Improved cosine and cotangent function-based similarity measures for q-rung orthopair fuzzy sets and TOPSIS method

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Abstract

Despite the importance of cosine and cotangent function-based similarity measures, the literature has not provided a satisfactory formulation for the case of q-rung orthopair fuzzy set (qROFS). This paper criticizes the existing attempts in terms of respect of the basic axioms of a similarity measure and strict inclusion relation. In addition, the maximum operator-based similarity measures are criticized. Then, new improved, axiomatically supported cosine and cotangent function-based similarity measures for qROFSs are proposed. Additional properties of the new similarity measures are discussed to guarantee their good performance. Two algorithmic procedures of TOPSIS method that based on fixed and relative ideal solutions are discussed. The numerical examples are provided to support the findings

Keywords QROFSs · Similarity measures · TOPSIS method · Decision analysis/MCDM

Introduction

Fuzziness, as developed in [1], is a kind of uncertainty which appears often in human decision-making problems. The fuzzy set theory deals with daily life uncertainties successfully. The membership degree is assigned to each element in a fuzzy set. The membership degrees can effectively be taken by fuzzy sets. However, in real-life situations, the non-membership degrees should be considered in many cases as well, and it is not necessary that the non-membership degree be equal to the one minus the membership degree. Thus, Atanassov [2] introduced the concept of intuitionistic fuzzy set (IFS) that considers both membership and non-membership degrees. Here the non-membership degree is not always obtained from a membership degree, which leads to the concept of hesitancy degree. For each element, the sum of its membership (ξ), non-membership (ν), and hesitancy (π) degrees should be equal to one for IFSs, that is

$$\xi + \nu + \pi = 1. \quad (1)$$

Condition (1) suggests that the region to choose membership and non-membership degrees is a proper subset of $[0, 1] \times [0, 1]$. Although IFS has been implemented in numerous areas, there were issues. Especially, when a decision maker provides such types of information but the sum of membership and non-membership degrees exceeds 1, then

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the IFS is not able to cope with it. For this reason, the theory of Pythagorean fuzzy set (PyFS) was proposed by Yager [3,4] by modifying the ideology of IFS. Moreover, a PyFS contains two grades that are called truth (membership) and falsity (non-membership), with a rule that is: the sum of the squares of both grades is restricted to the unit interval. PyFS has been implemented in numerous areas, but there were issues, when a decision maker provides such types of information whose sum of the squares is exceeded from the unit interval, then the PyFS is not able to cope with it. For this reason, the theory of q -rung orthopair fuzzy set (qROFS) was proposed by Yager by modifying the ideology of PyFS [5–7]. Moreover, a qROFS contains two grades are called truth and falsity with a rule that is the sum of the q^{th} power of both grades is restricted to the unit interval. qROFS has been implemented in numerous areas due to its ability to handle uncertain information more effectively [8–22,24]. It provides the more space to choose orthopair for representing the information than IFSs and PyFSs. It provides the more space to choose orthopair for representing the information than IFSs and PyFSs (see Fig. 1).

While dealing with real life situations qROFSs provide more flexible ways to define membership and non-membership grades for the elements of a set. That is why now a day many researchers are showing keen interest in qROFSs. The MCDM method based on normalized bidirectional projection model and generalized knowledge-based entropy measure in q -rung orthopair fuzzy environment was discussed by Liu et al. [25]. The stock investment evaluation problem was discussed with respect to decision-theoretic rough set model with q -rung orthopair fuzzy information by Tang et al. [26]. The associated probabilities' aggregations operators in interactive MCDM problems were defined by Sirbiladze [27]. The additive consistency-based priority generating method of q -rung orthopair fuzzy preference relation was extended by Zhang et al. [28]. Krishankumar et al. [30] debated the renewable energy source selection problems using qROFSs. The hybrid aggregation operator-based TOPSIS method for qROFSs was extended by Riaz et al. [33]. Zhang and Chen [34] proposed the method to solve the group decision making with incomplete q -rung orthopair fuzzy preference relations. Liu and Huang [35] discussed the consensus reaching process for fuzzy behavioral TOPSIS method with probabilistic linguistic q -rung orthopair fuzzy set based on correlation measure. The superiority and inferiority ranking methods to solve multiple attribute decision making problems within the q -rung orthopair fuzzy environment was extended by Zhu et al. [36]. Recently, Khan et al. [37] discussed the theoretical justifications of empirically successful VIKOR method.

To measure the similarity between any form of data is an important topic. The measures used to find the resemblance between data is called similarity measure. It has

Fig. 1 Comparison of IFS, PyFS and qROFSs ($q = 3, 10$)

different applications in classification, medical diagnosis, pattern recognition, data mining, clustering, decision making, and in image processing. The qROF multi-parametric similarity measure and combinative distance-based assessment were used to assess the classroom teaching quality [38]. The distance, similarity, entropy, and inclusion measures for qROFSs were defined by Peng et al. [39]. The Minkowski distance measures (Hamming, Euclidean and Chebyshev distances) for qROFSs were discussed in [40]. A new dissimilarity measure was defined with a nice interpretation. The TOPSIS and ELECTRE were described based on this novel dissimilarity measure for qROFSs in [41].

The trigonometric functions plays an important rule for defining the similarity and dissimilarity measures. In the following, the cosine and cotangent function-based similarity measures for qROFSs were discussed by Wang et al. [43]. They have discussed ten different similarity measures and their weighted versions. To remove the shortcomings of cosine similarity measure, Liu et al. [44] combined cosine similarity and Euclidean distance measures for qROFSs and discussed TOPSIS method.

The positive and negative ideal solutions are the best available and worst avoidable solutions in MCDM problems. The TOPSIS (technique for order preference by similarity to an ideal solution) method was presented by Chen and Hwang [45]. This method finds the solution which is closest to the positive ideal solution and farthest from the negative ideal solution. The TOPSIS method in q -rung orthopair fuzzy environment was already presented by Pinar and Boran [41] and Liu et al. [44]. The reason to reconsider this decision method

again is the shortcomings in their defined similarity measures.

Problem formulation: For any two qROFSs $A = (\xi_A, \nu_A)$ and $B = (\xi_B, \nu_B)$, the inclusion relation is defined as: $A \subseteq B \iff \xi_A \leq \xi_B \ \& \ \nu_A \geq \nu_B$. In the same way, the inclusion relation for three qROFSs A , B , and C , where $C = (\xi_C, \nu_C)$, is defined as: $\mu_A \leq \mu_B \leq \mu_C$ and $\nu_A \geq \nu_B \geq \nu_C$.

To understand the problem, we discriminate the different situations in inclusion relation for three qROFSs as follows: Equal case: $\mu_A = \mu_B = \mu_C$ and $\nu_A = \nu_B = \nu_C$, that is, $A = B = C$. Strictly included cases: $\mu_A < \mu_B < \mu_C$ and $\nu_A \geq \nu_B \geq \nu_C$ or $\mu_A \leq \mu_B \leq \xi_C$ and $\nu_A > \nu_B > \nu_C$.

The similarity measure of two qROFSs is non-negative, symmetric, and becomes one when they coincide. In addition, it increases when two sets become more identical. The first three characteristics are easy to analytically formulate and the fourth property uses the fact that if there exists an inclusion relation between three qROFSs then the similarity between extreme should be less than the similarity between means. Mathematically, for three qROFSs A , B , and C with $A \subseteq B \subseteq C$ and a similarity measure (Sim), then

$$\text{Sim}(A, C) \leq \max\{\text{Sim}(A, B), \text{Sim}(B, C)\}. \quad (2)$$

A measure of similarity between qROFSs considers all the axioms of the similarity measure. It is important for a similarity measure to discriminate between the two above-mentioned situations (that is, equal and strictly included cases).

Motivation: We have seen that the existing cosine and cotangent function-based similarity measures have no ability to discriminate between equal and strictly included cases. That is, it is possible for them to have equality in (2) for strictly included cases. In addition, motivation behind the origination of fuzzy and non-standard fuzzy models is their intimacy with human thinking. Therefore, if a similarity measure does not meet some cognitive aspect, we call it as a counter intuitive case.

Aims and objectives: The aim of this study to present new similarity measures based on cosine and cotangent functions that would clearly discriminate the equal and strictly included situations. In addition, to propose the TOPSIS method for qROFSs using the newly define similarity measures. To improve the notion, we propose modified similarity measures based on cosine and cotangent functions for qROFSs. The new similarity measures clearly differentiate between equal and strictly included cases. That is, the inequality in (2) becomes equality for the equal case and there is strict inequality in (2) for the strictly included cases. In addition, the two algorithmic procedures based on fixed and relative ideal solutions are proposed for the TOPSIS method.

The remaining article is structured as follows: Basic definitions are mentioned in the next section. The existing similarity measures based on cosine and cotangent functions are criticized in the following section. The improved cosine and cotangent function-based similarity measures for qROFSs and their properties are discussed in the next section. Two algorithmic procedures for TOPSIS method and the numerical example are discussed in the following sections, respectively. The comparison and concluding remarks are debated in the last sections, respectively.

Preliminaries

This section contains the definitions of fuzzy set, IFS, PyFS and qROFS. Let $Y = \{y_1, y_2, \dots, y_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ stand for non-empty, discrete and finite universal set and the criteria set, respectively.

Definition 1 [1] A fuzzy set R on a universal set Y is defined as

$$R = \{(y, \xi_R(y)) \mid y \in Y\},$$

where $\xi_R : Y \rightarrow [0, 1]$ is a membership degree.

Definition 2 [2] An IFS R on a universal set Y is defined as

$$R = \{(y, \xi_R(y), \nu_R(y)) \mid y \in Y\},$$

where $\xi_R : Y \rightarrow [0, 1]$ is a membership function and $\nu_R : Y \rightarrow [0, 1]$ is a non-membership function, with the constraint $\xi_R(y) + \nu_R(y) \leq 1$ for each $y \in Y$. The quantity $\pi_R(y) = 1 - (\xi_R(y) + \nu_R(y))$ is called the hesitancy degree of the element $y \in Y$.

Definition 3 [3,4] A PyFS R on a universal set Y is defined as

$$R = \{(y, \xi_R(y), \nu_R(y)) \mid y \in Y\},$$

where $\xi_R : Y \rightarrow [0, 1]$ and $\nu_R : Y \rightarrow [0, 1]$ are the membership and non-membership degrees under constraint $\xi_R^2(y) + \nu_R^2(y) \leq 1$, respectively. The quantity $\pi_R(y) = \sqrt{1 - (\xi_R^2(y) + \nu_R^2(y))}$ is called the hesitancy degree of the element $y \in Y$.

In qROFSs, the term “rung” is used synonymously with the term “level” and the parameter q is any real number that is greater than or equal to one.

Definition 4 [5,6] The membership (ξ_R) and non-membership (ν_R) functions from the universal set to the unit interval, with

the condition $\xi_R^q(y) + v_R^q(y) \leq 1$ for each $y \in Y$, define the qROFS R over a universal set Y as follows:

$$R = \{(y, \xi_R(y), v_R(y)) \mid y \in Y\}.$$

The hesitancy index of the element $y \in Y$ is defined as $\pi_R(y) = (1 - \xi_R^q(y) - v_R^q(y))^{1/q}$. For a single $y \in Y$, the triple $r = (y, \xi_R(y), v_R(y)) \in R$ is called a q-rung orthopair fuzzy value (qROFV). When there is no confusion, we write qROFV r as (ξ_r, v_r) instead of $(y, \xi_R(y), v_R(y))$.

Henceforth, $R_1, R_2, R_3, \dots$ represent qROFSs on Y , and the set of all qROFSs on Y will be denoted as $qR(Y)$.

Definition 5 [5,6] For two qROFVs $r_1 = (\xi_{r_1}, v_{r_1})$ and $r_2 = (\xi_{r_2}, v_{r_2})$, the following operations are defined:

- The complement operation: $r_1^c = (v_{r_1}, \xi_{r_1})$
- The partial order: $r_1 \subseteq r_2 \iff \xi_{r_1} \leq \xi_{r_2} \ \& \ v_{r_1} \geq v_{r_2}$
- The max operation: $r_1 \vee r_2 = (\max\{\xi_{r_1}, \xi_{r_2}\}, \min\{v_{r_1}, v_{r_2}\})$
- The min operation: $r_1 \wedge r_2 = (\min\{\xi_{r_1}, \xi_{r_2}\}, \max\{v_{r_1}, v_{r_2}\})$

To conclude this section, we recall the axiomatic definition of similarity measure for qROFSs.

Definition 6 A similarity measure between qROFSs on Y is a mapping $\text{Sim} : qR(Y) \times qR(Y) \rightarrow [0, 1]$ that satisfies the following properties:

- (S1) $0 \leq \text{Sim}(R_1, R_2) \leq 1$, where $R_1, R_2 \in qR(Y)$.
- (S2) $\text{Sim}(R_1, R_2) = 1 \iff R_1 = R_2$.
- (S3) $\text{Sim}(R_1, R_2) = \text{Sim}(R_2, R_1)$.

- (S4) If $R_1 \subseteq R_2 \subseteq R_3$, then $\text{Sim}(R_1, R_3) \leq \text{Sim}(R_1, R_2)$ and $\text{Sim}(R_1, R_3) \leq \text{Sim}(R_2, R_3)$, for every $R_3 \in qR(Y)$.

The authors in [41,44] tried to define the similarity measures based on cosine and cotangent functions for qROFSs. However, some of the proposed similarity measures do not hold the axiom (S4) of Definition 6 and all of them cannot differentiate the equal and strictly included cases, as discussed in the next section. This drawback motivates us to propose a better tool with the core of performance in “Improved cosine and cotangent function-based similarity measures”.

Review of existing cosine and cotangent function-based similarity measures for qROFSs

In this section, we mention the existing cosine and cotangent function-based similarity measures for qROFSs. The counter-intuitive cases are discussed. The counter-intuitive cases are based on the fact that the similarity of relatively similar qROFSs should be greater than the similarity of less similar qROFSs. In addition, the existing cosine and cotangent function-based similarity measures have no ability to discriminate between equal and strictly included cases. That is, it is possible for them to have equality in (2) for strictly included cases. Analysis is given about the inappropriateness of the maximum operator technique for similarity measures.

Wang et al. [43] expounded the cosine and cotangent function-based similarity measures for qROFSs.

Definition 7 The cosine and cotangent function-based similarity measures for qROFSs were defined as follows:

$$\begin{aligned} S_c^1(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \frac{\xi_{R_1}^q(y_j) \xi_{R_2}^q(y_j) + v_{R_1}^q(y_j) v_{R_2}^q(y_j)}{\sqrt{\xi_{R_1}^{2q}(y_j) + v_{R_1}^{2q}(y_j)} \sqrt{\xi_{R_2}^{2q}(y_j) + v_{R_2}^{2q}(y_j)}} \\ S_c^2(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cos \left[\frac{\pi}{2} \left(\max \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right|, \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| \right) \right) \right] \\ S_c^3(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cos \left[\frac{\pi}{4} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| \right) \right] \\ S_c^4(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right|, \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| \right) \right) \right] \\ S_c^5(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| \right) \right] \\ S_c^6(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \frac{\xi_{R_1}^q(y_j) \xi_{R_2}^q(y_j) + v_{R_1}^q(y_j) v_{R_2}^q(y_j) + \eta_{R_1}^q(y_j) \eta_{R_2}^q(y_j)}{\sqrt{\xi_{R_1}^{2q}(y_j) + v_{R_1}^{2q}(y_j) + \eta_{R_1}^{2q}(y_j)} \sqrt{\xi_{R_2}^{2q}(y_j) + v_{R_2}^{2q}(y_j) + \eta_{R_2}^{2q}(y_j)}} \end{aligned}$$

$$\begin{aligned}
S_c^7(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cos \left[\frac{\pi}{2} \left(\max \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right|, \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right|, \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right) \right] \\
S_c^8(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cos \left[\frac{\pi}{4} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| + \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right] \\
S_c^9(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right|, \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right|, \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right) \right] \\
S_c^{10}(R_1, R_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right| + \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right]
\end{aligned}$$

The similarity measures $S_c^1, S_c^3, S_c^4, S_c^5$ and S_c^6 were defined without considering hesitancy index. We know that the third parameter is calculated from the two parameters (membership and non-membership degrees), but the third parameter is important while calculating the similarity or dissimilarity measures between two qROFVs. Thus, leave the hesitancy index will an important part of information. By analysis, we can see that why third parameter is important. Let us verify if we can discard the values of the third parameter, we have

$$\begin{aligned}
& \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \\
&= \left| \left(\left(1 - \xi_{R_1}^q(y_j) - v_{R_1}^q(y_j) \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \xi_{R_2}^q(y_j) - v_{R_2}^q(y_j) \right)^{\frac{1}{q}} \right)^q \right| \\
&= \left| \left(1 - \xi_{R_1}^q(y_j) - v_{R_1}^q(y_j) \right) - \left(1 - \xi_{R_2}^q(y_j) - v_{R_2}^q(y_j) \right) \right| \\
&= \left| \left(\xi_{R_2}^q(y_j) - \xi_{R_1}^q(y_j) \right) + \left(v_{R_2}^q(y_j) - v_{R_1}^q(y_j) \right) \right| \\
&\leq \left| \left(\xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right) \right| + \left| \left(v_{R_1}^q(y_j) - v_{R_2}^q(y_j) \right) \right| \quad (3)
\end{aligned}$$

The inequality (3) ensures that the results are not same when we discard the hesitancy index for S_c^3, S_c^4, S_c^5 and S_c^6 . Similarly, it can be easily seen that the term $\eta_{R_1}^q(y_j) \times \eta_{R_2}^q(y_j)$ is not the constant multiple of the first two terms in S_c^1 .

In the following, we shall observe that the most of the existing cosine similarity measures not differentiate the equal and strictly included cases. The MCDM methods based on these similarity measures compromise the quality and lose their reliability, because these measures have not an ability to discriminate between equal and strictly included cases. To understand completely, we provide some explanatory examples.

Example 1 Consider three qROFSs R_1, R_2 and R_3 (for $q = 3$) on $Y = \{y_1, y_2, y_3\}$, such that

$$\begin{aligned}
R_1 &= \{(y_1, 0.6, 0.55), (y_2, 0.6, 0.6), (y_3, 0.5, 0.4)\}, \\
R_2 &= \{(y_1, 0.8, 0.50), (y_2, 0.8, 0.5), (y_3, 0.9, 0.3)\} \\
R_3 &= \{(y_1, 0.8, 0.40), (y_2, 0.8, 0.4), (y_3, 0.9, 0.2)\}
\end{aligned}$$

then $R_1 \subseteq R_2 \subseteq R_3$ with especially $v_{R_1}(y_i) > v_{R_2}(y_i) > v_{R_3}(y_i)$, but $S(R_1, R_2) = S(R_1, R_3)$, where S represents the similarity measure in Table 1. It is evident from Table 1 that the most of the existing similarity measures that based on the cosine and cotangent functions give us the equal similarity for strict inclusion cases.

Example 2 Consider three qROFSs R_1, R_2 and R_3 (for $q = 3$) on $Y = \{y_1, y_2, y_3\}$, such that

$$\begin{aligned}
R_1 &= \{(y_1, 0.20, 0.6), (y_2, 0.3, 0.7), (y_3, 0.3, 0.9)\}, \\
R_2 &= \{(y_1, 0.30, 0.6), (y_2, 0.4, 0.7), (y_3, 0.4, 0.9)\} \\
R_3 &= \{(y_1, 0.35, 0.4), (y_2, 0.5, 0.5), (y_3, 0.5, 0.6)\}
\end{aligned}$$

then $R_1 \subseteq R_2 \subseteq R_3$ with especially $\xi_{R_1}(y_i) < \xi_{R_2}(y_i) < \xi_{R_3}(y_i)$, but $S(R_2, R_3) = S(R_1, R_3)$, where S represents the similarity measure in Table 2. It is evident from Table 2 that the most of the existing similarity measures that based on the cosine and cotangent functions give us the equal distance for strict inclusion cases.

Remark 1 Is maximum operator suitable to define similarity measures?

In the following analysis, we observe that the maximum operator-based similarity measures are not suitable for practical applications. The reliability and validity of any similarity measure depend on axioms of Definition 6. If for a similarity measure, any axiom not satisfies, then it is not valid and reliable. The first three axioms of Definition 6 are straightforward but the fourth axiom is not. Now, we can see that the maximum operator remains ineffective for different situations and thus not a reliable way to define the similarity measure. For three distinct qROFSs, we have

$$R_1 \subseteq R_2 \subseteq R_3$$

Table 1 Calculated values for similarity measures in Example 1

Similarity measures	(R_1, R_2)	(R_2, R_3)	(R_1, R_3)	Results
S_c^2	0.790124	0.995653	0.790124	Not valid
S_c^4	0.518570	0.910893	0.518570	Not valid
S_c^6	0.705059	0.993089	0.725242	Not valid
S_c^7	0.790124	0.995653	0.790124	Not valid
S_c^8	0.790124	0.995653	0.790124	Not valid
S_c^9	0.518570	0.910893	0.518570	Not valid
S_c^{10}	0.518570	0.910893	0.518570	Not valid

Bold values indicate counter-intuitive results

Table 2 Calculated values for similarity measures in Example 2

Similarity measures	(R_1, R_2)	(R_2, R_3)	(R_1, R_3)	Results
S_c^2	0.868698	0.998726	0.868698	Not valid
S_c^4	0.631085	0.952535	0.631085	Not valid
S_c^6	0.825619	0.998511	0.836868	Not valid
S_c^7	0.868698	0.998726	0.868698	Not valid
S_c^8	0.868698	0.998726	0.868698	Not valid
S_c^9	0.631085	0.952535	0.631085	Not valid
S_c^{10}	0.631085	0.952535	0.631085	Not valid

Bold values indicate counter-intuitive results

$$\begin{aligned} \xi_{R_1} \leq \xi_{R_2} \leq \xi_{R_3} \quad \& \quad \nu_{R_1} \geq \nu_{R_2} \geq \nu_{R_3} \\ |\xi_{R_1} - \xi_{R_3}| \geq \max \{ |\xi_{R_1} - \xi_{R_2}|, |\xi_{R_2} - \xi_{R_3}| \} \quad \& \\ |\nu_{R_1} - \nu_{R_3}| \geq \max \{ |\nu_{R_1} - \nu_{R_2}|, |\nu_{R_2} - \nu_{R_3}| \} \end{aligned}$$

Then, the following inequality obtained

$$\begin{aligned} \max \{ |\xi_{R_1} - \xi_{R_3}|, |\nu_{R_1} - \nu_{R_3}| \} \geq \max \\ \left\{ \begin{aligned} &\max \{ |\xi_{R_1} - \xi_{R_2}|, |\nu_{R_1} - \nu_{R_2}| \} \quad \& \\ &\max \{ |\xi_{R_2} - \xi_{R_3}|, |\nu_{R_2} - \nu_{R_3}| \} \end{aligned} \right. \end{aligned} \quad (4)$$

The inequality 4 have greater than or equal to symbol and a reliable technique should differentiate the equal and unequal cases. Equality holds for equal qROFSs and inequality for distinct qROFSs. If there is no discrimination between these two cases, then the technique is not reliable and valid for practical applications.

But we have seen different situations for which the equality holds for distinct qROFSs. For three distinct qROFSs with $R_1 \subseteq R_2 \subseteq R_3$, we have the following counter-intuitive situations.

1. If $\xi_{R_1} < \xi_{R_2} < \xi_{R_3}$ and $\nu_{R_3} < \nu_{R_2} = \nu_{R_1}$ with $|\xi_{R_1} - \xi_{R_3}| < |\nu_{R_1} - \nu_{R_3}|$. Then, for three distinct qROFSs, inequality (4) becomes equality.
2. If $\xi_{R_1} < \xi_{R_2} < \xi_{R_3}$ and $\nu_{R_3} = \nu_{R_2} < \nu_{R_1}$ with $|\xi_{R_1} - \xi_{R_3}| < |\nu_{R_1} - \nu_{R_3}|$. Then, for three distinct qROFSs, inequality (4) becomes equality.

3. If $\xi_{R_1} = \xi_{R_2} < \xi_{R_3}$ and $\nu_{R_3} < \nu_{R_2} < \nu_{R_1}$ with $|\xi_{R_1} - \xi_{R_3}| > |\nu_{R_1} - \nu_{R_3}|$. Then, for three distinct qROFSs, inequality (4) becomes equality.
4. If $\xi_{R_1} < \xi_{R_2} = \xi_{R_3}$ and $\nu_{R_3} < \nu_{R_2} < \nu_{R_1}$ with $|\xi_{R_1} - \xi_{R_3}| > |\nu_{R_1} - \nu_{R_3}|$. Then, for three distinct qROFSs, inequality (4) becomes equality.

Thus, the above-mentioned cases reveal that this technique is not valid and reliable for practical applications and the similarity measures S_c^2 , S_c^4 , S_c^7 and S_c^9 are based on this technique.

Liu et al. [44] defined similarity measure as a combination of cosine similarity and Euclidean dissimilarity measures for qROFSs.

Definition 8 Let R_1 and R_2 be two qROFSs in Y . Then, the similarity measure between R_1 and R_2 is defined as follows:

$$D_L(R_1, R_2) = \frac{1 + C_L(R_1, R_2) - E_L(R_1, R_2)}{2}, \quad (5)$$

where

$$C_L(R_1, R_2) = \frac{1}{n} \sum_{j=1}^n \frac{\xi_{R_1}^q(y_j) \xi_{R_2}^q(y_j) + \nu_{R_1}^q(y_j) \nu_{R_2}^q(y_j)}{\sqrt{\xi_{R_1}^{2q}(y_j) + \nu_{R_1}^{2q}(y_j)} \sqrt{\xi_{R_2}^{2q}(y_j) + \nu_{R_2}^{2q}(y_j)}} \quad (6)$$

$$\begin{aligned} E_L(R_1, R_2) \\ = \left(\frac{1}{2n} \sum_{j=1}^n (|\xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j)|^2 + |\nu_{R_1}^q(y_j) - \nu_{R_2}^q(y_j)|^2) \right)^{\frac{1}{2}} \end{aligned} \quad (7)$$

There are some deficiencies in Definition 8 which should be removed for making the similarity measure more effective for qROFSs.

Example 3 Let $R_1 = (0, 0)$, $R_2 = (1, 0)$ and $R_3 = (0, 1)$ be three qROFSs in Y . Then, the similarities between them are:

$$\begin{aligned} D_L(R_1, R_2) &= \infty \\ D_L(R_2, R_3) &= 0 \\ D_L(R_1, R_3) &= \infty. \end{aligned}$$

Remark 2 From above example, we have seen that we cannot find similarity from the point $R_1 = (0, 0)$. Actually, for any arbitrary qROFV, we cannot find the similarity from the point $(0, 0)$.

For calculating similarity and dissimilarity measures, Liu et al. [44] used two parameters only, i.e., membership and non-membership degrees. They have not considered the hesitancy index while defining similarity measure. Since we know that it carry an important information, so leaving hesitancy index for similarity measures is not a good idea. By analysis, we can see that why third parameter is important. Let us verify if we can discard the values of η from (7), we have

$$\begin{aligned} & \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right|^2 \\ &= \left[\left(\left(1 - \xi_{R_1}^q(y_j) - \nu_{R_1}^q(y_j) \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \xi_{R_2}^q(y_j) - \nu_{R_2}^q(y_j) \right)^{\frac{1}{q}} \right)^q \right]^2 \\ &= \left[\left(1 - \xi_{R_1}^q(y_j) - \nu_{R_1}^q(y_j) \right) - \left(1 - \xi_{R_2}^q(y_j) - \nu_{R_2}^q(y_j) \right) \right]^2 \\ &= \left[\left(\xi_{R_2}^q(y_j) - \xi_{R_1}^q(y_j) \right) + \left(\nu_{R_2}^q(y_j) - \nu_{R_1}^q(y_j) \right) \right]^2 \\ &= \left(\xi_{R_2}^q(y_j) - \xi_{R_1}^q(y_j) \right)^2 + \left(\nu_{R_2}^q(y_j) - \nu_{R_1}^q(y_j) \right)^2 \\ &\quad + 2 \left(\xi_{R_2}^q(y_j) - \xi_{R_1}^q(y_j) \right) \left(\nu_{R_2}^q(y_j) - \nu_{R_1}^q(y_j) \right). \end{aligned}$$

Since $2 \left(\xi_{R_2}^q(y_j) - \xi_{R_1}^q(y_j) \right) \left(\nu_{R_2}^q(y_j) - \nu_{R_1}^q(y_j) \right) \neq 0$ in general. Hence, with the hesitation margin the result is not the constant multiple of the (7). Therefore, by ignoring the hesitation margin the result will be influenced. In addition, by adding the hesitancy index in (6) the result is not the constant multiple of the (6). Hence, the overall result will be influenced by ignoring the hesitancy degree.

Improved cosine and cotangent function-based similarity measures

In this section, we define improved cosine and cotangent function-based similarity measures. Additional properties of newly propose similarity measures are discussed. In the previous section, we have seen that for distinct qROFSs, the similarity values are equal, that is, the two situations when sets are equal and are not equal cannot differentiate by existing cosine or cotangent function-based similarity measures. The improved similarity measure clearly differentiate the two situations of equal and unequal cases (strict inclusion).

The inclusion relation discussed in Definition 5 is a partial order and the condition (S4) in Definition 6 is based on this inclusion relation. However, the inclusion relation in Definition 5 not differentiate equal and unequal cases. Therefore, we define the new partial order for qROFVs.

Definition 9 Let R_1 and R_2 be two qROFSs in Y . Then, the total inclusion between R_1 and R_2 is defined as

$$\begin{aligned} R_1 \sqsubset R_2 &\Leftrightarrow \begin{cases} \xi_{R_1}(y_i) = \xi_{R_2}(y_i), \nu_{R_1}(y_i) = \nu_{R_2}(y_i) \quad \forall y_i \in Y & \text{or} & (\text{Case 1}) \\ \xi_{R_1}(y_i) < \xi_{R_2}(y_i), \nu_{R_1}(y_i) \geq \nu_{R_2}(y_i) \quad \forall y_i \in Y & \text{or} & (\text{Case 2}) \\ \xi_{R_1}(y_i) \leq \xi_{R_2}(y_i), \nu_{R_1}(y_i) > \nu_{R_2}(y_i) \quad \forall y_i \in Y & & (\text{Case 3}) \end{cases} \end{aligned} \quad (8)$$

The situation in (8) is more clear, convincing and conveying more information than situation discussed in Definition 5, when $R_1 \subseteq R_2$. We will see that the total inclusion relation refines the similarity measures for qROFSs.

The condition (S2) in Definition 6 deals with the situation when two qROFVs are equal (Case 1 in (8)). Thus, we consider second and third cases in (8) as follows:

$$R_1 \sqsubset^t R_2 \Leftrightarrow \begin{cases} \xi_{R_1}(y_i) < \xi_{R_2}(y_i), \nu_{R_1}(y_i) \geq \nu_{R_2}(y_i) & \text{or} \\ \xi_{R_1}(y_i) \leq \xi_{R_2}(y_i), \nu_{R_1}(y_i) > \nu_{R_2}(y_i) \end{cases} \quad (9)$$

Based on this new inclusion relation defined in (9), we modified the fourth axiom of similarity measure as follows.

Definition 10 An improved similarity measure between two qROFSs R_1 and R_2 is a mapping $S: R_1 \times R_2 \rightarrow [0, 1]$, which satisfies the following properties:

- (MS1) $0 \leq S(R_1, R_2) \leq 1$.
- (MS2) $S(R_1, R_2) = 1 \iff R_1 = R_2$.
- (MS3) $S(R_1, R_2) = S(R_2, R_1)$.
- (MS4) For all $R_1, R_2, R_3 \in qROFSs$, if $R_1 \sqsubset^t R_2 \sqsubset^t R_3$ then

$$S(R_1, R_3) < \max \{S(R_1, R_2), S(R_2, R_3)\}$$

Any similarity measure that fulfills (MS4) also satisfies (S4) but the converse is not true. Thus, an improved similarity measure is also a similarity measure but a similarity measure not necessarily to be an improved similarity measure.

Definition 11 For two qROFSs R_1 and R_2 in Y , a cosine function-based similarity measure between R_1 and R_2 is define as follows:

$$S_q(R_1, R_2) = \frac{1}{n} \sum_{i=1}^n \text{Cos} \left[\frac{\pi}{4} \left(\left| \xi_{R_1}^q(y_i) - \xi_{R_2}^q(y_i) \right| + \left| \nu_{R_1}^q(y_i) - \nu_{R_2}^q(y_i) \right| \right) \times \left(1 - \frac{1}{2} \left| \eta_{R_1}^q(y_i) - \eta_{R_2}^q(y_i) \right| \right) \right] \quad (10)$$

For any two qROFVs $\rho = (\xi_\rho, \nu_\rho)$ and $\sigma = (\xi_\sigma, \nu_\sigma)$, the (10) becomes

$$S_q(\rho, \sigma) = \text{Cos} \left[\frac{\pi}{4} \left(\left| \xi_\rho^q - \xi_\sigma^q \right| + \left| \nu_\rho^q - \nu_\sigma^q \right| \right) \times \left(1 - \frac{1}{2} \left| \xi_\rho^q + \nu_\rho^q - \xi_\sigma^q - \nu_\sigma^q \right| \right) \right] \quad (11)$$

Theorem 1 The function S_q defined in (10) is an improved similarity measure.

Proof To complete this proof, we need to verify conditions (MS1–MS4) of Definition 10 for S_q . Let $Y = \{y\}$ and $R = \{(y, \xi, \nu)\}$ be an arbitrary qROFS and $\Psi = \{(y, \xi_\Psi, \nu_\Psi)\}$ be any particular qROFS in Y . S_q has characteristic function h , which can be represented as

$$h(\xi, \nu) = \text{Cos} \left[\frac{\pi}{4} \left(\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| \right) \times \left(1 - \frac{1}{2} \left| (\xi^q + \nu^q) - (\xi_\Psi^q + \nu_\Psi^q) \right| \right) \right] \quad (12)$$

(MS1) Since we have $0 \leq \left| \xi^q - \xi_\Psi^q \right| \leq 1, 0 \leq \left| \nu^q - \nu_\Psi^q \right| \leq 1$ and $\frac{1}{2} \leq 1 - \frac{1}{2} \left| \pi^q - \pi_\Psi^q \right| \leq 1$. Then, it is easy to see

$$0 \leq h(\xi, \nu) \leq 1$$

(MS2) For two qROFs R and Ψ , the characteristic function h satisfies

$$h(\xi, \nu) = 1 \Leftrightarrow R = \Psi,$$

If $h(\xi, \nu) = 1$ then from (12),

$$\text{Cos} \left[\frac{\pi}{4} \left(\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| \right) \times \left(1 - \frac{1}{2} \left| (\xi^q + \nu^q) - (\xi_\Psi^q + \nu_\Psi^q) \right| \right) \right] = 1$$

$$\begin{aligned} & \frac{\pi}{4} \left(\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| \right) \\ & \times \left(1 - \frac{1}{2} \left| (\xi^q + \nu^q) - (\xi_\Psi^q + \nu_\Psi^q) \right| \right) = 0 \\ & \left(\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| \right) \\ & \times \left(1 - \frac{1}{2} \left| (\xi^q + \nu^q) - (\xi_\Psi^q + \nu_\Psi^q) \right| \right) = 0 \end{aligned} \quad (13)$$

Equation (13) will satisfy only when $\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| = 0$ or $\left(1 - \frac{1}{2} \left| \pi^q - \pi_\Psi^q \right| \right) = 0$. However, $\left(1 - \frac{1}{2} \left| \pi^q - \pi_\Psi^q \right| \right) \neq 0$ and thus $\left(\left| \xi^q - \xi_\Psi^q \right| + \left| \nu^q - \nu_\Psi^q \right| \right) = 0$. This is only possible when $\xi = \xi_\Psi$ and $\nu = \nu_\Psi$ and hence $R = \Psi$. The sufficient condition is straightforward.

(MS3) Symmetric property for h is easy to prove.

(MS4) To complete the proof, we separately treat the two conditions: Case 1. $\xi_\Psi^q + \nu_\Psi^q \leq \xi^q + \nu^q$. Case 2. $\xi_\Psi^q + \nu_\Psi^q > \xi^q + \nu^q$. For Case 1 with $\xi_\Psi \leq \xi$ and $\nu_\Psi \geq \nu$, (12) is reduce to

$$\begin{aligned} h(\xi, \nu) &= \text{Cos} \left[\frac{\pi}{4} \left(\left(\xi^q - \xi_\Psi^q \right) + \left(\nu_\Psi^q - \nu^q \right) \right) \right. \\ & \quad \times \left. \left(1 - \frac{1}{2} \left(\xi^q + \nu^q - \xi_\Psi^q - \nu_\Psi^q \right) \right) \right] \\ h'_\xi(\xi, \nu) &= \frac{\partial h(\xi, \nu)}{\partial \xi} \\ &= - \left(\frac{\pi}{4} q \xi^{q-1} \left(\frac{1}{2} \left(\xi_\Psi^q + \nu_\Psi^q - \nu^q - \xi^q \right) + 1 \right) \right. \\ & \quad \left. - \frac{\pi}{8} q \xi^{q-1} \left(-\xi_\Psi^q + \nu_\Psi^q - \nu^q + \xi^q \right) \right) \\ & \quad \times \text{Sin} \left[\frac{\pi}{4} \left(-\xi_\Psi^q + \nu_\Psi^q - \nu^q + \xi^q \right) \right. \\ & \quad \times \left. \left(1 + \frac{1}{2} \left(\xi_\Psi^q + \nu_\Psi^q - \nu^q - \xi^q \right) \right) \right] \\ &= \frac{\pi}{4} q \xi^{q-1} \left(1 + \xi_\Psi^q - \xi^q \right) \\ & \quad \text{Sin} \left[\frac{\pi}{8} \left(2 + \xi_\Psi^q + \nu_\Psi^q - \nu^q - \xi^q \right) \right. \\ & \quad \left. \left(\xi_\Psi^q - \nu_\Psi^q + \nu^q - \xi^q \right) \right] \\ &\leq 0 \end{aligned} \quad (14)$$

$$\begin{aligned} h'_\nu(\xi, \nu) &= \frac{\partial h(\xi, \nu)}{\partial \nu} \\ &= \left(\frac{\pi}{8} q \nu^{q-1} \left(-\xi_\Psi^q + \nu_\Psi^q - \nu^q + \xi^q \right) \right. \\ & \quad \left. + \frac{\pi}{4} q \nu^{q-1} \left(1 + \frac{1}{2} \left(\xi_\Psi^q + \nu_\Psi^q - \nu^q - \xi^q \right) \right) \right) \end{aligned}$$

$$\begin{aligned}
& \times \sin \left[\frac{\pi}{4} \left(-\xi_{\Psi}^q + v_{\Psi}^q - v^q + \xi^q \right) \right. \\
& \left. \left(\frac{1}{2} \left(1 + \xi_{\Psi}^q + v_{\Psi}^q - v^q - \xi^q \right) \right) \right] \\
& = -\frac{\pi}{4} q v^{q-1} \left(1 + v_{\Psi}^q - v^q \right) \\
& \sin \left[\frac{\pi}{8} \left(2 + \xi_{\Psi}^q + v_{\Psi}^q - v^q - \xi^q \right) \right. \\
& \left. \left(\xi_{\Psi}^q - v_{\Psi}^q + v^q - \xi^q \right) \right] \\
& \geq 0
\end{aligned} \tag{15}$$

The four real numbers ξ_1, ξ_2, ξ_3 and v' are consider that satisfy $\xi_{\Psi} < \xi_1 < \xi_2 \leq 1, \xi_3 = \frac{\xi_1 + \xi_2}{2}$ and $0 \leq v' \leq v_{\Psi}$. From (14), we have $h'_{\xi}(\xi_1) \leq 0$ and $\xi_{\Psi} < \xi_1 < \xi_3 < \xi_2 \leq 1$, then we have $h(\xi_1, v') > h(\xi_3, v') > h(\xi_2, v')$. This implies $h(\xi_1, v') > h(\xi_2, v')$. Consequently, $h(\xi, v)$ is a strictly decreasing function of ξ when $\xi_{\Psi} \leq \xi$ and $v_{\Psi} > v$. Similarly, using (15), we reach that the $h(\xi, v)$ strictly increasing function of v . Now, we discuss Case 2 with $\xi_{\Psi} \leq \xi$ and $v_{\Psi} > v$, we have

$$\begin{aligned}
& h(\xi, v) \\
& = \cos \left[\frac{\pi}{4} \left(\left(\xi^q - \xi_{\Psi}^q \right) + \left(v_{\Psi}^q - v^q \right) \right) \right. \\
& \quad \times \left(1 - \frac{1}{2} \left(\xi_{\Psi}^q + v_{\Psi}^q - v^q - \xi^q \right) \right) \Big] \\
& h'_{\xi}(\xi, v) = \frac{\partial h(\xi, v)}{\partial \xi} \\
& = -\left(\frac{\pi}{8} q \xi^{q-1} \left(-\xi_{\Psi}^q + v_{\Psi}^q - v^q + \xi^q \right) \right. \\
& \quad \left. + \frac{\pi}{4} q \xi^{q-1} \left(1 + \frac{1}{2} \left(-\xi_{\Psi}^q - v_{\Psi}^q + v^q + \xi^q \right) \right) \right) \\
& \quad \sin \times \left[\frac{\pi}{4} \left(-\xi_{\Psi}^q + v_{\Psi}^q - v^q + \xi^q \right) \right. \\
& \quad \left. \left(1 + \frac{1}{2} \left(-\xi_{\Psi}^q - v_{\Psi}^q + v^q + \xi^q \right) \right) \right] \\
& = -\frac{\pi}{4} q \xi^{q-1} \left(1 - \xi_{\Psi}^q + \xi^q \right) \\
& \sin \left[\frac{\pi}{8} \left(\xi_{\Psi}^q + v_{\Psi}^q - v^q - \xi^q - 2 \right) \right. \\
& \quad \left. \left(\xi_{\Psi}^q - v_{\Psi}^q + v^q - \xi^q \right) \right] \\
& \leq 0 \\
& h'_v(\xi, v) = \frac{\partial h(\xi, v)}{\partial v}
\end{aligned} \tag{16}$$

$$\begin{aligned}
& = -\left(\frac{\pi}{8} q v^{q-1} \left(-\xi_{\Psi}^q + v_{\Psi}^q - v^q + \xi^q \right) \right. \\
& \quad \left. - \frac{\pi}{4} q v^{q-1} \right. \\
& \quad \left. \left(1 + \frac{1}{2} \left(-\xi_{\Psi}^q - v_{\Psi}^q + v^q + \xi^q \right) \right) \right) \\
& \quad \times \sin \left[\frac{\pi}{4} \left(-\xi_{\Psi}^q + v_{\Psi}^q - v^q + \xi^q \right) \right. \\
& \quad \left. \left(1 + \frac{1}{2} \left(-\xi_{\Psi}^q - v_{\Psi}^q + v^q + \xi^q \right) \right) \right] \\
& = \frac{\pi}{4} q v^{q-1} \left(1 - v_{\Psi}^q + v^q \right) \\
& \sin \left[\frac{\pi}{8} \left(\xi_{\Psi}^q + v_{\Psi}^q - v^q - \xi^q - 2 \right) \right. \\
& \quad \left. \left(\xi_{\Psi}^q - v_{\Psi}^q + v^q - \xi^q \right) \right] \\
& \geq 0
\end{aligned} \tag{17}$$

Using (16) and (17) and the same procedure as in condition first, we prove that $h(\xi, v)$ is a strictly decreasing function of ξ and strictly increasing function of v . In addition, with the conditions $\xi_{\Psi} \geq \xi$ and $v_{\Psi} < v$ in Case 1 & Case 2, we can easily prove that $h(\xi, v)$ is a strictly increasing function of ξ and strictly decreasing function of v . Thus, the function S_q based on cosine function is an improved similarity measure.

Similarly, it is not difficult to extend to any $Y = \{y_1, y_2, \dots, y_m\}$. $\square$

It is important to discuss the situation when the membership degree is equal to the non-membership degree and the graphical representation of such situation is called an equal line. Each qROFVs on equal line have same membership and non-membership degrees.

Corollary 1 S_q is a decreasing function on equal line from $(1, 0)$.

Proof Since we know that the membership degree is equal to non-membership degree on equal line. If we represent the qROFVs on equal line by $\rho = (\xi, v)$ and $(1, 0)$ by σ , then (11) becomes

$$\begin{aligned}
& S_q(\rho, \sigma) \\
& = \cos \left[\frac{\pi}{4} \left((1 - \xi^q) + (v^q - 0) \right) \times \left(1 - \frac{1}{2} (\eta^q - 0) \right) \right] \\
& = \cos \left[\frac{\pi}{4} \left(1 - \frac{1}{2} (\eta^q) \right) \right]
\end{aligned}$$

The hesitancy index is maximum at $(0, 0)$ and decreases on equal line and attains minimum at $\xi^q + v^q = 1$. Thus, S_q is a strictly decreasing on equal line. $\square$

Theorem 2 For each parallel line to equal line, the similarity measure S_q is a decreasing function from $(1, 0)$.

Proof If we represents the any qROFV on parallel line by $\rho = (\xi, \nu)$ and $(1, 0)$ by σ , then from (11), we have

$$S_q(\rho, \sigma) = \text{Cos} \left[\frac{\pi}{4} (1 - \xi^q + \nu^q) \times \left(1 - \frac{1}{2} (\eta^q) \right) \right] \quad (18)$$

If the parallel line start from ξ -axis, then $(\xi, 0)$ is the starting point and $\eta^q = 1 - \xi^q$. Then, (18) takes the form

$$S_q(\rho, \sigma) = \text{Cos} \left[\frac{\pi}{4} (\eta^q) \times \left(1 - \frac{1}{2} (\eta^q) \right) \right] \quad (19)$$

Now, if we consider any point on line and denoted as $(\xi + \Delta\xi, \Delta\nu)$, where $\Delta\xi$ and $\Delta\nu$ are the changes in first and second coordinates, respectively. Then, from (18) and (19), we have

$$S_q(\rho, \sigma) = \text{Cos} \left[\frac{\pi}{4} (\eta^q + 2(\Delta\nu)^q) \times \left(1 - \frac{1}{2} (\eta^q) \right) \right]$$

Since the hesitancy index is maximum at $(\xi, 0)$ and decreases on the parallel line and attains minimum at $\xi^q + \nu^q = 1$. Thus, S_q is a strictly decreasing on the parallel line. $\square$

Definition 12 For two qROFSS R_1 and R_2 in Y , the weighted cosine function-based similarity measure between R_1 and R_2 is define as follows:

$$WS_q(R_1, R_2) = \sum_{j=1}^n \omega_j \text{Cos} \left[\frac{\pi}{4} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| \nu_{R_1}^q(y_j) - \nu_{R_2}^q(y_j) \right| \right) \times \left(1 - \frac{1}{2} \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right] \quad (20)$$

where w_j are the weights with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$.

In the following, we use cotangent function to define the similarity measure. The propose similarity measure is the improve version of the existing similarity measures.

Definition 13 For two qROFSS R_1 and R_2 in Y , the cotangent function-based similarity measure between R_1 and R_2 is define as follows:

$$S_q^{\text{cot}}(R_1, R_2) = \frac{1}{n} \sum_{i=1}^n \text{Cot} \left[\frac{\pi}{4} + \frac{\pi}{8} \left(\left| \xi_{R_1}^q(y_i) - \xi_{R_2}^q(y_i) \right| + \left| \nu_{R_1}^q(y_i) - \nu_{R_2}^q(y_i) \right| \right) \times \left(1 - \frac{1}{2} \left| \eta_{R_1}^q(y_i) - \eta_{R_2}^q(y_i) \right| \right) \right] \quad (21)$$

For any two qROFVs $\rho = (\xi_\rho, \nu_\rho)$ and $\sigma = (\xi_\sigma, \nu_\sigma)$, (21) becomes

$$S_q^{\text{cot}}(\rho, \sigma) = \text{Cot} \left[\frac{\pi}{4} + \frac{\pi}{8} \left(\left| \xi_\rho^q - \xi_\sigma^q \right| + \left| \nu_\rho^q - \nu_\sigma^q \right| \right) \times \left(1 - \frac{1}{2} \left| \xi_\rho^q + \nu_\rho^q - \xi_\sigma^q - \nu_\sigma^q \right| \right) \right]. \quad (22)$$

Theorem 3 The function S_q^{cot} is an improved similarity measure.

Proof The proof is similar to the proof of the Theorem 1. $\square$

Definition 14 For two qROFSS R_1 and R_2 in Y , the weighted cotangent function-based similarity measure between R_1 and R_2 is define as follows:

$$WS_q^{\text{cot}}(R_1, R_2) = \sum_{j=1}^n \omega_j \text{Cot} \left[\frac{\pi}{4} + \frac{\pi}{8} \left(\left| \xi_{R_1}^q(y_j) - \xi_{R_2}^q(y_j) \right| + \left| \nu_{R_1}^q(y_j) - \nu_{R_2}^q(y_j) \right| \right) \times \left(1 - \frac{1}{2} \left| \eta_{R_1}^q(y_j) - \eta_{R_2}^q(y_j) \right| \right) \right] \quad (23)$$

where w_j are the weights with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$.

q-rung orthopair fuzzy TOPSIS (qROF-TOPSIS) method

We have seen from “Review of existing cosine and cotangent function-based similarity measures for qROFSS” that the existing similarity measures have counter intuitive cases and they not differentiate between equal and strictly included cases. We have provided the improved similarity measures based on cosine and cotangent functions in “Improved cosine and cotangent function-based similarity measures”. The proposed similarity measures have ability to differentiate between equal and strictly included cases. Thus, it is necessary to discuss the MCDM method based on improved similarity measures. In the following, we propose two algorithmic procedures for a well known TOPSIS method. The TOPSIS method gives a solution which is closest to the positive ideal solution and farthest to the negative ideal solution. The Algorithm 1 deals with fixed positive and negative ideal solutions, while Algorithm 2 based on relative positive and negative ideal solutions. The relative ideal solutions are based on the assessments of the alternatives against criteria and it vary as the preferences of the decision makers changes.

Let the set of alternatives and attributes (criteria) are represented by $Y = \{y_1, y_2, \dots, y_m\}$ and $C = \{c_1, \dots, c_n\}$, respectively.

Numerical example

The data have very much importance for any organization and to secure the data is a key task for them. The organization have different business activities and to access of an organization system or data, the cyber attackers modified their attacking techniques every time. Thus, this is impossible for any organization to be completely sure that they

Algorithm 1 qROF-TOPSIS method with fixed ideals

T1. At the first stage, the experts formulate the q-rung orthopair fuzzy decision matrix $M = [r_{ij}]_{m \times n} = (\xi_{ij}, v_{ij})$ ($1 \leq i \leq m, 1 \leq j \leq n$), where r_{ij} represents the evaluation of i^{th} alternative against j^{th} criterion. To ease the calculations, the cost criteria are converted into benefit criteria and thus normalized q-rung orthopair fuzzy decision matrix is obtained as

$$r_{ij} = \begin{cases} (\xi_{ij}, v_{ij}) & \text{if } c_j \in C_b \\ (v_{ij}, \xi_{ij}) & \text{if } c_j \in C_c \end{cases}$$

T2. Since we know that the qROFV $(1, 0)$ is the best available solution and $(0, 1)$ is the worst avoidable solution. Thus the qROFVs $(1, 0)$ and $(0, 1)$ are used to define the fixed positive ideal solution (FPIS) and fixed negative ideal solution (FNIS), respectively.

$$y^+ = \{r_j^+ = (1, 0)\}, \quad 1 \leq j \leq n$$

$$y^- = \{r_j^- = (0, 1)\}, \quad 1 \leq j \leq n$$

T3. The similarity between each alternative with FPIS and FNIS are calculated and represented as $S(y^+, y_i)$ and $S(y^-, y_i)$, respectively. Then for each alternative, the closeness index χ_i is formulated as follows

$$\chi_i = \frac{S(y^+, y_i)}{S(y^+, y_i) + S(y^-, y_i)}, \quad 1 \leq i \leq m. \quad (24)$$

T4. The ranking of alternatives is obtained by the closeness index. The alternative with the maximum value of closeness index stated as best or suitable alternative.

are safe from cyber security threats or attacks. In the following, we use the qROF-TOPSIS method to resolve this issue. The issue is resolved upto extent that the organization know their most weaker point, where the cyber attackers can easily attack. We consider only four different techniques of cyber attackers, that is, computer viruses, spyware, denial-of-service (DoS) attacks and data breach, and represented as $C = \{c_1, c_2, c_3, c_4\}$. However, the method can be used for any number of cyber attacks techniques. If an organization have five different possible entrances, where attackers can attack, we can take as alternatives, and represented as $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Then, each alternative is assessed against each criterion and the information is secured as qROFSs, and display in Table 3. We solve this problem using proposed algorithms.

A. Algorithm 1 with weighted cosine function-based similarity measure: First we solve this problem using Algorithm 1 and weighted cosine function-based similarity measure defined in (20). Let the weight vector of the criteria is $\omega = \{0.22, 0.20, 0.33, 0.25\}$. Since the FPIS and FNIS are defined as $y^+ = \{r_j^+ = (1, 0)\}$ and $y^- = \{r_j^- = (0, 1)\}$, ($1 \leq j \leq 4$). The similarity between each alternative and FPIS is calculated and represented as $d_i^+ = W S_q(y_i, y^+)$, and the similarity between each alternative and FNIS is calculated and represented as $d_i^- = W S_q(y_i, y^-)$. The closeness index for each alternative is formulated using

Algorithm 2 qROF-TOPSIS method with relative ideals

T1. At the first stage, the experts formulate the q-rung orthopair fuzzy decision matrix $M = [r_{ij}]_{m \times n} = (\xi_{ij}, v_{ij})$ ($1 \leq i \leq m, 1 \leq j \leq n$), where r_{ij} represents the evaluation of i^{th} alternative against j^{th} criterion. To ease the calculations, the cost criteria are converted into benefit criteria and thus normalized q-rung orthopair fuzzy decision matrix is obtained as

$$r_{ij} = \begin{cases} (\xi_{ij}, v_{ij}) & \text{if } c_j \in C_b \\ (v_{ij}, \xi_{ij}) & \text{if } c_j \in C_c \end{cases}$$

T2. The relative positive ideal solution (RPIS) and relative negative ideal solution (RNIS) are based on the evaluation data provided in the normalized q-rung orthopair fuzzy decision matrix and define as

$$y^+ = \{r_j^+ = (\max_{i=1}^m \xi_{ij}, \min_{i=1}^m v_{ij})\}, \quad 1 \leq j \leq n \quad (25)$$

$$y^- = \{r_j^- = (\min_{i=1}^m \xi_{ij}, \max_{i=1}^m v_{ij})\}, \quad 1 \leq j \leq n \quad (26)$$

T3. The similarity between each alternative with RPIS and RNIS are calculated and represented as $S(y^+, y_i)$ and $S(y^-, y_i)$, respectively. Then for each alternative, the closeness index χ_i is formulated as follows

$$\chi_i = \frac{S(y^+, y_i)}{S(y^+, y_i) + S(y^-, y_i)}, \quad 1 \leq i \leq m.$$

T4. The ranking of alternatives is obtained by the closeness index. The alternative with the maximum value of closeness index stated as best or suitable alternative.

Table 3 The q-rung orthopair fuzzy decision matrix

	c_1	c_2	c_3	c_4
y_1	(0.8, 0.1)	(0.3, 0.4)	(0.5, 0.3)	(0.2, 0.6)
y_2	(0.7, 0.2)	(0.6, 0.4)	(0.5, 0.2)	(0.3, 0.3)
y_3	(0.5, 0.5)	(0.6, 0.3)	(0.7, 0.1)	(0.8, 0.2)
y_4	(0.3, 0.5)	(0.5, 0.4)	(0.4, 0.4)	(0.6, 0.2)
y_5	(0.6, 0.2)	(0.7, 0.1)	(0.3, 0.5)	(0.5, 0.4)

(24). The summary of the results is presented in Table 4. The maximum closeness index value is correspond to alternative y_3 , and thus, the entrance y_3 is more sensitive for organization. The order of the alternatives using Algorithm 1 is $y_3 > y_2 > y_1 > y_5 > y_4$.

For different values of the parameter q ($q \in \{1, 2, \dots, 10\}$), the optimal alternative remain same. However, there is slight difference in the ranking of remaining alternatives. The closeness indexes and order of the alternatives are displayed in Table 5.

B. Algorithm 2 with weighted cosine function-based similarity measure: Now, we solve this problem by using Algorithm 2 and weighted cosine function-based similarity measure defined in (20). Let the weight vector of the criteria is $\omega = \{0.22, 0.20, 0.33, 0.25\}$. The RPIS and RNIS are calculated using (25) and (26) as follows:

Table 4 Similarity measure values and closeness indexes

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
d_i^+	0.903296	0.922302	0.927977	0.904749	0.908783
d_i^-	0.842578	0.853451	0.767745	0.885070	0.858689
χ_1	χ_2	χ_3	χ_4	χ_5	
	0.517389	0.519386	0.547246	0.505497	0.514171

$$y^+ = \{r_1^+ = (0.8, 0.1), r_2^+ = (0.7, 0.1),$$

$$r_3^+ = (0.7, 0.1), r_4^+ = (0.8, 0.2)\}$$

$$y^- = \{r_1^- = (0.3, 0.5), r_2^- = (0.3, 0.4),$$

$$r_3^- = (0.3, 0.5), r_4^- = (0.2, 0.6)\}$$

The similarity between each alternative and RPIS is calculated and represented as $rd_i^+ = WS_q(y_i, y^+)$, and the similarity between each alternative and RNIS is calculated and represented as $rd_i^- = WS_q(y_i, y^-)$. The closeness index for each alternative is formulated using (24). The summary of the results is presented in Table 6. The maximum closeness index value is correspond to alternative y_3 , and thus, the entrance y_3 is more sensitive for organization. The order of the alternatives by using Algorithm 1 is $y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$.

For different values of the parameter q ($q \in \{1, 2, \dots, 10\}$), the optimal alternative remain same. However, there is slight difference in the ranking of remaining alternatives. The closeness indexes and order of the alternatives are displayed in Table 7.

Remark 3 We solve this problem by weighted cotangent function-based similarity measure defined in (23) and Algorithm 1. The weights of attributes and ideal solutions remain same. The optimal alternative by this setup remain same. The summary of the results for different values of the parameter q is displayed in Table 8.

Table 5 Sensitivity analysis for weighted cosine function-based similarity measure in Algorithm 1

	χ_1	χ_2	χ_3	χ_4	χ_5	Ranking
$q = 1$	0.5351	0.5785	0.6449	0.5236	0.5518	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 2$	0.5243	0.5374	0.5797	0.5109	0.5259	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 3$	0.5173	0.5193	0.5472	0.5054	0.5141	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 4$	0.5125	0.5107	0.5297	0.5029	0.5082	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 5$	0.5091	0.5063	0.5196	0.5016	0.5049	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 6$	0.5067	0.5038	0.5134	0.5009	0.5031	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 7$	0.5050	0.5024	0.5093	0.5005	0.5020	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 8$	0.5038	0.5015	0.5066	0.5003	0.5013	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 9$	0.5029	0.5009	0.5048	0.5002	0.5008	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 10$	0.5022	0.5007	0.5035	0.5001	0.5005	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$

Table 6 Similarity measure values and closeness indexes

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
rd_i^+	0.9605	0.9806	0.9855	0.9649	0.969
rd_i^-	0.9794	0.9806	0.9529	0.9853	0.982
χ_1	χ_2	χ_3	χ_4	χ_5	
	0.4951	0.5000	0.5084	0.4948	0.4967

In addition, the problem is solved using weighted cotangent function-based similarity measure in Algorithm 2. The alternative y_3 is the optimal alternative using WS_q^{cot} in Algorithm 2. The summary of the results for different values of the parameter q is displayed in Table 9.

Remark 4 To compare our proposed MCDM method, we solve the above-mentioned problem by existing MCDM methods. The considered existing methods result optimal alternative y_3 . The ranking of alternatives are slightly different with different methods. The summary of the comparison with existing methods displayed in Table 10.

Comparison analysis

In this section, we compare our propose similarity measures with existing ones. Our comparison method is somehow different from the literature, because the pattern recognition-based comparison provided in the literature. Our comparison analysis is based on the axioms of Definitions 6 and 10, and where the similarity measures are unable to find the similarity between qROFSs. Different authors defined the similarity measures for qROFSs. As the cosine and cotangent function-based similarity measures are already mentioned in “Review of existing cosine and cotangent function-based similarity measures for qROFSs”, so we mention here (Table 11) the similarity measures defined in [38–41].

Table 7 Sensitivity analysis for weighted cosine function-based similarity measure in Algorithm 2

	χ_1	χ_2	χ_3	χ_4	χ_5	Ranking
$q = 1$	0.4894	0.5078	0.5269	0.4875	0.4952	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 2$	0.4921	0.5039	0.5168	0.4929	0.4966	$y_3 \succ y_2 \succ y_5 \succ y_4 \succ y_1$
$q = 3$	0.4951	0.5000	0.5084	0.4948	0.4967	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 4$	0.4971	0.4986	0.5040	0.4960	0.4971	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 5$	0.4984	0.4985	0.5019	0.4971	0.4978	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 6$	0.4991	0.4987	0.5009	0.4980	0.4984	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 7$	0.4995	0.4991	0.5005	0.4986	0.4988	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 8$	0.4997	0.4993	0.5002	0.4991	0.4992	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 9$	0.4999	0.4995	0.5001	0.4994	0.4995	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 10$	0.4999	0.4997	0.5001	0.4996	0.4996	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$

Table 8 Sensitivity analysis for cotangent function-based similarity measure in Algorithm 1

	χ_1	χ_2	χ_3	χ_4	χ_5	Ranking
$q = 1$	0.55054	0.61550	0.70481	0.53622	0.579337	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 2$	0.53722	0.56737	0.63354	0.52031	0.54610	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 3$	0.52819	0.53922	0.58721	0.51146	0.52778	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 4$	0.52176	0.52349	0.55874	0.50666	0.51743	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 5$	0.51695	0.51445	0.54071	0.50395	0.51124	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 6$	0.51329	0.50910	0.52889	0.50237	0.50737	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 7$	0.51046	0.50584	0.52089	0.50143	0.50489	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 8$	0.50827	0.50380	0.51533	0.50086	0.50327	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 9$	0.50655	0.50251	0.51140	0.50052	0.50220	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 10$	0.50520	0.50168	0.50856	0.50031	0.50149	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$

Table 9 Sensitivity analysis for cotangent function-based similarity measure in Algorithm 2

	χ_1	χ_2	χ_3	χ_4	χ_5	Ranking
$q = 1$	0.46459	0.52294	0.58125	0.45188	0.48745	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 2$	0.47234	0.51480	0.56407	0.46054	0.48704	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 3$	0.47726	0.50056	0.54345	0.46487	0.48348	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 4$	0.48106	0.49092	0.52807	0.46763	0.48156	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$
$q = 5$	0.48471	0.48646	0.51791	0.47108	0.48161	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$
$q = 6$	0.48808	0.48539	0.51150	0.47508	0.48291	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 7$	0.49096	0.48612	0.50739	0.47910	0.48485	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 8$	0.49328	0.48763	0.50484	0.48281	0.48697	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 9$	0.49507	0.48939	0.50322	0.48604	0.48902	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$
$q = 10$	0.49641	0.49110	0.50217	0.48877	0.49088	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$

Table 10 Comparison with existing MCDM methods

Reference	Method	Ranking	Optimal alternative
Wang et al. [46]	MABAC method	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$	y_3
Peng et al. [47]	WDBA method	$y_3 \succ y_1 \succ y_5 \succ y_2 \succ y_4$	y_3
Peng and Dai [48]	CODAS method	$y_3 \succ y_1 \succ y_4 \succ y_2 \succ y_5$	y_3
Peng and Dai [48]	Similarity measure	$y_3 \succ y_2 \succ y_5 \succ y_4 \succ y_1$	y_3
Liu and Wang [11]	qROFWA operator	$y_3 \succ y_1 \succ y_2 \succ y_5 \succ y_4$	y_3
Liu and Wang [11]	qROFWG operator	$y_3 \succ y_2 \succ y_5 \succ y_4 \succ y_1$	y_3
Proposed	Algorithm 1 with WS_q	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$	y_3
Proposed	Algorithm 2 with WS_q	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$	y_3
Proposed	Algorithm 1 with WS_q^{cot}	$y_3 \succ y_2 \succ y_1 \succ y_5 \succ y_4$	y_3
Proposed	Algorithm 2 with WS_q^{cot}	$y_3 \succ y_2 \succ y_5 \succ y_1 \succ y_4$	y_3

Table 11 Existing similarity measures for qROFSs

Authors	Similarity measures
Du [40]	$S_d^1(R_1, R_2) = 1 - \left(\frac{1}{3} \sum_{i=1}^n (\xi_{R_1,j} - \xi_{R_2,j} + v_{R_1,j} - v_{R_2,j} + \eta_{R_1,j} - \eta_{R_2,j})\right)$
Du [40]	$S_d^2(R_1, R_2) = 1 - \left(\frac{1}{3} \sum_{i=1}^n (\xi_{R_1,j} - \xi_{R_2,j} ^2 + v_{R_1,j} - v_{R_2,j} ^2 + \eta_{R_1,j} - \eta_{R_2,j} ^2)\right)^{\frac{1}{2}}$
Du [40]	$S_d^3(R_1, R_2) = 1 - \frac{1}{3} \max (\xi_{R_1,j} - \xi_{R_2,j} , v_{R_1,j} - v_{R_2,j} , \eta_{R_1,j} - \eta_{R_2,j})$
Peng and Liu [39]	$S_p^1(R_1, R_2) = 1 - \frac{1}{2n} \sum_{j=1}^m (\xi_{R_1,j}^q - \xi_{R_2,j}^q + v_{R_1,j}^q - v_{R_2,j}^q + \eta_{R_1,j}^q - \eta_{R_2,j}^q)$
Peng and Liu [39]	$S_p^2(R_1, R_2) = 1 - \frac{1}{2n} \sum_{j=1}^m \xi_{R_1,j}^q - \xi_{R_2,j}^q - (v_{R_1,j}^q - v_{R_2,j}^q) $
Peng and Liu [39]	$S_p^3(R_1, R_2) = 1 - \frac{1}{4n} \left(\sum_{j=1}^m (\xi_{R_1,j}^q - \xi_{R_2,j}^q + v_{R_1,j}^q - v_{R_2,j}^q + \pi_{R_1,j}^q - \pi_{R_2,j}^q) \right. \\ \left. + \sum_{j=1}^m \xi_{R_1,j}^q - \xi_{R_2,j}^q - (v_{R_1,j}^q - v_{R_2,j}^q) \right)$
Peng and Liu [39]	$S_p^4(R_1, R_2) = 1 - \frac{1}{n} \sum_{j=1}^m (\xi_{R_1,j}^q - \xi_{R_2,j}^q \vee v_{R_1,j}^q - v_{R_2,j}^q)$
Peng and Liu [39]	$S_p^5(R_1, R_2) = \frac{1}{n} \sum_{j=1}^m \frac{1 - \xi_{R_1,j}^q - \xi_{R_2,j}^q \vee v_{R_1,j}^q - v_{R_2,j}^q }{1 + \xi_{R_1,j}^q - \xi_{R_2,j}^q \vee v_{R_1,j}^q - v_{R_2,j}^q }$
Peng and Liu [39]	$S_p^6(R_1, R_2) = \frac{\sum_{j=1}^m (1 - \xi_{R_1,j}^q - \xi_{R_2,j}^q \vee v_{R_1,j}^q - v_{R_2,j}^q)}{\sum_{j=1}^m (1 + \xi_{R_1,j}^q - \xi_{R_2,j}^q \vee v_{R_1,j}^q - v_{R_2,j}^q)}$
Peng and Liu [39]	$S_p^7(R_1, R_2) = L_1 \frac{\sum_{j=1}^m (\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q)}{\sum_{j=1}^m (\xi_{R_1,j}^q \vee \xi_{R_2,j}^q)} + L_2 \frac{\sum_{j=1}^m (v_{R_1,j}^q \wedge v_{R_2,j}^q)}{\sum_{j=1}^m (v_{R_1,j}^q \vee v_{R_2,j}^q)}, \quad L_1 + L_2 = 1, \quad L_1, L_2 \in [0, 1]$
Peng and Liu [39]	$S_p^8(R_1, R_2) = \frac{L_1}{n} \sum_{j=1}^m \frac{(\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q)}{(\xi_{R_1,j}^q \vee \xi_{R_2,j}^q)} + \frac{L_2}{n} \sum_{j=1}^m \frac{(v_{R_1,j}^q \wedge v_{R_2,j}^q)}{(v_{R_1,j}^q \vee v_{R_2,j}^q)},$
Peng and Liu [39]	$S_p^9(R_1, R_2) = \frac{1}{n} \sum_{j=1}^m \frac{(\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q) + (v_{R_1,j}^q \wedge v_{R_2,j}^q)}{(\xi_{R_1,j}^q \vee \xi_{R_2,j}^q) + (v_{R_1,j}^q \vee v_{R_2,j}^q)}$
Peng and Liu [39]	$S_p^{10}(R_1, R_2) = \frac{\sum_{j=1}^m ((\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q) + (v_{R_1,j}^q \wedge v_{R_2,j}^q))}{\sum_{j=1}^m ((\xi_{R_1,j}^q \vee \xi_{R_2,j}^q) + (v_{R_1,j}^q \vee v_{R_2,j}^q))}$
Peng and Liu [39]	$S_p^{11}(R_1, R_2) = \frac{1}{n} \sum_{j=1}^m \frac{(\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q) + (1 - v_{R_1,j}^q \wedge 1 - v_{R_2,j}^q)}{(\xi_{R_1,j}^q \vee \xi_{R_2,j}^q) + (1 - v_{R_1,j}^q \vee 1 - v_{R_2,j}^q)}$
Peng and Liu [39]	$S_p^{12}(R_1, R_2) = \frac{\sum_{j=1}^m ((\xi_{R_1,j}^q \wedge \xi_{R_2,j}^q) + (1 - v_{R_1,j}^q \wedge 1 - v_{R_2,j}^q))}{\sum_{j=1}^m ((\xi_{R_1,j}^q \vee \xi_{R_2,j}^q) + (1 - v_{R_1,j}^q \vee 1 - v_{R_2,j}^q))}$
Peng and Liu [39]	$S_p^{13}(R_1, R_2) = 1 - \left(\frac{1}{2n(1+t_1)^p} \sum_{j=1}^n \left(t_1 (\xi_{R_1,j}^q - \xi_{R_2,j}^q) - (v_{R_1,j}^q - v_{R_2,j}^q) ^p \right) \right. \\ \left. + \frac{1}{2n(1+t_2)^p} \sum_{j=1}^n \left(t_2 (v_{R_1,j}^q - v_{R_2,j}^q) - (\xi_{R_1,j}^q - \xi_{R_2,j}^q) ^p \right) \right)^{\frac{1}{p}}$
Pinar and Boran [41]	$S_{pb}(R_1, R_2) = - \left(\frac{1}{2n} \sum_{j=1}^n \left(\left (1 - \kappa) (\xi_{R_1,j} - \xi_{R_2,j}) - \kappa (\sqrt[q]{1 - v_{R_1,j}^q} - \sqrt[q]{1 - v_{R_2,j}^q}) \right ^p \right. \right. \\ \left. \left. + \left (1 - \kappa) (v_{R_1,j} - v_{R_2,j}) - \kappa (\sqrt[q]{1 - \xi_{R_1,j}^q} - \sqrt[q]{1 - \xi_{R_2,j}^q}) \right ^p \right) \right)^{\frac{1}{p}} + 1$ where $\kappa = \frac{(\frac{q^2}{2} + \frac{3q}{2} - \frac{1}{3})}{(q^2 + 3q + 1)} \in [\frac{1}{3}, \frac{1}{2}]$.
Peng and Dai [38]	$S_{mp}(R_1, R_2) = 1 - \left(\frac{1}{2nt^p} \sum_{j=1}^n \left(\left (t - 1) (\xi_{R_1,j}^q - \xi_{R_2,j}^q) - (v_{R_1,j}^q - v_{R_2,j}^q) \right ^p \right. \right. \\ \left. \left. + \left (t - \kappa) (v_{R_1,j}^q - v_{R_2,j}^q) - \kappa (\xi_{R_1,j}^q - \xi_{R_2,j}^q) \right ^p \right) \right)^{\frac{1}{p}}$ where p, κ and t represents the L_p norm, slop and level of hesitancy related to κ , respectively, with the constraints $t \geq \kappa + 1$ and $\kappa \geq 0$

Table 12 Comparison analysis of similarity measures for qROFSs ($q = 3$)

Similarity measure	Case I		Case II		Case III		S4	MS4
	Similarity between R_1 & R_2	R_1 & R_3	Similarity between R_1 & R_2	R_1 & R_3	Similarity between R_1 & R_2	R_1 & R_3		
S_d^1	0.8002	0.8156	0.7439	0.7476	0.6890	0.7004	Not valid	Not valid
S_d^2	0.9535	0.9655	0.9108	0.9349	0.8964	0.8984	Not valid	Not valid
S_d^3	0.9002	0.9333	0.8606	0.9000	0.8667	0.8667	Not valid	Not valid
S_p^1	0.6140	0.6140	0.4870	0.4870	0.3960	0.3960	Valid	Not valid
S_p^4	0.6140	0.6140	0.4870	0.4870	0.3960	0.3960	Valid	Not valid
S_p^5	0.4430	0.4430	0.3219	0.3219	0.2469	0.2469	Valid	Not valid
S_c^6	0.7751	0.7776	0.5199	0.5971	0.4544	0.4195	Not valid	Not valid
S_c^7	0.8217	0.8217	0.6925	0.6925	0.5827	0.5827	Valid	Not valid
S_c^8	0.8217	0.8217	0.6925	0.6925	0.5827	0.5827	Valid	Not valid
S_c^9	0.5235	0.5235	0.4023	0.4023	0.3214	0.3214	Valid	Not valid
S_c^{10}	0.5235	0.5235	0.4023	0.4023	0.3214	0.3214	Valid	Not valid
S_{pb}	0.9342	0.9618	0.8747	0.9313	0.9058	0.9080	Not valid	Not valid

Table 13 Comparison analysis of similarity measures

Similarity measure	Case IV $S(R_1, R_2)$	Case V $S(R_1, R_2)$	Case VI $S(R_1, R_2)$	Results
S_c^1	1	Undefined	Not valid	(S2) violates and issues in Cases V & VI
S_{mp}	1	Valid	Valid	(S2) violates
S_p^1	Valid	Valid	Not valid	Case VI as counter case
S_p^2	1	Valid	Valid	(S2) violates
S_p^4	Valid	Valid	Not valid	Case VI as counter case
S_p^5	Valid	Valid	Not valid	Case VI as counter case
S_p^6	Valid	Valid	Not valid	Case VI as counter case
S_c^7	Valid	0	Not valid	Cases V & VI as counter cases
S_c^8	Valid	0	Not valid	Cases V & VI as counter cases
S_c^9	Valid	0	Not valid	Cases V & VI as counter cases
S_c^{10}	Valid	0	Not valid	Cases V & VI as counter cases
S_c^{11}	Valid	Valid	Not valid	Case VI as counter case
S_c^{12}	Valid	Valid	Not valid	Case VI as counter case
S_p^{13}	1	Valid	Valid	(S2) violates

- Case I: The orthopairs $R_1 = (0.7, 0.7)$, $R_2 = (0.9, 0.6)$ and $R_3 = (0.9, 0.5)$ are qROFVs for $q = 3$ and $R_1 \sqsubset^t R_2 \sqsubset^t R_3$ with $\nu_{R_1} = 0.7 > \nu_{R_2} = 0.6 > \nu_{R_3} = 0.5$.
- Case II: The orthopairs $R_1 = (0.6, 0.65)$, $R_2 = (0.9, 0.6)$ and $R_3 = (0.9, 0.4)$ are qROFVs for $q = 3$ and $R_1 \sqsubset^t R_2 \sqsubset^t R_3$ with $\nu_{R_1} = 0.65 > \nu_{R_2} = 0.6 > \nu_{R_3} = 0.4$.
- Case III: The orthopairs $R_1 = (0.5, 0.8)$, $R_2 = (0.9, 0.6)$ and $R_3 = (0.9, 0.45)$ are qROFVs for $q = 3$ and $R_1 \sqsubset^t R_2 \sqsubset^t R_3$ with $\nu_{R_1} = 0.8 > \nu_{R_2} = 0.6 > \nu_{R_3} = 0.45$. The details of similarity measures in Case I to Case III are shown in Table 12. The table reveals that the similarity measures S_d^1 , S_d^2 , S_d^3 , S_p^6 and S_{pb} violates the axiom

- (S4), and thus, they are not the similarity measures. In addition, the similarity measures S_d^1 , S_d^2 , S_d^3 , S_p^1 , S_p^4 , S_p^5 , S_p^6 , S_p^7 , S_p^8 , S_p^9 , S_p^{10} and S_{pb} violates the axiom (MS4), and thus, they are not the improved similarity measures.
- Case IV: If $R_1 = (a, a)$ and $R_2 = (b, b)$ ($a \neq b$) be two qROFSs having same membership and non-membership degrees. Then, (a, a) and (b, b) are two distinct qROFSs. Some existing similarity measures violate condition (S2) of Definition 6 and thus not reliable. The similarity measures S_c^1 and S_p^2 formulate the maximum similarity between R_1 and R_2 , that is, one. In addition, the multiparametric similarity measure for $t = 2$ and $k = 1$

work out the maximum similarity between them that is $S_{mp}(R_1, R_2) = 1$. For fix value of $t_1 = t_2 = 1$, the similarity measure S_p^{13} will leads to the same results. Thus, the second axiom of similarity definition is not satisfies for S_c^1, S_p^2, S_{mp} and S_p^{13} . The details could be seen in Table 13. The similarity measure S_p^3 is just the addition of S_p^1 and S_p^2 . Both components of S_p^3 have counter intuitive cases in Case I–Case IV. Thus, the resultant similarity measure is not reliable.

- Case V: If $R_1 = (0, 0)$ and $R_2 = (x, y)$ be two qROFSs. Then, the similarity measures $S_c^1, S_p^7, S_p^8, S_p^9$ and S_p^{10} are not able to find similarity between R_1 and R_2 . The details could be seen in Table 13.
- Case VI: If $R_1 = (0, 1)$ and $R_2 = (x, f_1)$ be two qROFSs, where f_1 is any fix value available for orthopairs. Then, the similarity measures $S_p^1, S_p^4, S_p^5, S_p^6, S_p^7, S_p^8, S_p^{11}$ and S_p^{12} are not able to find similarity between R_1 and R_2 . That is, $S((0, 0), (x, f_1)) = f_2$, where f_2 is any fix value obtain for similarity measures even we change the value of x . The details could be seen in Table 13. It is important to note that the similarity measures having this kind problems do not satisfy the condition (MS4) and thus not the improved similarity measures.

Conclusion

The reason to write this article is to point out the shortcomings in the existing cosine and cotangent functions, and maximum operator-based similarity measures. We have criticized the existing similarity measures and observed that they have failed to provide the intuitive results for strict inclusions and thus not reliable for practical applications (Examples 1 and 2, and “Comparision analysis”). We have defined an improved cosine and cotangent function-based similarity measures for qROFSs and discussed their properties. The fixed and relative ideal solution-based algorithmic procedures of the TOPSIS method have provided to solve MCDM problems. A numerical example to save organizational data from cyber attackers have discussed. In the future, we will apply these similarity measures to different real life applications and analyze other similarity measure-based MCDM methods. We will also formulate this kind of similarity measures for spherical fuzzy sets [49–51], T-spherical fuzzy sets [52], picture fuzzy sets [53–55], neutrosophic sets [56] and linear Diophantine sets [57].

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Declarations

Conflict of interest The authors declare no conflict of interest.

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References

1. Zadeh LA (1965) Fuzzy sets. *Inf Control* 8:338–353
2. Atanassov K (1999) Intuitionistic fuzzy sets: theory and applications. Springer, Heidelberg
3. Yager RR (2013) Pythagorean fuzzy subsets. In: *Proc Joint IFSA world congress and NAFIPS, annual meeting*, Edmonton, Canada June 24–28, pp 57–61. <https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375>
4. Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making: pythagorean membership grades and fuzzy subsets. *Int J Intell Syst* 28:436–452
5. Yager RR (2017) Generalized orthopair fuzzy sets. *IEEE Trans Fuzzy Syst* 25(5):1222–1230. <https://doi.org/10.1109/TFUZZ.2016.2604005>
6. Yager RR, Alajlan N (2017) Approximate reasoning with generalized orthopair fuzzy sets. *Inf Fusion* 38:65–73. <https://doi.org/10.1016/j.inffus.2017.02.005>
7. Ali MI (2018) Another view on q-rung orthopair fuzzy sets. *Int J Intell Syst*. <https://doi.org/10.1002/int.22007>
8. Hussain A, Ali MI, Mahmood T, Munir M (2020) q-rung orthopair fuzzy soft average aggregation operators and their application in multicriteria decision-making. *Int J Intell Syst* 35(4):571–599. <https://doi.org/10.1002/int.22217>
9. Liu P, Wang Y (2020) Multiple attribute decision making based on q-rung orthopair fuzzy generalized Maclaurin symmetric mean operators. *Inf Sci* 518:181–210
10. Peng X, Dai J, Garg H (2018) Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function. *Int J Intell Syst* 33(1):2255–2282. <https://doi.org/10.1002/int.22028>
11. Liu P, Wang P (2018) Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *Int J Intell Syst* 33(2):259–280
12. Xing Y, Zhang R, Wang J, Bai K, Xue J (2019) A new multi-criteria group decision-making approach based on q-rung orthopair fuzzy

- interaction Hamy mean operators. *Neural Comput Appl.* <https://doi.org/10.1007/s00521-019-04269-8>
13. Zhong Y, Gao H, Guo X, Qin Y, Huang M, Luo X (2019) Dombi power partitioned Heronian mean operators of q-rung orthopair fuzzy numbers for multiple attribute group decision making. *PLoS One.* <https://doi.org/10.1371/journal.pone.0222007>
 14. Liu P, Chen S, Wang P (2018) Multiple-attribute group decision-making based on q-rung orthopair fuzzy power Maclaurin symmetric mean operators. *Syst Man Cybern Syst IEEE Trans.* <https://doi.org/10.1109/TSMC.2018.2852948>
 15. Wei G, Gao H, Wei Y (2018) Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. *Int J Intell Syst* 33(7):1426–1458. <https://doi.org/10.1002/int.21985>
 16. Garg H, Chen SM (2020) Multiattribute group decision making based on neutrality aggregation operators of q-rung orthopair fuzzy sets. *Inf Sci* 517:427–447
 17. Riaz M, Pamucar D, Athar Farid HM, Hashmi MR (2020) q-Rung orthopair fuzzy prioritized aggregation operators and their application towards green supplier chain management. *Symmetry* 12:976
 18. Garg H, Gwak J, Mahmood T, Ali Z (2020) Power aggregation operators and VIKOR methods for complex q-rung orthopair fuzzy sets and their applications. *Mathematics* 8:538
 19. Yang Z, Garg H, Li J (2020) Investigation of multiple heterogeneous relationships using a q-rung orthopair fuzzy multi-criteria decision algorithm. *Neural Comput Appl.* <https://doi.org/10.1007/s00521-020-05003-5>
 20. Khan MJ, Kumam P, Shutaywi M (2021) Knowledge measure for the q-rung orthopair fuzzy sets. *Int J Intell Syst* 36(2):628–655. <https://doi.org/10.1002/int.22313>
 21. Khan MJ, Ali MI, Kumam P (2021) A new ranking technique for q-rung orthopair fuzzy values. *Int J Intell Syst* 36(1):558–592
 22. Tolga AC, Parlak IB, Castillo O (2020) Finite-interval-valued type-2 Gaussian fuzzy numbers applied to fuzzy TODIM in a healthcare problem. *Eng Appl Artif Intell* 87:103352
 23. Castillo O, Muhuri PK, Melin P, Pulkkinen P (2020) Emerging issues and applications of type-2 fuzzy sets and systems, *Engineering Applications of Artificial Intelligence*, Volume 90, April 2020, 103596. <https://doi.org/10.1016/j.engappai.2020.103596>
 24. Castillo O, Muhuri PK, Melin P, Pulkkinen P (2020) Emerging issues and applications of type-2 fuzzy sets and systems
 25. Liu Z, Wang X, Li L, Zhao X, Liu P (2021) Q-rung orthopair fuzzy multiple attribute group decision-making method based on normalized bidirectional projection model and generalized knowledge-based entropy measure. *J Ambient Intell Humaniz Comput* 12(2):2715–2730
 26. Tang G, Chiclana F, Liu P (2020) A decision-theoretic rough set model with q-rung orthopair fuzzy information and its application in stock investment evaluation. *Appl Soft Comput J.* <https://doi.org/10.1016/j.asoc.2020.106212>
 27. Sirbiladze G (2020) Associated probabilities' aggregations in interactive multiattribute decision making for q-rung orthopair fuzzy discrimination environment. *Int J Intell Syst* 35:335–372. <https://doi.org/10.1002/int.22206>
 28. Zhang C, Liao H, Luo L (2019) Additive consistency-based priority generating method of q-rung orthopair fuzzy preference relation. *Int J Intell Syst* 34:2151–2176. <https://doi.org/10.1002/int.22137>
 29. Riaz M, Farid HMA, Karaaslan F, Hashmi MR (2020) Some q-rung orthopair fuzzy hybrid aggregation operators and TOPSIS method for multi-attribute decision-making. *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 1, pp. 1227–1241, 2020
 30. Krishankumar R, Nimmagadda SS, Rani P, Mishra AR, Ravichandran KS, Gandomi AH (2020) Solving renewable energy source selection problems using a q-rung orthopair fuzzy based integrated decision-making approach. *J Clean Prod.* <https://doi.org/10.1016/j.jclepro.2020.123329>
 31. Liu D, Huang A (2020) Consensus reaching process for fuzzy behavioral TOPSIS method with probabilistic linguistic q-rung orthopair fuzzy set based on correlation measure. *Int J Intell Syst*, Volume35, Issue3, March 2020, Pages 494–528
 32. Zhu H, Zhao J, Li H (2021) q-ROF-SIR methods and their applications to multiple attribute decision making. *Int J Mach Learn Cybern*, <https://doi.org/10.1007/s13042-020-01267-4>
 33. Riaz M, Farid HMA, Karaaslan F, Hashmi MR (2020) Some q-rung orthopair fuzzy hybrid aggregation operators and TOPSIS method for multi-attribute decision-making. *J Intell Fuzzy Syst* 1–15 (**preprint**)
 34. Zhang Z, Chen S-M (2020) Group decision making with incomplete q-rung orthopair fuzzy preference relations. *Inf Sci.* <https://doi.org/10.1016/j.ins.2020.10.015>
 35. Liu D, Huang A (2020) Consensus reaching process for fuzzy behavioral TOPSIS method with probabilistic linguistic q-rung orthopair fuzzy set based on correlation measure. *Int J Intell Syst* 1–35
 36. Zhu H, Zhao J, Li H (2021) q-ROF-SIR methods and their applications to multiple attribute decision making. *Int J Mach Learn Cybern* 1–13
 37. Khan MJ, Kumam P, Kumam W (2021) Theoretical justifications for the empirically successful VIKOR approach to multi-criteria decision making. *Soft Comput.* <https://doi.org/10.1007/s00500-020-05548-6>
 38. Peng X, Dai J (2019) Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. *Int J Intell Syst* 34(7):1588–1630. <https://doi.org/10.1002/int.22109>
 39. Peng X, Liu L (2019) Information measures for q-rung orthopair fuzzy sets. *Int J Intell Syst* 34:1795–1834. <https://doi.org/10.1002/int.22115>
 40. Du WS (2018) Minkowski-type distance measures for generalized orthopair fuzzy sets. *Int J Intell Syst.* <https://doi.org/10.1002/int.21968>
 41. Pinar A, Boran F (2020) A q-rung orthopair fuzzy multi-criteria group decision making method for supplier selection based on a novel distance measure. *Int J Mach Learn Cybern.* <https://doi.org/10.1007/s13042-020-01070-1>
 42. Peng X, Dai J (2019) Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. *Int J Intell Syst.* Volume34, Issue7, July 2019, Pages 1588–1630
 43. Peng X, Dai J (2019) Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. *Int J Intell Syst.* Volume34, Issue7, July 2019, Pages 1588–1630
 44. Liu D, Chen X, Peng D (2019) Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *Int J Intell Syst* 34(7):1572–1587
 45. Chen SJ, Hwang CL (1992) *Fuzzy multiple attribute decision making: methods and applications*. Springer, Berlin
 46. Wang J, Wei G, Wei C, Wei Y (2020) MABAC method for multiple attribute group decision making under q-rung orthopair fuzzy environment. *Def Technol* 16(1):208–216
 47. Peng X, Ravichandran K, Krishankumar R (2019) Generalized orthopair fuzzy weighted distance-based approximation algorithm in emergency decision making. *Int J Intell Syst.* <https://doi.org/10.1002/int.22140>
 48. Peng X, Dai J (2019) Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. *Int J Intell Syst.* 1–43

49. Ashraf S, Abdullah S, Mahmood T, Ghani F, Mahmood T (2019) Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J Intell Fuzzy Syst* 36(3):2829–2844
50. Ashraf S, Abdullah S (2019) Spherical aggregation operators and their application in multi-attribute group decision-making. *J Int Fuzzy Syst* 34(3):493–523
51. Khan MJ, Kumam P, Deebani W, Kumam W, Shah Z (2020) Distance and similarity measures for spherical fuzzy sets and their applications in selecting mega projects. *Mathematics* 8(4):519. <https://doi.org/10.3390/math8040519>
52. Ullah K, Mahmood T, Jan N (2018) Similarity measures for T-spherical fuzzy sets with applications in pattern recognition. *Symmetry* 10:193
53. Cuong BC (2013) Picture fuzzy sets-first results. Part 1, seminar “neuro-fuzzy systems with applications institute of mathematics, Hanoi, Vietnam
54. Khan MJ, Kumam P, Ashraf S, Kumam W (2019) Generalized picture fuzzy soft sets and their application in decision support systems. *Symmetry* 11(3):415. <https://doi.org/10.3390/sym11030415>
55. Khan MJ, Kumam P, Liu P, Kumam W, Rehman H (2020) An adjustable weighted soft discernibility matrix based on generalized picture fuzzy soft set and its applications in decision making. *J Int Fuzzy Syst* 38(2):2103–2118. <https://doi.org/10.3233/JIFS-190812>
56. Smarandache F (1999) A unifying field in logics. In: *Neutrosophy: neutrosophic probability, set and logic*. American Research Press, Rehoboth
57. Riaz M, Hashmi MR (2019) Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *J Int Fuzzy Syst* 37(4):5417–5439

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