, we obtain $\mathbb{I}^{j}_{(\alpha\otimes\beta)^{\lambda}}=\mathbb{I}^{j}_{\alpha^{\lambda}\otimes\beta^{\lambda}}$ and $\mathbb{F}^{j}_{(\alpha\otimes\beta)^{\lambda}}=\mathbb{F}^{j}_{\alpha^{\lambda}\otimes\beta^{\lambda}}$ for any $j=1,...,p$ which proves the claim.

viii) We have for any $j=1,...,p$ that

$$\mathbb{T}^{j}_{\alpha^{\lambda}\otimes\alpha^{\gamma}}=\left(T\left(T_{\alpha^{\lambda}}^{j,t},T_{\alpha^{\gamma}}^{j,t}\right),S\left(T_{\alpha^{\lambda}}^{j,f},T_{\alpha^{\gamma}}^{j,f}\right)\right)$$
$$=\left(T\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)\right),g^{-1}\left(\gamma g\left(T_{\alpha}^{j,t}\right)\right)\right),\right.$$
$$\left.S\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)\right),h^{-1}\left(\gamma h\left(T_{\alpha}^{j,f}\right)\right)\right)\right)$$
$$=\left(g^{-1}\left(g\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)\right)\right)\right.\right.$$
$$\left.+g\left(g^{-1}\left(\gamma g\left(T_{\alpha}^{j,t}\right)\right)\right)\right),$$
$$h^{-1}\left(h\left(h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)\right)\right)\right.$$
$$\left.\left.+h\left(h^{-1}\left(\gamma h\left(T_{\alpha}^{j,f}\right)\right)\right)\right)\right)$$
$$=\left(g^{-1}\left(\lambda g\left(T_{\alpha}^{j,t}\right)+\gamma g\left(T_{\alpha}^{j,t}\right)\right)\right.$$
$$\left.h^{-1}\left(\lambda h\left(T_{\alpha}^{j,f}\right)+\gamma h\left(T_{\alpha}^{j,f}\right)\right)\right)$$
$$=\left(g^{-1}\left((\lambda+\gamma)\,g\left(T_{\alpha}^{j,t}\right)\right),\right.$$
$$\left.h^{-1}\left((\lambda+\gamma)\,h\left(T_{\alpha}^{j,f}\right)\right)\right)$$
$$=\mathbb{T}^{j}_{\alpha^{\lambda+\gamma}}.$$

Similarly, we obtain $\mathbb{I}^{j}_{\alpha^{\lambda}\otimes\alpha^{\gamma}}=\mathbb{I}^{j}_{\alpha^{\lambda+\gamma}}$ and $\mathbb{F}^{j}_{\alpha^{\lambda}\otimes\alpha^{\gamma}}=\mathbb{F}^{j}_{\alpha^{\lambda+\gamma}}$ for any $j=1,...,p$ which proves the claim.

ix) Let $\alpha\preceq\beta$. Then, for any $j=1,...,p$, we have

$$\mathbb{T}^{j}_{\alpha}\preceq_{(int)}\mathbb{T}^{j}_{\beta}$$

which yields that

$$T_{\alpha}^{j,t}\leq T_{\beta}^{j,t},T_{\alpha}^{j,f}\geq T_{\beta}^{j,f}$$

and so $\mathcal{T}^{*}\left(\mathbb{T}^{j}_{\alpha},\mathbb{T}^{j}_{\varphi}\right)\preceq_{(int)}\mathcal{T}^{*}\left(\mathbb{T}^{j}_{\beta},\mathbb{T}^{j}_{\varphi}\right)$. Similarly $\mathbb{I}^{j}_{\alpha}\succeq_{(int)}\mathbb{I}^{j}_{\beta}$ and $\mathbb{F}^{j}_{\alpha}\succeq_{(int)}\mathbb{F}^{j}_{\beta}$ imply that $\mathcal{T}\left(\mathbb{I}^{j}_{\alpha},\mathbb{I}^{j}_{\varphi}\right)\succeq_{(int)}\mathcal{T}\left(\mathbb{I}^{j}_{\beta},\mathbb{I}^{j}_{\varphi}\right)$ and $\mathcal{T}\left(\mathbb{F}^{j}_{\alpha},\mathbb{F}^{j}_{\varphi}\right)\succeq_{(int)}\mathcal{T}\left(\mathbb{F}^{j}_{\beta},\mathbb{F}^{j}_{\varphi}\right)$. Hence, $\alpha\oplus\varphi\preceq\beta\oplus\varphi$.

x) The proof is similar to the proof of (ix).

xi) We have

$$\alpha\oplus\overline{0}$$
$$=\left\langle\left(\mathcal{T}^{*}\left(\mathbb{T}^{j}_{\alpha},(1,0)\right)\right)_{j=1}^{p},\left(\mathcal{T}\left(\mathbb{I}^{j}_{\alpha},(0,1)\right)\right)_{j=1}^{p},\right.$$
$$\left.\left(\mathcal{T}\left(\mathbb{F}^{j}_{\alpha},(0,1)\right)\right)_{j=1}^{p}\right\rangle.$$

On the other hand, for any $j=1,...,p$, we obtain

$$\mathcal{T}^{*}\left(\mathbb{T}^{j}_{\alpha},(1,0)\right)=h^{-1}\left(h\left(T_{\alpha}^{j,t}\right)+h\left(0\right)\right)$$
$$=h^{-1}\left(h\left(T_{\alpha}^{j,t}\right)\right)$$
$$=T_{\alpha}^{j,t}$$

which yields that $\mathbb{T}^{j}_{\alpha\oplus\overline{0}}=\mathbb{T}^{j}_{\alpha}$. Similarly, we have $\mathbb{I}^{j}_{\alpha\oplus\overline{0}}=\mathbb{F}^{j}_{\alpha}$ and $\mathbb{F}^{j}_{\alpha\oplus\overline{0}}=\mathbb{T}^{j}_{\alpha}$. Hence, we get $\alpha\oplus\overline{0}=\alpha$.

xii) The proof is similar to the proof of (xi). □

Remark 3 From Theorem 3, it is clear that "⊗" and "⊕" define a t-norm and a t-conorm on the set of IFVNMVs, respectively.

Weighted aggregation operators

Aggregation operators are very crucial while transforming the data that is represented by a fuzzy set to a more compact form. In this section, using operations discussed in Sect. 3, we give some weighted aggregation operators for classes of IFVNMVs. Throughout this section, we study with t-norm T, the dual t-conorm S of T and intuitionistic fuzzy t-norm $\mathcal{T}$ that are generated by an algebraic generator g.

Weighted arithmetic aggregation operator

In the following, we define a weighted arithmetic aggregation operator for collections of IFVNMVs given with equal sequence lengths.

Definition 5 Let $\left\{\alpha_i=\left\langle(\mathbb{T}^{j}_{\alpha_i})_{j=1}^{p},(\mathbb{I}^{j}_{\alpha_i})_{j=1}^{p},(\mathbb{F}^{j}_{\alpha_i})_{j=1}^{p}\right\rangle\right.$ $\left.: i=1,...,n\right\}$ be a collection of IFVNMVs with equal sequence lengths $p\in\mathbb{Z}^{+}$. Then, a weighted arithmetic aggregation operator $WA-IFVNMV$ is defined by

$$WA-IFVNMV(\alpha_1,...,\alpha_n):=\bigoplus_{i=1}^{n}\omega_i\alpha_i$$

where $0\leq\omega_i\leq 1$ for any $i=1,...,n$ with $\sum_{i=1}^{n}\omega_i=1$.

Theorem 4 *Let* $\left\{\alpha_i=\left\langle(\mathbb{T}^{j}_{\alpha_i})_{j=1}^{p},(\mathbb{I}^{j}_{\alpha_i})_{j=1}^{p},(\mathbb{F}^{j}_{\alpha_i})_{j=1}^{p}\right\rangle\right.$ $\left.: i=1,...,n\right\}$ *be a collection of IFVNMVs with equal sequence lengths* $p\in\mathbb{Z}^{+}$ *and let* $0\leq\omega_i\leq 1$ *for* *any* $i = 1, ..., n$ *with* $\sum_{i=1}^{n}\omega_i = 1$. *Then,* $WA - IFVNMV(\alpha_1, ..., \alpha_n)$ *is an IFVNMV and we have*

$$\begin{aligned}&WA - IFVNMV(\alpha_1, ..., \alpha_n)\\&= \left(\left(\left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p}, \left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\right),\right.\\&\left(\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(I_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p}, \left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(I_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\right),\\&\left.\left(\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(F_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p}, \left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(F_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\right)\right).\end{aligned}$$

Proof From Proposition 2 and Theorem 2, $WA-IFVNMV$ $(\alpha_1, ..., \alpha_n)$ is an IFVNMV. For the second part of the proof, we conduct mathematical induction. For any $j = 1, ..., p$, we obtain

$$\begin{aligned}\mathbb{T}_{\omega_1\alpha_1\oplus\omega_2\alpha_2}^{j} &= \mathcal{T}^*\left(\mathbb{T}_{\omega_1\alpha_1}^{j}, \mathbb{T}_{\omega_2\alpha_2}^{j}\right)\\&= \mathcal{T}^*\left(\left(T_{\omega_1\alpha_1}^{j,t}, T_{\omega_1\alpha_1}^{j,f}\right), \left(T_{\omega_2\alpha_2}^{j,t}, T_{\omega_2\alpha_2}^{j,f}\right)\right)\\&= \left(S\left(T_{\omega_1\alpha_1}^{j,t}, T_{\omega_2\alpha_2}^{j,t}\right), T\left(T_{\omega_1\alpha_1}^{j,f}, T_{\omega_2\alpha_2}^{j,f}\right)\right)\\&= \left(h^{-1}\left(h\left(T_{\omega_1\alpha_1}^{j,t}\right) + h\left(T_{\omega_2\alpha_2}^{j,t}\right)\right)\right.\\&\quad\left., g^{-1}\left(g\left(T_{\omega_1\alpha_1}^{j,f}\right) + g\left(T_{\omega_2\alpha_2}^{j,f}\right)\right)\right)\\&= \left(h^{-1}\left(h\left(h^{-1}\left(\omega_1 h\left(T_{\alpha_1}^{j,t}\right)\right)\right)\right.\right.\\&\quad\left.+h\left(h^{-1}\left(\omega_2 h\left(T_{\alpha_2}^{j,t}\right)\right)\right)\right),\\&\quad g^{-1}\left(g\left(g^{-1}\left(\omega_1 g\left(T_{\alpha_1}^{j,f}\right)\right)\right)\right.\\&\quad\left.\left.+g\left(\omega_2 g\left(T_{\alpha_2}^{j,f}\right)\right)\right)\right)\\&= \left(h^{-1}\left(\omega_1 h\left(T_{\alpha_1}^{j,t}\right) + \omega_2 h\left(T_{\alpha_2}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(\omega_1 g\left(T_{\alpha_1}^{j,f}\right) + \omega_2 g\left(T_{\alpha_2}^{j,f}\right)\right)\right)\\&= \left(h^{-1}\left(\sum_{i=1}^{2}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(\sum_{i=1}^{2}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right)\right).\end{aligned}$$

Similar proof is valid for $\mathbb{I}_{\omega_1\alpha_1\oplus\omega_2\alpha_2}^{j}$ and $\mathbb{F}_{\omega_1\alpha_1\oplus\omega_2\alpha_2}^{j}$ for any $j = 1, ..., p$ which yield that the statement is true for $n = 2$. Assume that

$$\mathbb{T}_{A_{n-1}}^{j} = \left(h^{-1}\left(\sum_{i=1}^{n-1}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right), g^{-1}\left(\sum_{i=1}^{n-1}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right)\right)$$

where $A_n := \bigoplus_{i=1}^{n}\omega_i\alpha_i$. Then, we have for any $j = 1, ..., p$ that

$$\begin{aligned}\mathbb{T}_{A_n}^{j} &= \mathbb{T}_{A_{n-1}\oplus\omega_n\alpha_n}^{j}\\&= \mathcal{T}^*\left(\mathbb{T}_{A_{n-1}}^{j}, \mathbb{T}_{\omega_n\alpha_n}^{j}\right)\\&= \mathcal{T}^*\left(\left(T_{A_{n-1}}^{j,t}, T_{A_{n-1}}^{j,f}\right), \left(T_{\omega_n\alpha_n}^{j,t}, T_{\omega_n\alpha_n}^{j,f}\right)\right)\\&= \left(S\left(T_{A_{n-1}}^{j,t}, T_{\omega_n\alpha_n}^{j,t}\right), T\left(T_{A_{n-1}}^{j,f}, T_{\omega_n\alpha_n}^{j,f}\right)\right)\\&= \left(h^{-1}\left(h\left(T_{A_{n-1}}^{j,t}\right) + h\left(T_{\omega_n\alpha_n}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(g\left(T_{A_{n-1}}^{j,f}\right) + g\left(T_{\omega_n\alpha_n}^{j,f}\right)\right)\right)\\&= \left(h^{-1}\left(h\left(h^{-1}\left(\sum_{i=1}^{n-1}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right)\right)\right.\right.\\&\quad\left.+h\left(h^{-1}\left(\omega_n h\left(T_{\alpha_n}^{j,t}\right)\right)\right)\right),\\&\quad g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n-1}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right)\right)\right.\\&\quad\left.\left.+g\left(g^{-1}\left(\omega_n g\left(T_{\alpha_n}^{j,f}\right)\right)\right)\right)\right)\\&= \left(h^{-1}\left(\sum_{i=1}^{n-1}\omega_i h\left(T_{\alpha_i}^{j,t}\right) + \omega_n h\left(T_{\alpha_n}^{j,t}\right)\right),\right.\\&\quad\left.g^{-1}\left(\sum_{i=1}^{n-1}\omega_i g\left(T_{\alpha_i}^{j,f}\right) + \omega_n g\left(T_{\alpha_n}^{j,f}\right)\right)\right)\\&= h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right).\end{aligned}$$

Similar proof is valid for $\mathbb{I}_{A_n}^{j}$ and $\mathbb{F}_{A_n}^{j}$ for any $j = 1, ..., p$ which finishes the proof. □

Proposition 3 $WA - IFVNMV(\bar{\mathbf{0}}, ..., \bar{\mathbf{0}}) = \bigoplus_{i=1}^{n}\omega_i\bar{\mathbf{0}}$ *and* $WA - IFVNMV(\bar{\mathbf{1}}, ..., \bar{\mathbf{1}}) = \bigoplus_{i=1}^{n}\omega_i\bar{\mathbf{1}} = \bar{\mathbf{1}}$.

Proof From Theorem 4, we have

$$\begin{aligned}&WA - IFVNMV(\bar{\mathbf{0}}, ..., \bar{\mathbf{0}}) = \bigoplus_{i=1}^{n}\omega_i\bar{\mathbf{0}}\\&= \left(\left(\left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h(0)\right)\right\}_{j=1}^{p}, \left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g(1)\right)\right\}_{j=1}^{p}\right),\right.\\&\left(\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g(1)\right)\right\}_{j=1}^{p}, \left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h(0)\right)\right\}_{j=1}^{p}\right),\\&\left.\left(\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g(1)\right)\right\}_{j=1}^{p}, \left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h(0)\right)\right\}_{j=1}^{p}\right)\right).\\&= (((0,1), ...(0,1)), ((1,0), ..., (1,0)), ((1,0), ..., (1,0)))\\&= \bar{\mathbf{0}}.\end{aligned}$$

Note here that, $g(0)=\infty$ and $g^{-1}(\infty)=0$ stand for $\lim_{t\to 0^+} g(t)=\infty$ and $h(1)=\infty$ and $h^{-1}(\infty)=1$ stand for $\lim_{t\to 1^-} h(t)=\infty$. Similarly, it is seen that

$$WA-IFVNMV(\overline{\mathbf{1}},...,\overline{\mathbf{1}})=\bigoplus_{i=1}^{n}\omega_i\overline{\mathbf{1}}=\overline{\mathbf{1}}.$$

□

The following theorem shows that $WA-IFVNMV$ is monotone.

Theorem 5 *Let* $\left\{\alpha_i=\left\langle(\mathbb{T}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{I}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{F}_{\alpha_i}^{j})_{j=1}^{p}\right\rangle: i=1,...,n\right\}$ *and* $\left\{\beta_i=\left\langle(\mathbb{T}_{\beta_i}^{j})_{j=1}^{p},(\mathbb{I}_{\beta_i}^{j})_{j=1}^{p},(\mathbb{F}_{\beta_i}^{j})_{j=1}^{p}\right\rangle: i=1,...,n\right\}$ *be two families of IFVNMVs such that* $\alpha_i\preceq\beta_i$ *for any* $i=1,...,n$. *Then,*

$$WA-IFVNMV(\alpha_1,...,\alpha_n)\preceq WA-IFVNMV(\beta_1,...,\beta_n)$$

and

$$WG-IFVNMV(\alpha_1,...,\alpha_n)\preceq WG-IFVNMV(\beta_1,...,\beta_n).$$

Proof Assume that $\alpha_i\preceq\beta_i$ for any $i=1,...,n$. Then, we have

$$T_{\alpha_i}^{j,t}\le T_{\beta_i}^{j,t} \text{ and } T_{\alpha_i}^{j,f}\ge T_{\beta_i}^{j,f}.$$

Since h, h^{-1}, g and g^{-1} are increasing, we have

$$h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(T_{\alpha_i}^{j,t}\right)\right)\le h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(T_{\beta_i}^{j,t}\right)\right)$$

and

$$g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(T_{\alpha_i}^{j,f}\right)\right)\ge g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(T_{\beta_i}^{j,f}\right)\right)$$

which implies that $\mathbb{T}_{WA-IFVNMV(\alpha_1,...,\alpha_n)}\le_{(int)}\mathbb{T}_{WA-IFVNMV(\beta_1,...,\beta_n)}$.

Similarly, we obtain

$$\mathbb{I}_{WA-IFVNMV(\alpha_1,...,\alpha_n)}\ge_{(int)}\mathbb{I}_{WA-IFVNMV(\beta_1,...,\beta_n)}$$

and

$$\mathbb{F}_{WA-IFVNMV(\alpha_1,...,\alpha_n)}\ge_{(int)}\mathbb{F}_{WA-IFVNMV(\beta_1,...,\beta_n)}.$$

Therefore, we get $WA-IFVNMV(\alpha_1,...,\alpha_n)\preceq WA-IFVNMV(\beta_1,...,\beta_n)$. Hence, $WA-IFVNMV$ is monotone with respect to the partial order "$\preceq$". □

Weighted geometric aggregation operator

Definition 6 Let $\left\{\alpha_i=\left\langle(\mathbb{T}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{I}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{F}_{\alpha_i}^{j})_{j=1}^{p}\right\rangle: i=1,...,n\right\}$ be a collection of IFVNMVs with equal sequence lengths $p\in\mathbb{Z}^+$. Then, a weighted geometric aggregation operator $WG-IFVNMV$ is defined by

$$WG-IFVNMV(\alpha_1,...,\alpha_n):=\bigotimes_{i=1}^{n}\alpha_i^{\omega_i}$$

where $0\le\omega_i\le 1$ for any $i=1,...,n$ with $\sum_{i=1}^{n}\omega_i=1$.

Theorem 6 *Let* $\left\{\alpha_i=\left\langle(\mathbb{T}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{I}_{\alpha_i}^{j})_{j=1}^{p},(\mathbb{F}_{\alpha_i}^{j})_{j=1}^{p}\right\rangle: i=1,...,n\right\}$ *be a collection of IFVNMVs with equal sequence lengths* $p\in\mathbb{Z}^+$ *and let* $0\le\omega_i\le 1$ *for any* $i=1,...,n$ *with* $\sum_{i=1}^{n}\omega_i=1$. *Then,* $WG-IFVNMV(\alpha_1,...,\alpha_n)$ *is an IFVNMV and we have*

$$\begin{aligned}&WG-IFVNMV(\alpha_1,...,\alpha_n)\\&=\Bigg(\Bigg(\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(T_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p},\\&\qquad\left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(T_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\Bigg),\\&\Bigg(\left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(I_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p},\\&\qquad\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(I_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\Bigg),\\&\Bigg(\left\{h^{-1}\left(\sum_{i=1}^{n}\omega_i h\left(F_{\alpha_i}^{j,t}\right)\right)\right\}_{j=1}^{p},\\&\qquad\left\{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(F_{\alpha_i}^{j,f}\right)\right)\right\}_{j=1}^{p}\Bigg)\Bigg).\end{aligned}$$

Proof The proof is similar to the proof of Theorem 4. Therefore, we omit it. □

Proposition 4 $WG-IFVNMV(\overline{\mathbf{0}},...,\overline{\mathbf{0}})=\bigoplus_{i=1}^{n}\omega_i\overline{\mathbf{0}}$ *and* $WG-IFVNMV(\overline{\mathbf{1}},...,\overline{\mathbf{1}})=\bigoplus_{i=1}^{n}\omega_i\overline{\mathbf{1}}=\overline{\mathbf{1}}$.

Proof It can be proved similar to Proposition 3. □

Remark 4 Similar to Theorem 5, it can be proved that

$$WG-IFVNMV(\alpha_1,...,\alpha_n)\preceq WG-IFVNMV(\beta_1,...,\beta_n)$$

whenever $\alpha_i\preceq\beta_i$ for any $i=1,...,n$. Hence, $WG-IFVNMV$ is monotone with respect to partial order "$\preceq$".

Some particular cases of *WA − IFVNMV* and *WG − IFVNMV*

In this subsection, we study some particular cases of $WA-IFVNMV$ and $WG-IFVNMV$ by choosing particular additive generators.

(1) Consider the generator g recalled in Example 1 and defined by $g(t)=-\log t$. Then, $WA-IFVNMV$ and $WG-IFVNMV$ turn into following Algebraic weighted aggregation operators:

$$\begin{aligned}&WA_A-IFVNMV(\alpha_1,...,\alpha_n)\\&:=\Bigg(\Bigg(\Bigg\{1-\prod_{i=1}^{n}\left(1-T_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{\prod_{i=1}^{n}\left(T_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{\prod_{i=1}^{n}\left(I_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{1-\prod_{i=1}^{n}\left(1-I_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{\prod_{i=1}^{n}\left(F_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{1-\prod_{i=1}^{n}\left(1-F_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg)\Bigg)\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&WG_A-IFVNMV(\alpha_1,...,\alpha_n)\\&:=\Bigg(\Bigg(\Bigg\{\prod_{i=1}^{n}\left(T_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{1-\prod_{i=1}^{n}\left(1-T_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{1-\prod_{i=1}^{n}\left(1-I_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{\prod_{i=1}^{n}\left(I_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{1-\prod_{i=1}^{n}\left(1-F_{\alpha_i}^{j,t}\right)^{\omega_i}\Bigg\}_{j=1}^{p},\Bigg\{\prod_{i=1}^{n}\left(F_{\alpha_i}^{j,f}\right)^{\omega_i}\Bigg\}_{j=1}^{p}\Bigg)\Bigg),\end{aligned}\tag{4.2}$$

respectively.

(2) Consider the additive generator g defined by $g(t)=\log\frac{2-t}{t}$. Then, $WA-IFVNMV$ and $WG-IFVNMV$ turn into following Einstein weighted aggregation operators:

$$\begin{aligned}&WA_E-IFVNMV(\alpha_1,...,\alpha_n)\\&:=\Bigg(\Bigg(\Bigg\{\frac{\prod_{i=1}^{n}\left(1+T_{\alpha_i}^{j,t}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-T_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+T_{\alpha_i}^{j,t}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-T_{\alpha_i}^{j,t}\right)^{\omega_i}}\Bigg\}_{j=1}^{p},\Bigg\{\frac{2\prod_{i=1}^{n}\left(T_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-T_{\alpha_i}^{j,f}\right)^{\omega_i}+\prod_{i=1}^{n}\left(T_{\alpha_i}^{j,f}\right)^{\omega_i}}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{\frac{2\prod_{i=1}^{n}\left(I_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-I_{\alpha_i}^{j,t}\right)^{\omega_i}+\prod_{i=1}^{n}\left(I_{\alpha_i}^{j,t}\right)^{\omega_i}}\Bigg\}_{j=1}^{p},\Bigg\{\frac{\prod_{i=1}^{n}\left(1+I_{\alpha_i}^{j,f}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-I_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+I_{\alpha_i}^{j,f}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-I_{\alpha_i}^{j,f}\right)^{\omega_i}}\Bigg\}_{j=1}^{p}\Bigg),\\&\Bigg(\Bigg\{\frac{2\prod_{i=1}^{n}\left(F_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-F_{\alpha_i}^{j,t}\right)^{\omega_i}+\prod_{i=1}^{n}\left(F_{\alpha_i}^{j,t}\right)^{\omega_i}}\Bigg\}_{j=1}^{p},\Bigg\{\frac{\prod_{i=1}^{n}\left(1+F_{\alpha_i}^{j,f}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-F_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+F_{\alpha_i}^{j,f}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-F_{\alpha_i}^{j,f}\right)^{\omega_i}}\Bigg\}_{j=1}^{p}\Bigg)\Bigg)\end{aligned}\tag{4.3}$$

and

$$WG_E - IFVNMV(\alpha_1, ..., \alpha_n)$$
$$:= \left(\left(\left\{\frac{2\prod_{i=1}^n \left(T_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^n \left(2 - T_{\alpha_i}^{j,t}\right)^{\omega_i} + \prod_{i=1}^n \left(T_{\alpha_i}^{j,t}\right)^{\omega_i}}\right\}_{j=1}^p, \left\{\frac{\prod_{i=1}^n \left(1 + T_{\alpha_i}^{j,f}\right)^{\omega_i} - \prod_{i=1}^n \left(1 - T_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^n \left(1 + T_{\alpha_i}^{j,f}\right)^{\omega_i} + \prod_{i=1}^n \left(1 - T_{\alpha_i}^{j,f}\right)^{\omega_i}}\right\}_{j=1}^p\right),\right.$$
$$\left(\left\{\frac{\prod_{i=1}^n \left(1 + I_{\alpha_i}^{j,t}\right)^{\omega_i} - \prod_{i=1}^n \left(1 - I_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^n \left(1 + I_{\alpha_i}^{j,f}\right)^{\omega_i} + \prod_{i=1}^n \left(1 - I_{\alpha_i}^{j,f}\right)^{\omega_i}}\right\}_{j=1}^p, \left\{\frac{2\prod_{i=1}^n \left(I_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^n \left(2 - I_{\alpha_i}^{j,f}\right)^{\omega_i} + \prod_{i=1}^n \left(I_{\alpha_i}^{j,f}\right)^{\omega_i}}\right\}_{j=1}^p\right),$$
$$\left.\left(\left\{\frac{\prod_{i=1}^n \left(1 + F_{\alpha_i}^{j,t}\right)^{\omega_i} - \prod_{i=1}^n \left(1 - F_{\alpha_i}^{j,t}\right)^{\omega_i}}{\prod_{i=1}^n \left(1 + F_{\alpha_i}^{j,t}\right)^{\omega_i} + \prod_{i=1}^n \left(1 - F_{\alpha_i}^{j,t}\right)^{\omega_i}}\right\}_{j=1}^p, \left\{\frac{2\prod_{i=1}^n \left(F_{\alpha_i}^{j,f}\right)^{\omega_i}}{\prod_{i=1}^n \left(2 - F_{\alpha_i}^{j,f}\right)^{\omega_i} + \prod_{i=1}^n \left(F_{\alpha_i}^{j,f}\right)^{\omega_i}}\right\}_{j=1}^p\right)\right), \tag{4.4}$$

respectively.

An application of *IFVNMV*s to a classification problem

In this section, we give a classification method using $IFVNMS$s and aggregation operators defined in Section "Weighted aggregation operators".

A simplified neutrosophic valued cosine similarity measure for IFVNMVs

Similarity measures are convenient tools in the classification. A cosine similarity measure between IFSs depends on the cosine of the angle of the vector representations of the membership and non-membership degrees (see e.g., [45]). Now, we propose a simplified neutrosophic valued cosine similarity measure. First, let us recall the notion of SNS and simplified neutrosophic value (SNV). A SNS [16] on a universal set $X = \{x_1, ..., x_n\}$ is given by

$$A = \{\langle x_i, (T_A(x_i), I_A(x_i), F_A(x_i))\rangle : i = 1, ..., n\}$$

where $T_A, I_A, F_A : X \to [0, 1]$ are the truth, indeterminacy and falsity functions. For a fixed $x \in X$,

$$\tau = \langle T_\tau, I_\tau, F_\tau\rangle := \langle T_A(x), I_A(x), F_A(x)\rangle$$

is called a simplified neutrosophic value (SNV).

Definition 7 A simplified neutrosophic valued weighted cosine similarity measure between IFVNMVs α and β with same sequence lengths p is given with

$$CSN(\alpha, \beta)$$
$$:= \left(\sum_{j=1}^p \omega_j \frac{T_\alpha^{j,t} T_\beta^{j,t} + T_\alpha^{j,f} T_\beta^{j,f}}{\sqrt{\left(T_\alpha^{j,t}\right)^2 + \left(T_\alpha^{j,f}\right)^2}\sqrt{\left(T_\beta^{j,t}\right)^2 + \left(T_\beta^{j,f}\right)^2}},\right.$$
$$1 - \sum_{j=1}^p \omega_j \frac{I_\alpha^{j,t} I_\beta^{j,t} + I_\alpha^{j,f} I_\beta^{j,f}}{\sqrt{\left(I_\alpha^{j,t}\right)^2 + \left(I_\alpha^{j,f}\right)^2}\sqrt{\left(I_\beta^{j,t}\right)^2 + \left(I_\beta^{j,f}\right)^2}},$$
$$\left.1 - \sum_{j=1}^p \omega_j \frac{F_\alpha^{j,t} F_\beta^{j,t} + F_\alpha^{j,f} F_\beta^{j,f}}{\sqrt{\left(F_\alpha^{j,t}\right)^2 + \left(F_\alpha^{j,f}\right)^2}\sqrt{\left(F_\beta^{j,t}\right)^2 + \left(F_\beta^{j,f}\right)^2}}\right) \tag{5.1}$$

where $0 \le \omega_j \le 1$ for any $j = 1, ..., p$ and $\sum_{j=1}^p \omega_j = 1$.

It is clear that $CSN(\alpha, \beta)$ is a SNV. Therefore, we need a score function to rank the values of CSN. Let $A = (a, b, c)$ be a SNV. An improved score function N defined by

$$N(A) = \frac{1 + (a - 2b - c)(2 - a - c)}{2} \tag{5.2}$$

has been proposed by Nancy and Garg [25]. We use the score function N to rank the results of CSN.

Table 3 IFVNMV representations of value sizes of iris plant

Value size	$\{\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^3\}$	$\{\mathbb{I}^1, \mathbb{I}^2, \mathbb{I}^3\}$	$\{\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3\}$
Extremely large	{(1, 0) , (0.9, 0.1) , (0.95, 0.05)}	{(1, 0) , (1, 0) , (1, 0)}	{(0.05, 0.95) , (0.1, 0.9) , (0.1, 0.8)}
Very very large	{(0.85, 0.1) , (0.8, 0.1) , (0.9, 0.05)}	{(0.85, 0.1) , (0.9, 0.1) , (0.9, 0.05)}	{(0.1, 0.9) , (0.1, 0.8) , (0.15, 0.8)}
Very large	{(0.8, 0.1) , (0.8, 0.15) , (0.8, 0.1)}	{(0.75, 0.2) , (0.8, 0.15) , (0.8, 0.15)}	{(0.15, 0.8) , (0.1, 0.75) , (0.2, 0.75)}
Large	{(0.75, 0.1) , (0.7, 0.2) , (0.75, 0.15)}	{(0.7, 0.25) , (0.7, 0.2) , (0.75, 0.20)}	{(0.2, 0.75) , (0.15, 0.75) , (0.25, 0.7)}
Medium large	{(0.6, 0.3) , (0.65, 0.3) , (0.7, 0.2)}	{(0.65, 0.35) , (0.6, 0.35) , (0.7, 0.25)}	{(0.3, 0.6) , (0.25, 0.65) , (0.3, 0.65)}
Fair	{(0.55, 0.4) , (0.5, 0.4) , (0.6, 0.3)}	{(0.5, 0.5) , (0.5, 0.5) , (0.55, 0.40)}	{(0.35, 0.55) , (0.4, 0.55) , (0.35, 0.6)}
Medium small	{(0.45, 0.5) , (0.4, 0.5) , (0.5, 0.4)}	{(0.6, 0.35) , (0.65, 0.35) , (0.70, 0.25)}	{(0.5, 0.4) , (0.5, 0.45) , (0.4, 0.55)}
Small	{(0.25, 0.55) , (0.35, 0.5) , (0.45, 0.45)}	{(0.65, 0.3) , (0.7, 0.25) , (0, 85, 0.15)}	{(0.55, 0.25) , (0.55, 0.3) , (0.45, 0.5)}
Very small	{(0.2, 0.75) , (0.15, 0.75) , (0.25, 0.65)}	{(0.85, 0.1) , (0.9, 0.05) , (0.9, 0.05)}	{(0.75, 0.15) , (0.7, 0.2) , (0.65, 0.3)}
Very very small	{(0.1, 0.9) , (0.1, 0.85) , (0.15, 0.8)}	{(1, 0) , (1, 0) , (1, 0)}	{(0.9, 0.05) , (0.9, 0.1) , (0.85, 0.1)}

Numerical example

In this sub-section, we consider a real-life example of classification. Before, we summarize the proposed method by the following simple algorithm:

Step 1: Consider the whole data set as $X = \{x_1, ..., x_n\}$, where n is the sample size of the data set.

Step 2: Separate the database as the training set $T = \{t_1, ..., t_m\}$ and testing set $Y = \{y_1, ..., y_l\}$, where m and l are the sample sizes of the training and testing sets, respectively.

Step 3: According to the view of p experts, the value sizes are expressed as IFVNMVs.

Step 4: For each attribute x_i in data set, calculate the range as $R = x_{(m)} - x_{(1)}$ where $x_{(m)}$ is the maximum value and $x_{(1)}$ is the minimum value of the data set. The range is divided into c categories using percentiles. Determine the levels and each level corresponds to an IFVNMV from Step 3.

Step 5: For each class in the training set, calculate the mean value $\bar{X}$ for each attribute. The interval that $\bar{X}$ is coming from determines the level. Obtain the IFVNMS representations that consists of IFVNMVs from Step 4 of each class using these levels and using a weighted aggregation operator, obtain the IFVNMV representation of each class.

Step 6: With the same algorithm, determine the IFVNMV representation of the each sample from testing set.

Step 7: Calculate the similarity measures using CSN (5.1) and obtain the scores from score function N (5.2). Then, classify each sample according to the maximum score function value.

It should be also noted that, these steps are repeated using different aggregation operator.

As an application, a real example of classification from UCI Machine Learning Repository is conducted to show the performance of the method. Recently, researchers have carried out some studies on iris dataset with the help of fuzzy logic. For example, Singh and Ganie [47–49] have proposed similarity and correlation measures for some various fuzzy sets and have applied them to a classification problem with Iris database. In these studies, the authors have introduced the conversion formula from crisp data to fuzzy data. Also, to assess the performance of the proposed measures a performance index, namely "Degree of confidence (DoC)", has been introduced. In view of DoC, these proposed measures are found to outperform the existing compatibility measures and they have not proposed accuracy mean for result of classification. Moreover, Fei et al. [5] have introduced a vector valued similarity measure for IFSs and they have applied it to the same classification problem. Di Martino and Sessa [50] have proposed classification algorithm based on direct and inverse fuzzy transforms.

The data are for iris plants and consists of 150 samples. These samples are divided into three categories, namely Setosa, Versicolour and Virginica and each sample has four

Table 4 IFVNMV representations of each class with respect to aggregation operators (1=WA$_A$-IFVNMV, 2=WG$_A$-IFVNMV, 3=WA$_E$-IFVNMV, 4=WG$_E$-IFVNMV)

A.O.	Class	IFVNMV
	Setosa	{(0.38 0.46), (0.37 0.52), (0.44 0.46)} {(0.82 0.17), (0.84 0.12), (0.89 0.09)} {(0.55 0.36), (0.51 0.39), (0.53 0.41)}
1	Versicolour	{(0.61 0.26), (0.61 0.31), (0.67 0.22)} {(0.65 0.33), (0.64 0.32), (0.71 0.24)} {(0.31 0.61), (0.26 0.64), (0.31 0.64)}
	Virginica	{(0.73 0.15), (0.71 0.22), (0.73 0.16)} {(0.70 0.25), (0.73 0.22), (0.76 0.19)} {(0.22 0.72), (0.17 0.70), (0.25 0.70)}
	Setosa	{(0.21 0.75), (0.22 0.69), (0.30 0.63)} {(1.00 0.00), (1.00 0.00), (1.00 0.00)} {(0.76 0.15), (0.75 0.22), (0.69 0.24)}
2	Versicolour	{(0.59 0.31), (0.59 0.33), (0.65 0.24)} {(0.65 0.32), (0.64 0.30), (0.71 0.24)} {(0.33 0.57), (0.30 0.61), (0.31 0.64)}
	Virginica	{(0.68 0.22), (0.65 0.27), (0.70 0.20)} {(0.71 0.24), (0.75 0.20), (0.77 0.18)} {(0.27 0.66), (0.23 0.66), (0.27 0.68)}
	Setosa	{(0.34 0.51), (0.34 0.55), (0.41 0.48)} {(0.83 0.16), (0.85 0.11), (0.90 0.09)} {(0.58 0.32), (0.55 0.35), (0.56 0.38)}
3	Versicolour	{(0.61 0.27), (0.61 0.31), (0.67 0.22)} {(0.65 0.33), (0.64 0.31), (0.71 0.24)} {(0.31 0.60), (0.27 0.64), (0.31 0.64)}
	Virginica	{(0.72 0.15), (0.70 0.22), (0.73 0.16)} {(0.70 0.25), (0.74 0.21), (0.76 0.19)} {(0.22 0.72), (0.17 0.69), (0.25 0.70)}
	Setosa	{(0.22 0.72), (0.24 0.67), (0.31 0.61)} {(1.00 0.00), (1.00 0.00), (1.00 0.00)} {(0.74 0.16), (0.73 0.23), (0.67 0.26)}
4	Versicolour	{(0.59 0.31), (0.59 0.33), (0.66 0.24)} {(0.65 0.32), (0.64 0.31), (0.71 0.24)} {(0.33 0.58), (0.29 0.62), (0.31 0.64)}
	Virginica	{(0.69 0.21), (0.66 0.26), (0.71 0.19)} {(0.70 0.24), (0.74 0.20), (0.77 0.18)} {(0.26 0.67), (0.22 0.67), (0.26 0.68)}

Table 5 Scores of similarity measures between IFVNMVs of testing sets and all classes

Setosa			Versicolour			Virginica		
Setosa	Versicolour	Virginica	Setosa	Versicolour	Virginica	Setosa	Versicolour	Virginica
0.99	0.70	0.66	0.92	**0.93**	0.85	0.71	0.93	**0.99**
0.97	0.91	0.88	0.91	**0.94**	0.84	0.69	0.91	**0.97**
0.93	0.92	0.92	0.84	**0.95**	0.86	0.76	0.97	**0.99**
0.97	0.69	0.59	0.83	**0.99**	0.98	0.74	0.95	**0.99**
0.98	0.75	0.65	0.84	**0.99**	0.94	0.70	0.92	**0.98**
0.99	0.86	0.81	0.79	**0.99**	0.97	0.68	0.93	**0.98**
0.99	0.78	0.72	0.77	**0.99**	0.98	0.64	0.91	**0.99**
0.95	0.65	0.54	0.90	**0.98**	0.93	0.71	0.91	**0.98**
0.99	0.77	0.69	0.85	**0.98**	0.85	0.74	0.90	**0.99**
0.99	0.86	0.80	0.79	**0.98**	0.91	0.70	0.97	**0.98**
0.89	0.49	0.39	0.84	**0.99**	0.92	0.69	0.96	**0.99**
0.95	0.65	0.54	0.79	**0.96**	0.90	0.70	0.95	**0.99**
0.95	0.92	0.87	0.90	**0.98**	0.94	0.82	**0.99**	0.98
0.99	0.91	0.87	0.85	**0.99**	0.90	0.70	0.91	**0.99**
0.98	0.90	0.84	0.86	**0.95**	0.92	0.65	0.93	**0.98**
0.95	0.71	0.69	0.75	**0.95**	0.90	0.64	0.90	**0.99**
0.95	0.65	0.54	0.78	**0.96**	0.93	0.70	0.92	**0.99**
0.99	0.90	0.82	0.90	**0.99**	0.95	0.72	0.89	**0.98**
0.98	0.89	0.81	0.85	**0.99**	0.95	0.69	0.91	**0.98**
0.95	0.65	0.52	0.77	**0.98**	0.91	0.80	**0.99**	0.98

Table 6 Comparison of classification study for Iris plant

Study	Methodology	Accuracy mean
Fei et al. [5]	Vector valued similarity measure	90%
Di Martino and Sessa [50]	MFC classifier	98.15%
	Decision tree J48	98.38%
	Multilayer perceptron	98.22%
	Naive Bayes	96.55%
	Lazy IBK	97.17%
This study	Aggregation and score function, similarity measure	97%

attributes, namely Sepal Length (SL), Sepal Width (SW), Petal Length (PL) and Petal Width (PW). In this example, 30 samples are randomly selected from each category as a training set and 20 of them are used as a testing set. First, each value size of the iris plant is represented by an IFVNMV by adapting Table 2 of [5]. For a fixed $i = 1, 2, 3$; $\mathbb{T}^i$, $\mathbb{I}^i$ and $\mathbb{F}^i$ is considered the truth, indeterminacy and falsity IFV, respectively, given by the ith expert. The adapted values of IFVNMs are shown in Table 3.

The IFVNMVs of three classes are obtained as shown in Table 4 with four different aggregation operators.

Next, similarities are calculated between IFVNMVs of the three classes and the 20 testing samples of each class and score functions are calculated based on different aggregation operators (see Table 4). Table 5 is constructed using aggregation operator WA_A-IFVNMV for the sake of brevity. It can be seen that other aggregation operators give the same result as well.

It can be easily seen that, the accuracy is 100% in Setosa, 100% in Versicolour and 90% in Virginica class. The average accuracy is 97%.

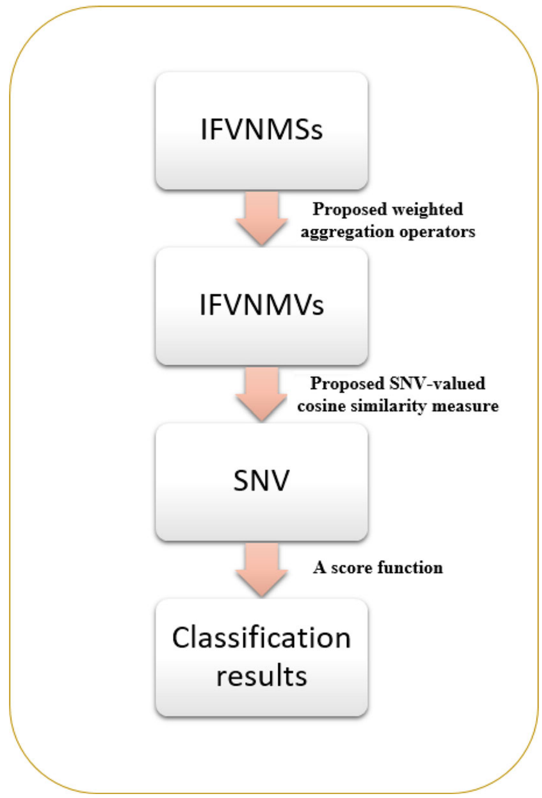

Fig. 1 Application of the proposed theory to the classification

Conclusion

In this study, we introduce the concept of intuitionistic fuzzy-valued neutrosophic multi-set (IFVNMS) by considering the sequences of intuitionistic fuzzy values instead of numbers. Therefore, more detailed information can be kept when transforming the data to fuzzy sets. In this manner, it prevents the loss of information. We present some set theoretic operations between IFVNMSs. Using general t-norms and t-conorms, we define some fundamental algebraic operations between intuitionistic fuzzy-valued neutrosophic values (IFVNMVs). With the help of these algebraic operations, we introduce some weighted arithmetic and geometric aggregation operators. These aggregation operators allow us to aggregate IFVNMSs to IFVNMVs. Thus, one may obtain more compact and rich representation of real data. By defining a simplified neutrosophic valued cosine similarity measure, we rank IFVNMVs in the simplified neutrosophic environment. After this, with the help of an improved score function for simplified neutrosophic values existing in the literature [25], we manage to rank results of the cosine similarity measure. We give an algorithm to give the applicability of the proposed theory to the classification. We also apply the algorithm to a real classification from UCI Machine Learning Repository and we show that the average accuracy is 97%. The steps how we used the theoretic part of this paper in the classification can be seen in Fig. 1.

The proposed theory is applicable in any multi-criteria decision making problem, pattern recognition and classification problems, especially problems with more than one decision makers. Therefore, this new theory will be a useful tool in decision and ranking problems such as robot selection, software selection, green suppliers selection or solid waste landfill site selection problems etc. In the future, we give some applications of the proposed theory to some multi-criteria decision making problems such as medical diagnosis and pattern recognition.

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