A novel complex fuzzy N-soft sets and their decision-making algorithm

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Abstract

Complex fuzzy N-soft set (CFN-SS) is an important technique to manage awkward and unreliable information in realistic decision-making problems. CFN-SS is a blend of two separate theories, called N-soft sets (N-SSs) and complex fuzzy sets (CFSs), which are the modified versions of soft sets (SSs) and fuzzy sets (FSs) to depict vague and uncertain information in daily life problems. In this manuscript, the novel concept of CFN-SS is explored and their fundamental laws are discussed. CFN-SS contains the grade of truth in the form of a complex number whose real and imaginary parts are limited to the unit interval. Besides, we examine some algebraic properties for CFN-SS like union, intersections and justify these properties with the help of some numerical examples. To examine the superiority and effectiveness of the proposed approaches, the special cases of the investigated approaches are also discussed. A decision-making procedure is developed by using the investigated ideas based on CFN-SSs. Further, some numerical examples are also illustrated with the help of explored ideas to find the reliability and effectiveness of the proposed approaches. Finally, the comparative analysis of the investigated ideas with some existing ideas is also demonstrated to prove the quality of the proposed works. The graphical expressions of the obtained results are also discussed.

Keywords Soft sets · N-soft sets · Fuzzy sets · Fuzzy N-soft sets · Complex fuzzy sets · Complex fuzzy N-soft sets · Decision-making

Introduction

With the development of the information age, the decision-making problems and decision-making environments are more and more complex. Thus, it becomes more and more difficult to express attribute values of alternatives. Based on this, Zadeh firstly developed the definition of fuzzy sets (FSs) [1], which can easily express fuzzy information for multi-attribute decision-making (MADM) problems and multi-attribute group decision-making (MAGDM) problems. The probability theory is also one of the most important and useful techniques to cope with awkward and complicated information in realistic issues, which were explored by numerous scholars [2, 3]. Later, FSs got a lot of attention

from researchers and got extensions such as interval-valued FSs (IvFSs) [4], intuitionistic FSs (IFSs) [5], interval-valued IFSs (IvIFSs) [6], etc. Historically, the set of real numbers was extended to the set of complex numbers, this extension motivated Ramot et al. [7] to introduce the notion of CFSs. Mahmood et al. [8] interpreted the concept of complex hesitant fuzzy sets.

The volume and intricacy of the gathered information in our advanced society are developing quickly. There regularly exist different sorts of vagueness in that information is identified with complex issues in different fields such as engineering, economics, social science, environmental science, and biology, etc. To portray and extract useful data covered up with uncertain information, researchers in computer science, mathematics, and related areas interpreted various theories like fuzzy set theory [1], vague set theory [9], probability theory, rough set theory [10–14], and interval mathematics [15]. Moreover, the theory of SS was developed by Molodtsove [16] in 1999, as another mathematical instrument to cope with the vagueness which is liberated from the hurdles influencing existing methods. A wide scope of applications of SSs has been established in various fields, such as game theory,

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probability theory, smoothness off functions, Riemann integration, operations research, and perron integration [16, 17]. SS and its applications got a lot of attention from researchers in recent years. Ali et al. [18] defined some new operations such as union, intersection for SS. In SS theory the application in DM problem was discussed Maji et al. [19]. Maji et al. [20] extended the SS, called Fuzzy SS (FSS). Roy and Maji [21] discussed a FSS theoretic approach to DM problems. Zhang and Zhan [22] described fuzzy soft β -covering based fuzzy rough sets and corresponding DM applications. The certain types of soft coverings based rough sets with applications were established by Zhan and Wang [23]. Jiang et al. [24] defined the MADM approach to covering based variable precision fuzzy rough set. Yang et al. [25] combined an interval-valued fuzzy set with a SS model to develop a new concept of interval-valued fuzzy soft sets (IvFSSs). Soft set relations and functions were developed by Babitha and Sunil [26]. Feng et al. [27] combined SSs with FSs and roughs sets. Cagman and Enginoglu [28] interpreted a DM method namely, the uni-int DM method, and redefined SS operations to utilize effectively in DM problems. Zhan and Alcantud [29] presented a novel type of soft rough covering and its application to multicriteria group decision-making. A survey of parameter reduction of SSs and corresponding algorithms were given by Zhan and Alcantud [30]. Zhan et al. [31] presented covering based variable precision fuzzy rough sets with PROMETHEE-EDAS method. An application to rating problem by using TOPSIS-WAA method based on a covering-based fuzzy rough set is introduced in [32]. Zhan and Xu [33] introduced a novel approach based on three-way decisions in the fuzzy information system. The notion of complex FSS was interpreted by Thirunavukarasu et al. [34].

From the most recent studies of SS, one can see that most of the researchers in SS theory worked on a binary evaluation $\{0, 1\}$ or closed interval $[0, 1]$ [35, 36]. However, in real-world problems, we mostly find information with a non-binary yet discrete structure. For instance, in social judgment frameworks, Alcantud and Laruelle [37] determined the ternary voting framework. Non-binary evaluations are likewise expected in ranking and rating positions. Inspired by these concerns, Fatimah et al. [38] proposed an extended SS model, namely N-SS and they described the significance of the ordered grades in real-world problems. Motivated by N-SS set al [39] interpreted the notion of fuzzy N-SS (FN-SS) which is the combination of FS theory with N-SS. They consider the fuzzy nature of parameterization of the universe.

Many of the features are jointly desirable but happen in separate formal models of knowledge. To overcome this drawback, in this article we combine complex fuzzy set theory with N-soft sets to introduce a new hybrid model called complex fuzzy N-soft sets. This model takes in the uncertainties concerning two aspects of data: what specific

grades are given to objects when parameterizations attributes are graded, which can be assigned as a partial degree of membership. The proposed model provides complete information about the occurrence of ratings and uncertainty under periodic function. It is also useful to get optimistic and pessimistic responses by decision-makers. Therefore, for the purpose of the modulization of decision-making problems it provides more flexibility when hesitation and complexity in the parameterizations are involved. We make the model fully applicable by developing decision-making algorithms that appeal to methodologies that have been validated in related frameworks.

In CFN-SS hypothesis, membership degree is unpredictable esteemed and are addressed in polar directions. The abundancy term comparing the membership degree gives the degree of belonging of an item in a CFN-SS and the stage term related with membership degree gives the extra data, for the most part related to periodicity. Since, in the current FN-SSs hypothesis, it is noticed that there is just a single boundary to address the data which brings about data misfortune in certain examples. Nonetheless, in everyday life, we go over complex characteristic wonders where we need to add the second measurement to the statement of membership grade. By presenting this subsequent measurement, the total data can be projected in one set, and henceforth loss of data can be stayed away from. For example, assume a specific organization chooses to set up biometric-based participation gadgets (BBPGs) in the entirety of its workplaces spread everywhere in the country. For this, the organization counsels a specialist who gives the data with respect to the two-measurements specifically, models of BBPGs and their comparing creation dates of BBPGs. The errand of the organization is to choose the most ideal model of BBPGs with its creation date all the while. It is clearly seen that such sort of issues can't be displayed precisely by thinking about both the measurements at the same time utilizing the conventional FN-SS hypotheses. Along these lines, for such sorts of issue, there is a need to improve the current hypotheses and thus a CFN-SS climate gives us a proficient method to deal with the two-venture judgment situations in which an abundancy term might be utilized to give an organization's choice with respect to demonstrate of BBPGs and the stage terms might be utilized to address organization's choice as to creation date of BBPGs in the choice making measure. Likewise, some different sorts of models under CFN-SSs incorporate a lot of informational indexes that are created from clinical exploration, just as government data sets for biometric and facial acknowledgment, sound, and pictures, and so on Hereafter, a CFN-SS is a more summed up expansion of the current hypotheses, for example, FSs, FN-SSs, CFSs. Obviously, the upside of the CFN-SSs is that it can contain considerably more information to communicate the data. Hence, maintaining the benefits of this set and tak-

ing the significance of connection measure of the CFN-SSs, the structure of this manuscript is discussed in the following ways:

1. To introduce a more generalized concept i.e., complex fuzzy N-Soft sets (CFN-SSs).
2. In this notion, we suppose the possibility that the parameterized nature of the universe is complex fuzzy.
3. To merge CFS theory with N-SSs to interpret a new model called CFN-SSs.
4. To give this model to cope with vagueness in information concerning which values are allocated to the objects, in the parameterization of attributes. With CFN-SSs we also deal with the two-dimensional and complex information given to the object of the universe, so it is more helpful in the DM problems.
5. As CFN-SSs is the generalization so one can get FN-SSs, N-SSs, and SSs from CFN-SSs.

This manuscript is settled as follows: In Section “**Preliminaries**” of this manuscript, we provide some fundamental definitions and their properties of FS, CFSs, SSs, FSSs, CFSs, N-SSs, and FN-SSs for the readers. In Section “**Complex Fuzzy N-Soft Sets**” of this manuscript, we propose the notion of CFN-SSs and their functional representation. The fundamental properties of the proposed method are also discussed in this section. In Section “**Relationships**” of this manuscript, we discussed the relationship between CF-NSSs and some existing theories. In Section “**Applications**”, we present applications of CFN-SSs into decision-making (DM) problems and present three algorithms to solve DM problems by utilizing CFN-SSs. In Section “**Comparison**” of this manuscript, we compare our new model with FN-SS. Finally, in Section “**Conclusion**”, the conclusion of the work done in this manuscript is given along with future directions.

Preliminaries

In this section, we review some basic notions of FSs, CFSs, SSs, N-SSs, and fuzzy N-SSs (FN-SSs). The fundamental properties of the existing methods are also discussed in detail. Throughout this manuscript, the symbols $M \neq \emptyset$ and Q are denoted the finite set and set of parameters, respectively.

Definition 1 [1] A fuzzy set (FS) is designated and defined by:

$$\mathcal{A} = \{(x, \mu(x)) : x \in M\},$$

where $\mu : M \rightarrow [0, 1]$ with a condition $0 \leq \mu(x) \leq 1$. Throughout, this manuscript the fuzzy numbers (FNs) is demonstrated by $\mathcal{A} = (x, \mu_{\mathcal{A}}(x))$. When you are taking a

single element x from a universal set M and assign it to a function the resultant values are restarted to unit interval is called the fuzzy number and the collection of all fuzzy numbers under the middle brackets is called fuzzy sets. Let $\mathcal{A} = (x, \mu_{\mathcal{A}}(x))$ and $\mathcal{B} = (x, \mu_{\mathcal{B}}(x))$ are two fuzzy numbers (FNs). Then

1. $\mathcal{A} = \mathcal{B} \iff \mu_{\mathcal{A}}(x) = \mu_{\mathcal{B}}(x)$;
2. $\mathcal{A} \subseteq \mathcal{B} \iff \mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$;
3. $\mathcal{A}^c = 1 - \mu_{\mathcal{A}}(x)$;
4. $\mathcal{A} \cup \mathcal{B} = \mathcal{A} \vee \mathcal{B} = \text{Max}(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x))$;
5. $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \wedge \mathcal{B} = \text{Min}(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x))$.

Definition 2 [7] A complex fuzzy set (CFS) is designated and defined by:

$$S = \{(x, \mu(x)) : x \in U\},$$

where $\mu = \mu' e^{i2\pi(\omega_{\mu'})}$ with a condition $0 \leq \mu', \omega_{\mu'} \leq 1$. Throughout, this manuscript the complex fuzzy numbers (CFNs) is demonstrated by $\mathcal{A} = (x, \mu_{\mathcal{A}}(x)) = (x, \mu'_{\mathcal{A}}(x) e^{i2\pi(\omega_{\mu'_{\mathcal{A}}}(x))})$. Let $\mathcal{A} = (x, \mu'_{\mathcal{A}}(x) e^{i2\pi(\omega_{\mu'_{\mathcal{A}}}(x))})$ and $\mathcal{B} = (x, \mu'_{\mathcal{B}}(x) e^{i2\pi(\omega_{\mu'_{\mathcal{B}}}(x))})$ are two complex fuzzy numbers (CFNs). Then.

1. $\mathcal{A} = \mathcal{B} \iff \mu_{\mathcal{A}}(x) = \mu_{\mathcal{B}}(x) \text{ i.e. } \mu'_{\mathcal{A}}(x) = \mu'_{\mathcal{B}}(x), \omega_{\mu'_{\mathcal{A}}}(x) = \omega_{\mu'_{\mathcal{B}}}(x)$;
2. $\mathcal{A} \subseteq \mathcal{B} \iff \mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x) \text{ i.e. } \mu'_{\mathcal{A}}(x) \leq \mu'_{\mathcal{B}}(x), \omega_{\mu'_{\mathcal{A}}}(x) \leq \omega_{\mu'_{\mathcal{B}}}(x)$;
3. $\mathcal{A}^c = 1 - \mu(x) = (1 - \mu'_{\mathcal{A}}(x)) e^{i2\pi(1 - (\omega_{\mu'_{\mathcal{A}}}(x)))}$;
4. $\mathcal{A} \cup \mathcal{B} = \mathcal{A} \vee \mathcal{B} = \text{Max}(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)) = \text{Max}(\mu'_{\mathcal{A}}(x), \mu'_{\mathcal{B}}(x)) e^{i2\pi \text{Max}(\omega_{\mu'_{\mathcal{A}}}(x), \omega_{\mu'_{\mathcal{B}}}(x))}$;
5. $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \wedge \mathcal{B} = \text{Min}(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)) = \text{Min}(\mu'_{\mathcal{A}}(x), \mu'_{\mathcal{B}}(x)) e^{i2\pi \text{Min}(\omega_{\mu'_{\mathcal{A}}}(x), \omega_{\mu'_{\mathcal{B}}}(x))}$.

Definition 3 [16] A pair (F, D) represents a SS over M if $F : D \rightarrow P(M) = 2^M, D \subseteq Q$ i.e. if $d \in D$, then $F(d) \subseteq P(M)$ is called d -approximation elements of (F, D) . Next, we discussed the modified operations of SS, which was developed by Ali et al. [18]. The existing operations called restricted and extended unions, intersections are discussed below.

Definition 4 [18] For any two SSs (F_1, D_1) and (F_2, D_2) with $D_1 \cap D_2 \neq \emptyset$, then the restricted union and intersection are denoted and defined by:

$$(F_1, D_1) \cup_r (F_2, D_2) = (\mathcal{H}, D_1 \cap D_2) \quad \forall d \in D_1 \cap D_2 \\ \Rightarrow \mathcal{H}(d) = F_1(d) \cup F_2(d).$$

$$(F_1, D_1) \cap_r (F_2, D_2) = (\mathcal{H}, D_1 \cap D_2) \quad \text{where } \mathcal{H}(d) \\ = F_1(d) \cap F_2(d) \quad \forall d \in D_1 \cap D_2.$$

Definition 5 [18] For any two SSs (F_1, D_1) and (F_2, D_2) , then the extended union and intersection are designated and defined by:

$$(F_1, D_1) \cup_e (F_2, D_2) = (\mathcal{H}, D_1 \cup D_2) \quad \text{where } \forall d \\ \in D_1 \cup D_2 \Rightarrow \mathcal{H}(d) \\ = \begin{cases} F_1(d) & \text{if } d \in D_1 \setminus D_2 \\ F_2(d) & \text{if } d \in D_2 \setminus D_1 \\ F_1(d) \cup F_2(d) & d \in D_1 \cap D_2 \end{cases}$$

$$(F_1, D_1) \cap_e (F_2, D_2) = (\mathcal{H}, D_1 \cup D_2) \quad \text{where } \forall d \\ \in D_1 \cup D_2 \Rightarrow \mathcal{H}(d) \\ = \begin{cases} F_1(d) & \text{if } d \in D_1 \setminus D_2 \\ F_2(d) & \text{if } d \in D_2 \setminus D_1 \\ F_1(d) \cap F_2(d) & d \in D_1 \cap D_2 \end{cases}.$$

Definition 6 [38] A pair (F, D, N) represents an N-soft set (N-SS) over M if $F : D \rightarrow 2^{M \times Z} = P(M \times Z)$, $D \subseteq Q$ defines by $F(d) = (m, z_d) \in P(M \times Z)$ and $Z = \{0, 1, 2, \dots, N-1\}$. if $d \in D$, then $F(d) \subseteq P(M \times Z)$ is called d -approximation elements of (F, D, N) . Next, the existing operations, called restricted and extended unions, intersections are discussed below. The existing notion and their operational laws are initiated by Fatimah et al. [38].

Definition 7 [38] For any two N-SSs (F, D_1, N_1) and (F_2, D_2, N_2) with $D_1 \cap D_2 \neq \emptyset$, then the restricted union and intersection are denoted and defined by:

$$(F, D_1, N_1) \cup_r (F_2, D_2, N_2) \\ = (\mathcal{K}, D_1 \cap D_2, \text{Max}(N_1, N_2)),$$

where $\forall d \in D_1 \cap D_2 \& m \in M \Rightarrow (m, z_d) \in \mathcal{H}(d) \Leftrightarrow z_d = \text{Max}(z_d^1, z_d^2)$, if $(m, z_d^1) \in F_1(d) \& (m, z_d^2) \in F_2(d)$.

$$(F, D_1, N_1) \cap_r (G, B, N_2) = (\mathcal{H}, A \cap B, \text{Min}(N_1, N_2))$$

where $\forall d \in D_1 \cap D_2 \& m \in M \Rightarrow (m, z_d) \in \mathcal{H}(d) \Leftrightarrow z_d = \text{Min}(z_d^1, z_d^2)$, if $(m, z_d^1) \in F_1(d) \& (m, z_d^2) \in F_2(d)$.

Definition 8 [38] For any two N-SSs (F_1, D_1, N_1) and (F_2, D_2, N_2) , then the extended union and intersection are designated and defined by:

$$(F_1, D_1, N_1) \cup_e (F_2, D_2, N_2) = (\mathcal{L}, A \cup B, \text{Max}(N_1, N_2))$$

$$\text{where } \mathcal{L}(d) = \begin{cases} F(d) & \text{if } d \in D_1 \setminus D_2 \\ G(d) & \text{if } d \in D_2 \setminus D_1 \\ (m, z_d)/z_d = \text{Max}(z_d^1, z_d^2) & (m, z_d^1) \in F_1(d) \& (m, z_d^2) \in F_2(d) \end{cases}$$

$$(F_1, D_1, N_1) \cap_e (F_2, D_2, N_2) = (\mathcal{J}, A \cup B, \text{Min}(N_1, N_2))$$

$$\text{where } \mathcal{J}(d) = \begin{cases} F(d) & \text{if } d \in D_1 \setminus D_2 \\ G(d) & \text{if } d \in D_2 \setminus D_1 \\ (m, z_d)/z_d = \text{Min}(z_d^1, z_d^2) & (m, z_d^1) \in F_1(d) \& (m, z_d^2) \in F_2(d) \end{cases}.$$

Definition 9 [20] A pair (F', D) represents a fuzzy soft set (FSS) over M if $F' : D \rightarrow P'(M)$, $D \subseteq Q$, where $P'(M)$ contains the set of all fuzzy sets of M . if $d \in D$, then $F'(d) \subseteq P'(M)$ is called d -approximation elements of (F', D) . Next, the existing operations, called union and intersection, are discussed below. The existing notion and their operational laws are initiated by Maji et al. [20].

Definition 10 [20] For any two FSSs (F'_1, D_1) and (F'_2, D_2) , then the union and intersection are designated and defined by:

$$(F'_1, D_1) \cup (F'_2, D_2) = (\mathbb{H}, D_1 \cup D_2) \quad \text{where } \forall d \\ \in D_1 \cup D_2 \Rightarrow \mathbb{H}(d) \\ = \begin{cases} F'_1(d) & \text{if } d \in D_1 \setminus D_2 \\ F'_2(d) & \text{if } d \in D_2 \setminus D_1 \\ F'_1(d) \cup F'_2(d) & d \in D_1 \cap D_2 \end{cases}$$

$$(F'_1, D_1) \cap (F'_2, D_2) = (\mathbb{H}, D_1 \cup D_2) \quad \text{where } \forall d \in D_1 \cup D_2 \\ \Rightarrow \mathbb{H}(d) = F'_1(d) \cap F'_2(d).$$

Definition 11 [34] A pair (F'', D) represents a complex fuzzy soft set (CFSS) over M if $F'' : D \rightarrow P''(U)$, $D \subseteq Q$, where $P''(M)$ contains the set of all complex fuzzy sets of M .

Definition 12 [39] A pair $(\mu', (F, A, N))$ represents a fuzzy N-soft set (FN-SS) over M if $\mu' : D \rightarrow \bigcup_{d \in D} \mathcal{F}(F(d))$, $d \in D \subseteq Q$ defines by $\mu'(d) \in \mathcal{F}(F(d))$ for each $d \in D$ and $Z = \{0, 1, 2, \dots, N-1\}$. if $d \in D$, then $\mu'(d) \subseteq \mathcal{F}(F(d))$ is called d -approximation elements of $(\mu', (F, A, N))$. Next, the existing operations, called restricted and extended unions, intersections are discussed below. The existing notion and their operational laws are initiated by Akram et al. [39].

Definition 13 [39] For any two FN-SSs $(\mu'_1, (F_1, D, N_1))$ and $(\mu'_2, (F_2, D_2, N_2))$, then the restricted union and intersection are designated and defined by:

$$(\mu'_1, (F_1, D_1, N_1)) \cup_R (\mu'_2, (F_2, D_2, N_2)) \\ = (\mathbb{K}, D_1 \cap D_2, \text{Max}(N_1, N_2)).$$

where $\forall d_k \in D_1 \cap D_2 \& m_j \in M \Rightarrow (z_{jk}, \mu_{jk}) \in \mathbb{K}$
 $(d) \Leftrightarrow z_{jk} = \text{Max}(z_{jk}^1, z_{jk}^2), \mu_{jk} = \text{Max}(\mu_{jk}^1, \mu_{jk}^2), \text{ if } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k), (z_{jk}^2, \mu_{jk}^2) \in \mu_1(d_k), \text{ where } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2.$

$$\begin{aligned} & (\mu'_1, (F_1, D_1, N_1)) \cap_R (\mu'_2, (F_2, D_2, N_2)) \\ & = (\mathbb{I}, D_1 \cap D_2, \text{Min}(N_1, N_2)). \end{aligned}$$

where $\forall d_k \in D_1 \cap D_2 \& m_j \in M \Rightarrow (z_{jk}, \mu_{jk}) \in \mathbb{I}$
 $(d) \Leftrightarrow z_{jk} = \text{Min}(z_{jk}^1, z_{jk}^2), \mu_{jk} = \text{Min}(\mu_{jk}^1, \mu_{jk}^2), \text{ if } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k), (z_{jk}^2, \mu_{jk}^2) \in \mu_1(d_k), \text{ where } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2.$

Definition 14 [39] For any two FN-SSs $(\mu'_1, (F_1, D_1, N_1))$ and $(\mu'_2, (F_2, D_2, N_2))$, then the extended union and intersection are denoted and defined by:

$$\begin{aligned} & (\mu'_1, (F_1, D_1, N_1)) \cup_E (\mu'_2, (F_2, D_2, N_2)) \\ & = (\mathbb{L}, D_1 \cup D_2, \text{Max}(N_1, N_2)) \text{ where } \forall d_k \\ & \in D_1 \cup D_2, m_j \in M \end{aligned}$$

$$\mathbb{L}(d_k) = \begin{cases} \mu'_1(d_k) & \text{if } d_k \in D_1 \setminus D_2 \\ \mu'_2(d_k) & \text{if } d_k \in D_2 \setminus D_1 \\ (z_{jk}, \mu_{jk}) \left(\begin{array}{l} s.t z_{jk} = \text{Max}(z_{jk}^1, z_{jk}^2), \mu_{jk} = \text{Max}(\mu_{jk}^1, \mu_{jk}^2) \\ \text{if } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k), (z_{jk}^2, \mu_{jk}^2) \in \mu_1(d_k), \text{ where } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2. \end{array} \right) & \text{if } d_k \in D_1 \cap D_2 \end{cases}$$

$$(\mu'_1, (F_1, D_1, N_1)) \cap_E (\mu'_2, (F_2, D_2, N_2)) = (\mathbb{T}, D_1 \cup D_2, \text{Min}(N_1, N_2)) \text{ where } \forall d_k \in D_1 \cup D_2, m_j \in M$$

$$\mathbb{T}(d_k) = \begin{cases} \mu'_1(d_k) & \text{if } d_k \in D_1 \setminus D_2 \\ \mu'_2(d_k) & \text{if } d_k \in D_2 \setminus D_1 \\ (z_{jk}, \mu_{jk}) \left(\begin{array}{l} s.t z_{jk} = \text{Min}(z_{jk}^1, z_{jk}^2), \mu_{jk} = \text{Min}(\mu_{jk}^1, \mu_{jk}^2) \\ \text{if } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k), (z_{jk}^2, \mu_{jk}^2) \in \mu_1(d_k), \text{ where } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2. \end{array} \right) & \text{if } d_k \in D_1 \cap D_2 \end{cases}$$

Complex fuzzy N-soft sets

In this section, we propose the notion of complex fuzzy N-soft sets (CFN-SSs) and their functional representation. The fundamental properties of the proposed method are also discussed in detail.

The notion of complex fuzzy N-soft sets

Definition 15 Let M be a universe of objects, Q be the set of attributes, $D \subseteq Q$, and $Z = \{0, 1, 2, \dots, N-1\}$ with $N \in \{2, 3, \dots\}$. A pair $(\mu, \mathcal{K})$ is said to be CFN-SS when $\mathcal{K} = (F, D, N)$ is an N -soft set on M , and μ is a mapping $\mu : D \rightarrow \bigcup_{d \in D} \mathcal{CF}(F(d))$ such that $\mu(d) = \mu'(d)e^{i2\pi(\omega_{\mu'}(d))} \in \mathcal{CF}(F(d))$ for each $d \in D$.

As stated by Definition (15), with every attribute the mapping μ allocates a complex fuzzy set on the image of that attributes by the mapping F . Consequently, for every $d \in D$ and $m \in M$ there exists a unique $(m, z_d) \in M \times Z$ such that $z_d \in Z$ and $((m, z_d), \mu_d(m)) \in \mu(d)$, which is a notation that reduces to $\mu_d(m) = \mu(m)(m, z_d)$. The accompanying example explains this formal definition. Furthermore, It presents a helpful tabular portrayal for CFN-SSs.

Example 1 a university wants to appoint a faculty member for the department of mathematics based on star ranking and ratings awarded by the selection board of the university including the president, vice president, director academic, and dean of the faculty. Let $M = \{m_1, m_2, m_3, m_4, m_5\}$ be the universe set of applicants appearing in an interview and Q be set of attributes “evaluation of applicants by selection board”. The subset $D \subseteq Q$ such that $D = \{d_1, d_2, d_3, d_4\}$ is used. We can get 5-SSs from Table 1, where.

Table 1 Information obtained from the associated data and tabular form of the related 5-SS

M/D	d_1	d_2	d_3	d_4	Stars	Representations
m_1	$* = 1$	$*** = 3$	$*** = 3$	$**** = 4$	$o = 0$	Poor
m_2	$**** = 4$	$** = 2$	$** = 2$	$** = 2$	$* = 1$	Normal
m_3	$*** = 3$	$*** = 3$	$**** = 4$	$** = 2$	$** = 2$	Good
m_4	$** = 2$	$o = 0$	$** = 2$	$o = 0$	$*** = 3$	Very Good
m_5	$o = 0$	$o = 0$	$* = 1$	$* = 1$	$**** = 4$	Excellent

1. Four stars appear for ‘Excellent’,
2. Three stars appear for ‘Very Good’,
3. Two stars appear for ‘Good’,
4. One star appears for ‘Normal’,
5. Hole appear for ‘Poor’.

This graded evaluation by stars can undoubtedly be related to numbers as $Z = \{0, 1, 2, 3, 4\}$, where.

- 0 serves as “o”,
 1 serves as “*”,
 2 serves as “**”,
 3 serves as “***”,
 4 serves as “****”.

Table 1 presented the information obtained from associated data, and also presented the tabular representation of its related 5-soft set.

In association with Definition 15, we clarify for example $(m_1, z_{d_2} = 3) \in F(d_2)$ and $(m_3, z_{d_4} = 2) \in F(d_3)$. This information is sufficient when it is obtained from real data, however, when the data is uncertain and fuzzy we require (F, N) -soft sets that give us $((m_1, z_{d_2} = 3), (0.78)) \in \mu(d_1)$ and $((m_3, z_{d_4} = 2), (0.54)) \in \mu(d_3)$. When the data is complicated and two-dimensional, then we require CFN-SS, which give us how these grades are given to applicants. The selection board follows this criterion based on the abilities of the applicants as follows:

- $0.0 \leq \Delta\mu_d(m) < 0.2$ when $z_d = 0$;
 $0.2 \leq \Delta\mu_d(m) < 0.4$ when $z_d = 1$;
 $0.4 \leq \Delta\mu_d(m) < 0.6$ when $z_d = 2$;
 $0.6 \leq \Delta\mu_d(m) < 0.8$ when $z_d = 3$;
 $0.8 \leq \Delta\mu_d(m) < 1.0$ when $z_d = 4$,

where $\Delta\mu_d(m) = \frac{\mu'_d(m) + \omega_{\mu'_d}(m)}{2}$. Therefore, the following CF5-SS by using Definition (15), is defined:

$$\mu(d_1) = \left\{ \begin{array}{l} ((m_1, 1), 0.3e^{i2\pi(0.25)}), \\ ((m_2, 4), 0.9e^{i2\pi(0.85)}), \\ ((m_3, 3), 0.7e^{i2\pi(0.77)}), \\ ((m_4, 2), 0.55e^{i2\pi(0.45)}), \\ ((m_5, 0), 0.14e^{i2\pi(0.15)}) \end{array} \right\} \in C\mathcal{F}(F(d_1))$$

$$\mu(d_2) = \left\{ \begin{array}{l} ((m_1, 3), 0.78e^{i2\pi(0.66)}), \\ ((m_2, 2), 0.42e^{i2\pi(0.56)}), \\ ((m_3, 3), 0.61e^{i2\pi(0.77)}), \\ ((m_4, 0), 0.1e^{i2\pi(0.2)}), \\ ((m_5, 0), 0.1e^{i2\pi(0.24)}) \end{array} \right\} \in C\mathcal{F}(F(d_2))$$

$$\mu(d_3) = \left\{ \begin{array}{l} ((m_1, 3), 0.68e^{i2\pi(0.72)}), \\ ((m_2, 2), 0.4e^{i2\pi(0.61)}), \\ ((m_3, 4), 0.82e^{i2\pi(0.8)}), \\ ((m_4, 2), 0.5e^{i2\pi(0.52)}), \\ ((m_5, 1), 0.25e^{i2\pi(0.41)}) \end{array} \right\} \in C\mathcal{F}(F(d_3))$$

$$\mu(d_4) = \left\{ \begin{array}{l} ((m_1, 4), 0.90e^{i2\pi(0.94)}), \\ ((m_2, 2), 0.54e^{i2\pi(0.48)}), \\ ((m_3, 2), 0.44e^{i2\pi(0.57)}), \\ ((m_4, 0), 0.13e^{i2\pi(0.23)}), \\ ((m_5, 1), 0.1e^{i2\pi(0.3)}) \end{array} \right\} \in C\mathcal{F}(F(d_4))$$

The tabular form of the information can display in Table 2.

The above example, motivated by the interpretations. We guess that the information is given in Table 2, are capture in any CFN-SSs over a finite universal set of alternatives, the general form of the attributes denotes in Table 3.

Remark 1 We have the following observations.

1. It isn't compulsory to set a scale for membership values to pick a grade as we picked in Example (1), one can also take arbitrary membership values to select a grade.
2. Grade $0 \in D$ in Definition (15), speaks about the lowest score. It doesn't imply that there is incomplete information or an absence of assessment.
3. Any CFN-SS can be considered as complex fuzzy $N + 1$ SS. Generally, it tends to be considered as an N' -SS with $N' > N$.

For example, the CF5-SS in Example (1) can be considered as CF6-SS over the same fuzzy parameterizations and parameters.

Motivated by Point 3 in Remark (1) we interpret the Definition (16)

Table 2 The tabular form of the CF5-SS in Example 1

$(\mu, (F, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(3, 0.68e^{i2\pi(0.72)})$	$(4, 0.90e^{i2\pi(0.94)})$
m_2	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(2, 0.4e^{i2\pi(0.61)})$	$(2, 0.54e^{i2\pi(0.48)})$
m_3	$(3, 0.7e^{i2\pi(0.77)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$	$(2, 0.44e^{i2\pi(0.57)})$
m_4	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(2, 0.5e^{i2\pi(0.52)})$	$(0, 0.13e^{i2\pi(0.23)})$
m_5	$(0, 0.14e^{i2\pi(0.15)})$	$(0, 0.1e^{i2\pi(0.24)})$	$(1, 0.25e^{i2\pi(0.41)})$	$(1, 0.1e^{i2\pi(0.3)})$

Table 3 The tabular form of the general CFN-SS

$(\mu, (F, D, N))$	d_1	d_2	$\dots$	d_q
m_1	(z_{11}, μ_{11})	(z_{11}, μ_{11})	$\dots$	(z_{11}, μ_{11})
m_2	(z_{11}, μ_{11})	(z_{11}, μ_{11})	$\dots$	(z_{11}, μ_{11})
$\vdots$				
m_p	(z_{11}, μ_{11})	(z_{11}, μ_{11})	$\dots$	(z_{11}, μ_{11})

Definition 16 A CFN-SS $(\mu, \mathcal{K})$ is called efficient if $((m_j, N-1), \mu_d(m_j)) \in \mu(d_k)$, for some $d_k \in D, m_j \in M$, where $\mu(d) = \mu'(d)e^{i2\pi(\omega_{\mu'}(d))}$ and $\mu'(d), \omega_{\mu'}(d) \in [0, 1]$.

Example 2 The CF5-SS in Example (1) is efficient.

Definition 17 If $(\mu, \mathcal{K})$ is a CFN-SS, then its minimized CFV-SS on M is designated by $(\mu_V, \mathcal{K}_V)$, where $\mathcal{K}_V = (F_V, D, V)$ is given as $V = \max_{\substack{m_j \in M \\ d_k \in D}} F(d_k)(m_j) + 1, F_v(d_k)(m_j) = F(d_k)(m_j)$ for all $d_k \in D, m_j \in M$ and $\mu_v(d) = \mu(d)$ for all $d_k \in D$.

Example 3 Consider $(\mu, \mathcal{K})$ be the CF6-SS is given in the tabular form in Table 4.

The minimized CFV-SS of CF6-SS is CF5-SS with $V = 5$, given in Table 5.

Remark 2 Any efficient CFN-SS coincides with its minimized CFN-SS.

Functional representation of CFN-SSs

The tabular form of CFN-SSs provides that we can use a function to express a more suitable representation of this concept.

Definition 18 The functional representation of a CFN-SS is a mapping $\Gamma : M \times D \rightarrow Z \times C$ such that $\Gamma(m, d) = (z, c)$ when $(m, z) \in F(d), c = \mu(d)(m, z)$ for all $d \in D, m \in M$ and C is the family of all complex fuzzy numbers.

Table 3 shows this general representation and relates it with the tabular representation described in portion 2.1. We can retrieve the original formulation of the CFN-SS presented in Definition (15) from functional representation.

Fundamental operations for CF-NSSs

Definition 19 Let $(\mu_1, (F_1, D_1, N_1))$ and $(\mu_2, (F_2, D_2, N_2))$ be two CFN-SSs. Then $(\mu_1, (F_1, D_1, N_1)) = (\mu_2, (F_2, D_2, N_2))$ iff $\mu_1 = \mu_2$ ($\mu'_1 = \mu'_2$ and $\omega_{\mu'_1} = \omega_{\mu'_2}$) and $(F_1, D_1, N_1) = (F_2, D_2, N_2)$.

Definition 20 Let $(\mu, (F, D, N))$ be a CFN-SS. Then the weak complement of the CFN-SS is $(\mu, (F^c, D, N))$ if (F^c, D, N) is the weak complement, i.e. $F^c(d) \cap F(d) = \emptyset, \forall d \in D$.

Example 4 Consider CF5-SS of Example (1) then the week complement of CF5-SS is given in Table 6.

Ramot et al. [7] defined the complex fuzzy complement of the complex fuzzy set as $\mu = \{(x_1\mu(x_1)), (x_2\mu(x_2)), (x_3\mu(x_3)), \dots, (x_n\mu(x_n))\} = \left\{ \left(x_1, \mu'(x_1)e^{i2\pi(\omega_{\mu'}(x_1))} \right), \left(x_2, \mu'(x_2)e^{i2\pi(\omega_{\mu'}(x_2))} \right), \dots, \left(x_n, \mu'(x_n)e^{i2\pi(\omega_{\mu'}(x_n))} \right) \right\}$

is a complex fuzzy set then complement of μ is

$$\begin{aligned} \mu^c &= \{(x_1, 1 - \mu(x_1)), (x_2, 1 - \mu(x_2)), (x_3, 1 - \mu(x_3)), \dots, (x_n, 1 - \mu(x_n))\} \\ &= \left\{ \left(x_1, 1 - \mu'(x_1)e^{i2\pi(1-\omega_{\mu'}(x_1))} \right), \left(x_2, 1 - \mu'(x_2)e^{i2\pi(1-\omega_{\mu'}(x_2))} \right), \dots, \left(x_n, 1 - \mu'(x_n)e^{i2\pi(1-\omega_{\mu'}(x_n))} \right) \right\} \end{aligned}$$

Table 4 The tabular form of CF6-SS

$(\mu, \mathcal{K} = (F, D, 6))$	d_1	d_2	d_3	d_4
m_1	$(4, 0.90e^{i2\pi(0.94)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.68e^{i2\pi(0.72)})$
m_2	$(3, 0.54e^{i2\pi(0.48)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.4e^{i2\pi(0.61)})$
m_3	$(2, 0.44e^{i2\pi(0.57)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(2, 0.7e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$
m_4	$(0, 0.13e^{i2\pi(0.23)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.5e^{i2\pi(0.52)})$

Table 5 The minimized CFV-SS of CF6-SS

$(\mu, \mathcal{K} = (F, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(4, 0.90e^{i2\pi(0.94)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.68e^{i2\pi(0.72)})$
m_2	$(3, 0.54e^{i2\pi(0.48)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.4e^{i2\pi(0.61)})$
m_3	$(2, 0.44e^{i2\pi(0.57)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(2, 0.7e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$
m_4	$(0, 0.13e^{i2\pi(0.23)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.5e^{i2\pi(0.52)})$

Table 6 The weak complement of CF5-SS

$(\mu, (F^c, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(2, 0.3e^{i2\pi(0.25)})$	$(4, 0.78e^{i2\pi(0.66)})$	$(4, 0.68e^{i2\pi(0.72)})$	$(1, 0.90e^{i2\pi(0.94)})$
m_2	$(3, 0.9e^{i2\pi(0.85)})$	$(4, 0.42e^{i2\pi(0.56)})$	$(1, 0.4e^{i2\pi(0.61)})$	$(3, 0.54e^{i2\pi(0.48)})$
m_3	$(1, 0.7e^{i2\pi(0.77)})$	$(2, 0.61e^{i2\pi(0.77)})$	$(3, 0.82e^{i2\pi(0.8)})$	$(3, 0.44e^{i2\pi(0.57)})$
m_4	$(4, 0.55e^{i2\pi(0.45)})$	$(3, 0.1e^{i2\pi(0.2)})$	$(0, 0.5e^{i2\pi(0.52)})$	$(3, 0.13e^{i2\pi(0.23)})$
m_5	$(4, 0.14e^{i2\pi(0.15)})$	$(1, 0.1e^{i2\pi(0.24)})$	$(0, 0.25e^{i2\pi(0.41)})$	$(4, 0.1e^{i2\pi(0.3)})$

We now interpret the complex fuzzy weak complement.

Definition 21 Let $(\mu, (F, D, N))$ be a CFN-SS on M . The complex fuzzy complement of $(\mu, (F, D, N))$ is $(\mu^c, (F, D, N))$ where $\mu^c : D \rightarrow \bigcup_{d \in D} C\mathcal{F}(F(d))$ such that $\mu^c(d) \in C\mathcal{F}(F(d))$ and $\mu^c(d)(m, z_d) = 1 - \mu(d)(m, z_d)$ for each $d \in D$.

In a fuzzy complement, the complex fuzzy grades are complementary and the parameterizations of the universe remain the same.

Definition 22 Let $(\mu, (F, D, N))$ be a CFN-SS on M . if (F^c, D, N) is weak complement and $(\mu^c, (F, D, N))$ is complex fuzzy complement then $(\mu^c, (F^c, D, N))$ is called complex fuzzy weak complement.

Example 5 Consider CF5-SS of Example (1) then the complex fuzzy weak complement of CF5-SS are given in Table 7.

Definition 23 Let $(\mu, (F, D, N))$ be a CFN-SS on M . The top complex fuzzy weak complement of $(\mu, (F, D, N))$ is $(\mu^c, (F^T, D, N))$, where.

$$F^T(d)(m) \begin{cases} N - 1 & \text{if } F(d)(m) < N - 1 \\ 0, & \text{if } F(d)(m) = N - 1 \end{cases}$$

Example 6 The top complex fuzzy weak complement of the CF5-SS in Example (1) is provided in Table 8.

Definition 24 Let $(\mu, (F, D, N))$ be a CFN-SS on M . The bottom complex fuzzy weak complement of $(\mu, (F, D, N))$ is $(\mu^c, (F^B, D, N))$, where.

$$F^B(d)(m) \begin{cases} 0 & \text{if } F(d)(m) < 0 \\ N - 1 & \text{if } F(d)(m) = 0 \end{cases}.$$

Example 7 The bottom complex fuzzy weak complement of the CF5-SS in Example (1) is provided in Table 9.

Definition 25 Let $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ be two CFN-SSs on M , where $\mathcal{K}_1 = (F_1, D_1, N_1)$ and $\mathcal{K}_2 = (F_2, D_2, N_2)$ are N-SSs on M . Then the restricted intersection of $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ is designated and defined as $(\mu_1, \mathcal{K}_1) \cap_{\mathcal{R}} (\mu_2, \mathcal{K}_2) = (\eta, \mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2)$, where $\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2 = (H, D_1 \cap D_2, \min(N_1, N_2)) \forall d_k \in D_1 \cap D_2$ and $m_j \in M$, $(z_{jk}, \mu_{jk}) \in \eta(d_k) \Leftrightarrow z_{jk} = \min(z_{jk}^1, z_{jk}^2)$ and $\min(\mu_{jk}^1, \mu_{jk}^2) = \left(\min(\mu_{jk}^1, \mu_{jk}^2) e^{i2\pi \left(\min(\omega_{\mu_{jk}^1}, \omega_{\mu_{jk}^2}) \right)} \right)$ if $(z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k)$ and $(z_{jk}^2, \mu_{jk}^2) \in \mu_2(d_k)$ where $d_k^1 \in D_1$ and $d_k^2 \in D_2$.

Example 8 Consider $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$, the CF5-SS and CF6-SS are given in Tables 10 and 11. Then their restricted intersection $(\mu_1, \mathcal{K}_1) \cap_{\mathcal{R}} (\mu_2, \mathcal{K}_2)$ is presented in Table 12.

Table 7 The complex fuzzy weak complement of CF5-SS

$(\mu^c, (F^c, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(2, 0.7e^{i2\pi(0.75)})$	$(4, 0.22e^{i2\pi(0.34)})$	$(4, 0.32e^{i2\pi(0.28)})$	$(1, 0.1e^{i2\pi(0.06)})$
m_2	$(3, 0.1e^{i2\pi(0.15)})$	$(4, 0.58e^{i2\pi(0.44)})$	$(1, 0.6e^{i2\pi(0.39)})$	$(3, 0.46e^{i2\pi(0.52)})$
m_3	$(1, 0.3e^{i2\pi(0.23)})$	$(2, 0.39e^{i2\pi(0.23)})$	$(3, 0.18e^{i2\pi(0.2)})$	$(3, 0.56e^{i2\pi(0.43)})$
m_4	$(4, 0.45e^{i2\pi(0.55)})$	$(3, 0.9e^{i2\pi(0.8)})$	$(0, 0.5e^{i2\pi(0.48)})$	$(3, 0.87e^{i2\pi(0.77)})$
m_5	$(4, 0.86e^{i2\pi(0.85)})$	$(1, 0.9e^{i2\pi(0.76)})$	$(0, 0.75e^{i2\pi(0.59)})$	$(4, 0.9e^{i2\pi(0.7)})$

Table 8 The top complex fuzzy weak complement of the CF5-SS

$(\mu^c, (F^T, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(4, 0.7e^{i2\pi(0.75)})$	$(4, 0.22e^{i2\pi(0.34)})$	$(4, 0.32e^{i2\pi(0.28)})$	$(0, 0.1e^{i2\pi(0.06)})$
m_2	$(0, 0.1e^{i2\pi(0.15)})$	$(4, 0.58e^{i2\pi(0.44)})$	$(4, 0.6e^{i2\pi(0.39)})$	$(4, 0.46e^{i2\pi(0.52)})$
m_3	$(4, 0.3e^{i2\pi(0.23)})$	$(4, 0.39e^{i2\pi(0.23)})$	$(0, 0.18e^{i2\pi(0.2)})$	$(4, 0.56e^{i2\pi(0.43)})$
m_4	$(4, 0.45e^{i2\pi(0.55)})$	$(4, 0.9e^{i2\pi(0.8)})$	$(4, 0.5e^{i2\pi(0.48)})$	$(4, 0.87e^{i2\pi(0.77)})$
m_5	$(4, 0.86e^{i2\pi(0.85)})$	$(4, 0.9e^{i2\pi(0.76)})$	$(4, 0.75e^{i2\pi(0.59)})$	$(4, 0.9e^{i2\pi(0.7)})$

Table 9 The bottom complex fuzzy weak complement of the CF5-SS

$(\mu^c, (F^B, D, 5))$	d_1	d_2	d_3	d_4
m_1	$(0, 0.7e^{i2\pi(0.75)})$	$(0, 0.22e^{i2\pi(0.34)})$	$(0, 0.32e^{i2\pi(0.28)})$	$(0, 0.1e^{i2\pi(0.06)})$
m_2	$(0, 0.1e^{i2\pi(0.15)})$	$(0, 0.58e^{i2\pi(0.44)})$	$(0, 0.6e^{i2\pi(0.39)})$	$(0, 0.46e^{i2\pi(0.52)})$
m_3	$(0, 0.3e^{i2\pi(0.23)})$	$(0, 0.39e^{i2\pi(0.23)})$	$(0, 0.18e^{i2\pi(0.2)})$	$(0, 0.56e^{i2\pi(0.43)})$
m_4	$(0, 0.45e^{i2\pi(0.55)})$	$(4, 0.9e^{i2\pi(0.8)})$	$(0, 0.5e^{i2\pi(0.48)})$	$(4, 0.87e^{i2\pi(0.77)})$
m_5	$(4, 0.86e^{i2\pi(0.85)})$	$(4, 0.9e^{i2\pi(0.76)})$	$(0, 0.75e^{i2\pi(0.59)})$	$(0, 0.9e^{i2\pi(0.7)})$

Table 10 The tabular form of the CF5-SS

$(\mu_1, \mathcal{K}_1)$	d_1	d_2	d_3	d_4
m_1	$(4, 0.90e^{i2\pi(0.94)})$	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(3, 0.68e^{i2\pi(0.72)})$
m_2	$(3, 0.54e^{i2\pi(0.48)})$	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(2, 0.4e^{i2\pi(0.61)})$
m_3	$(2, 0.44e^{i2\pi(0.57)})$	$(2, 0.7e^{i2\pi(0.77)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$
m_4	$(0, 0.13e^{i2\pi(0.23)})$	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(0, 0.5e^{i2\pi(0.52)})$

Table 11 The tabular form of the CF6-SS

$(\mu_1, \mathcal{K}_2)$	d_1	d_3	d_6	d_7
m_1	$(4, 0.8e^{i2\pi(0.7)})$	$(2, 0.45e^{i2\pi(0.6)})$	$(0, 0.1e^{i2\pi(0.07)})$	$(3, 0.6e^{i2\pi(0.8)})$
m_2	$(5, 0.95e^{i2\pi(0.9)})$	$(5, 0.9e^{i2\pi(0.92)})$	$(4, 0.88e^{i2\pi(0.79)})$	$(4, 0.8e^{i2\pi(0.9)})$
m_3	$(0, 0.1e^{i2\pi(0.17)})$	$(3, 0.61e^{i2\pi(0.7)})$	$(4, 0.8e^{i2\pi(0.77)})$	$(2, 0.4e^{i2\pi(0.5)})$
m_4	$(1, 0.3e^{i2\pi(0.35)})$	$(2, 0.44e^{i2\pi(0.57)})$	$(5, 0.93e^{i2\pi(0.93)})$	$(1, 0.03e^{i2\pi(0.19)})$

Definition 26 Let Γ_i is the functional representation of $(\mu_i, \mathcal{K}_i)$, $i = 1, 2$. Then the functional representation of their restricted intersection is $\Gamma : M \times (D_1 \cap D_2) \rightarrow Z \times C$

such that $\Gamma(m, d) = (\min(z_1, z_2), \min(c_1, c_2))$ for all $d \in D_1 \cap D_2$, $m \in M$, and $\Gamma_i(m, d) = (z_i, c_i)$, $i = 1, 2$.

Table 12 The tabular form of the restricted intersection $(\mu_1, \mathcal{K}_1) \cap_{\mathcal{R}} (\mu_2, \mathcal{K}_2)$

$(\eta, \mathcal{K}_2 \cap_{\mathcal{R}} \mathcal{K}_2)$	d_1	d_3
m_1	$(4, 0.8e^{i2\pi(0.7)})$	$(2, 0.45e^{i2\pi(0.6)})$
m_2	$(3, 0.54e^{i2\pi(0.48)})$	$(2, 0.42e^{i2\pi(0.56)})$
m_3	$(0, 0.1e^{i2\pi(0.17)})$	$(3, 0.61e^{i2\pi(0.7)})$
m_4	$(0, 0.13e^{i2\pi(0.23)})$	$(0, 0.1e^{i2\pi(0.2)})$

Definition 27 Let $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ be two CFN-SSs on M , where $\mathcal{K}_1 = (F_1, D_1, N_1)$ and $\mathcal{K}_2 = (F_2, D_2, N_2)$ are N-SSs on M . Then extended intersection of $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ is designated and defined as $(\mu_1, \mathcal{K}_1) \cap_{\mathcal{E}} (\mu_2, \mathcal{K}_2) = (\phi, \mathcal{K}_1 \cap_{\mathcal{E}} \mathcal{K}_2)$, where $\mathcal{K}_1 \cap_{\mathcal{E}} \mathcal{K}_2 = (P, D_1 \cap D_2, \min(N_1, N_2)) \forall d_k \in D_1 \cap D_2$ and $m_j \in M$, and ϕ_{d_k} is presented by.

$$\phi_{d_k} = \begin{cases} \mu_{1d_k} & \text{if } d_k \in D_1 - D_2 \\ \mu_{2d_k} & \text{if } d_k \in D_2 - D_1 \\ (z_{jk}, \mu_{jk}) & \left(\begin{array}{l} \text{such that } z_{jk} = \min(z_{jk}^1, z_{jk}^2) \text{ and } \mu_{jk} = \min(\mu_{jk}^1, \mu_{jk}^2) \\ = \left(\min(\mu_{jk}^1, \mu_{jk}^2) e^{i2\pi \left(\min(\omega_{\mu_{jk}^1}, \omega_{\mu_{jk}^2}) \right)} \right), \text{ where } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k) \text{ and} \\ (z_{jk}^2, \mu_{jk}^2) \in \mu_2(d_k) \text{ } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2. \end{array} \right) \end{cases}$$

Example 9 Consider CF5-SS and CF6-SS given in Tables 10 and 11 of Example (8) then in this case the extended intersection $(\phi, \mathcal{K}_1 \cap_{\mathcal{E}} \mathcal{K}_2)$ is presented in Table 13).

Definition 28 Let Γ_i is the functional representation of $(\mu_i, \mathcal{K}_i)$, $i = 1, 2$. Then the functional representation of their extended intersection is $\Gamma : M \times (D_1 \cap D_2) \rightarrow Z \times C$ such that.

$$\Gamma(m, d) = \Gamma_1(m, d) = (z_1, c_1) \text{ when } d \in D_1 - D_2$$

$$\Gamma(m, d) = \Gamma_2(m, d) = (z_2, c_2) \text{ when } d \in D_1 - D_2,$$

$$\Gamma(m, d) = (\min(z_1, z_2), \min(c_1, c_2)) \text{ for all } d \in D_1 \cap D_2, m \in M, \text{ and } \Gamma_i(m, d) = (z_i, m_i), i = 1, 2.$$

Definition 29 Let $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ be two CFN-SSs on M , where $\mathcal{K}_1 = (F_1, D_1, N_1)$ and $\mathcal{K}_2 = (F_2, D_2, N_2)$ are N-SSs on M . Then restricted union of $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ is designated and defined as $(\mu_1, \mathcal{K}_1) \cup_{\mathcal{R}} (\mu_2, \mathcal{K}_2) = (\delta, \mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2)$, where $\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2 = (G, D_1 \cap D_2, \max(N_1, N_2)) \forall d_k \in D_1 \cap D_2$ and $m_j \in M$, $(z_{jk}, \mu_{jk}) \in \delta(d_k) \Leftrightarrow z_{jk} = \max(z_{jk}^1, z_{jk}^2)$ and $\mu_{jk} = \max(\mu_{jk}^1, \mu_{jk}^2) =$

$$\left(\max(\mu_{jk}^1, \mu_{jk}^2) e^{i2\pi \left(\max(\omega_{\mu_{jk}^1}, \omega_{\mu_{jk}^2}) \right)} \right) \text{ if } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k) \text{ and } (z_{jk}^2, \mu_{jk}^2) \in \mu_2(d_k) \text{ where } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2.$$

Example 10 Consider CF5-SS and CF6-SS given in Tables 10 and 11 of Example (8) then in this case the restricted union $(\delta, \mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2)$ is presented in Table 14).

Definition 30 Let Γ_i is the functional representation of $(\mu_i, \mathcal{K}_i)$, $i = 1, 2$. Then the functional representation of their restricted intersection is $\Gamma : M \times (D_1 \cap D_2) \rightarrow Z \times C$ such that $\Gamma(m, d) = (\max(z_1, z_2), \max(c_1, c_2))$ for all $d \in D_1 \cap D_2, m \in M$, and $\Gamma_i(m, d) = (z_i, c_i), i = 1, 2$.

Definition 31 Let $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ be two CFN-SSs on M , where $\mathcal{K}_1 = (F_1, D_1, N_1)$ and $\mathcal{K}_2 = (F_2, D_2, N_2)$ are N-SSs on M . Then extended union of $(\mu_1, \mathcal{K}_1)$ and $(\mu_2, \mathcal{K}_2)$ is designated and defined as $(\mu_1, \mathcal{K}_1) \cup_{\mathcal{E}} (\mu_2, \mathcal{K}_2) = (\gamma, \mathcal{K}_1 \cup_{\mathcal{E}} \mathcal{K}_2)$, where $\mathcal{K}_1 \cup_{\mathcal{E}} \mathcal{K}_2 = (L, D_1 \cap D_2, \max(N_1, N_2)) \forall d_k \in D_1 \cap D_2$ and $m_j \in M$, and ϕ_{d_k} is presented by.

Table 13 The tabular form of the extended intersection $(\mu_1, \mathcal{K}_1) \cap_{\mathcal{E}} (\mu_2, \mathcal{K}_2)$

$(\phi, \mathcal{K}_2 \cap_{\mathcal{E}} \mathcal{K}_2)$	d_1	d_2	d_3	d_4	d_6	d_7
m_1	$(4, 0.8e^{i2\pi(0.7)})$	$(1, 0.3e^{i2\pi(0.25)})$	$(2, 0.45e^{i2\pi(0.6)})$	$(3, 0.68e^{i2\pi(0.72)})$	$(0, 0.1e^{i2\pi(0.07)})$	$(3, 0.6e^{i2\pi(0.8)})$
m_2	$(3, 0.54e^{i2\pi(0.48)})$	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(2, 0.4e^{i2\pi(0.61)})$	$(4, 0.88e^{i2\pi(0.79)})$	$(4, 0.8e^{i2\pi(0.9)})$
m_3	$(0, 0.1e^{i2\pi(0.17)})$	$(2, 0.7e^{i2\pi(0.77)})$	$(3, 0.61e^{i2\pi(0.7)})$	$(4, 0.82e^{i2\pi(0.8)})$	$(4, 0.8e^{i2\pi(0.77)})$	$(2, 0.4e^{i2\pi(0.5)})$
m_4	$(0, 0.13e^{i2\pi(0.23)})$	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(0, 0.5e^{i2\pi(0.52)})$	$(5, 0.93e^{i2\pi(0.93)})$	$(1, 0.03e^{i2\pi(0.19)})$

Table 14 The tabular form of the restricted union $(\mu_1, \mathcal{K}_1) \cup_{\mathcal{R}} (\mu_2, \mathcal{K}_2)$

$(\delta, \mathcal{K}_2 \cup_{\mathcal{R}} \mathcal{K}_2)$	d_1	d_3
m_1	$(4, 0.9e^{i2\pi(0.94)})$	$(3, 0.78e^{i2\pi(0.66)})$
m_2	$(5, 0.95e^{i2\pi(0.9)})$	$(5, 0.95e^{i2\pi(0.92)})$
m_3	$(2, 0.44e^{i2\pi(0.57)})$	$(3, 0.61e^{i2\pi(0.77)})$
m_4	$(1, 0.3e^{i2\pi(0.35)})$	$(2, 0.44e^{i2\pi(0.57)})$

$$\gamma_{d_k} = \begin{cases} \mu_{1d_k} & \text{if } d_k \in D_1 - D_2 \\ \mu_{2d_k} & \text{if } d_k \in D_2 - D_1 \\ (z_{jk}, \mu_{jk}) & \left(\begin{array}{l} \text{such that } z_{jk} = \max(z_{jk}^1, z_{jk}^2) \text{ and } \mu_{jk} = \max(\mu_{jk}^1, \mu_{jk}^2) \\ = \left(\max(\mu_{jk}^1, \mu_{jk}^2) e^{i2\pi \left(\max(\omega_{\mu_{jk}^1}, \omega_{\mu_{jk}^2}) \right)} \right), \text{ where } (z_{jk}^1, \mu_{jk}^1) \in \mu_1(d_k) \text{ and} \\ (z_{jk}^2, \mu_{jk}^2) \in \mu_2(d_k) \text{ } d_k^1 \in D_1 \text{ and } d_k^2 \in D_2. \end{array} \right) \end{cases}$$

Example 11 Consider CF5-SS and CF6-SS given in Tables 10 and 11 of Example (8) then in this case the extended union $(\gamma, \mathcal{K}_1 \cup_{\mathcal{E}} \mathcal{K}_2)$ is presented in Table 15.

Definition 32 Let Γ_i is the functional representation of $(\mu_i, \mathcal{K}_i)$, $i = 1, 2$. Then the functional representation of their extended intersection is $\Gamma : M \times (D_1 \cap D_2) \rightarrow Z \times C$ such that.

$$\begin{aligned} \Gamma(m, d) &= \Gamma_1(m, d) = (z_1, c_1) \text{ when } d \in D_1 - D_2 \\ \Gamma(m, d) &= \Gamma_2(m, d) = (z_2, c_2) \text{ when } d \in D_2 - D_1 \\ \Gamma(m, d) &= (\max(z_1, z_2), \max(c_1, c_2)) \text{ for all } d \in D_1 \cap D_2, m \in M, \text{ and } \Gamma_i(m, d) = (z_i, m_i), i = 1, 2 \end{aligned}$$

Relationships

The notion of CFN-SS can be identified with both N-SSs and complex fuzzy SSs (CFSSs). In this portion, we are going to clarify these relationships. For that reason, let us fix the accompanying setting: M means the set of objects and $(\mu, \mathcal{K})$ is a CFN-SS, where $\mathcal{K} = (F, D, N)$ is an N-SS.

We state that the CF-NSSs are the generalization of fuzzy N-SSs, N-SSs, and soft sets. Through the following Definitions one can derive SSs and CFSSs from $(\mu, \mathcal{K})$.

Definition 33 Let $0 < I < N$ be a threshold. The CFSS over M related with $(\mu, \mathcal{K})$ and I is (μ^I, D) given by: for each $d \in D$, $\mu^I(d) \in C\mathcal{F}(M)$ is such that.

$$\mu^I(d)(m) \begin{cases} \mu(d)(m, z_d) = \mu_d(m) \text{ if } (m, z_d) \in F(d) \text{ and } z_d \geq I \\ \text{Otherwise} \end{cases}$$

As we know that $\mu(d) = \mu'(d)e^{i2\pi(\omega_{\mu'}(d))}$, by letting $\omega_{\mu'}(d) = 0$ we can relate the fuzzy SS with CF-NSS.

Definition 34 In sense of the functional representation of $(\mu, \mathcal{K})$ the CFSS (μ^I, D) is designated and defined as

$$\mu^I(d)(m) \begin{cases} \mu & \text{if } \Gamma(m, d) = (z, \mu) \text{ and } z \geq I \\ 0 & \text{otherwise} \end{cases}.$$

Definition 35 Let $0 < I < N$ and $\lambda \in [0, 1]$ be two thresholds. The SS over M related with $(\mu, \mathcal{K})$ and (I, λ) is $(f^{(I, \lambda)}, D)$ given by: for each $d \in D$, $(f^{(I, \lambda)}, D) = \{m \in M : \Delta\mu^I(d)(m) \in \lambda\}$, where.

Table 15 The tabular form of the extended union $(\mu_1, \mathcal{K}_1) \cup_{\mathcal{E}} (\mu_2, \mathcal{K}_2)$

$(\gamma, \mathcal{K}_2 \cup_E \mathcal{K}_2)$	d_1	d_2	d_3	d_4	d_6	d_7
m_1	$(4, 0.9e^{i2\pi(0.94)})$	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(3, 0.68e^{i2\pi(0.72)})$	$(0, 0.1e^{i2\pi(0.07)})$	$(3, 0.6e^{i2\pi(0.8)})$
m_2	$(5, 0.95e^{i2\pi(0.9)})$	$(4, 0.9e^{i2\pi(0.85)})$	$(5, 0.95e^{i2\pi(0.92)})$	$(2, 0.4e^{i2\pi(0.61)})$	$(4, 0.88e^{i2\pi(0.79)})$	$(4, 0.8e^{i2\pi(0.9)})$
m_3	$(2, 0.44e^{i2\pi(0.57)})$	$(2, 0.7e^{i2\pi(0.77)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$	$(4, 0.8e^{i2\pi(0.77)})$	$(2, 0.4e^{i2\pi(0.5)})$
m_4	$(1, 0.3e^{i2\pi(0.35)})$	$(2, 0.55e^{i2\pi(0.45)})$	$(2, 0.44e^{i2\pi(0.57)})$	$(0, 0.5e^{i2\pi(0.52)})$	$(5, 0.93e^{i2\pi(0.93)})$	$(1, 0.03e^{i2\pi(0.19)})$

Table 16 CFSS linked with $(\mu, \mathcal{K})$ with threshold 1

(μ^1, D)	d_1	d_2	d_3	d_4
m_1	$0.3e^{i2\pi(0.25)}$	$0.78e^{i2\pi(0.66)}$	$0.68e^{i2\pi(0.72)}$	$0.90e^{i2\pi(0.94)}$
m_2	$0.9e^{i2\pi(0.85)}$	$0.42e^{i2\pi(0.56)}$	$0.4e^{i2\pi(0.61)}$	$0.54e^{i2\pi(0.48)}$
m_3	$0.7e^{i2\pi(0.77)}$	$0.61e^{i2\pi(0.77)}$	$0.82e^{i2\pi(0.8)}$	$0.44e^{i2\pi(0.57)}$
m_4	$0.55e^{i2\pi(0.45)}$	0	$0.5e^{i2\pi(0.52)}$	0
m_5	0	0	$0.25e^{i2\pi(0.41)}$	$0.1e^{i2\pi(0.3)}$

Table 17 CFSS linked with $(\mu, \mathcal{K})$ with threshold 2

(μ^2, D)	d_1	d_2	d_3	d_4
m_1	0	$0.78e^{i2\pi(0.66)}$	$0.68e^{i2\pi(0.72)}$	$0.90e^{i2\pi(0.94)}$
m_2	$0.9e^{i2\pi(0.85)}$	$0.42e^{i2\pi(0.56)}$	$0.4e^{i2\pi(0.61)}$	$0.54e^{i2\pi(0.48)}$
m_3	$0.7e^{i2\pi(0.77)}$	$0.61e^{i2\pi(0.77)}$	$0.82e^{i2\pi(0.8)}$	$0.44e^{i2\pi(0.57)}$
m_4	$0.55e^{i2\pi(0.45)}$	0	$0.5e^{i2\pi(0.52)}$	0
m_5	0	0	0	0

Table 18 CFSS linked with $(\mu, \mathcal{K})$ with threshold 3

(μ^3, D)	d_1	d_2	d_3	d_4
m_1	0	$0.78e^{i2\pi(0.66)}$	$0.68e^{i2\pi(0.72)}$	$0.90e^{i2\pi(0.94)}$
m_2	$0.9e^{i2\pi(0.85)}$	0	0	0
m_3	$0.7e^{i2\pi(0.77)}$	$0.61e^{i2\pi(0.77)}$	$0.82e^{i2\pi(0.8)}$	0
m_4	0	0	0	0
m_5	0	0	0	0

Table 19 CFSS linked with $(\mu, \mathcal{K})$ with threshold 4

(μ^4, D)	d_1	d_2	d_3	d_4
m_1	0	0	0	$0.90e^{i2\pi(0.94)}$
m_2	$0.9e^{i2\pi(0.85)}$	0	0	0
m_3	0	0	$0.82e^{i2\pi(0.8)}$	0
m_4	0	0	0	0
m_5	0	0	0	0

Example 12 Consider the CF5-SS given in Example (1). By Definition (33) we can find the associated CFSSs with CF5-SS. Let $0 < I < 5$ be the threshold. Then the possible CFSSs related to thresholds 1, 2, 3, 4 are interpreted by Tables 16, 17, 18 and 19 respectively.

Applications

This segment clarifies the decision-making (DM) process that works on models that we have interpreted in previous segments. Consequently, we characterize particular algorithms for problems that are described by CFN-SSs. To demonstrate their significance also, achievability we apply them to real circumstances that are completely developed.

We interpreted the following three algorithms of CFN-SSs for DM.

$$\Delta\mu^I(d)(m) \begin{cases} \Delta\mu(d)(m, z_d) = \Delta\mu_d(m) & \text{if } (m, z_d) \in F(d) \text{ and } z_d \geq I \\ 0 & \text{otherwise} \end{cases},$$

$$\text{and } \Delta\mu_d(m) = \frac{\mu'_d(m) + \omega_{\mu'_d}(m)}{2}.$$

Algorithm 1 The algorithm of choice values of CFN-SSs

1. Input $M = \{m_1, m_2, m_3, \dots, m_p\}$ as universe set
2. Input $D = \{d_1, d_2, d_3, \dots, d_q\}$ as a set of attributes.
3. Input $\mathcal{K} = (F, D, N)$ with $Z = \{0, 1, 2, \dots, N - 1\}$, $N \in \{2, 3, \dots\}$, for each $m_j \in M$, $d_k \in D$, there exists $z_{jk} \in Z$.
4. Input CFN-SS $(\mu, \mathcal{K})$, where $\mathcal{K} = (F, D, N)$.
5. Compute $\mathfrak{Q}_\ell = (\sum_{k=1}^q z_{jk}, \sum_{k=1}^q \mu_{jk})$, where $(m_j, z_{jk}) \in F(d)$, $\mu = \mu(d)(m_j, z_{jk})$, $\forall m_j \in M$, and the addition of two CFNs is interpreted as, if $\mu_1(x) = \mu'_1(x)e^{i2\pi(\omega_{\mu'_1(x)})}$ and $\mu_2(x) = \mu'_2(x)e^{i2\pi(\omega_{\mu'_2(x)})}$ are two CFNs then

$$\mu_1(x) + \mu_2(x) = (\mu'_1(x) + \mu'_2(x) - (\mu'_1(x) \cdot \mu'_2(x))) e^{i2\pi(\omega_{\mu'_1(x)} + \omega_{\mu'_2(x)} - (\omega_{\mu'_1(x)} \cdot \omega_{\mu'_2(x)}))}$$

6. Compute all indices ℓ for which $\mathfrak{Q}_\ell = \max_{\ell=1,2,\dots,p} \mathfrak{Q}_\ell$
7. Here we will define lexicographical order on CFN's to compare them, i.e. let $\mu_1(x) = \mu'_1(x)e^{i2\pi(\omega_{\mu'_1(x)})}$ and $\mu_2(x) = \mu'_2(x)e^{i2\pi(\omega_{\mu'_2(x)})}$ be two CFNs. Then if $\mu'_1(x) < \mu'_2(x)$ then we say $\mu_1(x) < \mu_2(x)$. But if $\mu'_1(x) = \mu'_2(x)$ then we compare the amplitude part i.e. if $\omega_{\mu'_1(x)} < \omega_{\mu'_2(x)}$ then we say $\mu_1(x) < \mu_2(x)$. If both $\mu'_1(x) = \mu'_2(x)$ and $\omega_{\mu'_1(x)} = \omega_{\mu'_2(x)}$ then we say $\mu_1(x) = \mu_2(x)$.
8. Any of the alternatives for which $\mathfrak{Q}_\ell = \max_{\ell=1,2,\dots,p} \mathfrak{Q}_\ell$ can be picked.
9. In DM we have to two cases:
 - i. If the phase part of the membership grade is non-zero ($\omega'_\mu(x) \neq 0$) then the decisions will make based on membership grades.
 - ii. If the phase part of the membership grade is equal to zero ($\omega'_\mu(x) = 0$) then the decisions will make based on grades.

Algorithm 2 The algorithm of I -choice values (I -CVs) of CFN-SSs

1. Input $M = \{m_1, m_2, m_3, \dots, m_p\}$ as universe set
2. Input $D = \{d_1, d_2, d_3, \dots, d_q\}$ as a set of attributes.
3. Input $\mathcal{K} = (F, D, N)$ with $Z = \{0, 1, 2, \dots, N - 1\}$, $N \in \{2, 3, \dots\}$, for each $m_j \in M$, $d_k \in D$, there exists $z_{jk} \in Z$.
4. Input CFN-SS $(\mu, \mathcal{K})$, where $\mathcal{K} = (F, D, N)$.
5. Input I threshold.
6. Determine $\mu^I(d)(m) \begin{cases} \mu & \text{if } \Gamma(m, d) = (z, \mu) \text{ and } z \geq I \\ 0 & \text{otherwise} \end{cases}$
7. Determine all indices ℓ for which $\mathfrak{Q}_\ell^I = \max_{\ell=1,2,\dots,p} \mathfrak{Q}_\ell^I$
8. Any of the alternatives for which $\mathfrak{Q}_\ell^I = \max_{\ell=1,2,\dots,p} \mathfrak{Q}_\ell^I$ can be picked.

Algorithm 3 The algorithm of the comparison table of CFN-SSs

1. Input $M = \{m_1, m_2, m_3, \dots, m_p\}$ as universe set
2. Input $D = \{d_1, d_2, d_3, \dots, d_q\}$ as a set of attributes.
3. Consider a CFN-SS in tabular representation.
4. Determine the comparison table of information function.
5. Determine the information score.
6. Note the maximum outcome with the maximum grade, if it takes place in j th row then we will select option m_j , $1 \leq j \leq p$.

Selection of faculty member in a university

In an educational field, the selection of faculty members is the way toward placing the right persons in the right job. Choosing the right person for a job is a tough assignment for educational institutes. Effective choices can only be made when there is successful matching. Selecting the best applicant as a faculty member will provide quality education in the best way. Taking Example (1) in which a university wants to appoint a faculty member for the department of mathematics based on star ranking and ratings awarded by the selection board of university to the applicants, represented the tabular form Table 2.

Choice value (CV) of CF5-SSs

We can calculate the CV of CF5-SS of the applicant's selection as

$$\Omega_j = (\sum_{k=1}^q z_{jk}, \sum_{k=1}^q \mu_{jk}), \text{ when } (m_j, z_{jk}) \in F(d), \text{ and } \mu = \mu(d)(m_j, z_{jk}), \forall m_j \in M.$$

From CVs given in Table 20 we can see that the applicant m_3 has the highest grade given by the selection board of the university. So the university will appoint the applicant m_3 as a faculty member in the department of mathematics. We also note the ranking as $m_3 > m_1 > m_2 > m_4 > m_5$.

I-CV of CF5-SSs

It is additionally practical to estimate the objects in M from the information in $(\mu, \mathcal{K})$, where $\mathcal{K} = (F, D, N)$ is N -SS and a threshold I , by standard applications of CVs to the CFSS (μ^I, D) . For this reason, we speak to it Ω_j^I the CV at choice j of CFSS (μ^R, D) and call Ω_j^I the I -CV of $(\mu, \mathcal{K})$, where $\mathcal{K} = (F, D, N)$ is an N -SS at choice j . we get Table 21.

In Table 21, we supposed $I = 3$ for DM so we get the 3-CV of CF5-SS. It can be observed from Table 21) that the applicant m_1 has the highest grade so m_1 is selected as a faculty member.

Comparison table of CF5-SSs

Definition 36 [40] it is a square table in which the number of rows and columns are the same, and both are tag by the name of objects of the universe such as $m_1, m_2, m_3, \dots, m_p$ and ϵ_{jk} be the entries, where ϵ_{jk} = the number of parameters for which the value of m_j exceeds or equal to the value of m_k .

Membership grades of Table 2 are presented in tabular form in Table 22.

The comparison table of membership grades is presented in Table 23.

The membership grades of each applicant are derived by subtracting the column sum from the row sum of Table 24.

From Table 24 we observe that the highest score is 8 which obtained by m_3 and m_1 but m_3 has the highest grade. So applicant m_3 is selected as a faculty member of the department of mathematics.

Selection of a player in the national German football team

The German football association [or Deutscher Fußball-Bund (DFB)] is a sport's national governing body. 26, 000 football clubs in Germany are associated with DFB. These clubs have more than 170, 000 teams with over 2 million players. The selection committee of DFB does a selection of the players for the national German football team from these clubs. It is not an easy job to select a skillful and talented player for the national team from these clubs.

Suppose a selection committee of DFB has to select a player for the German football team in the 4 shortlisted players. Let $\mathfrak{u} = \{i_1, i_2, i_3, i_4\}$ be the universe of players and $D = \{d_1, d_2, d_3, d_4\}$ be set of attributes. A selection committee allocates grades or ratings to the players depend on the per-

Table 20 The tabular form of the CV of CF5-SSs

$(\mu, (F, D, 5))$	d_1	d_2	d_3	d_4	Ω_j
m_1	$(1, 0.3e^{i2\pi(0.25)})$	$(3, 0.78e^{i2\pi(0.66)})$	$(3, 0.68e^{i2\pi(0.72)})$	$(4, 0.90e^{i2\pi(0.94)})$	$(10, 0.995e^{i2\pi(0.987)})$
m_2	$(4, 0.9e^{i2\pi(0.85)})$	$(2, 0.42e^{i2\pi(0.56)})$	$(2, 0.4e^{i2\pi(0.61)})$	$(2, 0.54e^{i2\pi(0.48)})$	$(10, 0.984e^{i2\pi(0.97)})$
m_3	$(3, 0.7e^{i2\pi(0.77)})$	$(3, 0.61e^{i2\pi(0.77)})$	$(4, 0.82e^{i2\pi(0.8)})$	$(2, 0.44e^{i2\pi(0.57)})$	$(12, 0.988e^{i2\pi(0.98)})$
m_4	$(2, 0.55e^{i2\pi(0.45)})$	$(0, 0.1e^{i2\pi(0.2)})$	$(2, 0.5e^{i2\pi(0.52)})$	$(0, 0.13e^{i2\pi(0.23)})$	$(4, 0.824e^{i2\pi(0.797)})$
m_5	$(0, 0.14e^{i2\pi(0.15)})$	$(0, 0.1e^{i2\pi(0.24)})$	$(1, 0.25e^{i2\pi(0.41)})$	$(1, 0.1e^{i2\pi(0.3)})$	$(2, 0.478e^{i2\pi(0.649)})$

Table 21 The tabular form of 3-CV of CF5-SSs

(μ^3, D)	d_1	d_2	d_3	d_4	Ω_j^3
m_1	0	$0.78e^{i2\pi(0.66)}$	$0.68e^{i2\pi(0.72)}$	$0.90e^{i2\pi(0.94)}$	$0.99e^{i2\pi(0.994)}$
m_2	$0.9e^{i2\pi(0.85)}$	0	0	0	$0.9e^{i2\pi(0.85)}$
m_3	$0.7e^{i2\pi(0.77)}$	$0.61e^{i2\pi(0.77)}$	$0.82e^{i2\pi(0.8)}$	0	$0.979e^{i2\pi(0.989)}$
m_4	0	0	0	0	0
m_5	0	0	0	0	0

Table 22 The tabular form of the membership values

$\mu_{jk}(D)$	d_1	d_2	d_3	d_4
m_1	$0.3e^{i2\pi(0.25)}$	$0.78e^{i2\pi(0.66)}$	$0.68e^{i2\pi(0.72)}$	$0.90e^{i2\pi(0.94)}$
m_2	$0.9e^{i2\pi(0.85)}$	$0.42e^{i2\pi(0.56)}$	$0.4e^{i2\pi(0.61)}$	$0.54e^{i2\pi(0.48)}$
m_3	$0.7e^{i2\pi(0.77)}$	$0.61e^{i2\pi(0.77)}$	$0.82e^{i2\pi(0.8)}$	$0.44e^{i2\pi(0.57)}$
m_4	$0.55e^{i2\pi(0.45)}$	$0.1e^{i2\pi(0.2)}$	$0.5e^{i2\pi(0.52)}$	$0.13e^{i2\pi(0.23)}$
m_5	$0.14e^{i2\pi(0.15)}$	$0.1e^{i2\pi(0.24)}$	$0.25e^{i2\pi(0.41)}$	$0.1e^{i2\pi(0.3)}$

Table 23 Comparison table of membership grades

	m_1	m_2	m_3	m_4	m_5
m_1	4	3	2	3	4
m_2	1	4	2	3	4
m_3	2	2	4	4	4
m_4	1	1	0	4	2
m_5	0	0	0	1	4

formance of the player for their clubs. A 4-SS can be derived from Table 23 where

1. Three stars appear for ‘Excellent’,
2. Two stars appear for ‘Good’,
3. One star appears for ‘Normal’,
4. Hole appear for ‘Poor’,

This graded evaluation by stars can undoubtedly be related to numbers as $Z = \{0, 1, 2, 3\}$, where.

- 0 serves as “o”,
- 1 serves as “*”,
- 2 serves as “**”,
- 3 serves as “***”,

Table 25 presented the information obtained from related data, and also presented the tabular representation of its related 4-SS.

The selection committee follows this criterion based on the performance of the player as follows:

$$0.0 \leq \Delta\mu_d(m) < 0.3 \text{ when } z_d = 0;$$

$$0.3 \leq \Delta\mu_d(m) < 0.5 \text{ when } z_d = 1;$$

$$0.5 \leq \Delta\mu_d(m) < 0.8 \text{ when } z_d = 2;$$

$$0.8 \leq \Delta\mu_d(m) < 1.0 \text{ when } z_d = 3,$$

Where $\Delta\mu_d(m) = \frac{\mu'_d(m) + \omega'_d(m)}{2}$. Therefore, the following CF4-soft set by using definition (15), is defined:

Table 24 Membership score table

	Grade sum (gs)	Row sum (rs)	Column Sum (cs)	Final score = (rs-cs)
m_1	10	16	8	8
m_2	10	14	10	4
m_3	12	16	8	8
m_4	4	8	15	-7
m_5	2	5	18	-13

Table 25 Information obtained from the linked data tabular representation of its related 4-SS

i/D	d_1	d_2	d_3	d_4	Stars	Representations
i_1	$* = 1$	$** = 2$	$* = 1$	$o = 0$	$o = 0$	Poor
i_2	$** = 2$	$*** = 3$	$* = 3$	$* = 1$	$* = 1$	Normal
i_3	$** = 2$	$o = 0$	$** = 2$	$** = 2$	$** = 2$	Good
i_4	$*** = 3$	$*** = 3$	$** = 2$	$*** = 3$	$*** = 3$	Excellent

Table 26 The tabular form of CF4-SS

$(\mu, (F, D, 4))$	d_1	d_2	d_3	d_4
i_1	$(1, 0.4e^{i2\pi(0.35)})$	$(2, 0.4e^{i2\pi(0.6)})$	$(3, 0.95e^{i2\pi(0.99)})$	$(2, 0.8e^{i2\pi(0.6)})$
i_2	$(2, 0.75e^{i2\pi(0.65)})$	$(3, 0.6e^{i2\pi(0.9)})$	$(3, 0.88e^{i2\pi(0.96)})$	$(0, 0.12e^{i2\pi(0.19)})$
i_3	$(2, 0.7e^{i2\pi(0.4)})$	$(0, 0.01e^{i2\pi(0.1)})$	$(3, 0.9e^{i2\pi(0.83)})$	$(2, 0.56e^{i2\pi(0.63)})$
i_4	$(3, 0.89e^{i2\pi(0.9)})$	$(3, 0.81e^{i2\pi(0.85)})$	$(2, 0.79e^{i2\pi(0.75)})$	$(2, 0.51e^{i2\pi(0.57)})$

Table 27 The tabular form of the CV of CF4-SSs

$(\mu, (F, D, 4))$	d_1	d_2	d_3	d_4	Ω_j
i_1	$(1, 0.4e^{i2\pi(0.35)})$	$(2, 0.4e^{i2\pi(0.6)})$	$(3, 0.95e^{i2\pi(0.99)})$	$(2, 0.8e^{i2\pi(0.6)})$	$(8, 0.996e^{i2\pi(0.997)})$
i_2	$(2, 0.75e^{i2\pi(0.65)})$	$(3, 0.6e^{i2\pi(0.9)})$	$(3, 0.88e^{i2\pi(0.96)})$	$(0, 0.12e^{i2\pi(0.19)})$	$(8, 0.991e^{i2\pi(0.989)})$
i_3	$(2, 0.7e^{i2\pi(0.4)})$	$(0, 0.01e^{i2\pi(0.1)})$	$(3, 0.9e^{i2\pi(0.83)})$	$(2, 0.56e^{i2\pi(0.63)})$	$(7, 0.987e^{i2\pi(0.962)})$
i_4	$(3, 0.89e^{i2\pi(0.9)})$	$(3, 0.81e^{i2\pi(0.85)})$	$(2, 0.79e^{i2\pi(0.75)})$	$(2, 0.51e^{i2\pi(0.57)})$	$(10, 0.998e^{i2\pi(0.989)})$

Table 28 The tabular form of 3-CV of CF4-SSs

(μ^3, D)	d_1	d_2	d_3	d_4	Ω_j^3
i_1	0	0	$0.95e^{i2\pi(0.99)}$	0	$0.95e^{i2\pi(0.99)}$
i_2	0	$0.6e^{i2\pi(0.9)}$	$0.88e^{i2\pi(0.96)}$	0	$0.952e^{i2\pi(0.996)}$
i_3	0	0	$0.9e^{i2\pi(0.83)}$	0	$0.9e^{i2\pi(0.83)}$
i_4	$0.89e^{i2\pi(0.9)}$	$0.81e^{i2\pi(0.85)}$	0	0	$0.979e^{i2\pi(0.985)}$

$$\begin{aligned}\mu(d_1) &= \left\{ \left((i_1, 1), 0.4e^{i2\pi(0.35)} \right), \left((i_2, 2), 0.75e^{i2\pi(0.65)} \right) \right\} \in CF(F(d_1)) \\ \mu(d_2) &= \left\{ \left((i_1, 2), 0.4e^{i2\pi(0.6)} \right), \left((i_2, 3), 0.6e^{i2\pi(0.9)} \right) \right\} \in CF(F(d_2)) \\ \mu(d_3) &= \left\{ \left((i_1, 2), 0.8e^{i2\pi(0.6)} \right), \left((i_2, 0), 0.12e^{i2\pi(0.19)} \right) \right\} \in CF(F(d_3)) \\ \mu(d_4) &= \left\{ \left((i_1, 3), 0.95e^{i2\pi(0.99)} \right), \left((i_2, 3), 0.88e^{i2\pi(0.96)} \right) \right\} \in CF(F(d_4))\end{aligned}$$

The tabular form of the information can display in Table 26.

CV of CF4-SSs

The calculated CV of CF4-SS of the player's selection is presented in Table 27.

From CVs given in Table 27 we can note that the player i_4 has maximum grade given by the selection committee of the German football association. So the player i_4 is selected for the national team of Germany. We also note the ranking as $i_4 > i_1 > i_2 > i_3$.

Table 29 The tabular form of membership grades

$\mu_{jk}(d)$	d_1	d_2	d_3	d_4
i_1	$0.4e^{i2\pi(0.35)}$	$0.4e^{i2\pi(0.6)}$	$0.95e^{i2\pi(0.99)}$	$0.8e^{i2\pi(0.6)}$
i_2	$0.75e^{i2\pi(0.65)}$	$0.6e^{i2\pi(0.9)}$	$0.88e^{i2\pi(0.96)}$	$0.12e^{i2\pi(0.19)}$
i_3	$0.7e^{i2\pi(0.4)}$	$0.01e^{i2\pi(0.1)}$	$0.9e^{i2\pi(0.83)}$	$0.56e^{i2\pi(0.63)}$
i_4	$0.89e^{i2\pi(0.9)}$	$0.81e^{i2\pi(0.85)}$	$0.79e^{i2\pi(0.75)}$	$0.51e^{i2\pi(0.57)}$

Table 30 Comparison table of membership grades

	i_1	i_2	i_3	i_4
i_1	4	2	3	2
i_2	2	4	2	1
i_3	1	2	4	2
i_4	2	3	2	4

I-CV of CF4-SSs

The calculated 3-CV of CF4-SS of the player's selection is presented in Table 28.

Here we let $I = 3$ as threshold and Table 28 gives the 3-CV of CF4-SS. It can be observed from Table 28 that the player i_4 has the highest grade so the selection committee selected i_4 for the national football team.

Comparison table of CF4-SSs

Membership grades of Table 26 are presented in tabular form in Table 29.

The comparison table of membership grades is presented in Table 30.

The membership grades of each football player are derived by subtracting the column sum from the row sum of Table 31.

From Table 31 we observe that the highest score is 2 which is obtained by two player i_4 and i_1 but i_4 has the highest grade. So player i_4 is selected for the national German football team.

Evaluation criteria of cleaner production for gold mines

In this section, we applied the proposed ranking approach to evaluation criteria system of cleaner production including five criteria is utilized according to the particular Characteristics of gold mines. For selecting the suitable cleaner production, we consider the group of decision-makers to perform the evaluation, between the five alternatives i.e. $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$ and $\bar{U}_4$. According to four criteria, the decision-maker describes the cleaner production, which is follow as:

Information related to alternatives for cleaner assessment production in gold mines.

Symbols	Representations	Detailed
d_1	Management level	It specifies the management level of cleaner production, which contains the integrality of cleaner production regulations and execution of cleaner production regulations
d_2	Production process and equipment	It specifies the level of production process and equipment, which contains the mining technology and production equipment
d_3	Resource and energy consumption	It specifies the consumption of resource and energy, which contains the water consumption of unit product and comprehensive energy consumption of unit product
d_4	Waste utilization	It specifies the comprehensive utilization of waste, which contains the utilization rate of solid waste, utilization rate of waste water and utilization rate of associated resources

This graded evaluation by stars can undoubtedly related to numbers as $Z = \{0, 1, 2, 3, 4, 5\}$, where

- 0 serves as “o”,
- 1 serves as “*”,
- 2 serves as “**”,
- 3 serves as “***”,
- 4 serves as “****”,
- 5 serves as “*****”,

Table 32 presented the information obtained from related data, and also presented the tabular representation of its related 6-SS.

The geometrical expressions of the gold mines are discussed in the form of Fig. 1.

The group of decision-makers follows this criterion based on the attributes of the gold mines as follows:

Table 31 Membership score table

	Grade sum (gs)	Row sum (rs)	Column Sum (cs)	Final score = (rs-cs)
$\bar{U}_1$	8	11	9	2
$\bar{U}_2$	8	9	11	-2
$\bar{U}_3$	7	9	11	-2
$\bar{U}_4$	10	11	9	2

Table 32 Information obtained from linked data and tabular representation of 6-SSs

$\bar{U}/D$	d_1	d_2	d_3	d_4
$\bar{U}_1$	***** = 5	**** = 4	***** = 5	*** = 3
$\bar{U}_2$	* = 1	$o = 0$	** = 2	*** = 3
$\bar{U}_3$	** = 2	*** = 3	*** = 3	**** = 4
$\bar{U}_4$	* = 1	***** = 5	**** = 4	$o = 0$
$\bar{U}_5$	$o = 0$	* = 1	***** = 5	** = 2

$0.0 \leq \Delta\mu_d(m) < 0.1$ when $z_d = 1$;

$0.1 \leq \Delta\mu_d(m) < 0.25$ when $z_d = 2$;

$0.25 \leq \Delta\mu_d(m) < 0.4$ when $z_d = 3$;

$0.4 \leq \Delta\mu_d(m) < 0.6$ when $z_d = 4$;

$0.6 \leq \Delta\mu_d(m) < 0.8$ when $z_d = 5$;

$0.8 \leq \Delta\mu_d(m) < 1.0$ when $z_d = 6s$,

where $\Delta\mu_d(m) = \frac{\mu'_d(m) + \omega_{\bar{U}_I}(m)}{2}$. Therefore, the following CF6-SS by using definition (15), is defined:

$$\mu(d_1) = \left\{ \begin{array}{l} ((\bar{U}_1, 5), 1.0e^{i2\pi(0.96)}), \\ ((\bar{U}_2, 1), 0.1e^{i2\pi(0.2)}), \\ ((\bar{U}_3, 2), 0.31e^{i2\pi(0.35)}), \\ ((\bar{U}_4, 1), 0.11e^{i2\pi(0.19)}), \\ ((\bar{U}_5, 0), 0.05e^{i2\pi(0.09)}) \end{array} \right\} \in CF(F(d_1))$$

$$\mu(d_2) = \left\{ \begin{array}{l} ((\bar{U}_1, 4), 0.95e^{i2\pi(0.95)}), \\ ((\bar{U}_2, 0), 0.02e^{i2\pi(0.05)}), \\ ((\bar{U}_3, 3), 0.45e^{i2\pi(0.59)}), \\ ((\bar{U}_4, 5), 0.7e^{i2\pi(0.9)}), \\ ((\bar{U}_5, 1), 0.15e^{i2\pi(0.2)}) \end{array} \right\} \in CF(F(d_2))$$

$$\mu(d_3) = \left\{ \begin{array}{l} ((\bar{U}_1, 5), 0.99e^{i2\pi(0.99)}), \\ ((\bar{U}_2, 2), 0.4e^{i2\pi(0.4)}), \\ ((\bar{U}_3, 3), 0.45e^{i2\pi(0.55)}), \\ ((\bar{U}_4, 4), 0.64e^{i2\pi(0.61)}), \\ ((\bar{U}_5, 5), 0.85e^{i2\pi(0.87)}) \end{array} \right\} \in CF(F(d_3))$$

$$\mu(d_4) = \left\{ \begin{array}{l} ((\bar{U}_1, 3), 0.58e^{i2\pi(0.59)}), \\ ((\bar{U}_2, 3), 0.5e^{i2\pi(0.4)}), \\ ((\bar{U}_3, 4), 0.66e^{i2\pi(0.61)}), \\ ((\bar{U}_4, 0), 0.03e^{i2\pi(0.03)}), \\ ((\bar{U}_5, 2), 0.13e^{i2\pi(0.2)}) \end{array} \right\} \in CF(F(d_4))$$

The tabular form of the information can display in Table 33.

Procedure of CF6-SSs

The calculated the best way for finding the gold mines of CF6-SS for the best procedure to find out the gold mines is presented in Table 34.

From Table 34 we can note that the cleaner production $\bar{U}_1$ has the highest grade to find the best way for examining the gold mines. So the cleaner production $\bar{U}_1$ will be the best procedure for finding the gold mines. We also note the ranking as $\bar{U}_1 > \bar{U}_3 > \bar{U}_4 > \bar{U}_5 > \bar{U}_2$.

I-CV of CF6-SSs

The calculated 4-CV of CF4-SS of the player's selection is presented in Table 35.

Here we let $I = 4$ as threshold and Table 35 gives the 4-CV of CF6-SS. It can be observed from Table 35 that the cleaner production $\bar{U}_1$ has the highest grade so the Group of decision-makers will select $\bar{U}_1$ as the best cleaner production of the year.

Comparison table of CF6-SSs

Membership grades of Table 33 are presented in tabular form in Table 36.

The comparison table of membership grades is presented in Table 37.

Fig. 1 Geometrical procedure of the evaluations of the gold mines **Table 33** The tabular form of the CF6-SSs

$(\mu, (F, D, 6))$	d_1	d_2	d_3	d_4
$\mathcal{U}_1$	$(5, 1.0e^{i2\pi(0.96)})$	$(4, 0.95e^{i2\pi(0.95)})$	$(5, 0.99e^{i2\pi(0.99)})$	$(3, 0.58e^{i2\pi(0.59)})$
$\mathcal{U}_2$	$(1, 0.1e^{i2\pi(0.2)})$	$(0, 0.02e^{i2\pi(0.05)})$	$(2, 0.4e^{i2\pi(0.4)})$	$(3, 0.5e^{i2\pi(0.4)})$
$\mathcal{U}_3$	$(2, 0.31e^{i2\pi(0.35)})$	$(3, 0.45e^{i2\pi(0.59)})$	$(3, 0.45e^{i2\pi(0.55)})$	$(4, 0.66e^{i2\pi(0.61)})$
$\mathcal{U}_4$	$(1, 0.11e^{i2\pi(0.19)})$	$(5, 0.7e^{i2\pi(0.9)})$	$(4, 0.64e^{i2\pi(0.61)})$	$(0, 0.03e^{i2\pi(0.03)})$
$\mathcal{U}_5$	$(0, 0.05e^{i2\pi(0.09)})$	$(1, 0.15e^{i2\pi(0.2)})$	$(5, 0.85e^{i2\pi(0.87)})$	$(2, 0.13e^{i2\pi(0.2)})$

Table 34 The tabular form of the procedure of CF6-SSs

$(\mu, (F, D, 6))$	d_1	d_2	d_3	d_4	$\mathcal{Q}_j$
$\mathcal{U}_1$	$(5, 1.0e^{i2\pi(0.96)})$	$(4, 0.95e^{i2\pi(0.95)})$	$(5, 0.99e^{i2\pi(0.99)})$	$(3, 0.58e^{i2\pi(0.59)})$	$(17, 1.0e^{i2\pi(0.999)})$
$\mathcal{U}_2$	$(1, 0.1e^{i2\pi(0.2)})$	$(0, 0.02e^{i2\pi(0.05)})$	$(2, 0.4e^{i2\pi(0.4)})$	$(3, 0.5e^{i2\pi(0.4)})$	$(6, 0.735e^{i2\pi(0.712)})$
$\mathcal{U}_3$	$(2, 0.31e^{i2\pi(0.35)})$	$(3, 0.45e^{i2\pi(0.59)})$	$(3, 0.45e^{i2\pi(0.55)})$	$(4, 0.66e^{i2\pi(0.61)})$	$(12, 0.916e^{i2\pi(0.886)})$
$\mathcal{U}_4$	$(1, 0.11e^{i2\pi(0.19)})$	$(5, 0.7e^{i2\pi(0.9)})$	$(4, 0.64e^{i2\pi(0.61)})$	$(0, 0.03e^{i2\pi(0.03)})$	$(10, 0.907e^{i2\pi(0.694)})$
$\mathcal{U}_5$	$(0, 0.05e^{i2\pi(0.09)})$	$(1, 0.15e^{i2\pi(0.2)})$	$(5, 0.85e^{i2\pi(0.87)})$	$(2, 0.13e^{i2\pi(0.2)})$	$(8, 0.895e^{i2\pi(0.905)})$

The membership grades of each cleaner production are derived by subtracting the column sum from the row sum of Table 38.

From Table 38 we observe that the highest score is 15 which obtained the best way to find the gold mines $\mathcal{U}_1$ with the highest grade. So the best procedure is $\mathcal{U}_1$.

Table 35 The tabular form of 4-CV of CF6-SSs

(μ^4, D)	d_1	d_2	d_3	d_4	Ω_j^4
$\tilde{U}_1$	$1.0e^{i2\pi(0.96)}$	$0.95e^{i2\pi(0.95)}$	$0.99e^{i2\pi(0.99)}$	0	$1.0e^{i2\pi(0.989)}$
$\tilde{U}_2$	0	0	0	0	0
$\tilde{U}_3$	0	0	0	$0.66e^{i2\pi(0.61)}$	$0.66e^{i2\pi(0.61)}$
$\tilde{U}_4$	0	$0.7e^{i2\pi(0.9)}$	$0.64e^{i2\pi(0.61)}$	0	$0.892e^{i2\pi(0.961)}$
$\tilde{U}_5$	0	0	$0.85e^{i2\pi(0.87)}$	0	$0.85e^{i2\pi(0.87)}$

Table 36 The tabular form of membership grades

$\mu_{jk}(d)$	d_1	d_2	d_3	d_4
$\tilde{U}_1$	$1.0e^{i2\pi(0.96)}$	$0.95e^{i2\pi(0.95)}$	$0.99e^{i2\pi(0.99)}$	$0.58e^{i2\pi(0.59)}$
$\tilde{U}_2$	$0.1e^{i2\pi(0.2)}$	$0.02e^{i2\pi(0.05)}$	$0.4e^{i2\pi(0.4)}$	$0.5e^{i2\pi(0.4)}$
$\tilde{U}_3$	$0.31e^{i2\pi(0.35)}$	$0.45e^{i2\pi(0.59)}$	$0.45e^{i2\pi(0.55)}$	$0.66e^{i2\pi(0.61)}$
$\tilde{U}_4$	$0.11e^{i2\pi(0.19)}$	$0.7e^{i2\pi(0.9)}$	$0.64e^{i2\pi(0.61)}$	$0.03e^{i2\pi(0.03)}$
$\tilde{U}_5$	$0.05e^{i2\pi(0.09)}$	$0.15e^{i2\pi(0.2)}$	$0.85e^{i2\pi(0.87)}$	$0.13e^{i2\pi(0.2)}$

Table 37 The comparison table of membership grades

.	$\tilde{U}_1$	$\tilde{U}_2$	$\tilde{U}_3$	$\tilde{U}_4$	$\tilde{U}_5$
$\tilde{U}_1$	4	4	4	4	4
$\tilde{U}_2$	0	4	0	1	2
$\tilde{U}_3$	1	4	4	3	3
$\tilde{U}_4$	0	3	2	4	2
$\tilde{U}_5$	0	2	1	2	4

Table 38 Membership score table

.	Grade sum (gs)	Row sum (rs)	Column Sum (cs)	Final score = (rs-cs)
$\tilde{U}_1$	16	20	5	15
$\tilde{U}_2$	6	7	17	−10
$\tilde{U}_3$	12	15	11	4
$\tilde{U}_4$	10	11	14	−4
$\tilde{U}_5$	8	9	15	−6

Table 39 Information obtained from the associated data and tabular form of the related 5-SS

M/D	d_1	d_2	d_3	d_4	Stars	Representations
c_1	** = 2	** = 2	$o = 0$	* = 1	$o = 0$	Poor
c_2	*** = 4	*** = 3	** = 2	*** = 3	* = 1	Normal
c_3	*** = 3	** = 1	*** = 3	*** = 4	** = 2	Good
c_4	*** = 3	*** = 3	* = 2	$o = 0$	*** = 3	Very Good
c_5	*** = 4	$o = 0$	** = 2	$o = 0$	*** = 4	Excellent

Comparison

In this section, we do a comparison of our proposed work with the existing work done by Akram et al. [39].

Example

A family went to the car showroom to buy a new car of a model 2020 based on the rating and ranking given by the

experts of cars. Let $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$ be set of 5 cars in which one of the car family wants to purchase and $D = \{d_1 = \text{Price}, d_2 = \text{Fuel saving}, d_3 = \text{Control of a car}, d_4 = \text{Speed of a car}\}$ be set of attributes. We can get 5-SSs from Table 39, where.

Four stars appear for ‘Excellent’,
Three stars appear for ‘Very Good’,
Two stars appear for ‘Good’,

Table 40 The tabular form of F5-SS

$(\mu', (F, D, 5))$	d_1	d_2	d_3	d_4
c_1	(2, 0.5)	(2, 0.48)	(0, 0.08)	(1, 0.29)
c_2	(4, 0.85)	(3, 0.62)	(2, 0.49)	(3, 0.74)
c_3	(3, 0.75)	(1, 0.22)	(3, 0.78)	(4, 0.94)
c_4	(3, 0.65)	(3, 0.68)	(2, 0.47)	(0, 0.13)
c_5	(4, 0.99)	(0, 0.15)	(2, 0.43)	(0, 0.18)

One star appears for ‘Normal’,
Hole appear for ‘Poor’,

This graded evaluation by stars can undoubtedly be related to numbers as $Z = \{0, 1, 2, 3, 4\}$, where.

0 serves as “0”,

1 serves as “*”,

2 serves as “**”,

3 serves as “***”,

4 serves as “****”,

Table 39 presented the information obtained from associated data, and also presented the tabular representation of its related 5-soft set.

Now we get the F5-SS as defined by Akram et al. [39] as follows

$$0.0 \leq \mu'_d(c) < 0.2 \text{ when } z_d = 0;$$

$$0.2 \leq \mu'_d(c) < 0.2 \text{ when } z_d = 1;$$

$$0.4 \leq \mu'_d(c) < 0.2 \text{ when } z_d = 2;$$

$$0.0 \leq \mu'_d(c) < 0.2 \text{ when } z_d = 3;$$

$$0.08 \leq \mu'_d(c) < 0.2 \text{ when } z_d = 4.$$

Therefore, the following (F, 5)-soft set is defined:

$$\mu'(d_1) = \left\{ ((c_1, 2), 0.5), ((c_2, 4), 0.85), ((c_3, 3), 0.75), ((c_4, 3), 0.65), ((c_5, 4), 0.99) \right\} \in \mathcal{F}(F(d_1))$$

$$\mu'(d_2) = \left\{ ((c_1, 2), 0.48), ((c_2, 3), 0.62), ((c_3, 1), 0.22), ((c_4, 3), 0.68), ((c_5, 0), 0.15) \right\} \in \mathcal{F}(F(d_2))$$

$$\mu'(d_3) = \left\{ ((c_1, 0), 0.08), ((c_2, 2), 0.49), ((c_3, 3), 0.78), ((c_4, 2), 0.47), ((c_5, 2), 0.43) \right\} \in \mathcal{F}(F(d_3))$$

$$\mu'(d_4) = \left\{ ((c_1, 1), 0.29), ((c_2, 3), 0.74), ((c_3, 4), 0.94), ((c_4, 0), 0.13), ((c_5, 0), 0.18) \right\} \in \mathcal{F}(F(d_4))$$

The tabular form of the F5-SS are given in Table 40.

Table 41 CV of F6-SS calculated by the algorithm defined by Akram et al. [39]

$(\mu', (F, D, 5))$	d_1	d_2	d_3	d_4	Ω_j
c_1	(2, 0.5)	(2, 0.48)	(0, 0.08)	(1, 0.29)	(5, 1.35)
c_2	(4, 0.85)	(3, 0.62)	(2, 0.49)	(3, 0.74)	(12, 2.7)
c_3	(3, 0.75)	(1, 0.22)	(3, 0.78)	(4, 0.94)	(11, 2.69)
c_4	(3, 0.65)	(3, 0.68)	(2, 0.47)	(0, 0.13)	(8, 1.93)
c_5	(4, 0.99)	(0, 0.15)	(2, 0.43)	(0, 0.18)	(6, 1.75)

Table 42 CV of F6-SS calculated by our proposed algorithm

$(\mu', (F, D, 5))$	d_1	d_2	d_3	d_4	Ω_j
c_1	(2, 0.5)	(2, 0.48)	(0, 0.08)	(1, 0.29)	(5, 0.83)
c_2	(4, 0.85)	(3, 0.62)	(2, 0.49)	(3, 0.74)	(12, 0.992)
c_3	(3, 0.75)	(1, 0.22)	(3, 0.78)	(4, 0.94)	(11, 0.997)
c_4	(3, 0.65)	(3, 0.68)	(2, 0.47)	(0, 0.13)	(8, 0.948)
c_5	(4, 0.99)	(0, 0.15)	(2, 0.43)	(0, 0.18)	(6, 996)

Table 43 The tabular form of 3-CV of F5-SS was calculated by using the algorithm defined by Akram et al. [39]

(μ^3, D)	d_1	d_2	d_3	d_4	Ω_j^3
c_1	0	0	0	0	0
c_2	0.85	0.62	0	0.74	2.21
c_3	0.75	0	0.78	0.94	2.47
c_4	0.65	0.68	0	0	1.33
c_5	0.99	0	0	0	0.99

Table 44 Tabular representation of 3-CV of F5-SS calculated by using our proposed algorithm

(μ^3, D)	d_1	d_2	d_3	d_4	Ω_j^3
c_1	0	0	0	0	0
c_2	0.85	0.62	0	0.74	0.985
c_3	0.75	0	0.78	0.94	0.996
c_4	0.65	0.68	0	0	0.88
c_5	0.99	0	0	0	0.99

CV of F6-SSs

We can calculate the CV of F6-SS by both algorithms defined by Akram et al. [39], and the proposed algorithm defined in this article, which is presented in Tables 41 and 42, respectively.

From CVs given in Tables 41 and 42 we can see that the car c_2 has the highest grade and highest membership grade given by the experts of the cars. So the family will purchase the car c_2 . We also note the ranking as $c_2 > c_3 > c_4 > c_5 > c_1$.

I-CV of F5-SSs

We can calculate the 3-CV of F6-SS by both algorithms defined by Akram et al. [39], and the proposed algorithm defined in this article, which is presented in Tables 43 and 44, respectively.

In Tables 43 and 44 we supposed $I = 3$ for DM so we get the 3-CV of F5-SS. It can be observed from Tables 43 and 44 that the car c_3 has the highest membership grade so the family will purchase a car c_3 .

Comparison table of F5-SSs

Membership grades of Table 40 are presented in tabular form in Table 45.

The comparison table of membership grades is presented in Table 46.

The membership grades of each car are derived by subtracting the column sum from the row sum of Table 47.

Table 45 The tabular form of the membership values

$\mu'_{jk}(d)$	d_1	d_2	d_3	d_4
c_1	0.5	0.48	0.08	0.29
c_2	0.85	0.62	0.49	0.74
c_3	0.75	0.22	0.78	0.94
c_4	0.65	0.68	0.47	0.13
c_5	0.99	0.15	0.43	0.18

Table 46 Comparison table of membership grades

	c_1	c_2	c_3	c_4	c_5
c_1	4	0	1	1	2
c_2	4	4	2	3	3
c_3	3	2	4	3	3
c_4	3	1	1	4	2
c_5	2	1	1	2	4

From Table 47 we observe that the highest score is 8 which obtained by c_2 and c_2 has also highest grade So the family will get the car c_2 .

Now if the family wants to know additional information about these cars, e.g. they want to know what membership grades are given by the experts to the set of attributes D to these cars of model 2019. Then the above F5-SS cannot provide any information about the model 2019 of these cars. But our proposed model CFN-SS can give us this additional information, e.g. we can say that the amplitude part $(\mu'(x))$ carry information about the cars of model 2020 and phase part $(\omega_{\mu'}(x))$ of membership, grade carries the information about the cars of model 2019. This means that membership grades carry both information about cars.

Example 6.2 To explain the above paragraph we will consider example 6.1 along with additional information on the cars of model 2019. We get the following CF5-SS:

$$0.0 \leq \Delta\mu_d(c) < 0.2 \text{ when } z_d = 0;$$

$$0.2 \leq \Delta\mu_d(c) < 0.4 \text{ when } z_d = 1;$$

$$0.4 \leq \Delta\mu_d(c) < 0.6 \text{ when } z_d = 2;$$

$$0.6 \leq \Delta\mu_d(c) < 0.8 \text{ when } z_d = 3;$$

$$0.8 \leq \Delta\mu_d(c) < 1.0 \text{ when } z_d = 4,$$

where $\Delta\mu_d(j) = \frac{\mu'_d(j) + \omega_{\mu'_d}(j)}{2}$. Therefore, the following CF5-SS by using definition (15), is defined:

$$\mu(d_1) = \left\{ \begin{array}{l} ((c_1, 2), 0.5e^{i2\pi(0.45)}), \\ ((c_2, 4), 0.85e^{i2\pi(0.9)}), \\ ((c_3, 3), 0.75e^{i2\pi(0.79)}), \\ ((c_4, 3), 0.65e^{i2\pi(0.7)}), \\ ((c_5, 4), 0.99e^{i2\pi(0.95)}) \end{array} \right\} \in CF(F(d_1))$$

Table 47 Membership score table

	Grade sum (gs)	Row sum (rs)	Column Sum (cs)	Final score = (rs-cs)
c_1	5	8	16	−8
c_2	12	16	8	8
c_3	11	15	9	6
c_4	8	11	13	−2
c_5	6	10	14	−4

Table 48 The tabular form of CF5-SS

$(\mu, (F, D, 5))$	d_1	d_2	d_3	d_4
c_1	$(2, 0.5e^{i2\pi(0.45)})$	$(2, 0.48e^{i2\pi(0.5)})$	$(0, 0.08e^{i2\pi(0.05)})$	$(1, 0.29e^{i2\pi(0.5)})$
c_2	$(4, 0.85e^{i2\pi(0.9)})$	$(3, 0.62e^{i2\pi(0.65)})$	$(2, 0.49e^{i2\pi(0.55)})$	$(3, 0.74e^{i2\pi(0.79)})$
c_3	$(3, 0.75e^{i2\pi(0.79)})$	$(1, 0.22e^{i2\pi(0.3)})$	$(3, 0.78e^{i2\pi(0.6)})$	$(4, 0.94e^{i2\pi(0.9)})$
c_4	$(3, 0.65e^{i2\pi(0.7)})$	$(3, 0.68e^{i2\pi(0.75)})$	$(2, 0.47e^{i2\pi(0.4)})$	$(0, 0.13e^{i2\pi(0.1)})$
c_5	$(4, 0.99e^{i2\pi(0.95)})$	$(0, 0.15e^{i2\pi(0.1)})$	$(2, 0.43e^{i2\pi(0.4)})$	$(0, 0.18e^{i2\pi(0.15)})$

Table 49 CV of CF6-SS

$(\mu, (F, D, 5))$	d_1	d_2	d_3	d_4	$\mathfrak{Q}_j$
c_1	$(2, 0.5e^{i2\pi(0.45)})$	$(2, 0.48e^{i2\pi(0.5)})$	$(0, 0.08e^{i2\pi(0.05)})$	$(1, 0.29e^{i2\pi(0.5)})$	$(5, 0.83e^{i2\pi(0.739)})$
c_2	$(4, 0.85e^{i2\pi(0.9)})$	$(3, 0.62e^{i2\pi(0.65)})$	$(2, 0.49e^{i2\pi(0.55)})$	$(3, 0.74e^{i2\pi(0.79)})$	$(12, 0.992e^{i2\pi(0.991)})$
c_3	$(3, 0.75e^{i2\pi(0.79)})$	$(1, 0.22e^{i2\pi(0.3)})$	$(3, 0.78e^{i2\pi(0.6)})$	$(4, 0.94e^{i2\pi(0.9)})$	$(11, 0.997e^{i2\pi(0.992)})$
c_4	$(3, 0.65e^{i2\pi(0.7)})$	$(3, 0.68e^{i2\pi(0.75)})$	$(2, 0.47e^{i2\pi(0.4)})$	$(0, 0.13e^{i2\pi(0.1)})$	$(8, 0.948e^{i2\pi(0.838)})$
c_5	$(4, 0.99e^{i2\pi(0.95)})$	$(0, 0.15e^{i2\pi(0.1)})$	$(2, 0.43e^{i2\pi(0.4)})$	$(0, 0.18e^{i2\pi(0.15)})$	$(6, 0.996e^{i2\pi(0.975)})$

$$\mu(d_2) = \left\{ \begin{array}{l} ((c_1, 2), 0.48e^{i2\pi(0.5)}), \\ ((c_2, 3), 0.62e^{i2\pi(0.65)}), \\ ((c_3, 1), 0.22e^{i2\pi(0.3)}), \\ ((c_4, 3), 0.68e^{i2\pi(0.75)}), \\ ((c_5, 0), 0.15e^{i2\pi(0.1)}) \end{array} \right\} \in CF(F(d_2))$$

$$\mu(d_3) = \left\{ \begin{array}{l} ((c_1, 0), 0.08e^{i2\pi(0.05)}), \\ ((c_2, 2), 0.49e^{i2\pi(0.55)}), \\ ((c_3, 3), 0.78e^{i2\pi(0.6)}), \\ ((c_4, 2), 0.47e^{i2\pi(0.4)}), \\ ((c_5, 2), 0.43e^{i2\pi(0.4)}) \end{array} \right\} \in CF(F(d_3))$$

$$\mu(d_4) = \left\{ \begin{array}{l} ((c_1, 1), 0.29e^{i2\pi(0.5)}), \\ ((c_2, 3), 0.74e^{i2\pi(0.79)}), \\ ((c_3, 4), 0.94e^{i2\pi(0.9)}), \\ ((c_4, 0), 0.13e^{i2\pi(0.1)}), \\ ((c_5, 0), 0.18e^{i2\pi(0.15)}) \end{array} \right\} \in CF(F(d_4))$$

CV of CF6-SSs

We can calculate the CV of CF6-SS by the proposed algorithm defined in this article, which is presented in Table 49 and Akram et al. [39] cannot calculate CF6-SS.

From CVs given in Table 49 we can see that the car c_3 has the highest membership grade given by the experts of the cars. So the family will purchase the car c_3 . We also note the ranking as $c_3 > c_5 > c_2 > c_4 > c_1$.

I-CV of CF5-SSs

We can calculate 3-CV of CF6-SS by the proposed algorithm defined in this article, which is presented in Table 50 and Akram et al. [39] cannot calculate 3-CV of CF6-SS.

In Table 50 we supposed $I = 3$ for DM so we get the 3-CV of CF5-SS. It can be observed from Table 50 that the car c_3 has the highest membership grade so the family will purchase a car c_3 .

The tabular form of the CF5-SS are given in Table 48.

Table 50 The tabular form of 3-CV of CF5-SS

(μ^3, D)	d_1	d_2	d_3	d_4	Ω_j^3
c_1	0	0	0	0	0
c_2	$0.85e^{i2\pi(0.9)}$	$0.62e^{i2\pi(0.65)}$	0	$0.74e^{i2\pi(0.79)}$	$0.985e^{i2\pi(0.992)}$
c_3	$0.75e^{i2\pi(0.79)}$	0	$0.78e^{i2\pi(0.6)}$	$0.94e^{i2\pi(0.9)}$	$0.996e^{i2\pi(0.991)}$
c_4	$0.65e^{i2\pi(0.7)}$	$0.68e^{i2\pi(0.75)}$	0	0	$0.88e^{i2\pi(0.925)}$
c_5	$0.99e^{i2\pi(0.95)}$	0	0	0	$0.99e^{i2\pi(0.95)}$

Table 51 The tabular form of the membership values

$\mu'_{jk}(d)$	d_1	d_2	d_3	d_4
c_1	$0.5e^{i2\pi(0.45)}$	$0.48e^{i2\pi(0.5)}$	$0.08e^{i2\pi(0.05)}$	$0.29e^{i2\pi(0.5)}$
c_2	$0.85e^{i2\pi(0.9)}$	$0.62e^{i2\pi(0.65)}$	$0.49e^{i2\pi(0.55)}$	$0.74e^{i2\pi(0.79)}$
c_3	$0.75e^{i2\pi(0.79)}$	$0.22e^{i2\pi(0.3)}$	$0.78e^{i2\pi(0.6)}$	$0.94e^{i2\pi(0.9)}$
c_4	$0.65e^{i2\pi(0.7)}$	$0.68e^{i2\pi(0.75)}$	$0.47e^{i2\pi(0.4)}$	$0.13e^{i2\pi(0.1)}$
c_5	$0.99e^{i2\pi(0.95)}$	$0.15e^{i2\pi(0.1)}$	$0.43e^{i2\pi(0.4)}$	$0.18e^{i2\pi(0.15)}$

Table 52 Comparison table of membership grades

.	c_1	c_2	c_3	c_4	c_5
c_1	4	0	1	1	2
c_2	4	4	2	3	3
c_3	3	2	4	3	3
c_4	3	1	1	4	2
c_5	2	1	1	2	4

Table 53 Membership score table

.	Grade sum (gs)	Row sum (rs)	Column Sum (cs)	Final score = (rs-cs)
c_1	5	8	16	-8
c_2	12	16	8	8
c_3	11	15	9	6
c_4	8	11	13	-2
c_5	6	10	14	-4

Comparison table of CF5-SSs

Membership grades of Table 48 are presented in tabular form in Table 51.

The comparison table of membership grades is presented in Table 52.

The membership grades of each car are derived by subtracting the column sum from the row sum of Table 53.

From Table 53) we observe that the highest score is 8 which obtained by c_2 and c_2 has also highest grade So the family will get the car c_2 .

Conclusion

In this manuscript, we interpreted a new enlarged and appropriate version of SSs, called CFN-SS, which is the

combination of CFSs with N-SSs to handle the complicated data in DM. Further, we interpreted some handy algebraic properties of CFN-SSs and establish their basic operations in this manuscript. Our novel approach along with its properties is explained with the help of examples. Additionally, we described the relationship of our novel approach with some existing methods such as complex fuzzy soft sets (CFSSs), FSSs, and SSs. Moreover, three DM procedures have been interpreted and clarified with the help of examples. Finally, we compared our model CFN-SS with the existing model FN-SS. Our novel approach spread out new directions for research. Within this novel concept, it is workable to study similarity indices, entropy, and the correlation between CFN-SSs as a mechanism to help in DM processes.

In the future, we aim to discuss N-soft sets to generalize the notions of complex dual hesitant fuzzy sets [41], complex

q-rung orthopair fuzzy sets [42–44], bipolar fuzzy soft set [45] etc. [46, 47].

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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