New Hermite–Hadamard and Jensen inequalities for log- s -convex fuzzy-interval-valued functions in the second sense

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Abstract

In this paper, our aim is to consider the new class of log-convex fuzzy-interval-valued function known as log- s -convex fuzzy-interval-valued functions (log- s -convex fuzzy-IVFs). By this concept, we have introduced Hermite–Hadamard inequalities (HH-inequalities) by means of fuzzy order relation. This fuzzy order relation is defined level-wise through Kulisch–Miranker order relation defined on interval space. Moreover, some new Hermite–Hadamard–Fejér inequalities (HH–Fejér-inequalities) and Jensen’s inequalities via log- s -convex fuzzy-IVFs are also established and verified with the support of useful examples. Some special cases are also discussed which can be viewed as applications of fuzzy-interval HH-inequalities. The concepts and approaches of this paper may be the starting point for further research in this area.

Keywords Fuzzy-interval-valued functions · Log- s -convex fuzzy-interval-valued function · Hermite–Hadamard inequality · Hermite–Hadamard–Fejér inequality · Jensen’s inequality

Mathematics Subject Classification 26A33 · 26A51 · 26D10

Introduction

In last few decades much attention has been given in studying and distinguishing diverse directions of classical idea of convexity. Recently, many extensions and generalizations of convex functions have been established. For more useful details, see [1–3] and the references are therein. In classical approach, a real valued function $\mathcal{F} : K \rightarrow \mathbb{R}$ is called convex if

$$\mathcal{F}(\tau x + (1 - \tau)y) \leq \tau \mathcal{F}(x) + (1 - \tau)\mathcal{F}(y), \quad (1)$$

for all $x, y \in K$, $\tau \in [0, 1]$.

The concept of convexity with integral problem is an interesting area for research. Therefore, many inequalities have been introduced as applications of convex functions. Among those, the Hermite–Hadamard inequality (HH-inequality) is an interesting outcome in convex analysis. The HH-inequality [4, 5] for convex function $\mathcal{F} : K \rightarrow \mathbb{R}^+$ on an interval $K = [u, \vartheta]$

$$\mathcal{F}\left(\frac{u + \vartheta}{2}\right) \leq \frac{1}{\vartheta - u} \int_u^\vartheta \mathcal{F}(x) dx \leq \frac{\mathcal{F}(u) + \mathcal{F}(\vartheta)}{2}, \quad (2)$$

for all $u, \vartheta \in K$.

It is well known that log-convex functions have serious importance in convex theory because using these functions we can derive more accurate inequalities as compared to convex functions. Recently, some of the authors discussed different classes of log-convex and log-nonconvex functions, for example h -convexity [6], s -logarithmically convexity [7], and log-preinvexity [8]. Specifically, an important special class of convex functions is the log-convex functions proposed by Pečarić et al. [9]. A step forward, Noor et al.

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[10] also presented the class of log- h -convex functions and constructed the following HH-inequality for log- h -convex functions:

Let $\mathcal{F} : K \rightarrow \mathbb{R}^+$ be a log- h -convex function on a h convex set K with $h : [0, 1] \subseteq K \rightarrow \mathbb{R}^+$ such that $h \neq 0$ and $u, \vartheta \in K$ with $u \leq \vartheta$. Then,

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} \leq \exp\left[\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{F}(x) dx\right] \leq [\mathcal{F}(u)\mathcal{F}(\vartheta)]^{\int_0^1 h(\tau) d\tau}. \quad (3)$$

If $\mathcal{T}$ is concave then, inequality (3) is reversed. If $h(\tau) = \tau^s$ with $s \in (0, 1)$ then, inequality (3) reduces for log- s -With the assistance of inequality (3), some classical inequalities can be obtained through special log-convex function. In addition, these inequalities have a very significant role for log-convex functions in both pure and applied mathematics.

Jensen's inequality [11] is one of these inequalities for convex functions, which can be stated as follows.

Let $\omega_j \in [0, 1]$, $u_j \in [u, \vartheta]$, ($j = 1, 2, 3, \dots, k, k \geq 2$) and $\mathcal{T}$ be a convex function then,

$$\mathcal{T}\left(\sum_{j=1}^k \omega_j x_j\right) \leq \left(\sum_{j=1}^k \omega_j \mathcal{T}(x_j)\right), \quad (4)$$

with $\sum_{j=1}^k \omega_j = 1$. If $\mathcal{T}$ is concave then, inequality (4) is reversed.

Moreover, with the help of this concept, some Jensen's and HH-inequalities are defined [12–16].

On the other hand, the concept of interval analysis was proposed and investigated by Moore [17] and Kulish and Miranker [18]. It is a discipline in which an uncertain variable is represented by an interval of real numbers. Inspired by the above literature, in 2018, Guo et al. [19] introduced log- h -convex interval-valued functions (in short, log- h -convex-IVF) and prove the HH-inequality for log- h -convex IVFs as follows:

Let $\mathcal{F} : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ be a log- h -convex-IVF given by $\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$ for all $x \in [u, \vartheta]$, where $\mathcal{F}_*(x)$ is a log- h -convex function and $\mathcal{F}^*(x)$ is a log- h -concave function. If $\mathcal{F}$ is Riemann integrable, then

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} \supseteq \exp\left[\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{F}(x) dx\right] \supseteq [\mathcal{F}(u)\mathcal{F}(\vartheta)]^{\int_0^1 h(\tau) d\tau}. \quad (5)$$

We urge readers to make further analysis on literature about applications and properties of generalized convex functions and HH-integral inequalities [20–25] and the references therein.

In [26], the enormous research work fuzzy set and system has been dedicated on development of different fields and it plays an important role in the study of a wide class problems arising in pure mathematics and applied sciences including

operation research, computer science, managements sciences, artificial intelligence, control engineering and decision sciences. Similarly, the notions of convexity and non-convexity play a vital role in optimization under fuzzy domain because during characterization of the optimality condition of convexity, we obtain fuzzy variational inequalities so variational inequality theory and fuzzy complementary problem theory established powerful mechanism of the mathematical problems and they have a friendly relationship. Many authors contributed to this fascinating and interesting field. Besides, Nanda and Kar [27], and Chang [28] discussed the idea of convex fuzzy mapping and find its optimality condition with the support of fuzzy variational inequality. Generalization and extension of fuzzy convexity play a vital and significant implementation in diverse directions. So let's note that, one of the most considered classes of nonconvex fuzzy mapping is preinvex fuzzy mapping. Noor [29] introduced this idea and proved some results that distinguish the fuzzy optimality condition of differentiable fuzzy preinvex mappings by fuzzy variational-like inequality. We refer to the readers for further analysis of literature on the applications and properties of variational-like inequalities and generalized convex fuzzy mappings [30–35] and the references therein. The fuzzy mappings are fuzzy-interval valued functions (fuzzy-IVFs, in short). There are some integrals to deal with fuzzy-IVFs, where the integrands are fuzzy-IVFs. For instance, Osuna-Gómez et al. [36], and Costa [37] constructed Jensen's integral inequality for fuzzy-IVF. Using the same approach Costa and Floures presented Minkowski and Beckenbach's inequalities, where the integrands are fuzzy-IVFs. Motivated by [19, 37–39] and especially by Costa and Roman-Flores [40] because they established the relation between elements of fuzzy-interval space and interval space, and introducing level-wise fuzzy order relation on fuzzy-interval space through Kulisch–Miranker order relation defined on interval space. Using this concept on fuzzy-interval space, we generalize integral inequalities (3), (4) and (5) by constructing fuzzy-interval integral inequality for convex fuzzy-IVF, where the integrands are convex fuzzy-IVF.

Inspired by the ongoing research work, we have considered the class of generalized log-convex fuzzy-IVFs is known as log- s -convex fuzzy-IVFs. Using this class, we obtain fuzzy-interval HH-inequalities and verify these inequalities with the support of examples. Moreover, some Jensen's inequalities are also introduced via log- s -convex fuzzy-IVFs.

Preliminaries

Let $\mathcal{K}_C$ be the collection of all closed and bounded intervals of $\mathbb{R}$ that is $\mathcal{K}_C = \{[\omega_*, \omega^*] : \omega_*, \omega^* \in \mathbb{R} \text{ and } \omega_* \leq \omega^*\}$. If $\omega_* \geq 0$, then $[\omega_*, \omega^*]$ is called positive interval. The set

of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\omega_*, \omega^*] : [\omega_*, \omega^*] \in \mathcal{K}_C \text{ and } \omega_* \geq 0\}$.

We now discuss some properties of intervals under the arithmetic operations addition, multiplication and scalar multiplication. If $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$[\mu_*, \mu^*] + [\omega_*, \omega^*] = [\mu_* + \omega_*, \mu^* + \omega^*],$$

$$[\mu_*, \mu^*] \times [\omega_*, \omega^*] = [\min\{\mu_*\omega_*, \mu^*\omega_*, \mu_*\omega^*, \mu^*\omega^*\}, \max\{\mu_*\omega_*, \mu^*\omega_*, \mu_*\omega^*, \mu^*\omega^*\}],$$

$$\rho \cdot [\mu_*, \mu^*] = \begin{cases} [\rho\mu_*, \rho\mu^*] & \text{if } \rho \geq 0, \\ [\rho\mu^*, \rho\mu_*] & \text{if } \rho < 0. \end{cases}$$

For $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathcal{K}_C$, the inclusion “ $\subseteq$ ” is defined by

$$[\mu_*, \mu^*] \subseteq [\omega_*, \omega^*], \text{ if and only if } \omega_* \leq \mu_*, \mu^* \leq \omega^*.$$

Remark 2.1 The relation “ $\leq_I$ ” defined on $\mathcal{K}_C$ by

$$[\mu_*, \mu^*] \leq_I [\omega_*, \omega^*] \text{ if and only if } \mu_* \leq \omega_*, \mu^* \leq \omega^*,$$

for all $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathcal{K}_C$, it is an order relation, see [17]. For given $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathcal{K}_C$, we say that $[\mu_*, \mu^*] \leq_I [\omega_*, \omega^*]$ if and only if $\mu_* \leq \omega_*, \mu^* \leq \omega^*$ or $\mu_* \leq \omega_*, \mu^* < \omega^*$.

The concept of Riemann integral for IVF first introduced by Moore [16] is defined as follow:

Theorem 2.2 [16] If $\mathcal{F} : [c, d] \subset \mathbb{R} \rightarrow \mathcal{K}_C$ is an IVF on such that $[\mathcal{F}_*, \mathcal{F}^*]$. Then $\mathcal{F}$ is Riemann integrable over $[c, d]$ if and only if, $\mathcal{F}_*$ and $\mathcal{F}^*$ both are Riemann integrable over $[c, d]$ such that

$$(\mathcal{IR}) \int_c^d \mathcal{F}(x)dx = \left[(R) \int_c^d \mathcal{F}_*(u)dx, (R) \int_c^d \mathcal{F}^*(u)dx \right].$$

The collection of all Riemann integrable real valued functions and Riemann integrable IVFs is denoted by $\mathcal{R}_{[c,d]}$ and $\mathcal{IR}_{[c,d]}$, respectively.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset set $\mathcal{A}$ of $\mathbb{R}$ is distinguished by a function $\varphi : \mathbb{R} \rightarrow [0, 1]$ called the membership function. In this study this depiction is approved. Moreover, the collection of all fuzzy subsets of $\mathbb{R}$ is denoted by $\mathbb{F}(\mathbb{R})$.

A real fuzzy-interval φ is a fuzzy set in $\mathbb{R}$ with the following properties:

- (1) φ is normal i.e. there exists $x \in \mathbb{R}$ such that $\varphi(x) = 1$;

- (2) φ is upper semi continuous i.e., for given $x \in \mathbb{R}$, for every $x \in \mathbb{R}$ there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\varphi(x) - \varphi(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|x - y| < \delta$.
- (3) φ is fuzzy convex i.e., $\varphi((1 - \tau)x + \tau y) \geq \min(\varphi(x), \varphi(y))$, $\forall x, y \in \mathbb{R}$ and $\tau \in [0, 1]$;
- (4) φ is compactly supported i.e., $cl\{x \in \mathbb{R} | \varphi(x) > 0\}$ is compact.

The collection of all real fuzzy-intervals is denoted by $\mathbb{F}_C(\mathbb{R})$.

Since $\mathbb{F}_C(\mathbb{R})$ denotes the set of all real fuzzy-intervals and let $\varphi \in \mathbb{F}_C(\mathbb{R})$ be real fuzzy-interval, if and only if, α -levels $[\varphi]^\alpha$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\varphi]^\alpha = \{x \in \mathbb{R} | \varphi(x) \geq \alpha\},$$

from these definitions, we have

$$[\varphi]^\alpha = [\varphi_*(\alpha), \varphi^*(\alpha)],$$

where

$$\varphi_*(\alpha) = \inf\{x \in \mathbb{R} | \varphi(x) \geq \alpha\}, \quad \varphi^*(\alpha) = \sup\{x \in \mathbb{R} | \varphi(x) \geq \alpha\}.$$

Thus a real fuzzy-interval φ can be identified by parameterized triples

$$\{(\varphi_*(\alpha), \varphi^*(\alpha), \alpha) : \alpha \in [0, 1]\}.$$

This leads the following characterization of a real fuzzy-interval in terms of the two end point functions $\varphi_*(\alpha)$ and $\varphi^*(\alpha)$.

Theorem 2.3 [31, 38] Suppose that $\varphi_*(\alpha) : [0, 1] \rightarrow \mathbb{R}$ and $\varphi^*(\alpha) : [0, 1] \rightarrow \mathbb{R}$ satisfy the following conditions:

- (1) $\varphi_*(\alpha)$ is a non-decreasing function.
- (2) $\varphi^*(\alpha)$ is a non-increasing function.
- (3) $\varphi_*(1) \leq \varphi^*(1)$.
- (4) $\varphi_*(\alpha)$ and $\varphi^*(\alpha)$ are bounded and left continuous on $(0, 1]$ and right continuous at $\alpha = 0$.

Moreover, if $\varphi : \mathbb{R} \rightarrow [0, 1]$ is a real fuzzy-interval given by $[\varphi_*(\alpha), \varphi^*(\alpha)]$, then function $\varphi_*(\alpha)$ and $\varphi^*(\alpha)$ find the conditions (1)–(4).

Proposition 2.4 [40] Let $\varphi, \phi \in \mathbb{F}_C(\mathbb{R})$. Then relation “ $\preceq$ ” given on $\mathbb{F}_C(\mathbb{R})$ by

$$\varphi \preceq \phi \text{ if and only if, } [\varphi]^\alpha \leq_I [\phi]^\alpha \text{ for all } \alpha \in [0, 1],$$

it is partial order relation.

We now discuss some properties of real fuzzy-intervals under addition, scalar multiplication, multiplication and division. If $\varphi, \phi \in \mathbb{F}_C(\mathbb{R})$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$[\varphi \tilde{+} \phi]^\alpha = [\varphi]^\alpha + [\phi]^\alpha, \quad (6)$$

$$[\varphi \tilde{\times} \phi]^\alpha = [\varphi]^\alpha \times [\phi]^\alpha, \quad (7)$$

$$[\rho \cdot \varphi]^\alpha = \rho \cdot [\varphi]^\alpha. \quad (8)$$

For $\psi \in \mathbb{F}_C(\mathbb{R})$ such that $\varphi = \phi \tilde{+} \psi$, then by this result we have existence of Hukuhara difference of φ and ϕ , and we say that ψ is the H-difference of φ and ϕ , and denoted by $\varphi \tilde{-} \phi$. If H-difference exists, then

$$\begin{aligned} (\psi)^*(\alpha) &= (\varphi \tilde{-} \phi)^*(\alpha) = \varphi^*(\alpha) - \phi^*(\alpha), \\ (\psi)_*(\alpha) &= (\varphi \tilde{-} \phi)_*(\alpha) = \varphi_*(\alpha) - \phi_*(\alpha). \end{aligned} \quad (9)$$

Remark 2.5 Obviously, $\mathbb{F}_C(\mathbb{R})$ is closed under addition and nonnegative scalar multiplication and above defined properties on $\mathbb{F}_C(\mathbb{R})$ are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number $\rho \in \mathbb{R}$,

$$[\rho \tilde{+} \varphi]^\alpha = \rho + [\varphi]^\alpha. \quad (10)$$

Definition 2.6 [3] A fuzzy-interval-valued map $\mathcal{F} : K \subset \mathbb{R} \rightarrow \mathbb{F}_C(\mathbb{R})$ is called fuzzy-IVF. For each $\alpha \in [0, 1]$, whose α -levels define the family of IVFs $\mathcal{F}_\alpha : K \subset \mathbb{R} \rightarrow \mathcal{K}_C$ are given by $\mathcal{F}_\alpha(x) = [\mathcal{F}_*(x, \alpha), \mathcal{F}^*(x, \alpha)]$ for all $x \in K$. Here, for each $\alpha \in [0, 1]$, the end point real functions $\mathcal{F}_*(\cdot, \alpha), \mathcal{F}^*(\cdot, \alpha) : K \rightarrow \mathbb{R}$ are called lower and upper functions of $\mathcal{F}$.

Remark 2.7 Let $\mathcal{F} : K \subset \mathbb{R} \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF. Then $\mathcal{F}(x)$ is said to be continuous at $x \in K$, if for each $\alpha \in [0, 1]$, both end point functions $\mathcal{F}_*(x, \alpha)$ and $\mathcal{F}^*(x, \alpha)$ are continuous at $x \in K$.

From above literature review, following results can be concluded, see [3, 16, 38, 40]:

Definition 2.8 Let $\mathcal{F} : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}_C(\mathbb{R})$ is called fuzzy-IVF. The fuzzy integral of $\mathcal{F}$ over $[c, d]$, denoted by $(FR) \int_c^d \mathcal{F}(x) dx$, it is defined level-wise by

$$\begin{aligned} \left[(FR) \int_c^d \mathcal{F}(x) dx \right]^\alpha &= (IR) \int_c^d \mathcal{F}_\alpha(x) dx \\ &= \left\{ \int_c^d \mathcal{F}(x, \alpha) dx : \mathcal{F}(x, \alpha) \in \mathcal{R}_{[c, d]} \right\}, \end{aligned} \quad (11)$$

for all $\alpha \in [0, 1]$, where $\mathcal{R}_{[c, d]}$ is the collection of end point functions of IVFs. $\mathcal{F}$ is (FR) -integrable over $[c, d]$ if $(FR) \int_c^d \mathcal{F}(x) dx \in \mathbb{F}_C(\mathbb{R})$. Note that, if both end point functions are Lebesgue-integrable, then $\mathcal{F}$ is fuzzy Aumann-integrable, see [3].

Theorem 2.9 Let $\mathcal{F} : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF, whose α -levels define the family of IVFs $\mathcal{F}_\alpha : [c, d] \subset \mathbb{R} \rightarrow \mathcal{K}_C$ are given by $\mathcal{F}_\alpha(x) = [\mathcal{F}_*(x, \alpha), \mathcal{F}^*(x, \alpha)]$ for all $x \in [c, d]$ and for all $\alpha \in [0, 1]$. Then $\mathcal{F}$ is (FR) -integrable over $[c, d]$ if and only if, $\mathcal{F}_*(x, \alpha)$ and $\mathcal{F}^*(x, \alpha)$ both are R -integrable over $[c, d]$. Moreover, if $\mathcal{F}$ is (FR) -integrable over $[c, d]$, then

$$\begin{aligned} \left[(FR) \int_c^d \mathcal{F}(x) dx \right]^\alpha &= \left[(R) \int_c^d \mathcal{F}_*(x, \alpha) dx, (R) \int_c^d \mathcal{F}^*(x, \alpha) dx \right] \\ &= (IR) \int_c^d \mathcal{F}_\alpha(x) dx, \end{aligned} \quad (12)$$

for all $\alpha \in [0, 1]$.

The family of all (FR) -integrable fuzzy-IVFs and R -integrable functions over $[c, d]$ are denoted by $\mathcal{FR}_{([c, d], \alpha)}$ and $\mathcal{R}_{([c, d], \alpha)}$, for all $\alpha \in [0, 1]$.

Definition 2.10 [7] A function $\mathcal{T} : K \rightarrow \mathbb{R}$ is said to be log- s -convex function in the second sense if

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)^{\tau^s} \mathcal{T}(y)^{(1-\tau)^s}, \quad \forall x, y \in K, \tau \in [0, 1], \quad (13)$$

where $\mathcal{T}(x) \geq 0, s \in [0, 1]$. If (13) is reversed then, $\mathcal{T}$ is called log- s -concave in the second sense.

Definition 2.11 [9] A function $\mathcal{T} : K \rightarrow \mathbb{R}$ is said to be log-convex function if

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)^\tau \mathcal{T}(y)^{1-\tau}, \quad \forall x, y \in K, \tau \in [0, 1], \quad (14)$$

where $\mathcal{T}(x) \geq 0$. If (14) is reversed then, $\mathcal{T}$ is called log-concave.

Definition 2.12 [10] A function $\mathcal{T} : K \rightarrow \mathbb{R}$ is said to be log- P -convex function if

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)\mathcal{T}(y), \quad \forall x, y \in K, \tau \in [0, 1], \quad (15)$$

where $\mathcal{T}(x) \geq 0$. If (15) is reversed then, $\mathcal{T}$ is called log- P -concave.

Now we define the new class, log- s -convex fuzzy-IVF in the second by means of fuzzy order relation.

Definition 2.13 Let K be a convex set. Then fuzzy-IVF $\mathcal{T} : K \rightarrow \mathbb{F}_C(\mathbb{R})$ is said to be:

- log- s -convex fuzzy-IVF in the second sense on K if

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)^{\tau^s} \tilde{\times} \mathcal{T}(y)^{(1-\tau)^s}, \quad (16)$$

for all $x, y \in K, \tau \in [0, 1]$, where $\mathcal{T}(x) \geq \tilde{0}, s \in [0, 1]$.

- log- s -concave fuzzy-IVF in the second sense on K if inequality (16) is reversed.
- Affine log- s -convex fuzzy-IVF in the second sense on K if

$$\mathcal{T}(\tau x + (1 - \tau)y) = \mathcal{T}(x)^{\tau^s} \tilde{\times} \mathcal{T}(y)^{(1-\tau)^s}, \quad (17)$$

for all $x, y \in K, \tau \in [0, 1]$, where $\mathcal{T}(x) \geq \tilde{0}, s \in [0, 1]$.

The following remark discusses the classical log-convex functions as a special cases of log- s -convex fuzzy-IVF in the second sense.

Remark 2.14

- If $s = 1$, then log- s -convex fuzzy-IVF in the second sense becomes log-convex fuzzy-IVF in the second sense, see [27], that is

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)^{\tau} \tilde{\times} \mathcal{T}(y)^{1-\tau}, \quad \forall x, y \in K, \tau \in [0, 1], \quad (18)$$

- If $s \equiv 0$, then log- s -convex fuzzy-IVF in the second sense becomes log- P -convex fuzzy-IVF, that is

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x) \tilde{\times} \mathcal{T}(y), \quad \forall x, y \in K, \tau \in [0, 1]. \quad (19)$$

Theorem 2.15 Let K be a convex set let $\mathcal{T} : K \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF with $\mathcal{T}(x) \geq \tilde{0}$, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : K \subset \mathbb{R} \rightarrow \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by

$$\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)], \quad (20)$$

for all $x \in K$ and for all $\alpha \in (0, 1]$. Then $\mathcal{T}$ is log- s -convex in the second sense on K , if and only if, for all $\alpha \in (0, 1]$, $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ both are log- s -convex in the second sense.

Proof Let $\mathcal{T}$ is log- s -convex fuzzy-IVF in the second sense on K . Then for all $x, y \in K$ and $\tau \in [0, 1]$, we have

$$\mathcal{T}(\tau x + (1 - \tau)y) \leq \mathcal{T}(x)^{\tau^s} \tilde{\times} \mathcal{T}(y)^{(1-\tau)^s}.$$

Therefore, from inequality (20) and Proposition 2.4, we have

$$\begin{aligned} & [\mathcal{T}_*(\tau x + (1 - \tau)y, \alpha), \mathcal{T}^*(\tau x + (1 - \tau)y, \alpha)] \\ & \leq_l [\mathcal{T}_*(x, \alpha)^{\tau^s}, \mathcal{T}^*(x, \alpha)^{\tau^s}] \times [\mathcal{T}_*(y, \alpha)^{(1-\tau)^s}, \mathcal{T}^*(y, \alpha)^{(1-\tau)^s}]. \end{aligned} \quad (21)$$

It follows that

$$\mathcal{T}_*(\tau x + (1 - \tau)y, \alpha) \leq \mathcal{T}_*(x, \alpha)^{\tau^s} \mathcal{T}_*(y, \alpha)^{(1-\tau)^s},$$

and

$$\mathcal{T}^*(\tau x + (1 - \tau)y, \alpha) \leq \mathcal{T}^*(x, \alpha)^{\tau^s} \mathcal{T}^*(y, \alpha)^{(1-\tau)^s},$$

for each $\alpha \in (0, 1]$. This shows that $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ both are log- s -convex functions in the second sense.

Conversely, suppose that $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ both are log- s -convex functions in the second sense. Then from definition and inequality (21), it follows that $\mathcal{T}(x)$ is log- s -convex fuzzy-IVF in the second sense.

Example 2.16 We consider the fuzzy-IVF $\mathcal{T} : [1, 4] \rightarrow \mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{T}(x)(\sigma) = \begin{cases} \frac{\sigma}{1} & \sigma \in [0, \frac{1}{x}] \\ \frac{\frac{1}{x} - \sigma}{\frac{1}{x} - \frac{1}{x}} & \sigma \in (\frac{1}{x}, 2e^{x^2}] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\alpha \in (0, 1]$, we have $\mathcal{T}_\alpha(x) = [\alpha \frac{1}{x}, (2 - \alpha) \frac{1}{x}]$.

Since end point functions $\mathcal{T}_*(x, \alpha)$, $\mathcal{T}^*(x, \alpha)$ are log- s -convex functions in the second sense for each $\alpha \in (0, 1]$ then, by Theorem 2.15 $\mathcal{T}(x)$ is log- s -convex fuzzy-IVF in the second sense.

Corollary 2.17 Let K be a convex set let $\mathcal{T} : K \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF with $\mathcal{T}(x) \geq \tilde{0}$, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : K \subset \mathbb{R} \rightarrow \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by

$$\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)], \quad (22)$$

for all $x \in K$ and for all $\alpha \in (0, 1]$. Then $\mathcal{T}$ is log-convex in the second sense on K , if and only if, for all $\alpha \in (0, 1]$, $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ both are log-convex in the second sense.

Corollary 2.18 Let K be a convex set let $\mathcal{T} : K \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF with $\mathcal{T}(x) \geq \tilde{0}$, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : K \subset \mathbb{R} \rightarrow \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by

$$\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)], \quad (23)$$

for all $x \in K$ and for all $\alpha \in (0, 1]$. Then $\mathcal{T}$ is log- P -convex on K , if and only if, for all $\alpha \in (0, 1]$, $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ both are log- P -convex.

Remark 2.19

- (i) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ then log- s -convex fuzzy-IVF in the second sense becomes log- s -convex function in the second sense, see [7].
- (ii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ and $s = 1$ then log- s -convex function becomes log-convex function, see [9].
- (iii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ and $s = 0$ then log- s -convex function reduces to the log- P -convex function, see [10].

Theorem 2.20 Let K be convex set and $\mathcal{T}: K \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: K \subset \mathbb{R} \rightarrow \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by

$$\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$$

for all $x \in K$ and for all $\alpha \in (0, 1]$. Then $\mathcal{T}$ is log- s -concave in the second sense on K , if and only if, for all $\alpha \in (0, 1]$, $\mathcal{T}_*(x, \alpha)$ and $\mathcal{T}^*(x, \alpha)$ are log- s -concave functions in the second sense.

Proof Proof is similar to the proof of Theorem 2.15.

Example 2.21 We consider the fuzzy IVFs $\mathcal{T}: [u, \vartheta] = [0, 8] \rightarrow \mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{T}(x)(\sigma) = \begin{cases} \frac{\sigma}{x}, & \sigma \in [0, x], \\ \frac{2x-\sigma}{x}, & \sigma \in (x, 2x], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\alpha \in (0, 1]$, we have

$$\mathcal{T}_\alpha(x) = [\alpha x, (2 - \alpha)x].$$

Since end point functions $\mathcal{T}_*(x, \alpha) = \alpha x$, $\mathcal{T}^*(x, \alpha) = (2 - \alpha)x$ are log- s -concave functions in the second sense, for each $\alpha \in (0, 1]$ then, by Theorem 2.20 $\mathcal{T}(x)$ is log- s -concave fuzzy-IVF in the second sense.

Main results

This section presents Hermite–Hadamard, Hermite–Hadamard–Fejér and Jensen’s inequalities for log- s -convex fuzzy-interval-valued functions in the second sense, and verify with the help of useful examples. First of all, we prove

Hermite–Hadamard inequalities for log- s -convex fuzzy-interval-valued functions.

Theorem 3.1 Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log- s -convex fuzzy-IVF in the second sense, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $\mathcal{T} \in \mathcal{FR}_{([u, \vartheta], \alpha)}$, then

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right)^{2^{s-1}} \leq \exp\left[\frac{1}{\vartheta-u}(\mathcal{FR}) \int_u^\vartheta \ln \mathcal{T}(x) dx\right] \leq [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)]^{\frac{1}{s+1}}. \quad (24)$$

Proof Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$, log- s -convex fuzzy-IVF in the second sense. Then, by hypothesis, we have

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right) \leq [\mathcal{T}(\tau u + (1-\tau)\vartheta)]^{\frac{1}{2^s}} \tilde{\times} [\mathcal{T}((1-\tau)u + \tau\vartheta)]^{\frac{1}{2^s}}.$$

Therefore, for every $\alpha \in (0, 1]$, we have

$$\begin{aligned} \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq [\mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)]^{\frac{1}{2^s}} \times [\mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha)]^{\frac{1}{2^s}}, \\ \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq [\mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha)]^{\frac{1}{2^s}} \times [\mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)]^{\frac{1}{2^s}}. \end{aligned} \quad (25)$$

Taking logarithms on both sides of (25), then we obtain

$$\begin{aligned} \frac{1}{2^s} \ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha), \\ \frac{1}{2^s} \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha). \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{2^s} \int_0^1 \ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) d\tau &\leq \int_0^1 \ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) d\tau \\ &\quad + \int_0^1 \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha) d\tau, \\ \frac{1}{2^s} \int_0^1 \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) d\tau &\leq \int_0^1 \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) d\tau \\ &\quad + \int_0^1 \ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha) d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2^{\frac{1}{s}} 2^s} \ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx, \\ \frac{1}{2^{\frac{1}{s}} 2^s} \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx, \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right)^{2^{s-1}} &\leq \exp\left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx\right), \\ \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right)^{2^{s-1}} &\leq \exp\left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx\right). \end{aligned}$$

That is

$$\left[\mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right)^{2^{s-1}}, \mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right)^{2^{s-1}} \right] \leq_I \left[\exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx \right), \exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx \right) \right].$$

Thus,

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right)^{2^{s-1}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right]. \quad (26)$$

In a similar way as above, we have

$$\exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right] \leq [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)]^{\frac{1}{s+1}}. \quad (27)$$

Combining (26) and (27), we have

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right)^{2^{s-1}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right] \leq [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)]^{\frac{1}{s+1}}.$$

the required result.

Now we discuss some special cases of Theorem 3.1.

Note that the demonstration of proof of Corollary 3.2 is similar to proof of Corollary 3.3. If $s = 1$ then, from Theorem 3.1, we obtain the following result.

Corollary 3.2 Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log-convex fuzzy-IVF in the second sense, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $\mathcal{T} \in \mathcal{FR}_{([u, \vartheta], \alpha)}$, then

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right) \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right] \leq [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)]^{\frac{1}{2}}. \quad (28)$$

If $s = 0$ then, from Theorem 3.1, we obtain the following result for log- P -convex fuzzy-IVF.

Corollary 3.3 Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log- P -convex fuzzy-IVF, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $\mathcal{T} \in \mathcal{FR}_{([u, \vartheta], \alpha)}$, then

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right)^{\frac{1}{2}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right] \leq \mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta).$$

Proof Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$, log- P -convex fuzzy-IVF. Then, by hypothesis, we have

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right) \leq \mathcal{T}(\tau u + (1-\tau)\vartheta) \tilde{\times} \mathcal{T}((1-\tau)u + \tau\vartheta).$$

Therefore, for every $\alpha \in (0, 1]$, we have

$$\begin{aligned} \mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) \times \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha), \\ \mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) \times \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha). \end{aligned} \quad (29)$$

Taking logarithms on both sides of (29), then we obtain

$$\begin{aligned} \ln \mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha), \\ \ln \mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha). \end{aligned}$$

Then,

$$\begin{aligned} \int_0^1 \ln \mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) d\tau &\leq \int_0^1 \ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) d\tau \\ &\quad + \int_0^1 \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha) d\tau, \\ \int_0^1 \ln \mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) d\tau &\leq \int_0^1 \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) d\tau \\ &\quad + \int_0^1 \ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha) d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2} \ln \mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx, \\ \frac{1}{2} \ln \mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) &\leq \frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx, \end{aligned}$$

which implies that

$$\begin{aligned} \left[\mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) \right]^{\frac{1}{2}} &\leq \exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx \right), \\ \left[\mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) \right]^{\frac{1}{2}} &\leq \exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx \right), \end{aligned}$$

that is

$$\begin{aligned} &\left[\left[\mathcal{T}_* \left(\frac{u+\vartheta}{2}, \alpha \right) \right]^{\frac{1}{2}}, \left[\mathcal{T}^* \left(\frac{u+\vartheta}{2}, \alpha \right) \right]^{\frac{1}{2}} \right] \\ &\leq_I \left[\exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}_*(x, \alpha) dx \right), \exp \left(\frac{1}{\vartheta-u} \int_u^\vartheta \ln \mathcal{T}^*(x, \alpha) dx \right) \right]. \end{aligned}$$

Thus,

$$\mathcal{T} \left(\frac{u+\vartheta}{2} \right)^{\frac{1}{2}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right]. \quad (30)$$

In a similar way as above, we have

$$\exp \left[\frac{1}{\vartheta-u} (FR) \int_u^\vartheta \ln \mathcal{T}(x) dx \right] \leq \mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta). \quad (31)$$

Combining (30) and (31), we have

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^{\vartheta} \ln \mathcal{T}(x) dx \right] \leq \mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta).$$

the required result.

Remark 3.4 If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$, then Theorem 3.1 reduces to the result for log-s-convex fuzzy-IVF in the second sense, see [10]:

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right)^{2^{s-1}} \leq \exp \left[\frac{1}{\vartheta-u} (R) \int_u^{\vartheta} \ln \mathcal{T}(x) dx \right] \leq [\mathcal{T}(u) \times \mathcal{T}(\vartheta)]^{\frac{1}{s+1}}.$$

If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ and $s = 1$, then Theorem 3.1 reduces to the result for log-convex function see [23]:

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right) \leq \exp \left[\frac{1}{\vartheta-u} (R) \int_u^{\vartheta} \ln \mathcal{T}(x) dx \right] \leq \sqrt{\mathcal{T}(u) \times \mathcal{T}(\vartheta)}.$$

If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ and $s = 0$ then Theorem 3.1 reduces to the result for log-P-convex function, see [10]:

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2}} \leq \exp \left[\frac{1}{\vartheta-u} (R) \int_u^{\vartheta} \ln \mathcal{T}(x) dx \right] \leq \mathcal{T}(u) \times \mathcal{T}(\vartheta).$$

Example 3.5 We consider the fuzzy-IVF $\mathcal{T}: [u, \vartheta] = [1, 4] \rightarrow \mathbb{F}_C(\mathbb{R})$ defined by, $\mathcal{T}_\alpha(x) = [\alpha e^{x^2}, (2-\alpha)e^{x^2}]$, then $\mathcal{T}(x)$ is log-s-convex fuzzy-IVF in the second sense. Since, $\mathcal{T}_*(x, \alpha) = \alpha e^{x^2}$ and $\mathcal{T}^*(x, \alpha) = (2-\alpha)e^{x^2}$ then, we have

$$\mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right)^{2^{s-1}} = \alpha e^{\left(\frac{5}{2}\right)^2} = \alpha e^{\frac{25}{4}},$$

$$\begin{aligned} & \exp \left(\frac{1}{\vartheta-u} \int_u^{\vartheta} \ln \mathcal{T}_*(x, \alpha) dx \right) \\ &= \exp \left(\frac{1}{3} \int_1^4 \ln (\alpha e^{x^2}) dx \right) = e^{\ln(\alpha)+7}, \end{aligned}$$

$$[\mathcal{T}_*(u) \times \mathcal{T}_*(\vartheta)]^{\frac{1}{s+1}} = [(\alpha e)(4\alpha e^{16})]^{\frac{1}{2}} = 2\alpha e^{\frac{17}{2}},$$

for all $\alpha \in (0, 1]$. That means

$$\alpha e^{\frac{25}{4}} \leq e^{\ln(\alpha)+7} \leq 2\alpha e^{\frac{17}{2}}.$$

Similarly, it can be easily show that

$$\begin{aligned} \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right)^{2^{s-1}} &\leq \exp \left[\frac{1}{\vartheta-u} \int_u^{\vartheta} \ln \mathcal{T}^*(x, \alpha) dx \right] \\ &\leq [\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)]^{\frac{1}{s+1}}. \end{aligned}$$

for all $\alpha \in (0, 1]$, such that

$$\mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right)^{2^{s-1}} = (2-\alpha)e^{\left(\frac{5}{2}\right)^2} = (2-\alpha)e^{\frac{25}{4}},$$

$$\begin{aligned} & \exp \left(\frac{1}{\vartheta-u} \int_u^{\vartheta} \ln \mathcal{T}^*(x, \alpha) dx \right) \\ &= \exp \left(\frac{1}{3} \int_1^4 \ln ((2-\alpha)e^{x^2}) dx \right) = e^{\ln(2-\alpha)+7}, \end{aligned}$$

$$[\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)]^{\frac{1}{s+1}} = [(2-\alpha)e \cdot 4(2-\alpha)e^{16}]^{\frac{1}{2}} = 2(2-\alpha)e^{\frac{17}{2}}.$$

From which, it follows that

$$(2-\alpha)e^{\frac{25}{4}} \leq e^{\ln(2-\alpha)+7} \leq 2(2-\alpha)e^{\frac{17}{2}},$$

that is

$$\begin{aligned} & \left[\alpha e^{\frac{25}{4}}, (2-\alpha)e^{\frac{25}{4}} \right] \leq_I [e^{\ln(\alpha)+7}, e^{\ln(2-\alpha)+7}] \\ & \leq_I \left[2\alpha e^{\frac{17}{2}}, 2(2-\alpha)e^{\frac{17}{2}} \right], \quad \text{for all } \alpha \in (0, 1]. \end{aligned}$$

Hence,

$$\mathcal{T}\left(\frac{u+\vartheta}{2}\right)^{2^{s-1}} \leq \exp \left[\frac{1}{\vartheta-u} (FR) \int_u^{\vartheta} \ln \mathcal{T}(x) dx \right] \leq [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)]^{\frac{1}{s+1}}.$$

As we know that HH-Fejér inequality is generalized HH-Fejér inequality and with the help of this inequality, we can derive some special inequalities like HH-inequality. Firstly, we obtain the second HH-Fejér inequality for log-s-convex fuzzy-IVF in the second sense.

Theorem 3.6 Let $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log-s-convex fuzzy-IVF in the second sense with $u < \vartheta$, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $\mathcal{T} \in \mathcal{FR}_{([u, \vartheta], \alpha)}^{\frac{u+\vartheta}{2}}$ and $\Omega: [u, \vartheta] \rightarrow \mathbb{R}, \Omega(x) \geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, then

$$\frac{1}{\vartheta-u} (FR) \int_u^{\vartheta} [\ln \mathcal{T}(x)] \Omega(x) dx \leq \ln [\mathcal{T}(u) \tilde{\times} \mathcal{T}(\vartheta)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau. \quad (32)$$

Proof Let $\mathcal{T}$ be a log-s-convex fuzzy-IVF in the second sense. Then, for each $\alpha \in (0, 1]$, we have

$$\begin{aligned} & [\ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) \\ & \leq (\tau^s \ln \mathcal{T}_*(u, \alpha) + (1-\tau)^s \ln \mathcal{T}_*(\vartheta, \alpha)) \Omega(\tau u + (1-\tau)\vartheta), \\ & [\ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) \\ & \leq (\tau^s \ln \mathcal{T}^*(u, \alpha) + (1-\tau)^s \ln \mathcal{T}^*(\vartheta, \alpha)) \Omega(\tau u + (1-\tau)\vartheta). \end{aligned} \quad (33)$$

And

$$\begin{aligned} & [\ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) \\ & \leq ((1-\tau)^s \ln \mathcal{T}_*(u, \alpha) + \tau^s \ln \mathcal{T}_*(\vartheta, \alpha)) \Omega((1-\tau)u + \tau\vartheta), \\ & [\ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) \\ & \leq ((1-\tau)^s \ln \mathcal{T}^*(u, \alpha) + \tau^s \ln \mathcal{T}^*(\vartheta, \alpha)) \Omega((1-\tau)u + \tau\vartheta). \end{aligned} \quad (34)$$

After adding (33) and (34), and integrating over $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 [\ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau + \int_0^1 \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha) \Omega((1-\tau)u + \tau\vartheta) d\tau \\ & \leq \int_0^1 \left[\ln \mathcal{T}_*(u, \alpha) \{ \tau^s \Omega(\tau u + (1-\tau)\vartheta) + (1-\tau)^s \Omega((1-\tau)u + \tau\vartheta) \} \right. \\ & \quad \left. + \ln \mathcal{T}_*(\vartheta, \alpha) \{ (1-\tau)^s \Omega(\tau u + (1-\tau)\vartheta) + \tau^s \Omega((1-\tau)u + \tau\vartheta) \} \right] d\tau, \\ & \int_0^1 [\ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau + \int_0^1 \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ & \leq \int_0^1 \left[\ln \mathcal{T}^*(u, \alpha) \{ \tau^s \Omega(\tau u + (1-\tau)\vartheta) + (1-\tau)^s \Omega((1-\tau)u + \tau\vartheta) \} \right. \\ & \quad \left. + \ln \mathcal{T}^*(\vartheta, \alpha) \{ (1-\tau)^s \Omega(\tau u + (1-\tau)\vartheta) + \tau^s \Omega((1-\tau)u + \tau\vartheta) \} \right] d\tau. \\ & = 2 \ln \mathcal{T}_*(u, \alpha) \int_0^1 \tau^s \Omega(\tau u + (1-\tau)\vartheta) d\tau + 2 \ln \mathcal{T}_*(\vartheta, \alpha) \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ & = 2 \ln \mathcal{T}^*(u, \alpha) \int_0^1 \tau^s \Omega(\tau u + (1-\tau)\vartheta) d\tau + 2 \ln \mathcal{T}^*(\vartheta, \alpha) \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau. \end{aligned}$$

Since Ω is symmetric, then

$$\begin{aligned} & = 2 \ln [\mathcal{T}_*(u, \alpha) \times \mathcal{T}_*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ & = 2 \ln [\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau. \end{aligned} \quad (35)$$

Since

$$\begin{aligned} & \int_0^1 [\ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ & = \int_0^1 [\ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau = \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx \\ & \int_0^1 [\ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau \\ & = \int_0^1 [\ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau = \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx. \end{aligned} \quad (36)$$

From (35) and (36), we have

$$\begin{aligned} & \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx \leq \ln [\mathcal{T}_*(u, \alpha) \times \mathcal{T}_*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ & \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx \leq \ln [\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau, \end{aligned}$$

that is

$$\begin{aligned} & \left[\frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx, \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx \right] \\ & \leq_l [\ln [\mathcal{T}_*(u, \alpha) \times \mathcal{T}_*(\vartheta, \alpha)], \ln [\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)]] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau, \end{aligned}$$

hence

$$\frac{1}{\vartheta - u} (FR) \int_u^\vartheta [\ln \mathcal{T}(x)] \Omega(x) dx \leq \ln [\mathcal{T}(u) \times \mathcal{T}(\vartheta)] \int_0^1 \tau^s \Omega((1-\tau)u + \tau\vartheta) d\tau.$$

This concludes the proof.

Next, we construct first HH–Fejér inequality for log- s -convex fuzzy-IVF in the second sense, which generalizes first HH–Fejér inequality for log- s -convex function in the second sense.

Theorem 3.7 Let $\mathcal{T} : [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log- s -convex fuzzy-IVF in the second sense with $u < \vartheta$, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $\mathcal{T} \in \mathcal{FR}_{([u, \vartheta], \alpha)}$ and $\Omega : [u, \vartheta] \rightarrow \mathbb{R}, \Omega(x) \geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, and $\int_u^\vartheta \Omega(x)dx > 0$, then

$$\ln \mathcal{T}\left(\frac{u+\vartheta}{2}\right) \leq \frac{2^{1-s}}{\int_u^\vartheta \Omega(x)dx} (FR) \int_u^\vartheta [\ln \mathcal{T}(x)] \Omega(x)dx. \quad (37)$$

Proof Since $\mathcal{T}$ is a log- s -convex in the second sense, then for $\alpha \in (0, 1]$ we have

$$\begin{aligned} 2^s \ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha), \\ 2^s \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha) + \ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha). \end{aligned} \quad (38)$$

By multiplying (38) by $\Omega((1-\tau)u + \tau\vartheta) = \Omega(\tau u + (1-\tau)\vartheta)$ and integrate it by τ over $[0, 1]$, we obtain

$$\begin{aligned} 2^s \left[\ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) \right] \int_0^1 \Omega((1-\tau)u + \tau\vartheta) d\tau \\ \leq \int_0^1 [\ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ + \int_0^1 [\ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ 2^s \left[\ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) \right] \int_0^1 \Omega((1-\tau)u + \tau\vartheta) d\tau \\ \leq \int_0^1 [\ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ + \int_0^1 [\ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau. \end{aligned} \quad (39)$$

Since

$$\begin{aligned} \int_0^1 [\ln \mathcal{T}_*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ = \int_0^1 [\ln \mathcal{T}_*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ = \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx, \\ \int_0^1 [\ln \mathcal{T}^*(\tau u + (1-\tau)\vartheta, \alpha)] \Omega(\tau u + (1-\tau)\vartheta) d\tau \\ = \int_0^1 [\ln \mathcal{T}^*((1-\tau)u + \tau\vartheta, \alpha)] \Omega((1-\tau)u + \tau\vartheta) d\tau, \\ = \frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx. \end{aligned} \quad (40)$$

From (39) and (40), we have

$$\begin{aligned} \ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \frac{2^{1-s}}{\int_u^\vartheta \Omega(x)dx} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx, \\ \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) &\leq \frac{2^{1-s}}{\int_u^\vartheta \Omega(x)dx} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx. \end{aligned}$$

From which, we have

$$\begin{aligned} \left[\ln \mathcal{T}_*\left(\frac{u+\vartheta}{2}, \alpha\right), \ln \mathcal{T}^*\left(\frac{u+\vartheta}{2}, \alpha\right) \right] \\ \leq \frac{2^{1-s}}{\int_u^\vartheta \Omega(x)dx} \left[\int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx, \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx \right], \end{aligned}$$

that is

$$\ln \mathcal{T}\left(\frac{u+\vartheta}{2}\right) \leq \frac{2^{1-s}}{\int_u^\vartheta \Omega(x)dx} (FR) \int_u^\vartheta [\ln \mathcal{T}(x)] \Omega(x) dx.$$

Then we complete the proof.

Remark 3.8

- (i) If $s = 1$ then inequalities in Theorems 3.6 and 3.7 reduces for log-convex fuzzy-IVFs.
- (ii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(u, \alpha)$ with $\alpha = 1$, then Theorems 3.6 and 3.7 reduces to classical first and second HH–Fejér inequality for log- s -convex function in the second sense, see [19].
- (iii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(u, \alpha)$ with $\alpha = 1$ and $s = 1$ then Theorems 3.6 and 3.7 reduces to classical first and second HH–Fejér inequality for log-convex function, see [19].

Example 3.9 We consider $s = 1$ for $\tau \in [0, 1]$ and the fuzzy IVFs $\mathcal{T} : [u, \vartheta] = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \rightarrow \mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{T}(x)(\sigma) = \begin{cases} \frac{\sigma}{e^{\sin x}}, & \sigma \in [0, e^{\sin x}], \\ \frac{2e^{\sin x} - \sigma}{e^{\sin x}}, & \sigma \in (e^{\sin x}, 2e^{\sin x}], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\alpha \in (0, 1]$, we have

$$\mathcal{T}_\alpha(x) = [\alpha e^{\sin x}, (2-\alpha)e^{\sin x}].$$

Since end point functions $\mathcal{T}_*(x, \alpha) = \alpha e^{\sin x}$, $\mathcal{T}^*(x, \alpha) = (2-\alpha)e^{\sin x}$ are log-1-convex functions in the second sense, for each $\alpha \in (0, 1]$ then, by Theorem 2.15, $\mathcal{T}(x)$ is log- s -convex fuzzy-IVF in the second sense. If

$$\Omega(x) = \begin{cases} x - \frac{\pi}{4}, & \sigma \in \left[\frac{\pi}{4}, \frac{3\pi}{8}\right], \\ \frac{\pi}{2} - x, & \sigma \in \left(\frac{3\pi}{8}, \frac{\pi}{2}\right], \end{cases}$$

then, we have

$$\begin{aligned}
\frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx \\
&\quad + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \ln \mathcal{T}_*(x, \alpha) \Omega(x) dx, \\
\frac{1}{\vartheta - u} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx \\
&\quad + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \ln \mathcal{T}^*(x, \alpha) \Omega(x) dx, \\
&= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\ln (\alpha e^{\sin x})] \left(x - \frac{\pi}{4}\right) dx + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} [\ln (\alpha e^{\sin x})] \left(\frac{\pi}{2} - x\right) dx \approx \frac{1}{25\pi} \left[\frac{31}{2} \ln(\alpha) + 14\right], \\
&= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\ln (2 - \alpha) e^{\sin x}] \left(x - \frac{\pi}{4}\right) dx + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} [\ln ((2 - \alpha) e^{\sin x})] \left(\frac{\pi}{2} - x\right) dx \approx \frac{1}{25\pi} \left[\frac{31}{2} \ln(2 - \alpha) + 14\right],
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&\ln [\mathcal{T}_*(u, \alpha) \times \mathcal{T}_*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1 - \tau)u + \tau\vartheta) d\tau \\
&\ln [\mathcal{T}^*(u, \alpha) \times \mathcal{T}^*(\vartheta, \alpha)] \int_0^1 \tau^s \Omega((1 - \tau)u + \tau\vartheta) d\tau \\
&= \left[2 \ln(\alpha) + \frac{2 + \sqrt{2}}{2}\right] \left[\int_0^{\frac{1}{2}} \tau^2 dx + \int_{\frac{1}{2}}^1 \tau(1 + \tau) d\tau\right] \approx \frac{17}{24\pi} \left[\frac{63}{10} \ln(\alpha) + \frac{2 + \sqrt{2}}{2}\right], \\
&= \left[2 \ln(2 - \alpha) + \frac{2 + \sqrt{2}}{2}\right] \left[\int_0^{\frac{1}{2}} \tau^2 dx + \int_{\frac{1}{2}}^1 \tau(1 + \tau) d\tau\right] \approx \frac{17}{24\pi} \left[\frac{63}{10} \ln(2 - \alpha) + \frac{2 + \sqrt{2}}{2}\right].
\end{aligned} \tag{42}$$

From (41) and (42), we have

$$\left[\frac{1}{25\pi} \left[\frac{31}{2} \ln(\alpha) + 14\right], \frac{1}{25\pi} \left[\frac{31}{2} \ln(2 - \alpha) + 14\right]\right] \leq {}_I \left[\frac{17}{24\pi} \left[\frac{63}{10} \ln(\alpha) + \frac{2 + \sqrt{2}}{2}\right], \frac{17}{24\pi} \left[\frac{63}{10} \ln(2 - \alpha) + \frac{2 + \sqrt{2}}{2}\right]\right],$$

for all $\alpha \in (0, 1]$. Hence, Theorem 3.6 is verified.

For Theorem 3.7, we have

$$\begin{aligned}
\ln \mathcal{T}_*\left(\frac{u + \vartheta}{2}, \alpha\right) &= \ln \mathcal{T}_*\left(\frac{3\pi}{8}, \alpha\right) \approx \ln\left(\frac{5}{2}\alpha\right), \\
\ln \mathcal{T}^*\left(\frac{u + \vartheta}{2}, \alpha\right) &= \ln \mathcal{T}^*\left(\frac{3\pi}{8}, \alpha\right) \approx \ln\left(\frac{5}{2}(2 - \alpha)\right),
\end{aligned} \tag{43}$$

$$\int_u^\vartheta \Omega(x) dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} \left(x - \frac{\pi}{4}\right) dx + \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) dx \approx \frac{3}{20},$$

$$\begin{aligned}
\frac{2^{1-s}}{\int_u^\vartheta \Omega(x) dx} \int_u^\vartheta [\ln \mathcal{T}_*(x, \alpha)] \Omega(x) dx &\approx \frac{2}{15} \left[\frac{31}{4} \ln(\alpha) + 7\right] \\
\frac{2^{1-s}}{\int_u^\vartheta \Omega(x) dx} \int_u^\vartheta [\ln \mathcal{T}^*(x, \alpha)] \Omega(x) dx &\approx \frac{2}{15} \left[\frac{31}{4} \ln(2 - \alpha) + 7\right]
\end{aligned} \tag{44}$$

From (43) and (44), we have

$$\begin{aligned}
&\left[\ln\left(\frac{5}{2}\alpha\right), \ln\left(\frac{5}{2}(2 - \alpha)\right)\right] \\
&\leq {}_I \left[\frac{2}{15} \left[\frac{31}{4} \ln(\alpha) + 7\right], \frac{2}{15} \left[\frac{31}{4} \ln(2 - \alpha) + 7\right]\right].
\end{aligned}$$

Hence, Theorem 3.7 is verified.

Now, we prove the Jensen's inequality for log-s-convex fuzzy-IVF in the second sense.

Theorem 3.10 Let $\omega_j \in \mathbb{R}^+$, $x_j \in [u, \vartheta]$, ($j = 1, 2, 3, \dots, k, k \geq 2$) and $\mathcal{T} : [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log-s-convex fuzzy-IVF in the second sense, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. Then,

$$\mathcal{T}\left(\frac{1}{W_k} \sum_{j=1}^k \omega_j x_j\right) \leq \prod_{j=1}^k [\mathcal{T}(x_j)]^{\left(\frac{\omega_j}{W_k}\right)^s}, \quad (45)$$

where $W_k = \sum_{j=1}^k \omega_j$. If $\mathcal{T}$ is log-s-concave in the second sense then, inequality (45) is reversed.

Proof When $k = 2$ inequality (45) is true. Consider inequality (16) is true for $k = n - 1$, then

$$\mathcal{T}\left(\frac{1}{W_{n-1}} \sum_{j=1}^{n-1} \omega_j x_j\right) \leq \prod_{j=1}^{n-1} [\mathcal{T}(x_j)]^{\left(\frac{\omega_j}{W_{n-1}}\right)^s},$$

Now, let us prove that inequality (45) holds for $k = n$:

$$\begin{aligned} & \mathcal{T}\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j\right) \\ &= \mathcal{T}\left(\frac{1}{W_{n-1}} \sum_{j=1}^{n-1} \omega_j x_j + \frac{\omega_n}{W_n} \left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n} x_{n-1} + \frac{\omega_n}{\omega_{n-1} + \omega_n} x_n\right)\right). \end{aligned}$$

Therefore, for every $\alpha \in (0, 1]$, we have

$$\begin{aligned} & \mathcal{T}_*\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j, \alpha\right) \\ & \mathcal{T}^*\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j, \alpha\right) \\ & \leq \mathcal{T}_*\left(\frac{1}{W_n} \sum_{j=1}^{n-1} \omega_j x_j + \frac{\omega_{n-1} + \omega_n}{W_n} \left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n} x_{n-1} + \frac{\omega_n}{\omega_{n-1} + \omega_n} x_n\right), \alpha\right), \\ & \leq \mathcal{T}^*\left(\frac{1}{W_n} \sum_{j=1}^{n-1} \omega_j x_j + \frac{\omega_{n-1} + \omega_n}{W_n} \left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n} x_{n-1} + \frac{\omega_n}{\omega_{n-1} + \omega_n} x_n\right), \alpha\right), \end{aligned}$$

$$\begin{aligned} & \leq \left[\prod_{j=1}^{n-1} [\mathcal{T}_*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} \mathcal{T}_*\left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n} x_{n-1} + \frac{\omega_n}{\omega_{n-1} + \omega_n} x_n, \alpha\right) \right]^{\left(\frac{\omega_{n-1} + \omega_n}{W_n}\right)^s}, \\ & \leq \prod_{j=1}^{n-1} [\mathcal{T}^*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} \left[\mathcal{T}^*\left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n} x_{n-1} + \frac{\omega_n}{\omega_{n-1} + \omega_n} x_n, \alpha\right) \right]^{\left(\frac{\omega_{n-1} + \omega_n}{W_n}\right)^s}, \end{aligned}$$

$$\begin{aligned} & \leq \prod_{j=1}^{n-1} [\mathcal{T}_*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} \left[[\mathcal{T}_*(x_{n-1}, \alpha)]^{\left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n}\right)^s} [\mathcal{T}_*(x_n, \alpha)]^{\left(\frac{\omega_n}{\omega_{n-1} + \omega_n}\right)^s} \right]^{\left(\frac{\omega_{n-1} + \omega_n}{W_n}\right)^s}, \\ & \leq \prod_{j=1}^{n-1} [\mathcal{T}^*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} \left[[\mathcal{T}^*(x_{n-1}, \alpha)]^{\left(\frac{\omega_{n-1}}{\omega_{n-1} + \omega_n}\right)^s} [\mathcal{T}^*(x_n, \alpha)]^{\left(\frac{\omega_n}{\omega_{n-1} + \omega_n}\right)^s} \right]^{\left(\frac{\omega_{n-1} + \omega_n}{W_n}\right)^s}, \end{aligned}$$

$$\begin{aligned} & \leq \prod_{j=1}^{n-1} [\mathcal{T}_*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} [\mathcal{T}_*(x_{n-1}, \alpha)]^{\left(\frac{\omega_{n-1}}{W_n}\right)^s} [\mathcal{T}_*(x_n, \alpha)]^{\left(\frac{\omega_n}{W_n}\right)^s}, \\ & \leq \prod_{j=1}^{n-1} [\mathcal{T}^*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} [\mathcal{T}^*(x_{n-1}, \alpha)]^{\left(\frac{\omega_{n-1}}{W_n}\right)^s} [\mathcal{T}^*(x_n, \alpha)]^{\left(\frac{\omega_n}{W_n}\right)^s}, \\ & = \prod_{j=1}^n [\mathcal{T}_*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s}, \\ & = \prod_{j=1}^n [\mathcal{T}^*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s}, \end{aligned}$$

From which, we have

$$\begin{aligned} & \left[\mathcal{T}_*\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j, \alpha\right), \mathcal{T}^*\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j, \alpha\right) \right] \\ & \leq_I \left[\prod_{j=1}^n [\mathcal{T}_*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s}, \prod_{j=1}^n [\mathcal{T}^*(x_j, \alpha)]^{\left(\frac{\omega_j}{W_n}\right)^s} \right], \end{aligned}$$

that is,

$$\mathcal{T}\left(\frac{1}{W_n} \sum_{j=1}^n \omega_j x_j\right) \leq \prod_{j=1}^n [\mathcal{T}(x_j)]^{\left(\frac{\omega_j}{W_n}\right)^s},$$

and the result follows.

If $\omega_1 = \omega_2 = \omega_3 = \dots = \omega_k = 1$, then Theorem 3.10 reduces to the following result:

Corollary 3.11 Let $x_j \in [u, \vartheta]$, ($j = 1, 2, 3, \dots, k, k \geq 2$) and $\mathcal{T} : [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log-s-convex fuzzy-IVF in the second sense, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. Then,

$$\mathcal{T}\left(\frac{1}{k} \sum_{j=1}^k x_j\right) \leq \prod_{j=1}^k [\mathcal{T}(x_j)]^{\left(\frac{1}{k}\right)^s}. \quad (46)$$

If $\mathcal{T}$ is a log-s-concave in the second sense then, inequality (46) is reversed.

To obtain a refinement of Jensen's inequality for log-s-convex-IVFs firstly, we prove the following the result:

Theorem 3.12 Let $s \in (0, 1)$ and $\mathcal{T} : [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF, whose α -levels define the family of IVFs $\mathcal{T}_\alpha : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$

for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $(L, U) \subseteq [u, \vartheta]$ and $\mathcal{T}$ be a log-s-convex fuzzy-IVF in the second sense then, for $x_1, x_2, x_3 \in [u, \vartheta]$, $x_1 < x_2 < x_3$ such that $x_3 - x_1, x_3 - x_2, x_2 - x_1 \in [0, 1]$, we have

$$(x_3 - x_1)^s \mathcal{T}(x_2) \leq \mathcal{T}(x_1)^{(x_3-x_2)^s} \mathcal{T}(x_3)^{(x_2-x_1)^s}. \quad (47)$$

Proof Let $x_1, x_2, x_3 \in [u, \vartheta]$ and $(x_3 - x_2)^s > 0$. Then by hypothesis, we have

$$\left(\frac{x_3 - x_2}{x_3 - x_1}\right)^s = \frac{(x_3 - x_2)^s}{(x_3 - x_1)^s} \quad \text{and} \quad \left(\frac{x_2 - x_1}{x_3 - x_1}\right)^s = \frac{(x_2 - x_1)^s}{(x_3 - x_1)^s}.$$

Consider $\lambda = \frac{x_3 - x_2}{x_3 - x_1}$, then $x_2 = \lambda x_1 + (1 - \lambda)x_3$. Since $\mathcal{T}$ is a log-s-convex fuzzy-IVF then, by hypothesis, we have

$$\begin{aligned} \mathcal{T}_*(x_2, \alpha) &\leq [\mathcal{T}_*(x_1, \alpha)]^{\left(\frac{x_3-x_2}{x_3-x_1}\right)^s} [\mathcal{T}_*(x_3, \alpha)]^{\left(\frac{x_2-x_1}{x_3-x_1}\right)^s}, \\ \mathcal{T}^*(x_2, \alpha) &\leq [\mathcal{T}^*(x_1, \alpha)]^{\left(\frac{x_3-x_2}{x_3-x_1}\right)^s} [\mathcal{T}^*(x_3, \alpha)]^{\left(\frac{x_2-x_1}{x_3-x_1}\right)^s}, \end{aligned} \quad (48)$$

$$\begin{aligned} &= [\mathcal{T}_*(x_1, \alpha)]^{\left(\frac{x_3-x_2}{x_3-x_1}\right)^s} [\mathcal{T}_*(x_3, \alpha)]^{\frac{(x_2-x_1)^s}{(x_3-x_1)^s}}, \\ &= [\mathcal{T}^*(x_1, \alpha)]^{\left(\frac{x_3-x_2}{x_3-x_1}\right)^s} [\mathcal{T}^*(x_3, \alpha)]^{\frac{(x_2-x_1)^s}{(x_3-x_1)^s}}. \end{aligned} \quad (49)$$

Taking log on both sides of (49), we have

$$\begin{aligned} \mathcal{T}_*(x_2, \alpha)^{(x_3-x_1)^s} &\leq [\mathcal{T}_*(x_1, \alpha)]^{(x_3-x_2)^s} [\mathcal{T}_*(x_3, \alpha)]^{(x_2-x_1)^s}, \\ \mathcal{T}^*(x_2, \alpha)^{(x_3-x_1)^s} &\leq [\mathcal{T}^*(x_1, \alpha)]^{(x_3-x_2)^s} [\mathcal{T}^*(x_3, \alpha)]^{(x_2-x_1)^s}, \end{aligned}$$

that is

$$\begin{aligned} &[\mathcal{T}_*(x_2, \alpha)]^{(x_3-x_1)^s}, [\mathcal{T}^*(x_2, \alpha)]^{(x_3-x_1)^s} \\ &\leq_p [\mathcal{T}_*(x_1, \alpha)]^{(x_3-x_2)^s} [\mathcal{T}_*(x_3, \alpha)]^{(x_2-x_1)^s}, [\mathcal{T}^*(x_1, \alpha)]^{(x_3-x_2)^s} [\mathcal{T}^*(x_3, \alpha)]^{(x_2-x_1)^s}. \end{aligned}$$

Hence

$$[\mathcal{T}(x_2)]^{(x_3-x_1)^s} \leq [\mathcal{T}(x_1)]^{(x_3-x_2)^s} [\mathcal{T}(x_3)]^{(x_2-x_1)^s}.$$

Now we obtain a refinement of Jensen's inequality for log-s-convex-IVF which is given in the following results.

Theorem 3.13 Let $\omega_j \in \mathbb{R}^+$, $x_j \in [u, \vartheta]$, ($j = 1, 2, 3, \dots, k, k \geq 2$) and $\mathcal{T}: [u, \vartheta] \rightarrow \mathbb{F}_C(\mathbb{R})$ be a log-s-convex fuzzy-IVF in the second sense, whose α -levels define the family of IVFs $\mathcal{T}_\alpha: [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ are given by $\mathcal{T}_\alpha(x) = [\mathcal{T}_*(x, \alpha), \mathcal{T}^*(x, \alpha)]$ for all $x \in [u, \vartheta]$ and for all $\alpha \in (0, 1]$. If $(L, U) \subseteq [u, \vartheta]$ then,

$$\prod_{j=1}^k [\mathcal{T}(x_j)]^{\left(\frac{\omega_j}{W_k}\right)^s} \leq \prod_{j=1}^k \left([\mathcal{T}(L)]^{\left(\frac{U-x_j}{U-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} [\mathcal{T}(U)]^{\left(\frac{x_j-L}{M-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} \right), \quad (50)$$

where $W_k = \sum_{j=1}^k \omega_j$. If $\mathcal{T}$ is log-s-concave in the second sense then, inequality (50) is reversed.

Proof Consider $x_1, x_j = x_2$, ($j = 1, 2, 3, \dots, k$), $U = x_3$ in inequality (50). Then, for each $\alpha \in (0, 1]$, then

$$\begin{aligned} \mathcal{T}_*(x_j, \alpha) &\leq [\mathcal{T}_*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s} [\mathcal{T}_*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s}, \\ \mathcal{T}^*(x_j, \alpha) &\leq [\mathcal{T}^*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s} [\mathcal{T}^*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s}. \end{aligned}$$

Above inequality can be written as,

$$\begin{aligned} \mathcal{T}_*(x_j, \alpha)^{\left(\frac{\omega_j}{W_k}\right)^s} &\leq [\mathcal{T}_*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} [\mathcal{T}_*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s}, \\ \mathcal{T}^*(x_j, \alpha)^{\left(\frac{\omega_j}{W_k}\right)^s} &\leq [\mathcal{T}^*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} [\mathcal{T}^*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s}. \end{aligned} \quad (51)$$

Taking multiplication of all inequalities (51) for $j = 1, 2, 3, \dots, k$, we have

$$\begin{aligned} &\prod_{j=1}^k \mathcal{T}_*(x_j, \alpha)^{\left(\frac{\omega_j}{W_k}\right)^s} \\ &\leq \prod_{j=1}^k \left([\mathcal{T}_*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} [\mathcal{T}_*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} \right), \\ &\prod_{j=1}^k \mathcal{T}^*(x_j, \alpha)^{\left(\frac{\omega_j}{W_k}\right)^s} \\ &\leq \prod_{j=1}^k \left([\mathcal{T}^*(L, \alpha)]^{\left(\frac{U-x_j}{U-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} [\mathcal{T}^*(U, \alpha)]^{\left(\frac{x_j-L}{M-L}\right)^s \left(\frac{\omega_j}{W_k}\right)^s} \right), \end{aligned}$$

that is

$$\begin{aligned}
\prod_{j=1}^k \mathcal{T}_\alpha(x_j) \left(\frac{\omega_j}{w_k} \right)^s &= \left[\prod_{j=1}^k \mathcal{T}_*(x_j, \alpha) \left(\frac{\omega_j}{w_k} \right)^s, \prod_{j=1}^k \mathcal{T}^*(x_j, \alpha) \left(\frac{\omega_j}{w_k} \right)^s \right] \\
&\leq_l \left[\prod_{j=1}^k \left([\mathcal{T}_*(L, \alpha)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s [\mathcal{T}_*(U, \alpha)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \right), \prod_{j=1}^k \left([\mathcal{T}^*(L, \alpha)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s [\mathcal{T}^*(U, \alpha)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \right) \right], \\
&\leq_l \prod_{j=1}^k \left(\left[[\mathcal{T}_*(L, \alpha)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s, [\mathcal{T}^*(L, \alpha)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \right] \right) \cdot \prod_{j=1}^k \left(\left[[\mathcal{T}_*(U, \alpha)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s, [\mathcal{T}^*(U, \alpha)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \right] \right), \\
&= \prod_{j=1}^k [\mathcal{T}_\alpha(L)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \cdot \prod_{j=1}^k [\mathcal{T}_\alpha(U)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s.
\end{aligned}$$

Thus,

$$\prod_{j=1}^k [\mathcal{T}(x_j)] \left(\frac{\omega_j}{w_k} \right)^s \leq \prod_{j=1}^k \left([\mathcal{T}(L)] \left(\frac{U-x_j}{U-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s [\mathcal{T}(U)] \left(\frac{x_j-L}{M-L} \right)^s \left(\frac{\omega_j}{w_k} \right)^s \right),$$

this completes the proof.

Remark 3.14

- (i) If $s = 1$, then Theorems 3.12 and 3.13 reduces to the result for log convex fuzzy-IVF.
- (ii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$, then Theorems 3.12 and 3.13 reduce to the result for log- s -convex function in the second sense, see [6].
- (iii) If $\mathcal{T}_*(u, \alpha) = \mathcal{T}^*(\vartheta, \alpha)$ with $\alpha = 1$ and $s = 1$, then Theorems 3.12 and 3.13 reduce to the result for log-convex function, see [13].

Results and discussion

We derive some Jensen, Schur and HH-Inequalities for log- s -convex fuzzy-IVFs in the second sense by means of fuzzy order relation. Our results generalize most of the results which are proved for classical log-convex functions. We would like to mention that due to the lack of “fuzzy interval derivatives” with some good properties, we have not discussed those inequalities involving fuzzy interval derivatives.

Conclusion

HH-Inequalities are true for this concept of log- s -convex fuzzy-IVFs in the second sense. As a future research, we try to explore this concept for generalized of log- s -convex fuzzy-IVFs in the second sense and some applications in fuzzy-interval nonlinear programing. We hope that this

concept will be helpful for other authors to pay their roles in different fields of sciences.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

References

- Alomari M, Darus M, Dragomir SS, Cerone P (2010) Ostrowski type inequalities for functions whose derivatives are s -convex in the second sense. *Appl Math Lett* 23:1071–1076
- Cerone P, Dragomir SS (2004) Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstr Math* 37(2):299–308
- Kaleva O (1987) Fuzzy differential equations. *Fuzzy Sets Syst* 24:301–317
- Hadamard J (1893) Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann. *Journal de mathématiques pures et appliquées* 7:171–215
- Hermite C (1883) Sur deux limites d’une intégrale définie. *Mathesis* 3:82–97
- Varošaneć S (2007) On h -convexity. *J Math Anal Appl* 326(1):303–311
- Xi BY, Qi F (2015) Some integral inequalities of Hermite–Hadamard type for s -logarithmically convex functions. *Acta Math Sci Ser A (Chin Ed)* 35(3):515–524
- Noor MA (2007) Hermite–Hadamard integral inequalities for log-preinvex functions. *J Math Anal Approx Theory* 2(2):126–131
- Pečarić JE, Proschan F, Tong YL (1991) Convex functions, partial orderings and statistical applications. Academic Press

10. Noor MA, Qi F, Awan MU (2013) Some Hermite–Hadamard type inequalities for log- h -convex functions. *Analysis* 33(4):367–375
11. Adil Khan M, Pečarić J, Chu Y-M (2020) Refinements of Jensen's and McShane's inequalities with applications. *AIMS Math* 5(5):4931–4945
12. Dragomir SS (2001) Refinements of the Hermite–Hadamard integral inequality for log-convex functions. *Aust Math Soc Gaz* 28(3):129–134
13. Dragomir SS (2011) A survey of Jensen type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces. *Commun Math Anal* 10(1):82–104
14. Dragomir SS, Mond B (1998) Integral inequalities of Hadamard type for log convex functions. *Demonstr Math* 31(2):354–364
15. Dragomir SS (2017) New inequalities of Hermite–Hadamard type for log convex functions. *Khayyam J Math* 3(2):98–115
16. Niculescu CP (2012) The Hermite–Hadamard inequality for log convex functions. *Nonlinear Anal* 75(2):662–669
17. Moore RE (1966) *Interval analysis*. Prentice Hall, Englewood Cliffs
18. Kulish UW, Miranker W (1981) *Computer arithmetic in theory and practice*. Academic Press, New York
19. Guo Y, Ye G, Zhao D, Liu W (2019) Some integral inequalities for log- h -convex interval-valued functions. *IEEE Access* 7:86739–86745
20. Chalco-Cano Y, Flores-Franulić A, Román-Flores H (2012) Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput Appl Math* 31(3):457–472
21. Chalco-Cano Y, Lodwick WA, Condori-Equice W (2015) Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput* 19:3293–3300
22. Costa TM, Román-Flores H, Chalco-Cano Y (2019) Opial-type inequalities for interval-valued functions. *Fuzzy Sets Syst* 358:48–63
23. Dragomir SS, Pearce CEM (2000) *Selected topics on Hermite–Hadamard inequalities and applications*. RGMIA Monographs Victoria University
24. Román-Flores H, Chalco-Cano Y, Lodwick WA (2018) Some integral inequalities for interval-valued functions. *Comput Appl Math* 37(2):1306–1318
25. Zhao DF, An TQ, Ye GJ, Liu W (2018) New Jensen and Hermite–Hadamard type inequalities for h -convex interval-valued functions. *J Inequal Appl* 2018:1–14
26. Zadeh LA (1965) Fuzzy sets. *Inf Control* 8(3):338–353
27. Nanda S, Kar K (1992) Convex fuzzy mappings. *Fuzzy Sets Syst* 48:129–132
28. Chang SS (1991) *Variational inequality and complementarity problems theory and applications*. Shanghai Scientific and Technological Literature Publishing House, Shanghai
29. Noor MA (1994) Fuzzy preinvex functions. *Fuzzy Sets Syst* 64(1):95–104
30. Bede B, Gal SG (2005) Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets Syst* 151(3):581–599
31. Bede B (2013) Mathematics of fuzzy sets and fuzzy logic. In: *Studies in fuzziness and soft computing*, vol. 295. Springer
32. Cervelati J, Jiménez-Gamero MD, Vilca-Labra F, Rojas-Medar MA (2004) Continuity for s -convex fuzzy processes. In: *Soft methodology and random information systems*, vol 7, pp 653–660
33. Chalco-Cano Y, Rojas-Medar MA, Román-Flores H (2000) M -convex fuzzy mappings and fuzzy integral mean. *Comput Math Appl* 40(10–11):1117–1126
34. Khan MB, Noor MA, Noor KI, Chu Y-M (2021) New Hermite–Hadamard type inequalities for (h_1, h_2) -convex fuzzy-interval-valued functions. *Adv Differ Equ* 2021:6–20
35. Puri ML, Ralescu DA (1986) Fuzzy random variables. In: *Readings in fuzzy sets for intelligent systems*, vol 114, pp 409–422
36. Osuna-Gómez R, Jiménez-Gamero MD, Chalco-Cano Y, Rojas-Medar MA (2004) Hadamard and Jensen inequalities for s -convex fuzzy processes. In: *Soft methodology and random information systems*. Advances in soft computing, vol 126. Springer, Berlin, pp 1–15
37. Costa TM (2017) Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst* 327:1–47
38. Goetschel R Jr, Voxman W (1986) Elementary fuzzy calculus. *Fuzzy Sets Syst* 18:31–43
39. Diamond P, Kloeden PE (1994) *Metric spaces of fuzzy sets: theory and applications*. World Scientific
40. Costa TM, Roman-Flores H (2017) Some integral inequalities for fuzzy-interval-valued functions. *Inf Sci* 420:110–125

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