



Applications of graph's complete degree with bipolar fuzzy information

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Abstract

Due to the presence of two opposite directional thinking in relationships between countries and communication systems, the systems may not always be balanced. Therefore, the perfectness between countries relations are highly important. It comes from how much they were connected to each other for communication. In this study, first perfectly regular bipolar fuzzy graph is introduced and examined the regularity of nodes. Then, the relationship between the adjacent nodes and their regularity are visualized as a perfectly edge-regular bipolar fuzzy graphs. The totally accurate communication between all connected nodes is explained by introducing completely open neighborhood degree and completely closed neighborhood degree of nodes and edges in a bipolar fuzzy graph. Some algorithms and flowcharts of the proposed methods are given. Finally, two applications of these cogitation are exhibited in two bipolar fuzzy fields. The first one is in international relationships between some countries during cold-war era and the second one is in decision-making between teachers–students communication system for the improvement of teaching.

Keywords Bipolar fuzzy graph · Perfectly regular bipolar fuzzy graph · Completely open & Closed neighborhood degree · International relationship

Introduction

Research background

In daily life of humanity, decision structure is build upon the human thinking potency. Graph theory plays an essential part to maintain the relationship and communication in various types of connected fields including computer network, artificial intelligence, decision-making, engineering science, signal processing, pattern recognition, image segmentation, and medical science. If the nodes as well as edges in a graph become uncertain [33], then fuzzy graph theory [26] gives us more accuracy for decision-making in some real problems. Therefore, at present, fuzzy graph theory is one of the important research areas. But once in while human decision

generated two antithetical sides [35,37] which can come positive and negative sides, increasing and decreasing side, good and bad relation, profit and loss, effect, and side effect.

Due to the existence of bipolar judgemental thinking, the bipolar fuzzy sets and relations play a major role in extensive number of real-life bipolar fuzzy fields including qualitative model, cognitive mapping, cooperation, clustering, analysis of multi-agent data mining, strategic decision in international relationship, neurological modeling, analysis of diagnose of major depressive, granular computing, etc. When the effects of the nodes are positive and negative in the graphs, then the notion of bipolar fuzzy graphs gives us more accurate result for decision-making. During the recent time, the Wi-Fi network system has been placed in many places of a big town including office, court, educational field, town, bus stand, railway station, etc. The speed of Wi-Fi network becomes slow due to certain reasons like for excessive users, mechanical disturbances, and natural disaster. Therefore, the speed of Wi-Fi is uncertain. For natural disturbance and excessive user, the Internet speed is not fixed. Then, there exists a bipolarity in the Wi-Fi machines as well as between every pair of Wi-Fi machines. To handle these types of graphical systems, bipolar fuzzy graph plays an essential role. To control

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the total power of the Wi-Fi machine and the Internet speed between two Wi-Fi machines, open and closed neighborhood degrees of nodes and edges are introduced.

In fuzzy graphs [26], there is only one character of nodes and edges whose membership values lie in $[0, 1]$. However, if there is another opposite character for the nodes and edges whose membership values lie in $[-1, 0]$, then these types situations cannot be handled by means of fuzzy graphs. Bipolar fuzzy graphs (BFGs) [5,31] gives the two opposite side information about the nodes and edges. However, it cannot give any information whether the system is perfectly regular or not. The degree of nodes in BFGs gives the total information and contribution of the positive and negative characters separately. Therefore, we do not have any idea about the highest and lowest connected nodes and edges. Therefore, we cannot order the nodes and edges in this case.

Motivation

Many events related to our real life can be modeled using BFG. For instance, some people are connected by a group in social media. Their relationships and communications to each other are uncertain. Often, it may happen that between any two friends, one may agree with an incident and disagree with another incident. Therefore, there exists bipolarity between each pair of friends. To handle these type of situations, BFGs always conduct an essential part. It happens many times that some friends send ‘Good Morning’ or ‘Good Night’ message in the group and personally to others. However, others may not reply to these type of messages. All of these types of communication explain how much a man perfectly related and communicated to the others in that group and personally. This type of social network group is called a perfectly regular group if all people in this group are perfect and is called a perfectly irregular group if all people in this group are imperfect. This motivates us to introduce perfectly regular and edge-regular BFG. However, a man can agree or disagree about an information with his friends personally and also in locally. This can be explained by open and closed neighborhood degree of nodes and edges. Consequently, it motivates us to define completely open neighborhood degree and completely closed neighborhood degree.

Framework of this study

This manuscript is organized as follows: The next section contains literature of review and some basic definitions related to this manuscript. Perfectly regular and perfectly edge-regular bipolar fuzzy graph with their properties are initiated in “Perfectly regular and perfectly edge-regular BFG”. The terms completely open neighborhood degree and completely closed neighborhood degree are initiated with their properties in “Completely open neighborhood

degree (COND) and completely closed neighborhood degree (CCND) in a BFG”. Algorithms and flowcharts for calculating various types of degrees are described in “Algorithm and flowchart for calculating COND and CCND in BFG”. Applications of these substantive thoughts are conferred to maintain international relationship and decision-making in such real-life communication systems in “Applications of BFG using COND and CCND”. Finally, a conclusion of all these works is given in “Conclusion and future works”.

Literature review and preliminaries

Due to the presence of fuzziness or uncertainty in human thinking, Zadeh [33] originated the concept of fuzzy sets. Liao et al. [18] discussed about several types of fuzzy sets with operations and applications in various multi-attribute decision-making. Hesitancy degree-based correlation measures for hesitant fuzzy sets are described in [17]. Citil [8] investigated many fuzzy problems based on Laplace transform. Yuan et. al. [32] explained application of graph kernel which is based on link prediction for signed in social networks. Gao et al. [12,13], respectively, introduced different algorithms for partial multi-dividing ontology and Wiener number in bicyclic molecular structures. Nguyen et al. [21] established a notable techniques to handle knowledge graph fusion in smart systems. Imran et al. [16] introduced different types of network based on graph. Currently, fuzzy graph theory [26] is extensively used in numerous fields. Mordeson and Nair [19,20] created disjoint arcs and cycles in fuzzy graphs. Strength of arcs in fuzzy graphs is explained in [7]. Gross et al. [15] gives an approach of fuzzy graph matching in intelligent analysis to maintain continuous situational awareness. Different types of picture fuzzy graph and their application are described in [9,29]. Fan et al. [11] explained the intuition about bipartite fuzzy graph with its real-life approach. Based on bipolarity of fuzzy sets, Zhang [35] presented the conceptual idea of bipolar fuzzy sets. Different types of operations and properties of bipolar fuzzy sets and bipolar fuzzy relations with their applications in various real fields are briefly discussed in [30,37]. Akram [1,2] first introduced the concept of BFG, isomorphism on BFGs, complement, and its application. After that, Yang et. al. [31] gave the generalized concept of BFGs. Then, Akram et al. [3–5] defined trees, cycles, paths, metric, forest, and cut node in BFG. Different types of applications of BFGs founded in [6]. The concept of fuzzy soft and bipolar fuzzy soft graphs and their application in wireless Internet founded in [27,28]. Poulik and Ghorai [23] initiated geodesic distance and different types of nodes in BFGs with their applications. Different types of indices and degrees in bipolar fuzzy graphs with applications are explained in [22,25]. Poulik et al. [24] described different operations and applications of interval-valued fuzzy graph.

Ghorai [14] characterized regular bipolar fuzzy graphs using fundamental sequences.

Now, we represent some elementary definitions in the next sections.

Definition 1 [35] Let X be a non-empty set.

- The set $A = \{(x, v_A^p(x), v_A^n(x)) : x \in X\}$ is said to be a bipolar fuzzy set in X , where $v_A^p : X \rightarrow [0, 1]$ and $v_A^n : X \rightarrow [-1, 0]$ are mappings.
- A mapping $A = (v_A^p, v_A^n) : X \times X \rightarrow [0, 1] \times [-1, 0]$ is said to be a bipolar fuzzy relation on X , where $v_A^p(x, y) \in [0, 1]$ and $v_A^n(x, y) \in [-1, 0]$.

The positive membership value $v_A^p(x), x \in V$ is the positive satisfaction degree of x corresponding to bipolar fuzzy set A and the negative membership value $v_A^n(x), x \in V$ is the negative satisfaction degree of x .

Definition 2 [35] Let X be a non-empty set. Then, $B = (v_B^p, v_B^n) : X \times X \rightarrow [0, 1] \times [-1, 0]$ is called a bipolar fuzzy relation on X , such that $v_B^p(a, b) \in [0, 1]$ and $v_B^n(a, b) \in [-1, 0]$.

Let $A = (v_A^p, v_A^n)$ be a bipolar fuzzy set on a set X . If $B = (v_B^p, v_B^n)$ is a bipolar fuzzy relation on a set X , then $B = (v_B^p, v_B^n)$ is a bipolar fuzzy relation on $A = (v_A^p, v_A^n)$ if $v_B^p(a, b) \leq \min\{v_A^p(a), v_A^p(b)\}$ and $v_B^n(a, b) \geq \max\{v_A^n(a), v_A^n(b)\}$ for all $a, b \in X$.

In 2011, Akram [5] first defined BFG as follows.

Definition 3 [5] A BFG G on a non-empty set X is a pair $G = (A, B)$, where $A : X \rightarrow [0, 1] \times [-1, 0]$ is a bipolar fuzzy set on X and $B : X \times X \rightarrow [0, 1] \times [-1, 0]$ is a bipolar fuzzy relation in X , such that $v_B^p(ab) \leq \min\{v_A^p(a), v_A^p(b)\}$ and $v_B^n(ab) \geq \max\{v_A^n(a), v_A^n(b)\}$, for all $a, b \in X$.

For a given set V , an equivalence relation \sim on $V \times V - \{(x, x) | x \in V\}$ is defined as $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow$ either $(x_1, y_1) = (x_2, y_2)$ or $x_1 = x_2, x_2 \approx y_1$. The quotient set obtained in this way is denoted by \tilde{V}^2 , and the equivalent class which contains the element (x, y) is denoted by xy or yx . Yang et al. [31] defined the BFG in 2013.

Definition 4 [31] A BFG G is a triplet (V, A, B) of the crisp graph $G^* = (V, E)$, where $A = (v_A^p, v_A^n)$ and $B = (v_B^p, v_B^n)$ are the bipolar fuzzy sets in V and \tilde{V}^2 , respectively, for which

$$v_B^p(ab) \leq \min\{v_A^p(a), v_A^p(b)\}$$

and

$$v_B^n(ab) \geq \max\{v_A^n(a), v_A^n(b)\}$$

for all $ab \in \tilde{V}^2$ with $v_B^p(ab) = v_B^n(ab) = 0$ for all $ab \in (\tilde{V}^2 - E)$.

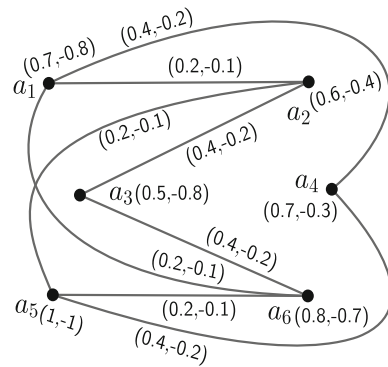


Fig. 1 Example of a BFG G

Example 1 In Fig. 1, we consider a graph $G^* = (V, E)$, where $V = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $E = \{a_1a_2, a_1a_4, a_1a_6, a_2a_3, a_2a_5, a_3a_6, a_4a_5, a_5a_6\}$. Every node and edge has two components, positive and negative. All the positive and negative components of all the nodes and edges lie in $[0, 1]$ and $[-1, 0]$, respectively. This means that $v_A^p(a_i), v_B^p(a_ja_j) \in [0, 1]$ and $v_A^n(a_i), v_B^n(a_ja_j) \in [-1, 0], \forall a_i \in V, \forall a_ia_j \in \tilde{V}^2$. Here $(v_A^p(a_1), v_A^n(a_1)) = (0.7, -0.8), (v_A^p(a_2), v_A^n(a_2)) = (0.6, -0.4), (v_A^p(a_1a_2), v_A^n(a_1a_2)) = (0.2, -0.1)$. Therefore, $v_B^p(a_1a_2) < \min\{v_A^p(a_1), v_A^p(a_2)\}$ and $v_B^n(a_1a_2) > \max\{v_A^n(a_1), v_A^n(a_2)\}$. Similarly, all the other edges and vertices satisfy the condition $v_B^p(a_ia_j) \leq \min\{v_A^p(a_i), v_A^p(a_j)\}$ and $v_B^n(a_ia_j) \geq \max\{v_A^n(a_i), v_A^n(a_j)\}, 1 \leq i \neq j \leq 6$. Therefore, G is a BFG.

The open neighborhood degree (OND) of a node $a \in V$ in a BFG G is denoted by $deg(a) = (deg^p(a), deg^n(a))$ and is defined as $deg^p(a) = \sum_{\substack{a \neq b \\ ab \in \tilde{V}^2}} v_B^p(ab)$ and $deg^n(a) = \sum_{\substack{a \neq b \\ ab \in \tilde{V}^2}} v_B^n(ab)$. If $deg(a) = (d_1, d_2), \forall a \in V, G$ is called (d_1, d_2) -regular.

For example, the BFG G of Fig. 1, we have $deg(a_i) = (0.8, -0.4) \forall a_i$. Therefore, G is a $(0.8, -0.4)$ -regular BFG. The order of a BFG G is denoted by $O(G) = (O^p(G), O^n(G))$ and is defined as $O^p(G) = \sum_{a \in V} v_A^p(a)$ and $O^n(G) = \sum_{a \in V} v_A^n(a)$. Also, $S(G) = (S^p(G), S^n(G))$ is the size of G , where $S^p(G) = \sum_{ab \in \tilde{V}^2} v_B^p(ab)$ and $S^n(G) = \sum_{ab \in \tilde{V}^2} v_B^n(ab)$. From the BFG G of Fig. 1, we see that $O(G) = (4.3, -4.0)$ and $S(G) = (2.4, -1.2)$.

The closed neighborhood degree (CND) of a node $a \in V$ in a BFG G is denoted by $deg[a] = (deg^p[a], deg^n[a])$ and is defined as $deg^p[a] = deg^p(a) + v_A^p(a)$ and $deg^n[a] = deg^n(a) + v_A^n(a)$. If $deg[a] = (f_1, f_2) \forall a \in V$, then G is called (f_1, f_2) -totally regular.

For the BFG G of Fig. 1, $deg[a_2] = (1.4, -0.8) \neq (1.8, -1.4) = deg[a_5]$. Therefore, G is not totally regular. The degree of an edge $ab \in E$ in a BFG G is denoted by $deg(ab) = (deg^p(ab), deg^n(ab))$ and is defined as

$$deg^p(ab) = \sum_{a \neq c} v_B^p(ac) + \sum_{b \neq c} v_B^p(bc) = \sum_{ac \in \tilde{V}^2} v_B^p(ac) + \sum_{bc \in \tilde{V}^2} v_B^p(bc) - 2v_B^p(ab)$$

$$\text{and } deg^n(ab) = \sum_{a \neq c} v_B^n(ac) + \sum_{b \neq c} v_B^n(bc) = \sum_{ac \in \tilde{V}^2} v_B^n(ac) + \sum_{bc \in \tilde{V}^2} v_B^n(bc) - 2v_B^n(ab).$$

If $deg(ab) = (k_1, k_2), \forall ab \in \tilde{V}^2$, then G is called (k_1, k_2) -edge regular.

For the BFG G of Fig. 1, $deg(a_1a_2) = (1.2, -0.6) \neq (0.8, -0.4) = deg(a_4a_5)$. Therefore, G is not edge regular.

The total degree of an edge ab in a BFG G is denoted by $deg[ab] = (deg^p[ab], deg^n[ab])$ and is defined as $deg^p[ab] = deg^p(ab) + v_B^p(ab)$ and $deg^n[ab] = deg^n(ab) + v_B^n(ab)$. If $deg[ab] = (t_1, t_2), \forall ab \in \tilde{V}^2$, G is called (t_1, t_2) -totally edge-regular.

For the BFG G of Fig. 1, $deg[a_1a_2] = (1.4, -0.7) \neq (1.2, -0.6) = deg[a_4a_5]$. Therefore, G is not totally edge-regular.

Definition 5 [22] The connectivity index of a BFG $G = (V, A, B)$ denoted by $CI_{BF}(G)$ is defined as

$$CI_{BF}(G) = (CI_{BF}^p(G), CI_{BF}^n(G)) = \left(\sum_{a_i, a_j \in V} v_A^p(a_i)v_A^p(a_j)CONN_G^p(a_i, a_j), \sum_{a_i, a_j \in V} v_A^n(a_i)v_A^n(a_j)CONN_G^n(a_i, a_j) \right),$$

where $CI_{BF}^p(G)$ and $CI_{BF}^n(G)$, respectively, denote the positive connectivity index and negative connectivity index of G and $CONN_G^p(a_i, a_j) = (v_B^p((a_i, a_j))^\infty)$ and $CONN_G^n(a_i, a_j) = (v_B^n((a_i, a_j))^\infty)$, respectively, denote the positive strength of connectedness and negative strength of connectedness between every pair of vertices (a_i, a_j) in G .

Also, the average connectivity index of G denoted by $ACI_{BF}(G)$ is defined as

$$ACI_{BF}(G) = (ACI_{BF}^p(G), ACI_{BF}^n(G)) = \frac{1}{mC_2} \left[\sum_{a_i, a_j \in V} v_A^p(a_i)v_A^p(a_j)CONN_G^p(a_i, a_j), \sum_{a_i, a_j \in V} v_A^n(a_i)v_A^n(a_j)CONN_G^n(a_i, a_j) \right],$$

where $|V| = m$.

Definition 6 [25] The Wiener index of a BFG $G = (V, A, B)$ is defined as

$$WI_{BF}(G) = (WI_{BF}^p(G), WI_{BF}^n(G)) = \left(\sum_{a_i, a_j \in V} v_A^p(a_i)v_A^p(a_j)d_{g(G)}^p(a_i, a_j), \sum_{a_i, a_j \in V} v_A^n(a_i)v_A^n(a_j)d_{g(G)}^n(a_i, a_j) \right),$$

where $WI_{BF}^p(G)$ and $WI_{BF}^n(G)$ represent the positive and negative Wiener index of G , respectively. $d_{g(G)}^p(a_i, a_j)$ and $d_{g(G)}^n(a_i, a_j)$ are, respectively, the minimum and maximum values of the sums of all positive and negative membership values of all the edges on the geodesics from a_i to a_j .

Perfectly regular and perfectly edge-regular BFG

In a BFG, we cannot justify a node, with a single node according to the exceptional and unexceptional contributions. It would be better, if the justification can be made with all the nodes at the same time. In this matter, the OND and CND of the nodes and their regularity in a BFG play an important factor. Therefore, the equal contribution of all the nodes and all the edges in a BFG system are extremely helpful to maintain the equivalency of the system. In this section, perfectly regular BFG is defined and some theorems connected to these are discussed with examples. Perfectly edge regular BFG is defined with its boundedness. Some properties related to this are also established with examples.

Definition 7 A BFG is said to be a perfectly regular if it is both regular BFG and totally regular BFG.

We consider a perfectly regular BFG $G = (V, A, B)$. Therefore, G must be regular BFG and totally regular BFG. Then, the OND and CND of all the nodes of G are equal. Hence, the positive and negative membership values of all the nodes of G must be equal. Thus, we have the following proposition.

Proposition 1 For a perfectly regular BFG $G = (V, A, B)$, $A = (v_A^p, v_A^n)$ is a constant function.

Proof Let G is perfectly regular BFG. Therefore, G is both (d_1, d_2) -regular and (f_1, f_2) -totally regular. Therefore, $deg^p[a] = deg^p(a) + v_A^p(a) = deg^p(b) + v_A^p(b) = deg^p[b]$ and $deg^n[a] = deg^n(a) + v_A^n(a) = deg^n(b) + v_A^n(b) = deg^n[b], \forall a, b \in V$.

However, $deg^p(a) = d_1 = deg^p(b), deg^n(a) = d_2 = deg^n(b)$ and $deg^p[a] = f_1 = deg^p[b], deg^n[a] = f_2 =$

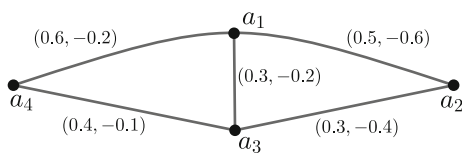


Fig. 2 A BFG G with membership values of all nodes as $(0.7, -0.8)$

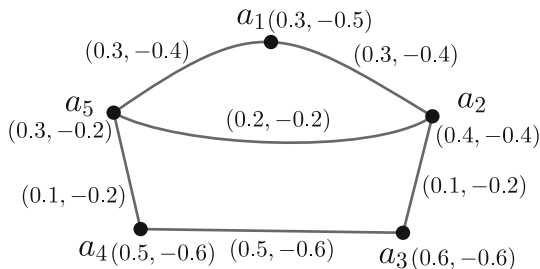


Fig. 3 A $(0.6, -0.8)$ -regular BFG G

$deg^n[b], \forall a, b \in V. \Rightarrow v_A^p(a) = v_A^p(b) = f_1 - d_1 = \text{constant}$ and $v_A^n(a) = v_A^n(b) = f_2 - d_2 = \text{constant}, \forall a, b \in V.$

Hence, $A = (v_A^p, v_A^n)$ is a constant function. \square

Example 2 From the BFG G of Fig. 2, we see that $A = (v_A^p, v_A^n)$ is constant. However, $deg(a_1) \neq deg(a_3)$. Hence, G is not perfectly regular. Therefore, the converse of the Proposition 1 is not true for all BFG.

Remark 1 The truthfulness of the converse of the Proposition 1 needs not be essential.

The measure of the total contribution between every pair of nodes in a BFG actually depend upon the size of a BFG. Therefore, when we want to determine the size of a perfectly regular BFG, every edge comes two times in total. This is explained in the next Theorem.

Theorem 1 Let $G = (V, A, B)$ be a perfectly regular BFG of the graph $G^* = (V, E)$. Then, size of G is $S(G) = \frac{|V|}{2}(d_1, d_2)$, where (d_1, d_2) is the OND of a node in G .

Proof Since G is a perfectly regular BFG and (d_1, d_2) is the OND of a node of G . Then, $deg(a) = (d_1, d_2) \forall a \in V$. Now, $\sum_{a \in V} deg(a) = |V|(d_1, d_2) \Rightarrow (\sum_{a \in V} \sum_{a \neq b} v_B^p(ab), \sum_{a \in V} \sum_{a \neq b} v_B^n(ab)) = |V|(d_1, d_2) \Rightarrow (2 \sum_{ab \in \tilde{V}^2} v_B^p(ab), 2 \sum_{ab \in \tilde{V}^2} v_B^n(ab)) = |V|(d_1, d_2) \Rightarrow S(G) = \frac{|V|}{2}(d_1, d_2).$ \square

Example 3 Consider the BFG G of Fig. 3. Here, $S(G) = \frac{|V|}{2}(d_1, d_2) = (1.5, -2.0)$. However, $deg[a_1] \neq deg[a_2]$. Therefore, G is not perfectly regular. Hence, the converse of the Proposition 1 is not true in this BFG and the general case is stated in Remark 2.

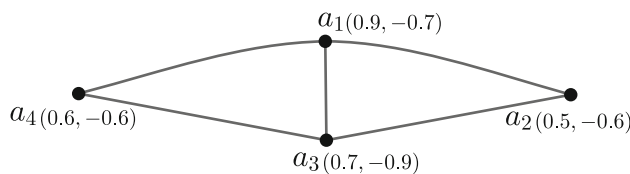


Fig. 4 A BFG G with membership values of all edges as $(0.4, -0.5)$

Remark 2 The converse of the Theorem 3 does not hold in general.

An edge in a BFG maintains the relationship between the two corresponding nodes. Therefore, the total contribution of every edge of the BFG has an important sustainability. The intuition about perfect regularity of a BFG in the sense of edges is definitely important. Thus, we have introduced the concept of perfectly edge-regular BFG in the next definition.

Definition 8 A BFG is said to be a perfectly edge-regular BFG if it is both edge-regular and totally edge-regular.

We consider now a perfectly edge-regular BFG $G = (V, A, B)$. Then, the degree and total degree of all the edges of G are equal. Therefore, the positive and negative membership values of all the edges of G remain same. Thus, we have the proposition.

Proposition 2 For a perfectly edge-regular BFG $G = (V, A, B)$ of the graph $G^* = (V, E)$, $B = (v_B^p, v_B^n)$ is a constant function.

Example 4 Consider the BFG G of Fig. 4. Here, $B = (v_B^p, v_B^n)$ is constant. However, $deg(a_1a_3) \neq deg(a_1a_2)$ and this implies that G is not perfectly regular. Thus, the converse of Proposition 2 does not hold in general.

The maximum and minimum values of the total positive and negative communication powers of all the nodes in a BFG G are important factor to control a bipolar fuzzy connected field. This means that the total of the positive membership values and the total of the negative membership values of all the nodes in a BFG G represent the order of the BFG G , i.e., $O(G) = (\sum_{a \in V} v^p(a), \sum_{a \in V} v^n(a))$. This implies that the lower and upper boundary of the order of a BFG is very important, which is stated in the next Theorem.

Theorem 2 (Boundedness property for the order of a perfectly edge-regular BFG) Let G be a perfectly edge-regular BFG and $|V| = k$. Then,

$$\sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} \leq O^p(G) \leq k \text{ and } -k \leq O^n(G) \leq \sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\}, 1 \leq i, j \leq k.$$

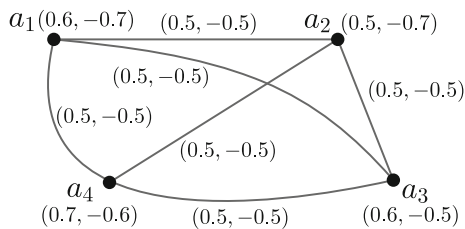


Fig. 5 Example of a BFG G which satisfies Theorem 2

Proof Since $|V| = k$, let $V = \{a_1, a_2, \dots, a_k\}$. $v_A^p(a_i) \leq 1$ and $-1 \leq v_A^n(a_i)$, $1 \leq i \leq k$.

Therefore, $\sum_{a_i \in V} v_A^p(a_i) \leq k$ and $-k \leq \sum_{a_i \in V} v_A^n(a_i)$.

This implies $O^p(G) \leq k$ and $-k \leq O^n(G)$ (i)

Now, $v_B^p(a_i a_j) \leq \min\{v_A^p(a_i), v_A^p(a_j)\}$ and $v_B^n(a_i a_j) \geq \max\{v_A^n(a_i), v_A^n(a_j)\}$,

$$1 \leq i, j \leq k, i \neq j$$

$\Rightarrow \max v_B^p(a_i a_j) \leq \{v_A^p(a_i), v_A^p(a_j)\}$ and $\min v_B^n(a_i a_j) \geq \{v_A^n(a_i), v_A^n(a_j)\}$, $1 \leq i, j \leq k, i \neq j$

$\Rightarrow \sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} \leq \sum_{a_i \in V} v_A^p(a_i)$ and

$\sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\} \geq \sum_{a_i \in V} v_A^n(a_i)$, $1 \leq i, j \leq k, i \neq j$

$\Rightarrow \sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} \leq O^p(G)$ and $\sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\} \geq O^n(G)$ (ii)

Combining (i) and (ii), we have

$$\sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} \leq O^p(G) \leq k \text{ and } -k \leq O^n(G) \leq \sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\}, 1 \leq i, j \leq k, i \neq j$$

□

Example 5 From the BFG of Fig. 5, it is clearly seen that $|V| = 4$, $(O^p(G), O^n(G)) = (2.4, -2.5)$, $\sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} = 0.5 + 0.5 + 0.5 + 0.5 = 2.0$, $\sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\} = -0.5 - 0.5 - 0.5 - 0.5 = -2.0$.

Now, $2.0 \leq 2.4 \leq 4$ and $-4 \leq -2.5 \leq -2.0$

$\Rightarrow \sum_{a_i \in V} \{\max v_B^p(a_i a_j) | i \neq j\} \leq O^p(G) \leq 4$ and $-4 \leq O^n(G) \leq \sum_{a_i \in V} \{\min v_B^n(a_i a_j) | i \neq j\}$, $1 \leq i, j \leq 4$. Thus, G satisfies Theorem 2.

In chemical graph theory, different types of indices are initiated likes connectivity index, Wiener index, Randic index, Zagreb index, etc. The concept of connectivity index and Wiener index of BFGs and real applications are presented in [22,25]. However, if BFGs are perfectly regular and perfectly edge regular, then the connectivity index and Wiener index can be found easily, which are explained in the following theorems.

Theorem 3 Let $G = (V, A, B)$ be a perfectly regular BFG, such that $|V| = m$ and $(v_B^p(a_i a_j), v_B^n(a_i a_j)) = (e_1, e_2), \forall a_i, a_j \in V$. Then $CI_{BF}^p(G) = \frac{m(m-1)}{2} v_1^2 e_1$ and $CI_{BF}^n(G) = \frac{m(m-1)}{2} v_2^2 e_2$, where $(v_1, v_2) = (v_A^p(a_i), v_A^n(a_i))$.

Proof Since G is a perfectly regular BFG, so by Proposition 1, we have $A = (v_a^p, v_a^n)$ must be constant. Since $(v_1, v_2) = (v_A^p(a_i), v_A^n(a_i)), \forall a_i \in V$.

Again, since $(v_B^p(a_i a_j), v_B^n(a_i a_j)) = (e_1, e_2), \forall a_i, a_j \in V$, then there exists an edge between every pair of nodes in G and the membership values of all the nodes are same i.e. (e_1, e_2) . Therefore, the total number of edges in G is ${}^m C_2 = \frac{m(m-1)}{2}$.

Again, the strength of connectedness between two vertices lies on an edge and is equal to the membership value of that edge. So, $CONN_G^p(a_i, a_j) = e_1$ and $CONN_G^n(a_i, a_j) = e_2, \forall a_i, a_j \in V$.

$$\text{Now, } CI_{BF}^p(G) = \sum_{a_i, a_j \in V} v_A^p(a_i) v_A^p(a_j) CONN_G^p(a_i, a_j) = \sum_{a_i, a_j \in V} v_1 \cdot v_1 \cdot CONN_G^p(a_i, a_j) = v_1^2 e_1 \cdot {}^m C_2 = \frac{m(m-1)}{2} v_1^2 e_1 \text{ and}$$

$$CI_{BF}^n(G) = \sum_{a_i, a_j \in V} v_A^n(a_i) v_A^n(a_j) CONN_G^n(a_i, a_j) = \sum_{a_i, a_j \in V} v_2 \cdot v_2 \cdot CONN_G^n(a_i, a_j) = v_2^2 e_2 \cdot {}^m C_2 = \frac{m(m-1)}{2} v_2^2 e_2. \quad \square$$

Note 1 If the perfectly regular BFG G of the Theorem 3 is complete, then $e_1 = v_B^p(a_i a_j) = \min\{v_A^p(a_i), v_A^p(a_j)\} = v_1$ and $e_2 = v_B^n(a_i a_j) = \max\{v_A^n(a_i), v_A^n(a_j)\} = v_2$. Then, $CI_{BF}^p(G) = \frac{m(m-1)}{2} v_1^3$ and $CI_{BF}^n(G) = \frac{m(m-1)}{2} v_2^3$.

Corollary 1 For the BFG G of Note 1, $ACI_{BF}(G) = (v_1^3, v_2^3)$.

Proof Since the G satisfies all the conditions of Note 1, so we have

$e_1 = v_1, e_2 = v_2, CONN_G^p(a_i, a_j) = e_1 = v_1$ and $CONN_G^n(a_i, a_j) = e_2 = v_2, \forall a_i, a_j \in V$. Since G is complete and $|V| = m$, so there is an edge between every pair of vertices and the total number of edges in G is ${}^m C_2$.

Now, from the Definition 5, we have

$$\begin{aligned} ACI_{BF}^p(G) &= \frac{1}{{}^m C_2} \sum_{a_i, a_j \in V} v_A^p(a_i) v_A^p(a_j) CONN_G^p(a_i, a_j) \\ &= \frac{{}^m C_2}{{}^m C_2} v_1 \cdot v_1 \cdot e_1 = v_1^3 \text{ and} \\ ACI_{BF}^n(G) &= \frac{1}{{}^m C_2} \sum_{a_i, a_j \in V} v_A^n(a_i) v_A^n(a_j) CONN_G^n(a_i, a_j) \\ &= \frac{{}^m C_2}{{}^m C_2} v_2 \cdot v_2 \cdot e_2 = v_2^3. \end{aligned}$$

$\therefore ACI_{BF}(G) = (ACI_{BF}^p(G), ACI_{BF}^n(G)) = (v_1^3, v_2^3). \quad \square$

Theorem 4 Let $G = (V, A, B)$ be perfectly regular complete BFG, such that $|V| = m$ and $(v_A^p(a_i), v_A^n(a_i)) = (v_1, v_2)$. Then, $WI_{BF}^p(G) = \frac{m(m-1)}{2} v_1^3$ and $WI_{BF}^n(G) = \frac{m(m-1)}{2} v_2^3$.

Proof Since G is perfectly regular complete BFG and $|V| = m$, so from the Note 1, we have

$$v_B^p(a_i a_j) = \min\{v_A^p(a_i), v_A^p(a_j)\} = v_1 \text{ and } v_B^n(a_i a_j) = \max\{v_A^n(a_i), v_A^n(a_j)\} = v_2.$$

Now, $d_{g(G)}^p(a, b) =$ minimum value of the sums of all positive membership values of all the edges on the geodesics from a to b is equal to v_1 and $d_{g(G)}^n(a, b) =$ maximum value of the sums of all negative membership values of all the edges on the geodesics from a to b is equal to v_2 .

$$\text{Therefore, } WI_{BF}^p(G) = \sum_{a_i, a_j \in V} v_A^p(a_i) v_A^p(a_j) d_{g(G)}^p(a_i, a_j) = {}^m C_2 v_1 \cdot v_1 \cdot v_1 = \frac{m(m-1)}{2} v_1^3 \text{ and}$$

$$WI_{BF}^n(G) = \sum_{a_i, a_j \in V} v_A^n(a_i) v_A^n(a_j) d_{g(G)}^n(a_i, a_j) = {}^m C_2 v_2 \cdot v_2 \cdot v_2 = \frac{m(m-1)}{2} v_2^3.$$

□

Completely open neighborhood degree (COND) and completely closed neighborhood degree (CCND) in a BFG

We can measure the importance of a nodes not only by its positive and negative characters but also by the sum of their positive and negative characters. The positive neighborhood degree of a node in a BFG equals or not-equals to the negative neighborhood degree of that node. Therefore, the complete neighborhood degrees, i.e., COND and CCND, of nodes are very essential. In this section, COND and CCND of nodes in a BFG are defined with their properties and examples.

Definition 9 The COND of a node a_1 in a BFG G is denoted by $COND(a_1)$ and is defined as $COND(a_1) = deg^p(a_1) + deg^n(a_1)$.

The CCND of a node a_1 in a BFG G is denoted by $CCND[a_1]$ and is defined as $CCND[a_1] = deg^p[a_1] + deg^n[a_1]$.

If all the nodes in a BFG G have equal contribution for communication means, then G is perfectly regular BFG. Then, using the Proposition 1, we can say that the positive and negative membership values of every node of G are equal. Now, using the Definition 9, we can effortlessly determine the COND and CCND of a perfectly regular BFG, which is stated in Proposition 3.

Proposition 3 For a perfectly regular BFG

- (i) the CONDs of all the nodes are same and
- (ii) the CCNDs of all the nodes are same.

Example 6 For the BFG G of Fig. 6, we have $COND(a_i) = 0.8$ and $CCND(a_i) = 0.7, i = 1, 2, 3, 4, 5$. Thus, the CONDs and CCNDs of all the nodes of G are same. However, the membership values of some nodes are different. Therefore, by Proposition 1, we have G which is not perfectly

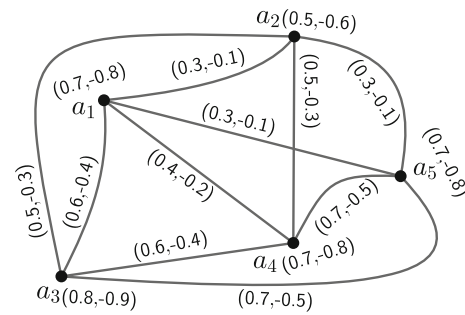


Fig. 6 A connected BFG G

regular BFG. Therefore, the converse part of the Proposition 3 is not true in this case.

We can also measure the edges not only their positive and negative communications but also their total communications. Therefore, the concepts of completely degrees of nodes in a BFG, i.e., COND and CCND, are very essential.

Definition 10 The COND of an edge ab in a BFG G is denoted by $COND(ab)$ and is defined as $COND(ab) = deg^p(ab) + deg^n(ab)$. The CCND of a edge ab in a BFG G is denoted by $CCND[ab]$ and is defined as $CCND[ab] = deg^p[ab] + deg^n[ab]$ for all $ab \in \widehat{V}^2$.

If the communication between every pair of nodes in a BFG G is same, i.e., if the BFG G is perfectly edge-regular BFG, then by Proposition 2 we have, the positive and negative membership values of every edge in G are same. Then, using the Definition 10, we can effortlessly determine the COND and CCND of all the edges which is stated in Proposition 4.

Proposition 4 For a perfectly edge-regular BFG

- (i) the CONDs of all the edges are same and
- (ii) the CCNDs of all the edges are same.

Example 7 For the BFG G of Fig. 6, we have $COND(a_i a_j) = 1.2, CCND(a_i a_j) = 1.4$, for all edges $a_i a_j$ in G . However, the membership values of some edges of G are different. Therefore, by Proposition 2, we have G which is not perfectly edge-regular BFG. Therefore, the converse part of the Proposition 4 is not true in this case.

Theorem 5 (Relation between COND and CCND in a BFG) Let $G = (V, A, B)$ be a BFG. Then

- (i) $COND(a_i) \geq CCND(a_i) \Leftrightarrow v_A^p(a_i) \leq |v_A^n(a_i)|,$
 $COND(a_i) \leq CCND(a_i) \Leftrightarrow v_A^p(a_i) \geq |v_A^n(a_i)|,$
 $\forall a_i \in V,$
 and
- (ii) $COND(a_i a_j) \geq CCND(a_i a_j) \Leftrightarrow v_B^p(a_i a_j) \leq |v_B^n(a_i a_j)|,$

$$COND(a_i a_j) \leq CCND(a_i a_j) \Leftrightarrow v_B^p(a_i a_j) \geq |v_B^n(a_i a_j)|, \forall a_i a_j \in \tilde{V}^2.$$

Proof Let $G = (V, A, B)$ be a BFG.

- (i) Let $a_i \in V$. Therefore, from the definition of $COND$ and $CCND$ of a node, we have $COND(a_i) - CCND(a_i) = -(v_A^p(a_i) + v_A^n(a_i)) \geq (\leq) 0 \Leftrightarrow v_A^p(a_i) \leq (or \geq) |v_A^n(a_i)|$.
 Since a_i is an arbitrary node in G , therefore $COND(a_i) \geq CCND(a_i) \Leftrightarrow v_A^p(a_i) \leq |v_A^n(a_i)|, \forall a_i \in V$.
 Similarly, $COND(a_i) \leq CCND(a_i) \Leftrightarrow v_A^p(a_i) \geq |v_A^n(a_i)|, \forall a_i \in V$
- (ii) Let $a_i a_j \in E$. Then, from the definition of $COND$ and $CCND$ of an edge, we have $COND(a_i a_j) - CCND(a_i a_j) = -(v_B^p(a_i a_j) + v_B^n(a_i a_j)) \geq (\leq) 0 \Leftrightarrow v_B^p(a_i a_j) \leq (or \geq) |v_B^n(a_i a_j)|$. Since $a_i a_j$ is an arbitrary edge in G , therefore $COND(a_i a_j) \geq CCND(a_i a_j) \Leftrightarrow v_B^p(a_i a_j) \leq |v_B^n(a_i a_j)|, \forall a_i a_j \in \tilde{V}^2$.
 Similarly, $COND(a_i a_j) \leq CCND(a_i a_j) \Leftrightarrow v_B^p(a_i a_j) \geq |v_B^n(a_i a_j)|, \forall a_i a_j \in \tilde{V}^2$.

□

Algorithm and flowchart for calculating COND and CCND in BFG

In this section, the algorithms to determine the CONDs and CCNDs of all the nodes and edges are proposed. Then, the flowcharts of the proposed algorithms are presented.

Algorithm and flowchart to determine most communicable nodes in a BFG using COND and CCND

Using the Dijkstra’s algorithm [10] first draws a crisp graph $G^* = (V, E)$, where V and E are, respectively, denote the set of all nodes and edges with $|V| = n$ and $|E| = m$.

Step 1. Put the membership values $v_A(a_i) = (v_A^p(a_i), v_A^n(a_i))$ of the nodes a_i ’s, $i = 1, 2, \dots, n$, $v_A^p(a_i) \in [0, 1]$, $v_A^n(a_i) \in [-1, 0]$.

Step 2. Write the membership values $v_B(a_i a_j) = (v_B^p(a_i a_j), v_B^n(a_i a_j))$ of the edges $a_i a_j \in E$, such that $v_B^p(a_i a_j) \leq \min\{v_A^p(a_i), v_A^p(a_j)\}$, $v_B^n(a_i a_j) \geq \max\{v_A^n(a_i), v_A^n(a_j)\}$, $v_B^p(a_i a_j) \in [0, 1]$, $v_B^n(a_i a_j) \in [-1, 0]$.

Step 3. Calculate ONDs of the nodes $a_i, i = 1, 2, \dots, n$, $deg(a_i) = (deg^p(a_i), deg^n(a_i))$, where $deg^p(a_i) = \sum_{\substack{i \neq j \\ a_i a_j \in \tilde{V}^2}} v_B^p(a_i a_j)$,

$$deg^n(a_i) = \sum_{\substack{i \neq j \\ a_i a_j \in \tilde{V}^2}} v_B^n(a_i a_j).$$

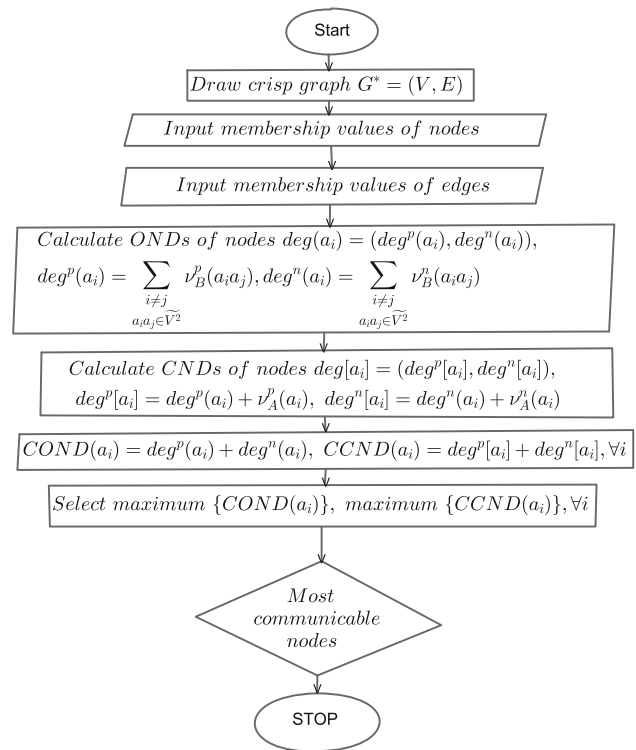


Fig. 7 Flowchart to determine most communicable nodes in BFG

Step 4. Calculate CNDs of the nodes $a_i, i = 1, 2, \dots, n$, $deg[a_i] = (deg^p[a_i], deg^n[a_i])$, where $deg^p[a_i] = deg^p(a_i) + v_A^p(a_i)$ and $deg^n[a_i] = deg^n(a_i) + v_A^n(a_i)$.

Step 5. Calculate $COND(a_i) = deg^p(a_i) + deg^n(a_i)$ and $CCND(a_i) = deg^p[a_i] + deg^n[a_i], i = 1, 2, \dots, n$.

Step 6. Select $maximum\{COND(a_i)\}$ and $maximum\{CCND(a_i)\}, i = 1, 2, \dots, n$.

Step 7. Most communicable nodes.

Algorithm and flowchart to determine most communicable edges in a BFG using COND and CCND

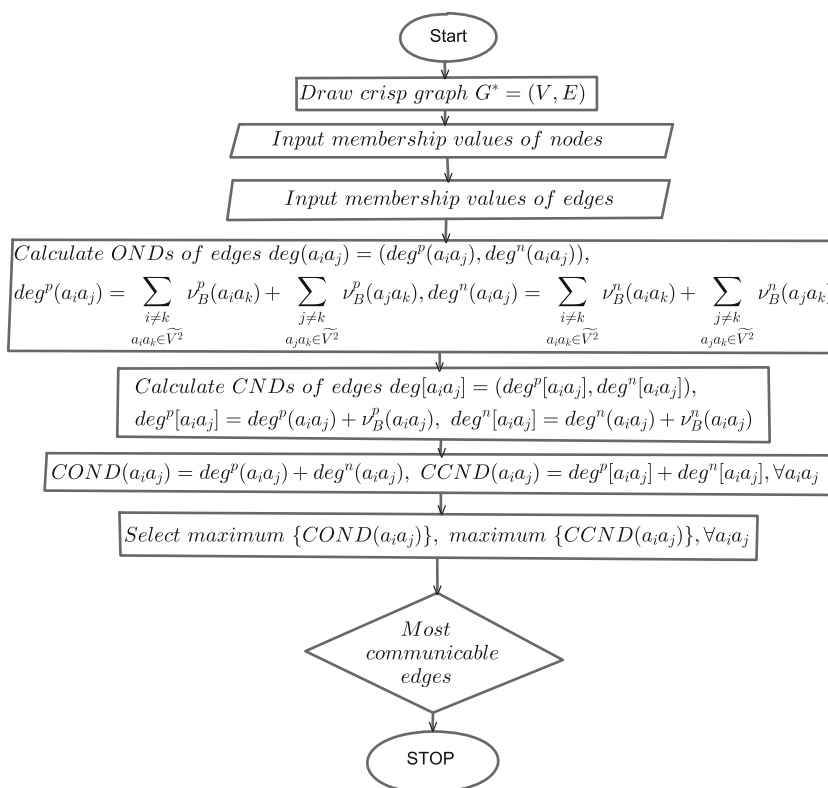
Using the Dijkstra’s algorithm [10] first draws a crisp graph $G^* = (V, E)$, where V and E are, respectively, denote the set of all nodes and edges with $|V| = n$ and $|E| = m$.

Step 1. Put the membership values $v_A(a_i) = (v_A^p(a_i), v_A^n(a_i))$ of the nodes a_i ’s, $i = 1, 2, \dots, n$, $v_A^p(a_i) \in [0, 1]$, $v_A^n(a_i) \in [-1, 0]$.

Step 2. Write the membership values $v_B(a_i a_j) = (v_B^p(a_i a_j), v_B^n(a_i a_j))$ of the edges $a_i a_j \in \tilde{V}^2$, such that $v_B^p(a_i a_j) \leq \min\{v_A^p(a_i), v_A^p(a_j)\}$, $v_B^n(a_i a_j) \geq \max\{v_A^n(a_i), v_A^n(a_j)\}$, $v_B^p(a_i a_j) \in [0, 1]$, $v_B^n(a_i a_j) \in [-1, 0]$.

Step 3. Calculate ONDs of the edges $a_i a_j \in \tilde{V}^2$, $deg(a_i a_j) = (deg^p(a_i a_j), deg^n(a_i a_j))$, where $deg^p(a_i a_j) =$

Fig. 8 Flowchart to determine most communicable edges in BFG



$$= \sum_{a_i a_k \in \tilde{V}^2} \nu_B^p(a_i a_k) + \sum_{a_j a_k \in \tilde{V}^2} \nu_B^p(a_j a_k), \text{deg}^n(a_i a_j) = \sum_{a_i a_k \in \tilde{V}^2} \nu_B^n(a_i a_k) + \sum_{a_j a_k \in \tilde{V}^2} \nu_B^n(a_j a_k).$$

Step 4. Calculate CNDs of the edges $a_i a_j \in E$, $\text{deg}[a_i a_j] = (\text{deg}^p[a_i a_j], \text{deg}^n[a_i a_j])$, where $\text{deg}^p[a_i a_j] = \text{deg}^p(a_i a_j) + \nu_B^p(a_i a_j)$ and $\text{deg}^n[a_i a_j] = \text{deg}^n(a_i a_j) + \nu_B^n(a_i a_j)$.

Step 5. Calculate $\text{COND}(a_i a_j) = \text{deg}^p(a_i a_j) + \text{deg}^n(a_i a_j)$ and $\text{CCND}(a_i a_j) = \text{deg}^p[a_i a_j] + \text{deg}^n[a_i a_j]$, for all $a_i a_j \in \tilde{V}^2$.

Step 6. Select $\text{maximum}\{\text{COND}(a_i a_j)\}$ and $\text{maximum}\{\text{CCND}(a_i a_j)\}$, for all $a_i a_j \in \tilde{V}^2$.

Step 7. Most communicable edges.

Applications of BFG using COND and CCND

In this section, we have constructed two real-life bipolar fuzzy models. First, one is international relationship between east and west countries during the cold-war era and second one is communication between teachers and students. Using the COND and CCND, the selection of the highest communicated countries and best pair of countries according to internal communication is determined for the first one and the best communicated teachers for the second one are shown.

International relationship between countries during the cold-war era

The original work is to find the most communicated countries in bipolar fuzzy decision-making problems. In a BFG $G = (V, A, B)$ with n nodes and m edges, $\text{COND}(a_i) = \text{deg}^p(a_i) + \text{deg}^n(a_i)$ and $\text{CCND}(a_i) = \text{deg}^p[a_i] + \text{deg}^n[a_i]$. Now, using $\text{max}\{\text{COND}(a_i)\}$ and $\text{max}\{\text{CCND}(a_i)\}$ for all i , we have to find the top most communicated nodes for which COND and CCND are maximum. Similarly, we have to find the edges between two corresponding nodes for which $\text{COND}(a_i a_j)$ and $\text{CCND}(a_i a_j)$, $i \neq j$ are maximum are the top most connected nodes according to communications.

At the time of cold war in 1950s, the race of confrontation and all armament between the east and west countries accrued a huge amount of energy. The energy could have been a major possible cause of the Third World War. This dangerous energetic relation and its computation have been discussed as a bipolar crisp relational graph in [36], before the enemies between USSR and China. The main interesting fact is that the releasing of this accrued energy was disequilibrium. The China card performed a main role at the end of cold war. However, China and USSR reduced their own enemies in the late 1960s. Since China and the U.S. were facing the same enemy with USSR, so this was a common enemy between China and the U.S. At that time, the president of U.S. understood that the Taiwan problem is a major

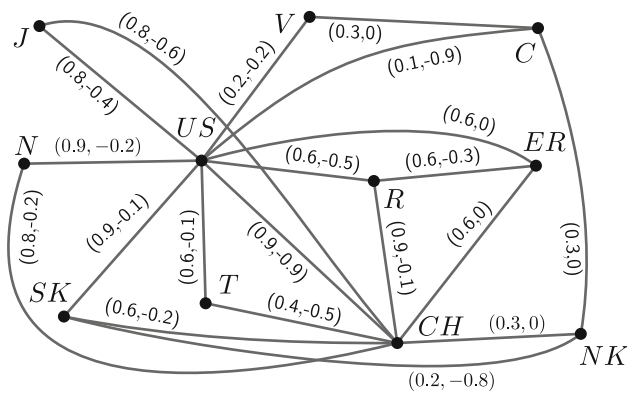


Fig. 9 Modeling of countries during the post-world war era as a BFG with membership degree of the nodes as $(1, -1)$

interruption between China and the U.S. He made a plan to open the door to China, and he fetched the relationship between China and the U.S. that is a normalization course, which was followed by western allies and Japan. After that, the USSR collapse into some parts. However, there arises a question that is “why the other countries agreed with this normalization course between China and the U.S. ?”

Representation of membership values

The reason depended upon the relationship and communication in the sense of power, facility, security, import–export, etc., and all these things depend upon a bipolar fuzzy relation that has been discussed in [34]. Here, we represent all the good and bad relationships between every pair of countries as a BFG, which is shown in Fig. 9, that has been taken from [34]. In Fig. 9, the nodes represent the countries Japan, Vietnam, Cuba, Nato, U.S., Russia, E. European countries, S. Korea, Taiwan, China, and N. Korea, and they are denoted by J, V, C, N, US, R, ER, SK, T, CH, and NK, respectively.

Decision-making

From Table 1, we see that the *COND* and *CCND* between the nodes *CH* and the *US* are more than the other nodes. This means that the total connection (direct and indirect) of China and the U.S. with other countries was better than the other countries. From Table 2, we have the *CCND* of the edge between the nodes *CH* and *US* is more than the other edges. Therefore, the total relationship and communication (direct and indirect) between China and the U.S. was at the top level from the other countries.

Why the other countries agreed with the normal relationship between China and the U.S.? The answer of the question is “the total aggregate relation and communication of China and the U.S. was at the highest level from the other countries. Also the total internal relationship (for showing normalized

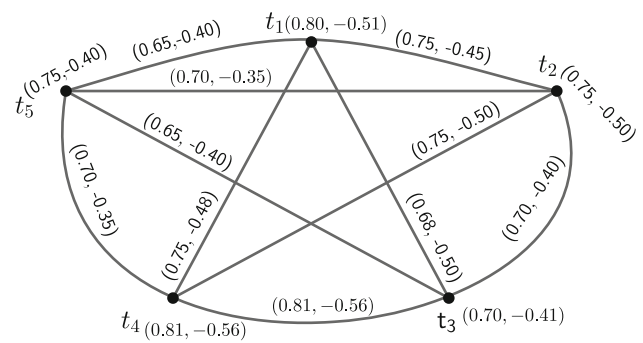


Fig. 10 Modeling of a communication system as a BFG *G*

to the other countries) between China and the U.S. was maximum than the other pair of countries.”

Therefore, China and the U.S was the most communicated countries and the total internal communication between China and the U.S was at the highest level.

Modeling of a communication system for improvement of students

Here, first, we will model the communications between teachers and students as a BFG. Then, using the *COND* and *CCND* of the nodes, we have to select that nodes for which the *COND* and *CCND* both are maximum. Then, we can find that the most communicated teachers for the corresponding nodes have the maximum *COND* and maximum *CCND*.

Model construction

During recent time, the improvement of student is very indispensable in educational field. Now, most of the improvement of a student depends upon (i) students’ skill, (ii) communication with each other, (iii) good results in various examinations, (iv) taking part in various competitive examinations, (v) study plan, etc. All these matters mainly depend upon the communication between the students with each other and the communication between the students and teachers. Therefore, communications are very needed in the present time. Now, we presume that, in an Institution, there are five teachers in Mathematics Department. Here, we demonstrate this communication system in this Institution as a BFG (Fig. 10) between teachers and students.

Description of membership values

Here, each node performs as a teacher. The positive membership value of each node represents the communication power of a teacher with students. The communication varies teacher to teacher due to lack of time, pressure of official works, etc. Therefore, this is uncertain. Its value is 0 if the communi-

Table 1 Nodes and their corresponding COND, CCND of the BFG G of Fig. 9

Node	COND	CCND	Node	COND	CCND	Node	COND	CCND
J	0.6	0.6	V	0.3	0.3	C	-0.2	-0.2
N	1.3	1.3	US	2.3	2.3	R	1.2	1.2
ER	1.5	1.5	SK	0.6	0.6	T	0.4	0.4
CH	2.8	2.8	NK	0	0			

Table 2 Edges and their corresponding CCND of the BFG G of Fig. 9

Edge	CCND	Edge	CCND	Edge	CCND	Edge	CCND
SK-NK	1.2	J-CH	3.2	J-US	2.5	V-US	2.6
V-C	-0.2	C-US	2.9	C-NK	-0.5	N-US	2.9
N-CH	3.5	US-SK	2.1	US-T	2.2	US-CH	5.1
US-ER	3.2	US-R	3.4	R-ER	2.4	R-CH	3.2
ER-CH	3.7	SK-CH	3.0	T-CH	3.3	CH-NK	2.5

ation is $\leq 10\%$, and its value is 1 if the communication is $\geq 90\%$. Therefore, this lies between 0 and 1. Also the negative membership value of each node represents the incapability of teachers to maintain this communication (because of lacks of time, pressure of official works, etc). Therefore, this is uncertain. Its value is -1 if the incapability is $\geq 85\%$ and its value is 0 if the incapability is $\leq 10\%$. Therefore, this lies between -1 and 0.

The positive membership value of each edge represents that the communication between two teachers for students and its varies also due to lack of time, pressure of official works, etc. Therefore, this is again uncertain. Its value is 0 if the communication is $\leq 15\%$, and its value is 1, if the communication is $\geq 80\%$. Therefore, this lies between 0 and 1. The negative membership value of each edge represents the incapability of teachers to maintain this communication. Therefore, this is uncertain. Its value is -1 if the incapability is $\geq 85\%$ and its value is 0 if the incapability is $\leq 10\%$. Therefore, this lies between -1 and 0.

Decision-making

From Table 3, we have $COND(t_1) < COND(t_3) < COND(t_4) < COND(t_2) = COND(t_5)$ and $CCND(t_1) < CCND(t_3) < CCND(t_4) < CCND(t_2) < CCND(t_5)$. $\max\{COND(t_i)\} = 1.20 = COND(t_2) = COND(t_5)$ and $\max\{CCND[t_i]\} = 1.55 = CCND[t_5]$, for all $i = 1, 2, 3, 4, 5$. Thus, for the node t_5 , both the value $COND(t_5)$ and $CCND[t_5]$ are maximum. $\min\{COND(t_i)\} = 0.98 = COND(t_3)$ and $\min\{CCND[t_i]\} = 1.27 = CCND[t_3]$, for all $i = 1, 2, 3, 4, 5$. Thus, for the node t_3 , both the value $COND(t_3)$ and $CCND[t_3]$ are minimum.

Since the COND and CCND of the node t_5 both are highest, therefore the totally accurate communication of the node t_5 is maximum. Hence, the teacher t_5 is the most communicable teacher for development of future of students.

Therefore, using the help of maximum value of COND and maximum value of CCND of the nodes, the selection of the t_5 teacher as a best communicate teacher is depicted.

Comperative analysis

In a fuzzy graphical model, there exists only one side membership value of each node and edge. Akram et al. [5,6] and Poulik and Ghorai [25] have described different types of applications of BFGs in wireless network, determination of journey’s order using path, connectivities, and indices. However, in this work, different types of degrees of nodes have been used. In our first model, there exists some good and bad relations between countries which are of opposite to each other and uncertain. Therefore, in this type of situations, bipolar fuzzy graph can be used to get better explanation. Also, the degree of vertices in a fuzzy graph gives the total contribution of the nodes in the system only. However, the degree of nodes in a bipolar fuzzy graph gives the total information and contribution of the positive and negative characters separately. Many decision-making problems on bipolar fuzzy graphs had been done previously. However, in our proposed methods, there are positive as well as negative relations. Here, the COND and CCND give the total combined (both positive and negative side) contribution of the countries in the system. Based on COND and CCND, the highest communicated countries in international communication systems and the most communicated teacher in teacher–students communication system are determined.

Advantages and limitations

The main advantages of the proposed method are as follows:

- The relation of communication between some countries have been analyzed here

Table 3 Nodes and their corresponding COND, CCND of the BFG G of Fig. 10

Node	COND	CCND	Node	COND	CCND	Node	COND	CCND
t_1	1.00	1.29	t_2	1.20	1.45	t_3	0.98	1.27
t_4	1.12	1.37	t_5	1.20	1.55			

- The most communicated countries are shown using COND and CCND in a BFG.
- This method can be applied to explain teacher–student communication in education system for the improvement of study under bipolar fuzzy information.

Some of the limitations of this work are as follows:

- This work mainly focused on BFG and its related network systems.
- This method is applicable only when two opposite-sided directional thinking exist in a connected bipolar fuzzy graphical system.
- If the membership values of the characters are given in different environment, then the concept of BFG cannot be applicable.
- Collection of real data may not be possible always.

Conclusion and future works

The degree of nodes and edges in a BFG have been applied to analyze the international relationship between some countries and teacher students' communication in education field. The regularity of nodes and edges in a BFG plays an important factor in many decision-making problem with bipolar fuzzy environment for instances, shortest path, boundary and interior stations in a wireless network, water and electric connection, etc. First, it contains perfectly regular BFG, which is an interpretation of the degrees of nodes with their characteristic. Second, regular interconnections between nodes are visualized as a perfectly edge-irregular BFG with their properties. Third, the total complete communication of nodes in a bipolar fuzzy environment has been described in terms of COND and CCND with some theorems. Some algorithms with corresponding flowcharts are exhibited to calculate the most communicable nodes and edges in a BFG. Two applications of our research work have been discussed using COND and CCND of BFGs. First one is to select the most communicated countries between countries at the time of cold-war era. The second one is in decision-making to select the best communicated teacher in a teacher–student communication system. At last, a comparative analysis and some advantages and limitations of this work have been described.

Based on the multiple characteristics of the nodes and edges in graphs, there are a lot of future research scope,

such as: (i) complete degree in intuitionistic fuzzy graphs, (ii) complete degree in pythagorean fuzzy graphs, (iii) complete degree in m -polar fuzzy graphs, etc.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

1. Akram M (2011) Bipolar fuzzy graphs. *Inform Sci* 181(24):5548–5564
2. Akram M (2013) Bipolar fuzzy graphs with applications. *Knowl-Based Syst* 39:1–8
3. Akram M, Farooq A (2016) Bipolar fuzzy tree. *New Trends Math Sci* 4(3):58–72
4. Akram M, Karunambigal MG (2011) Metric in bipolar fuzzy graphs. *World Appl Sci J* 14(12):1920–1927
5. Akram M, Sarwar M, Dudek WA (2021) Graphs for the Analysis of Bipolar Fuzzy Information, *Studies in Fuzziness and Soft Computing*. Springer 401. <https://doi.org/10.1007/978-981-15-8756-6>
6. Akram M, Waseem N (2018) Novel applications of bipolar fuzzy graphs to decision making problems. *J Appl Math Comput* 56:73–91
7. Bhutani KR, Rosenfeld A (2003) Strong arcs in fuzzy graphs. *Inform Sci* 152:319–322
8. Citiş HG (2019) Investigation of a fuzzy problem by the fuzzy Laplace transform. *Appl Math Nonlinear Sci* 4(2):407–416
9. Das S, Ghorai G, Pal M (2020) Certain competition graphs based on picture fuzzy environment with applications. *Artificial Intelligence Review*. <https://doi.org/10.1007/s10462-020-09923-5>
10. Douglas B (2002) West. *Introduction to Graph Theory*, Pearson Education India

11. Fan KC, Liu CW, Wang YK (1998) A fuzzy bipartite weighted graph matching approach to fingerprint verification, SMC'98 Conference Proceedings. 1998 IEEE International Conference on Systems, Man, and Cybernetics (Cat. No.98CH36218)
12. Gao W, Guirao JLG, Basavanagoud B, Wu J (2018) Partial multi-dividing ontology learning algorithm. *Inform Sci* 467:35–58
13. Gao W, Wang W (2015) The vertex version of weighted Wiener number for bicyclic molecular structures. *Computational and Mathematical Methods in Medicine*. <https://doi.org/10.1155/2015/418106>
14. Ghorai G (2021) Characterization of regular bipolar fuzzy graphs. *Afrika Matematika* 32(5):1043–1057
15. Gross G, Negi R, Sambhoos K (2014) A fuzzy graph matching approach in intelligence analysis and maintenance of continuous situational awareness. *Inform Fusion* 18:43–61
16. Imran M, Baig AQ, Ali H (2016) On molecular topological properties of hex-derived networks. *J Chemometrics* 30(3):121–129
17. Liao H, Gou X, Xu Z, Zeng X, Herrera F (2019) Hesitancy degree-based correlation measures for hesitant fuzzy linguistic term sets and their applications in multiple criteria decision making. *Inform Sci* 508:275–292
18. Liao H, Mi X, Xu Z, Xu J, Herrera F (2018) Intuitionistic fuzzy analytic network process. *IEEE Trans Fuzzy Syst* 26(5):2578–2590
19. Mordeson JN, Nair PS (1999) Arc disjoint fuzzy graphs, 18th International Conference of the North American Fuzzy Information Processing Society—NAFIPS (Cat. No99TH8397)
20. Mordeson JN, Nair PS (1996) Cycles and cocycles of fuzzy graphs. *Inform Sci* 90(1–4):39–49
21. Nguyen HL, Vu DT, Jung JJ (2020) Knowledge graph fusion for smart system: a survey. *Inform Fusion* 61:56–70
22. Poulik S, Ghorai G (2020) Certain indices of graphs under bipolar fuzzy environment with applications. *Soft Comput* 24(7):5119–5131
23. Poulik S, Ghorai G (2020) Detour g-interior nodes and detour g-boundary nodes in bipolar fuzzy graph with applications. *Hacettepe J Math Stat* 49(1):106–119. <https://doi.org/10.15672/HJMS.2019.666>
24. Poulik S, Ghorai G, Xin Q (2020) Pragmatic results in Taiwan education system based IVFG & IVNG. *Soft Computing*. <https://doi.org/10.1007/s00500-020-05180-4>
25. Poulik S, Ghorai G (2021) Determination of journeys order based on graph's Wiener absolute index with bipolar fuzzy information. *Information Sciences* 545:608–619
26. Rosenfield R (1975). In: Zadeh LA, Fu KS, Shimura M (eds) *Fuzzy graphs, Fuzzy Sets and Their Application*. Academic press, New York, pp 77–95
27. Shahzadi S, Rasool A, Sarwar M, Akram M (2021) A framework of decision making based on bipolar fuzzy competition hypergraphs. *Journal of Intelligent & Fuzzy Systems*. <https://doi.org/10.3233/JIFS-210216>
28. Sarwar M, Akram M, Shahzadi S (2021) Bipolar fuzzy soft information applied to hypergraphs. *Soft Comput* 25(2):1–23
29. Samanta S, Pal M (2015) Fuzzy planar graphs. *IEEE Trans Fuzzy Syst* 23(6):1936–1942
30. Yang HL, Li SG, Guo ZL, Ma CH (2012) Transformation of bipolar fuzzy rough set models. *Knowl-Based Syst* 27:60–68
31. Yang HL, Li SG, Yang WH, Lu Y (2013) Notes on bipolar fuzzy graphs. *Inform Sci* 242:113–121
32. Yuan W, He K, Guan D, Zhou L, Li C (2019) Graph kernel based link prediction for signed social networks. *Inform Fus* 46:1–10
33. Zadeh LA (1965) Fuzzy sets, information. *Control* 8(3):338–353
34. Zhang W (2002) Bipolar fuzzy cognitive mapping and bipolar visualization for OLAP/OLAM, 2002 Annual Meeting of the North American Fuzzy Information Processing Society Proceedings. NAFIPS-FLINT
35. Zhang W (1994) Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *Proceeding of IEEE Conf.* 305–309
36. Zhang W (2003) Equilibrium relations and bipolar cognitive mapping for online analytical processing with applications in international relations and strategic decision support, *IEEE Transactions on Systems, Man, and Cybernetics. Part B (Cybernetics)* 33(2):295–307
37. Zhang W (Yin) (Yang) (1998) bipolar fuzzy sets, 1998 IEEE International Conference on Fuzzy Systems Proceedings. IEEE World Congress on Computational Intelligence

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