Hybrid structures applied to ideals in near-rings

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Abstract

Human endeavours span a wide spectrum of activities which includes solving fascinating problems in the realms of engineering, arts, sciences, medical sciences, social sciences, economics and environment. To solve these problems, classical mathematics methods are insufficient. The real-world problems involve many uncertainties making them difficult to solve by classical means. The researchers world over have established new mathematical theories such as fuzzy set theory and rough set theory in order to model the uncertainties that appear in various fields mentioned above. In the recent days, soft set theory has been developed which offers a novel way of solving real world issues as the issue of setting the membership function does not arise. This comes handy in solving numerous problems and many advancements are being made now-a-days. Jun introduced hybrid structure utilizing the ideas of a fuzzy set and a soft set. It is to be noted that hybrid structures are a speculation of soft set and fuzzy set. In the present work, the notion of hybrid ideals of a near-ring is introduced. Significant work has been carried out to investigate a portion of their significant properties. These notions are characterized and their relations are established furthermore. For a hybrid left (resp., right) ideal, different left (resp., right) ideal structures of near-rings are constructed. Efforts have been undertaken to display the relations between the hybrid product and hybrid intersection. Finally, results based on homomorphic hybrid preimage of a hybrid left (resp., right) ideals are proved.

Keywords Hybrid structure · Near-ring · Hybrid product · Hybrid left ideal · Hybrid intersection · Hybrid image

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Introduction

An algebra $(N, +, \cdot)$ satisfying all the conditions of an associative ring is called a near-ring except the commutativity of addition and one of the two distributive laws. If N satisfies the left (right) distributive law $m \cdot (n + t) = m \cdot n + m \cdot t$, then N

is called a left (right) near-ring. In fact, near-rings are firmly identified with groups and rings and were first studied by Zassenhaus and Wielandt in 1930. Since then, the near ring theory has been developed a lot and currently it has turned out into a sophisticated theory with various applications in different areas, namely group theory, geometries, polynomials and matrices, interpolation theory and especially designs that are important applications of the near ring.

Zadeh has presented the Fuzzy set theory and it has been applied to different fields [1]. Furthermore in the literature, a number of generalizations and extensions of fuzzy sets have been introduced, for instance, interval-valued fuzzy sets, fuzzy multi-sets, intuitionistic fuzzy sets and so on. Different problems in diverse fields include data which contain uncertainties that are managed with the wide scope of existing assumptions such as theory of interval mathematics, (intuitionistic) fuzzy set theory, rough set theory, vague sets and the theory of probability. All of these hypotheses had their problems underlined in [2]. To defeat these troubles, Molodtsov [2] introduced the soft set theory as a

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new mathematical method for addressing trouble-free uncertainty. Molodtsov has implemented the soft set principle effectively in several ways, such as smoothness of functions, probability, theory of measurement, game theory, operations research, Perron integration, Riemann integration and so on [2–6]. Moreover, Muhiuddin et al., and several authors extended the Fuzzy set theory and similar ideas to several algebraic structures [7–19].

With the notion of fuzzy subgroups, Rosenfeld [20] started the analysis of fuzzy algebraic structures. In applying the fuzzy sets to algebra, Zaid in [21] introduced the concept of near-ring fuzzy ideals and then provided some explanation for the fuzzy subrings and examined the properties of fuzzy left (right) ideals.

Jun, Song and Muhiuddin [22] introduced the notion of hybrid structures in a set of parameters over an initial universe set, and investigated several properties. Using this notion, they introduced the concepts of a hybrid subalgebra, a hybrid field and a hybrid linear space.

Recently, Anis et al. [23] introduced the notions of hybrid sub-semigroups and hybrid left (resp., right) ideals in semigroups and obtained several properties. The notion of hybrid systems and their properties has been extended to semigroups and has provided several useful results [24–27]. As a follow up, in the present paper, the notion of hybrid ideals of a near-ring is introduced and some of their important properties are investigated. Furthermore, the characterizations of these notions are given and their relations are established. In this respect, for a hybrid left (resp., right) ideal of N over U , left (resp., right) ideals of a near-ring, we construct a new left (resp., right) ideal in near-rings. Finally, we show that a near-ring homomorphic preimage of a hybrid left (resp., right) ideal is a hybrid left (resp., right) ideal.

This research paper is structured as follows: - In the preliminary and hybrid sections, important pre-requisite concepts of near ring and basic definitions of hybrid structures of near rings are given. The subsequent section is based on hybrid sub-near rings and ideals. In this section, the notions of hybrid sub-near ring and hybrid ideal of a near ring over a universal set are defined and they are illustrated with examples. In the concluding part, the future research scope has been discussed.

Preliminaries

We collect a few concepts and results in this section which will be needed in our key results.

By a *near-ring* we say a non-empty set N with two binary operations “+” and “.” which satisfy the axioms:

- (a1) $(N, +)$ is a group,
- (a2) $(N, \cdot)$ is a semigroup,

$$(a3) (r + s) \cdot l = r \cdot l + s \cdot l \quad \forall r, s, l \in N.$$

Precisely because it satisfies the right distributive law, it is a right near-ring right. We would instead use the term “near-ring” of “near ring right”. We denote rs instead of $r \cdot s$. Note that $0(r) = 0$ and $(-r)s = -rs$ but in general $r(0) \neq 0$ for some $r \in N$. Recall that a subset I of N a *subnear-ring* if I itself is also a near-ring.

A non-empty subset I of a near-ring N is called an *ideal* of N if it satisfies:

- (a4) $(I, +)$ is a normal subgroup of $(N, +)$,
- (a5) $IN \subseteq I$,
- (a6) $s(i + r) - sr \in I \quad \forall i \in I; r, s \in N$.

We say that I is a *right ideal* of N if it satisfies (a4) and (a5), and I is *left ideal* of N if it satisfies (a4) and (a6). We note that the intersection of a family of left (resp. right) ideals is a left (resp. right) ideal, and that the onto homomorphic image of a left (resp. right) ideal is also a left (resp. right) ideal. Throughout this paper unless stated otherwise N is a near-ring. For the basic terminology and notations of near-ring, we refer to [28].

Hybrid structures

In our further discussion, we collect some basic notations and results on hybrid structures given by Jun et al. [22]. Let $\mathcal{P}(U)$ be the power set of an initial universal set U and I denote the unit interval.

Definition 1 [22] A hybrid structure in N over U is defined to be a mapping

$$\tilde{g}_\gamma := (\tilde{g}, \gamma) : N \rightarrow \mathcal{P}(U) \times I, l \mapsto (\tilde{g}(l), \gamma(l))$$

where $\tilde{g} : N \rightarrow \mathcal{P}(U)$ and $\gamma : N \rightarrow I$ are mappings.

Let $\mathbb{H}(N)$ be the set of all hybrid structures in N over U . Define a relation $\ll$ on as follows :

$$(\forall \tilde{f}_\lambda, \tilde{h}_\gamma \in \mathbb{H}(N)) \left(\tilde{f}_\lambda \ll \tilde{h}_\gamma \Leftrightarrow \tilde{f} \subseteq \tilde{h}, \lambda \geq \gamma \right)$$

where $\tilde{f} \subseteq \tilde{h}$ means that $\tilde{f}(l) \subseteq \tilde{h}(l)$ and $\lambda \geq \gamma$ means that $\lambda(l) \geq \gamma(l)$ for all $l \in N$. Then $(\mathbb{H}(N), \ll)$ is a partially ordered set.

Definition 2 [22] For $\tilde{f}_\nu, \tilde{g}_\lambda \in \mathbb{H}(N)$, the hybrid intersection of $\tilde{f}_\nu$ and $\tilde{g}_\lambda$ is denoted by $\tilde{f}_\nu \cap \tilde{g}_\lambda$ and is defined to be a hybrid structure

$$\tilde{f}_\nu \cap \tilde{g}_\lambda : N \rightarrow \mathcal{P}(U) \times I, l \mapsto ((\tilde{f} \cap \tilde{g})(l), (\nu \vee \lambda)(l)),$$

where

$$\begin{aligned}\tilde{f} \tilde{\cap} \tilde{h} : N &\longrightarrow \mathcal{P}(U), l \mapsto \tilde{f}(l) \cap \tilde{h}(l), \\ v \vee \lambda : N &\rightarrow I, l \mapsto v(l) \vee \lambda(l).\end{aligned}$$

Definition 3 [22] Let $\tilde{g}_\lambda \in \mathbb{H}(N)$, $\beta \in \mathcal{P}(U)$ and $\delta \in I$. Then the set $\tilde{g}_\lambda[\beta, \delta] = \{l \in N \mid \tilde{g}(l) \supseteq \beta, \lambda(l) \leq \delta\}$ is called the $[\beta, \delta]$ -hybrid cut of $\tilde{g}_\lambda$.

Definition 4 [22] Let $\tilde{f}_v, \tilde{h}_\lambda \in \mathbb{H}(N)$. Then the hybrid product of $\tilde{f}_v$ and $\tilde{h}_\lambda$ is denoted by $\tilde{f}_v \odot \tilde{h}_\lambda$ and is defined to be a hybrid structure $\tilde{f}_v \odot \tilde{h}_\lambda = (\tilde{f} \tilde{\circ} \tilde{h}, v \tilde{\circ} \lambda)$ in N over U , where

$$(\tilde{f} \tilde{\circ} \tilde{h})(t) = \begin{cases} \bigcup_{t=rs} \{\tilde{f}(r) \cap \tilde{h}(s)\} & \text{if } \exists r, s \in N : t = rs \\ \phi & \text{otherwise} \end{cases}$$

and

$$(v \tilde{\circ} \lambda)(t) = \begin{cases} \bigwedge_{t=rs} \{v(r) \vee \lambda(s)\} & \text{if } \exists r, s \in N : t = rs \\ 1 & \text{otherwise} \end{cases}$$

for all $t \in N$.

Definition 5 [22] Let $\tilde{h}_\lambda \in \mathbb{H}(N)$. For $\phi \neq A \subseteq N$, the characteristic hybrid structure in N over U is denoted by $\chi_A(\tilde{h}_\lambda)$ and is defined to be a hybrid structure

$$\begin{aligned}\chi_A(\tilde{h}_\lambda) &= (\chi_A(\tilde{h}), \chi_A(\lambda)) : N \longrightarrow \mathcal{P}(U) \times I, \\ l &\mapsto (\chi_A(\tilde{h})(l), \chi_A(\lambda)(l)),\end{aligned}$$

where

$$\chi_A(\tilde{h}) : N \rightarrow \mathcal{P}(U), l \mapsto \begin{cases} U & \text{if } l \in A \\ \phi & \text{otherwise,} \end{cases}$$

$$\chi_A(\lambda) : N \rightarrow I, l \mapsto \begin{cases} 0 & \text{if } l \in A \\ 1 & \text{otherwise.} \end{cases}$$

Hybrid subnear-rings and ideals

Definition 6 Let $\tilde{g}_\lambda \in \mathbb{H}(N)$. $\tilde{g}_\lambda$ is called a *hybrid subnear-ring* of N over U if the following assertions are valid:

$$\begin{aligned}\text{(H1)} \quad & (\forall l_1, l \in N) \left(\tilde{g}(l_1 - l) \supseteq \tilde{g}(l_1) \cap \tilde{g}(l) \right), \\ \text{(H2)} \quad & (\forall k, l \in N) \left(\tilde{g}(kl) \supseteq \tilde{g}(k) \cap \tilde{g}(l) \right), \\ & \left(\lambda(kl) \leq \lambda(k) \vee \lambda(l) \right).\end{aligned}$$

Lemma 1 Every hybrid subnear-ring $\tilde{g}_\lambda$ of N over U satisfies:

$$(\forall n' \in N) \left(\tilde{g}(n') \subseteq \tilde{g}(0), \lambda(n') \geq \lambda(0) \right).$$

Proof Note that $n' - n' = 0 \forall n' \in N$. Hence

$$\begin{aligned}\tilde{g}(0) &= \tilde{g}(n' - n') \supseteq \tilde{g}(n') \cap \tilde{g}(n') = \tilde{g}(n'), \\ \lambda(0) &= \lambda(n' - n') \leq \lambda(n') \vee \lambda(n') = \lambda(n')\end{aligned}$$

for all $n' \in N$. $\square$

Proposition 1 For a hybrid subnear-ring $\tilde{g}_\lambda$ of N over U , the below statements are equivalent:

- (i) $(\forall r, l \in N) \left(\tilde{g}(r - l) \supseteq \tilde{g}(l), \lambda(r - l) \leq \lambda(l) \right),$
- (ii) $(\forall r \in N) \left(\tilde{g}(0) = \tilde{g}(r), \lambda(0) = \lambda(r) \right).$

Proof If we take $l = 0$ in (i), then $\tilde{g}(0) \subseteq \tilde{g}(r - 0) = \tilde{g}(r)$ and $\lambda(0) \geq \lambda(r - 0) = \lambda(r)$ for all $r \in N$. Combining this and Lemma 1, we have $\tilde{g}(0) = \tilde{g}(r)$, $\lambda(0) = \lambda(r)$ for all $r \in N$.

Conversely, assume that (ii) is true. Then

$$\begin{aligned}\tilde{g}(l) &= \tilde{g}(0) \cap \tilde{g}(l) = \tilde{g}(r) \cap \tilde{g}(l) \subseteq \tilde{g}(r - l), \\ \lambda(l) &= \lambda(0) \vee \lambda(l) = \lambda(r) \vee \lambda(l) \geq \lambda(r - l)\end{aligned}$$

for all $r, l \in N$. $\square$

Definition 7 A hybrid subnear-ring $\tilde{g}_\lambda$ in N over U is called a *hybrid ideal* of N over U if the following assertions are valid:

$$\begin{aligned}\text{(H3)} \quad & (\forall r, s \in N) \left(\tilde{g}(r + s - r) \supseteq \tilde{g}(s) \right), \\ & \left(\lambda(r + s - r) \leq \lambda(s) \right), \\ \text{(H4)} \quad & (\forall r, s \in N) \left(\tilde{g}(sr) \supseteq \tilde{g}(s) \right), \\ & \left(\lambda(sr) \leq \lambda(s) \right), \\ \text{(H5)} \quad & (\forall r, m, z \in N) \left(\tilde{g}(r(z + m) - rm) \supseteq \tilde{g}(z) \right), \\ & \left(\lambda(r(z + m) - rm) \leq \lambda(z) \right).\end{aligned}$$

Note that hybrid structure $\tilde{g}_\lambda$ is a hybrid left ideal of N if it satisfies (H1), (H2), (H3) and (H5) and $\tilde{g}_\lambda$ is a hybrid right ideal of N if it satisfies (H1), (H2), (H3) and (H4).

We now give some examples of hybrid ideals of near-rings.

Example 1 Let $N = \{0, r, l, t\}$ be a set with two binary operations as follows:

+	0	r	l	t
0	0	r	l	t
r	r	0	t	l
l	l	t	r	0
t	t	l	0	r

.	0	r	l	t
0	0	0	0	0
r	0	0	0	0
l	0	0	0	r
t	0	0	0	r

Then $(N, +, \cdot)$ is a near-ring. Define a hybrid structure $\tilde{g}_\lambda$ in N over any U by $\tilde{g}(l) = \tilde{g}(t) \subset \tilde{g}(r) \subset \tilde{g}(0)$ and $\lambda : N \rightarrow I$ is any constant mapping. Then $\tilde{g}_\lambda$ is a hybrid ideal in N over U . $\square$

Example 2 Let $N = \{0, r, l, t\}$ be a set with two binary operations as follows:

$+$	$\begin{array}{c ccc} & 0 & r & l & t \\ \hline 0 & 0 & r & l & t \\ r & r & 0 & t & l \\ l & l & t & r & 0 \\ t & t & l & 0 & r \end{array}$	$\cdot$	$\begin{array}{c ccc} & 0 & r & l & t \\ \hline 0 & 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & r \\ l & 0 & 0 & 0 & l \\ t & 0 & 0 & 0 & r \end{array}$
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Then $(N, +, \cdot)$ is a near-ring. Define a hybrid structure $\tilde{g}_\lambda$ in N over any U by $\tilde{g}(l) = \tilde{g}(t) \subset \tilde{g}(r) \subset \tilde{g}(0)$ and $\lambda : N \rightarrow I$ is any constant mapping. Then $\tilde{g}_\lambda$ is a hybrid right ideal in N , but not a hybrid left ideal in N over U as $(\tilde{g}(l(r+t) - lt) = \tilde{g}(t) \subset \tilde{g}(r))$. $\square$

Proposition 2 For $\tilde{f}_\lambda \in \mathbb{H}(N)$, the following are equivalent:

- (i) $(\forall l, s \in N) \left(\begin{array}{l} \tilde{f}(l+s-l) \supseteq \tilde{f}(s) \\ \lambda(l+s-l) \leq \lambda(s) \end{array} \right)$,
- (ii) $(\forall l, s \in N) \left(\begin{array}{l} \tilde{f}(l+s) \supseteq \tilde{f}(s+l) \\ \lambda(l+s) \leq \lambda(s+l) \end{array} \right)$.

Proof If we take $s = s+l$ in (i), then $\tilde{f}(s+l) \subseteq \tilde{f}(l+s+l-l) = \tilde{f}(l+s)$ and $\lambda(s+l) \geq \lambda(l+s+l-l) = \lambda(l+s)$ for all $s, l \in N$.

Conversely, if we take $s = s-l$ in (ii), then $\tilde{f}(l+s-l) \supseteq \tilde{f}(s-l+l) = \tilde{f}(s)$ and $\lambda(l+s-l) \leq \lambda(s-l+l) = \lambda(s)$ for any $l, s \in N$. $\square$

Proposition 3 For $\tilde{f}_\lambda \in \mathbb{H}(N)$ and $\phi \neq L \subseteq N$, the below assertions are equivalent:

- (i) L is a left (resp., right) ideal of N ,
- (ii) $\chi_L(\tilde{f}_\lambda)$ is a hybrid left (resp., right) ideal in N .

Proof Assume that L is a left ideal of N and let $r, t \in N$.

If $r \notin L$ or $t \notin L$, then $\chi_L(\tilde{f})(r-t) \supseteq \phi = \chi_L(\tilde{f})(r) \cap \chi_L(\tilde{f})(t)$ and $\chi_L(\lambda)(r-t) \leq 1 = \chi_L(\lambda)(r) \vee \chi_L(\lambda)(t)$. If $r, t \in L$, then $r-t \in L$, and so $\chi_L(\tilde{f})(r-t) = U = \chi_L(\tilde{f})(r) \cap \chi_L(\tilde{f})(t)$ and $\chi_L(\lambda)(r-t) = 0 = \chi_L(\lambda)(r) \vee \chi_L(\lambda)(t)$.

If $r \in L$, then $t+r-t \in L$ and hence $\chi_L(\tilde{f})(t+r-t) = U = \chi_L(\tilde{f})(r)$ and $\chi_L(\lambda)(t+r-t) = 0 = \chi_L(\lambda)(r)$. Clearly $\chi_L(\tilde{f})(t+r-t) \supseteq \phi = \chi_L(\tilde{f})(r)$ and $\chi_L(\lambda)(t+r-t) \leq 1 = \chi_L(\lambda)(r)$.

Now let $r, u, v \in N$. If $r \in L$, then $u(r+v) - uv \in L$ and thus $\chi_L(\tilde{f})(u(r+v) - uv) = U = \chi_L(\tilde{f})(r)$ and $\chi_L(\lambda)(u(r+v) - uv) = 0 = \chi_L(\lambda)(r)$. If $r \notin L$, then

$$\chi_L(\tilde{f})(u(r+v) - uv) \supseteq \phi = \chi_L(\tilde{f})(r) \text{ and } \chi_L(\lambda)(u(r+v) - uv) \leq 1 = \chi_L(\lambda)(r).$$

Therefore $\chi_L(\tilde{f}_\lambda)$ is a hybrid left ideal in N .

Conversely, let $s, t \in L$. Then $\chi_L(\tilde{f})(s-t) \supseteq \chi_L(\tilde{f})(s) \cap \chi_L(\tilde{f})(t) = U$ and $\chi_L(\lambda)(s-t) \leq \chi_L(\lambda)(s) \vee \chi_L(\lambda)(t) = 0$, so $s-t \in L$.

For $t \in N$ and $s \in L$, we have $\chi_L(\tilde{f})(t+s-t) \supseteq \chi_L(\tilde{f})(s) = U$ and $\chi_L(\lambda)(t+s-t) \leq \chi_L(\lambda)(s) = 0$. Hence $t+s-t \in L$. Now let $s \in L$ and $k, v \in N$. Then $\chi_L(\tilde{f})(k(s+v) - kv) \supseteq \chi_L(\tilde{f})(s) = U$ and $\chi_L(\lambda)(k(s+v) - kv) \leq \chi_L(\lambda)(s) = 0$. Hence $k(s+v) - kv \in L$.

Therefore L is a left ideal of N . $\square$

Theorem 1 For a hybrid left (resp., right) ideal $\tilde{g}_\lambda$ in N over U , the set

$$N_{\tilde{g}_\lambda} := \{n \in N \mid \tilde{g}(n) = \tilde{g}(0), \lambda(n) = \lambda(0)\}$$

is a left (resp., right) ideal of N .

Proof Suppose that $\tilde{g}_\lambda$ is a hybrid left ideal in N over U .

Let $r, l \in N_{\tilde{g}_\lambda}$. Then $\tilde{g}(r) = \tilde{g}(0)$, $\tilde{g}(l) = \tilde{g}(0)$, $\lambda(r) = \lambda(0)$ and $\lambda(l) = \lambda(0)$. It follows from (H1) that $\tilde{g}(r-l) \supseteq \tilde{g}(r) \cap \tilde{g}(l) = \tilde{g}(0)$ and $\lambda(r-l) \leq \lambda(r) \vee \lambda(l) = \lambda(0)$. By Lemma 1, $\tilde{g}(r-l) \subseteq \tilde{g}(0)$ and $\lambda(r-l) \geq \lambda(0)$. Thus $\tilde{g}(r-l) = \tilde{g}(0)$ and $\lambda(r-l) = \lambda(0)$. Hence $r-l \in N_{\tilde{g}_\lambda}$.

Let $l \in N$ and $r \in N_{\tilde{g}_\lambda}$. Then $\tilde{g}(r) = \tilde{g}(0)$ and $\lambda(r) = \lambda(0)$. It follows from (H3) that $\tilde{g}(l+r-l) \supseteq \tilde{g}(r) = \tilde{g}(0)$ and $\lambda(l+r-l) \leq \lambda(r) = \lambda(0)$. By Lemma 1, $\tilde{g}(l+r-l) \subseteq \tilde{g}(0)$ and $\lambda(l+r-l) \geq \lambda(0)$. Thus $\tilde{g}(l+r-l) = \tilde{g}(0)$ and $\lambda(l+r-l) = \lambda(0)$. Hence $l+r-l \in N_{\tilde{g}_\lambda}$.

Let $l \in N_{\tilde{g}_\lambda}$; $u, v \in N$. Then $\tilde{g}(l) = \tilde{g}(0)$. It follows from (H5) that $\tilde{g}(u(l+v) - uv) \supseteq \tilde{g}(l) = \tilde{g}(0)$ and $\lambda(u(l+v) - uv) \leq \lambda(l) = \lambda(0)$. By Lemma 1, we have $\tilde{g}(u(l+v) - uv) \subseteq \tilde{g}(0)$ and $\lambda(u(l+v) - uv) \geq \lambda(0)$. Thus $\tilde{g}(u(l+v) - uv) = \tilde{g}(0)$ and $\lambda(u(l+v) - uv) = \lambda(0)$. Hence $u(l+v) - uv \in N_{\tilde{g}_\lambda}$.

Therefore $N_{\tilde{g}_\lambda}$ is a left ideal of N . $\square$

Proposition 4 For $\tilde{g}_\lambda, \tilde{h}_\mu \in \mathbb{H}(N)$, the below assertions are valid:

- (i) If $\tilde{g}_\lambda$ and $\tilde{h}_\mu$ are hybrid left (resp., right) ideals in N , then $\tilde{g}_\lambda \cap \tilde{h}_\mu$ is a hybrid left (resp., right, ideal) in N .
- (ii) If N is zero-symmetric and if $\tilde{g}_\lambda$ is a hybrid right ideal and $\tilde{h}_\mu$ is a hybrid left ideal of N , then $\tilde{g}_\lambda \odot \tilde{h}_\mu \ll \tilde{g}_\lambda \cap \tilde{h}_\mu$.

Proof (i) Assume that $\tilde{g}_\lambda$ and $\tilde{h}_\mu$ are hybrid left ideals in N over U . Let $l_1, r \in N$. Then

$$\begin{aligned} (\tilde{g} \cap \tilde{h})(l_1 - r) &= \tilde{g}(l_1 - r) \cap \tilde{h}(l_1 - r) \\ &\supseteq (\tilde{g}(l_1) \cap \tilde{g}(r)) \cap (\tilde{h}(l_1) \cap \tilde{h}(r)) \\ &= (\tilde{g}(l_1) \cap \tilde{h}(l_1)) \cap (\tilde{g}(r) \cap \tilde{h}(r)) \end{aligned}$$

$$\begin{aligned}
&= (\tilde{g}\tilde{h})(l_1) \cap (\tilde{g}\tilde{h})(r) \\
&(\lambda \vee \mu)(l_1 - r) = \lambda(l_1 - r) \vee \mu(l_1 - r) \\
&\leq \{\lambda(l_1) \vee \lambda(r)\} \vee \{\mu(l_1) \vee \mu(r)\} \\
&= \{\lambda(l_1) \vee \mu(l_1)\} \vee \{\lambda(r) \vee \mu(r)\} \\
&= (\lambda \vee \mu)(l_1) \vee (\lambda \vee \mu)(r).
\end{aligned}$$

Also

$$\begin{aligned}
(\tilde{g}\tilde{h})(r + l_1 - r) &= \tilde{g}(r + l_1 - r) \cap \tilde{h}(r + l_1 - r) \\
&\supseteq \tilde{g}(l_1) \cap \tilde{h}(l_1) = (\tilde{g}\tilde{h})(l_1) \\
(\lambda \vee \mu)(r + l_1 - r) &= \lambda(r + l_1 - r) \vee \mu(r + l_1 - r) \\
&\leq \lambda(l_1) \vee \mu(l_1) = (\lambda \vee \mu)(l_1).
\end{aligned}$$

Now let $u, r, l \in N$. Then

$$\begin{aligned}
&(\tilde{g}\tilde{h})(u(l + r) - ur) \\
&= \tilde{g}(u(l + r) - ur) \cap \tilde{h}(u(l + r) - ur) \\
&\supseteq \tilde{g}(l) \cap \tilde{h}(l) = (\tilde{g}\tilde{h})(l), \\
&(\lambda \vee \mu)(u(l + r) - ur) \\
&= \lambda(u(l + r) - ur) \vee \mu(u(l + r) - ur) \\
&\leq \lambda(l) \vee \mu(l) = (\lambda \vee \mu)(l).
\end{aligned}$$

Therefore $\tilde{g}_\lambda \cap \tilde{h}_\mu$ is a hybrid left ideal in N .

(ii) Assume that $\tilde{g}_\lambda$ and $\tilde{h}_\mu$ are hybrid right and hybrid left ideals, respectively, of a zero-symmetric near-ring N , and let $l \in N$.

If $\exists y, z \in N : l = yz$, then

$$\begin{aligned}
\tilde{g}(l) &= \tilde{g}(yz) \supseteq \tilde{g}(y), \\
\lambda(l) &= \lambda(yz) \leq \lambda(y), \\
\tilde{h}(l) &= \tilde{h}(yz) = \tilde{h}(y(z + 0) - y0) \supseteq \tilde{h}(z), \\
&\text{and} \\
\mu(l) &= \mu(yz) = \mu(y(z + 0) - y0) \leq \mu(z)
\end{aligned}$$

which imply

$$\begin{aligned}
(\tilde{g}\tilde{h})(l) &= \bigcup_{l=yz} \{\tilde{g}(y) \cap \tilde{h}(z)\} \\
&\supseteq \tilde{g}(l) \cap \tilde{h}(l) = (\tilde{g} \cap \tilde{h})(l), \\
(\lambda\tilde{o}\mu)(l) &= \bigwedge_{l=yz} \{\lambda(y) \vee \mu(z)\} \\
&\geq \lambda(l) \vee \mu(l) = (\lambda \vee \mu)(l).
\end{aligned}$$

Therefore $\tilde{g}_\lambda \odot \tilde{h}_\mu \ll \tilde{g}_\lambda \cap \tilde{h}_\mu$. $\square$

Definition 8 [23] Let $\tilde{f}_\lambda \in \mathbb{H}(N)$. For any $(\epsilon, \delta) \in \mathcal{P}(U) \times I$, we define

$$L_f^\epsilon := \{l \in N : \tilde{f}(l) \supseteq \epsilon\} \text{ and } L_\lambda^\delta := \{l \in N : \lambda(l) \leq \delta\}.$$

Note that

$$\begin{aligned}
L_f^\epsilon \cap L_\lambda^\delta &= \{l \in N \mid \tilde{f}(l) \supseteq \epsilon, \lambda(l) \leq \delta\} \\
&= \tilde{f}_\lambda[\epsilon, \delta].
\end{aligned}$$

Proposition 5 Let $\tilde{f}_\lambda \in \mathbb{H}(N)$. Then the below statements are equivalent:

- (i) $\tilde{f}_\lambda$ is hybrid left (resp., right) ideal in N over U ,
- (ii) For any $(\epsilon, \delta) \in \mathcal{P}(U) \times I$, the non-empty sets L_f^ϵ and L_λ^δ are left (resp., right) ideals of N .

Proof Assume that $\tilde{f}_\lambda$ is a hybrid left ideal in N . Let $\epsilon \in \mathcal{P}(U)$ and $\delta \in I$ be such that $L_f^\epsilon \neq \emptyset$ and $L_\lambda^\delta \neq \emptyset$.

If $l, s \in L_f^\epsilon \cap L_\lambda^\delta$, then $\tilde{f}(l) \supseteq \epsilon, \tilde{f}(s) \supseteq \epsilon, \lambda(l) \leq \delta$ and $\lambda(s) \leq \delta$. Since $\tilde{f}(l - s) \supseteq \tilde{f}(l) \cap \tilde{f}(s) \supseteq \epsilon$ and $\lambda(l - s) \leq \lambda(l) \vee \lambda(s) \leq \delta$, so $l - s \in L_f^\epsilon \cap L_\lambda^\delta$.

Let $s \in N$ and $l \in L_f^\epsilon \cap L_\lambda^\delta$. Then $\tilde{f}(s + l - s) \supseteq \tilde{f}(l) \supseteq \epsilon$ and $\lambda(s + l - s) \leq \lambda(l) \leq \delta$ which imply $s + l - s \in L_f^\epsilon \cap L_\lambda^\delta$.

Now for $l \in L_f^\epsilon \cap L_\lambda^\delta$ and $u, v \in N$, $\tilde{f}(u(l + v) - uv) \supseteq \tilde{f}(l) \supseteq \epsilon$ and $\lambda(u(l + v) - uv) \leq \lambda(l) \leq \delta$ which imply $u(l + v) - uv \in L_f^\epsilon \cap L_\lambda^\delta$.

Hence L_f^ϵ and L_λ^δ are left ideals.

Conversely assume that L_f^ϵ and L_λ^δ are left ideals of N for all $(\epsilon, \delta) \in \mathcal{P}(U) \times I$.

Let $r, s \in N$ be such that $\tilde{f}(r) = \epsilon_a$ and $\tilde{f}(s) = \epsilon_b$. If we put $\epsilon := \epsilon_a \cap \epsilon_b$, then $r, s \in L_f^\epsilon$ and $r - s \in L_f^\epsilon$ and so $\tilde{f}(r - s) \supseteq \epsilon = \epsilon_a \cap \epsilon_b = \tilde{f}(r) \cap \tilde{f}(s)$. Since $r \in L_f^{\epsilon_a}$, we have $s + r - s \in L_f^{\epsilon_a}$ and $u(r + v) - uv \in L_f^{\epsilon_a}$ for all $u, v \in N$ which imply $\tilde{f}(s + r - s) \supseteq \epsilon_a = \tilde{f}(r)$ and $\tilde{f}(u(r + v) - uv) \supseteq \epsilon_a = \tilde{f}(r)$.

Also, let $\lambda(r) = \delta_a$ and $\lambda(s) = \delta_b$. Then by taking $\delta := \delta_a \vee \delta_b$, we get $r, s \in L_\lambda^\delta$ and $r - s \in L_\lambda^\delta$ which imply $\lambda(r - s) \leq \delta = \delta_a \vee \delta_b = \lambda(r) \vee \lambda(s)$. Also $r \in L_\lambda^{\delta_a}$ gives $s + r - s \in L_\lambda^{\delta_a}$ and $u(r + v) - uv \in L_\lambda^{\delta_a}$ imply that $\lambda(s + r - s) \leq \delta_a = \lambda(r)$ and $\lambda(u(r + v) - uv) \leq \delta_a = \lambda(r)$.

Hence $\tilde{f}_\lambda$ is a hybrid left ideal in N . $\square$

Corollary 1 For a hybrid left (resp., right) ideal $\tilde{f}_\lambda$ in N , the nonempty $[\epsilon, r]$ -hybrid cut of $\tilde{f}_\lambda$ is a left (resp., right) ideal of N for all $\epsilon \in \mathcal{P}(U)$ and $r \in I$.

Proof It is trivial as $L_f^\epsilon \cap L_\lambda^r = \tilde{f}_\lambda[\epsilon, r]$. $\square$

Theorem 2 For a hybrid left (resp., right) ideal $\tilde{f}_\lambda$ in N , the set

$$\Omega := \{l \in N \mid \tilde{f}(l) \cap \alpha \neq \emptyset, \lambda(x) \leq \delta\}$$

is a left (resp., right) ideal of $N \forall (\alpha, \delta) \in \mathcal{P}(U) \times I$ with $\alpha \neq \emptyset$ whenever it is nonempty.

Proof Let $(\alpha, \delta) \in \mathcal{P}(U) \times I$ be such that $\Omega \neq \phi \neq \alpha$. Let $r, s \in \Omega$. Then $\tilde{f}(r) \cap \alpha \neq \phi \neq \tilde{f}(s) \cap \alpha, \lambda(r) \leq \delta$ and $\lambda(s) \leq \delta$. It follows from (H1) that $\tilde{f}(r-s) \cap \alpha \supseteq (\tilde{f}(r) \cap \tilde{f}(s)) \cap \alpha = (\tilde{f}(r) \cap \alpha) \cap (\tilde{f}(s) \cap \alpha) \neq \phi$ and $\lambda(r-s) \leq \lambda(r) \vee \lambda(s) \leq \delta$. So $r-s \in \Omega$.

Let $l \in \Omega$ and $s \in N$. Then $\tilde{f}(l) \cap \alpha \neq \phi$ and $\lambda(l) \leq \delta$. It follows from (H3) that $\tilde{f}(s+l-s) \cap \alpha \supseteq \tilde{f}(l) \cap \alpha \neq \phi$ and $\lambda(s+l-s) \leq \lambda(l) \leq \delta$. So $s+l-s \in \Omega$.

Let $l \in \Omega$ and $s, v \in N$. Then $\tilde{f}(l) \cap \alpha \neq \phi$ and $\lambda(l) \leq \delta$. It follows from (H5) that $\tilde{f}(s(l+v)-sv) \cap \alpha \supseteq \tilde{f}(l) \cap \alpha \neq \phi$ and $\lambda(s(l+v)-sv) \leq \lambda(l) \leq \delta$. So $s(l+v)-sv \in \Omega$.

Therefore Ω is a left ideal of N . $\square$

Theorem 3 For a left (resp., right) ideal A of N and $(\epsilon, \delta) \in (\mathcal{P}(U) \setminus \{\phi\}) \times [0, 1)$, there exists a hybrid left (resp., right) ideal $\tilde{f}_\lambda$ in N over U such that $L_\epsilon^\epsilon = A = L_\lambda^\delta$.

Proof Let $\tilde{f}_\lambda \in \mathbb{H}(N)$ defined as

$$\tilde{f}_\lambda(r) = \begin{cases} \epsilon & \text{if } r \in A \\ \phi & \text{otherwise} \end{cases} \text{ and } \lambda(r) = \begin{cases} \delta & \text{if } r \in A \\ 1 & \text{otherwise,} \end{cases}$$

for all $r \in N$, where $\epsilon \in \mathcal{P}(U) \setminus \{\phi\}$ and $\delta \in [0, 1)$. Then clearly $L_\epsilon^\epsilon = A = L_\lambda^\delta$.

We now prove that $\tilde{f}_\lambda$ is a hybrid left ideal in N .

Let $k, s \in N$. If $k \notin A$ or $s \notin A$, then $\tilde{f}_\lambda(k-s) \supseteq \phi = \tilde{f}_\lambda(k) \cap \tilde{f}_\lambda(s)$ and $\lambda(k-s) \leq 1 = \lambda(k) \vee \lambda(s)$. If $k, s \in A$, then $k-s \in A$ implies $\tilde{f}_\lambda(k-s) = \epsilon = \tilde{f}_\lambda(k) \cap \tilde{f}_\lambda(s)$ and $\lambda(k-s) = \delta = \lambda(k) \vee \lambda(s)$.

If $k \in A$, then $s+k-s \in A$ gives $\tilde{f}_\lambda(s+k-s) = \epsilon = \tilde{f}_\lambda(k)$ and $\lambda(s+k-s) = \delta = \lambda(k)$. If $k \notin A$, then $\tilde{f}_\lambda(s+k-s) \supseteq \phi = \tilde{f}_\lambda(k)$ and $\lambda(s+k-s) \leq 1 = \lambda(k)$.

Now let $k, u, r \in N$. If $k \in A$, then $u(k+r)-ur \in A$ implies $\tilde{f}_\lambda(u(k+r)-ur) = \epsilon = \tilde{f}_\lambda(k)$ and $\lambda(u(k+r)-ur) = \delta = \lambda(k)$. If $k \notin A$, then $\tilde{f}_\lambda(u(k+r)-ur) \supseteq \phi = \tilde{f}_\lambda(k)$ and $\lambda(u(k+r)-ur) \leq 1 = \lambda(k)$.

Therefore $\tilde{f}_\lambda$ is a hybrid left ideal in N . $\square$

Theorem 4 For $\tilde{f}_\lambda \in \mathbb{H}(N)$, we have $\tilde{f}_\lambda(x) = \bigcup \{\epsilon \mid \epsilon \in \mathcal{P}(U) \text{ and } a \in L_\epsilon^\epsilon\}$ and $\lambda(a) = \bigwedge \{t \mid t \in [0, 1] \text{ and } a \in L_\lambda^t\}$ for all $x \in N$.

Proof Observe that for any $a \in N$, $a \in L_\epsilon^\epsilon$ and $a \in L_\lambda^t$ for some $\epsilon \in \mathcal{P}(U)$ and $t \in [0, 1]$.

Let $l \in N$ and take $T := \bigcup \{\epsilon \mid \epsilon \in \mathcal{P}(U) \text{ and } l \in L_\epsilon^\epsilon\}$. Then $\tilde{f}_\lambda(l) \supseteq T$. If $\tilde{f}_\lambda(l) \supset T$, then $f(l) \supset \epsilon_i \forall \epsilon_i \in \mathcal{P}(U)$, which is not possible as $\tilde{f} : N \rightarrow \mathcal{P}(U)$ is a mapping. So $\tilde{f}_\lambda(l) = T = \bigcup \{\epsilon \mid \epsilon \in \mathcal{P}(U) \text{ and } l \in L_\epsilon^\epsilon\}$.

Also for $l \in N$, take $\delta := \bigwedge \{t \mid t \in [0, 1] \text{ and } l \in L_\lambda^t\}$, then $\lambda(l) \leq \delta$. If $\lambda(l) < \delta$, then $\lambda(l) < t$ for all $t \in [0, 1]$, which is also not possible as $\lambda : N \rightarrow I$ is a mapping. $\square$

We presently consider the converse of Theorem 4.

Theorem 5 Let $\{L_\epsilon^\epsilon \mid \epsilon \in \mathcal{P}(U)\}$ and $\{L_\lambda^t : t \in [0, 1]\}$ be collections of left (resp., right) ideals of N such that

- (i) $N = \bigcup_{\epsilon \in \mathcal{P}(U)} L_\epsilon^\epsilon = \bigcup_{t \in [0, 1]} L_\lambda^t$,
- (ii) $S \supset L$ if and only if $L_\epsilon^\epsilon \subset L_\lambda^t \forall S, L \in \mathcal{P}(U)$,
- (iii) $s < r$ if and only if $L_\lambda^s \subset L_\lambda^r \forall s, r \in I$.

Define a hybrid structure $\tilde{f}_\lambda$ in N over U by

$\tilde{f}_\lambda(a) = \bigcup \{\epsilon \mid \epsilon \in \mathcal{P}(U) \text{ and } a \in L_\epsilon^\epsilon\}$ and $\lambda(a) = \bigwedge \{t \mid t \in [0, 1] \text{ and } a \in L_\lambda^t\}$ for all $a \in N$. Then $\tilde{f}_\lambda$ is a hybrid left (resp., right) ideal in N .

Proof By Proposition 5, it is sufficient to prove that L_ϵ^ϵ and L_λ^t are left ideals of $N \forall (\epsilon, t) \in \mathcal{P}(U) \times I$.

We now prove L_ϵ^ϵ is a left ideal of $N \forall \epsilon \in \mathcal{P}(U)$.

Consider the following two cases:

- (i) $\epsilon = \bigcup \{\epsilon_i \mid \epsilon_i \in \mathcal{P}(U) \text{ and } a \in L_{\epsilon_i}^{\epsilon_i}\}$ and (ii) $\epsilon \neq \bigcup \{\epsilon_i \mid \epsilon_i \in \mathcal{P}(U) \text{ and } a \in L_{\epsilon_i}^{\epsilon_i}\}$.

Case (i) implies that $a \in L_{\epsilon_j}^{\epsilon_j}$ if and only if $a \in L_{\epsilon_j}^{\epsilon_j}$ for all $\epsilon_j \subset \epsilon$ if and only if $a \in \bigcap_{\epsilon_j \subset \epsilon} L_{\epsilon_j}^{\epsilon_j}$. Thus $L_\epsilon^\epsilon = \bigcap_{\epsilon_j \subset \epsilon} L_{\epsilon_j}^{\epsilon_j}$ and hence L_ϵ^ϵ is a left ideal in N .

For the case (ii), we prove that $L_\epsilon^\epsilon = \bigcup_{\epsilon_j \supseteq \epsilon} L_{\epsilon_j}^{\epsilon_j}$. If $a \in$

$\bigcup_{\epsilon_j \supseteq \epsilon} L_{\epsilon_j}^{\epsilon_j}$, then $a \in L_{\epsilon_j}^{\epsilon_j}$ for some $\epsilon_j \supseteq \epsilon$. By assumption, we have $a \in L_\epsilon^\epsilon$, so $\bigcup_{\epsilon_j \supseteq \epsilon} L_{\epsilon_j}^{\epsilon_j} \subseteq L_\epsilon^\epsilon$. For converse, if $a \notin \bigcup_{\epsilon_j \supseteq \epsilon} L_{\epsilon_j}^{\epsilon_j}$, then $a \notin L_{\epsilon_j}^{\epsilon_j}$ for all $\epsilon_j \supseteq \epsilon$. In particular, $a \notin L_\epsilon^\epsilon$, so $L_\epsilon^\epsilon = \bigcup_{\epsilon_j \supseteq \epsilon} L_{\epsilon_j}^{\epsilon_j}$ and L_ϵ^ϵ is a left ideal of N .

We now prove L_λ^δ is a left ideal of N for $\delta \in I$.

We consider the following two cases:

- (i) $\delta = \bigwedge \{t \mid t \in [0, 1] \text{ and } t > \delta\}$ and (ii) $\delta \neq \bigwedge \{t \mid t \in [0, 1] \text{ and } t > \delta\}$.

Case (i) implies that $a \in L_\lambda^\delta$ if and only if $a \in L_\lambda^t$ for all $t > \delta$ if and only if $a \in \bigcap_{t > \delta} L_\lambda^t$. Thus $L_\lambda^\delta = \bigcap_{t > \delta} L_\lambda^t$ and L_λ^δ is a left ideal of N .

For the case (ii), we prove that $L_\lambda^\delta = \bigcup_{t \geq \delta} L_\lambda^t$. If $a \in$

$\bigcup_{t \geq \delta} L_\lambda^t$, then $a \in L_\lambda^t$ for some $t \geq \delta$. By assumption, we have $a \in L_\lambda^\delta$, so $\bigcup_{t \geq \delta} L_\lambda^t \subseteq L_\lambda^\delta$. For converse, if $a \notin \bigcup_{t \geq \delta} L_\lambda^t$, then $a \notin L_\lambda^t$ for all $t \geq \delta$. In particular $a \notin L_\lambda^\delta$. So $L_\lambda^\delta = \bigcup_{t \geq \delta} L_\lambda^t$ and L_λ^δ is a left ideal of N . $\square$

Definition 9 [22] Let $\psi : L \rightarrow M$ be a mapping from a set L to a set M . For a hybrid structure $\tilde{g}_\mu$ in M over U , consider a hybrid structure $\psi^{-1}(\tilde{g}_\mu) := (\psi^{-1}(\tilde{g}), \psi^{-1}(\mu))$ in L over U where $\psi^{-1}(\tilde{g}(l)) = \tilde{g}(\psi(l))$ and $\psi^{-1}(\mu)(l) = \mu(\psi(l))$

for all $l \in L$. We say that $\psi^{-1}(\tilde{g}_\mu)$ is the hybrid preimage of $\tilde{g}_\mu$ under ψ .

For a hybrid structure $\tilde{f}_\lambda$ in L over U , the hybrid image of $\tilde{f}_\lambda$ under ψ is defined to be a hybrid structure $\psi(\tilde{f}_\lambda) := (\psi(\tilde{f}), \psi(\lambda))$ in M over U where

$$\psi(\tilde{f}(k)) = \begin{cases} \bigcup_{l \in \psi^{-1}(k)} \tilde{f}(l) & \text{if } \psi^{-1}(k) \neq \emptyset \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\psi(\lambda)(k) = \begin{cases} \bigcap_{l \in \psi^{-1}(k)} \lambda(l) & \text{if } \psi^{-1}(k) \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

for every $k \in M$.

Definition 10 Let N_1 and N_2 be near-rings. A map $\psi : N_1 \rightarrow N_2$ is called a near-ring homomorphism if $\psi(r+s) = \psi(r) + \psi(s)$ and $\psi(rs) = \psi(r)\psi(s)$ for any $r, s \in N_1$.

It is clear that every homomorphic preimage of a left (resp., right) ideal is also a left (resp., right) ideal. Similar way, we have the following theorem for hybrid preimage.

Theorem 6 Every homomorphic hybrid preimage of a hybrid left (resp., right) ideal is also a hybrid left (resp., right) ideal.

Proof Let $\psi : N_1 \rightarrow N_2$ be a near-ring homomorphism, and $\tilde{g}_\mu$ be a hybrid left ideal of N_2 over U and let $(k, s, t \in N_1)$. Then

$$\begin{aligned} \psi^{-1}(\tilde{g})(k-s) &= \tilde{g}(\psi(k-s)) \\ &= \tilde{g}(\psi(k) - \psi(s)) \\ &\supseteq \tilde{g}(\psi(k)) \cap \tilde{g}(\psi(s)) \\ &= \psi^{-1}(\tilde{g})(k) \cap \psi^{-1}(\tilde{g})(s), \\ \psi^{-1}(\mu)(k-s) &= \mu(\psi(k-s)) \\ &= \mu(\psi(k) - \psi(s)) \\ &\leq \mu(\psi(k)) \vee \mu(\psi(s)) \\ &= \psi^{-1}(\mu)(k) \vee \psi^{-1}(\mu)(s), \end{aligned}$$

and

$$\begin{aligned} \psi^{-1}(\tilde{g})(s+k-s) &= \tilde{g}(\psi(s+k-s)) \\ &= \tilde{g}(\psi(s) + \psi(k) - \psi(s)) \\ &\supseteq \tilde{g}(\psi(k)) = \psi^{-1}(\tilde{g})(k) \\ \psi^{-1}(\mu)(s+k-s) &= \mu(\psi(s+k-s)) \\ &= \mu(\psi(s) + \psi(k) - \psi(s)) \\ &\leq \mu(\psi(k)) = \psi^{-1}(\mu)(k). \end{aligned}$$

Also

$$\begin{aligned} \psi^{-1}(\tilde{g})(k(s+t) - kt) &= \tilde{g}(\psi(k(s+t) - kt)) \\ &= \tilde{g}(\psi(k)(\psi(s) + \psi(t)) - \psi(k)\psi(t)) \\ &\supseteq \tilde{g}(\psi(s) = \psi^{-1}(\tilde{g})(s)) \\ \psi^{-1}(\mu)(k(s+t) - kt) &= \mu(\psi(k(s+t) - kt)) \\ &= \mu(\psi(k)(\psi(s) + \psi(t)) - \psi(k)\psi(t)) \\ &\leq \mu(\psi(s) = \psi^{-1}(\mu)(s)). \end{aligned}$$

Therefore $\psi^{-1}(\tilde{g}_\mu)$ is a hybrid left ideal of N_1 .

Suppose that $\tilde{g}_\mu$ is a hybrid right ideal in N_1 . Then

$$\begin{aligned} \psi^{-1}(\tilde{g})(ks) &= \tilde{g}(\psi(ks)) \\ &= \tilde{g}(\psi(k)\psi(s)) \\ &\supseteq \tilde{g}(\psi(k)) = \psi^{-1}(\tilde{g})(k), \\ \psi^{-1}(\mu)(ks) &= \mu(\psi(ks)) \\ &= \mu(\psi(k)\psi(s)) \\ &\leq \mu(\psi(k)) = \psi^{-1}(\mu)(k) \end{aligned}$$

for any $k, s \in N_1$.

So $\psi^{-1}(\tilde{g}_\mu)$ is a hybrid right ideal of N_1 . $\square$

For an onto homomorphism $\psi : N_1 \rightarrow N_2$ of near-rings, let $\psi^{-1}(\tilde{g}_\gamma) := (\psi^{-1}(\tilde{g}), \psi^{-1}(\gamma))$ be a hybrid left ideal of N_1 over U where $\tilde{g}_\gamma$ is a hybrid structure in N_2 over U .

Let $r, s, t \in N_2$. Then $\psi(r_1) = r$, $\psi(s_1) = s$ and $\psi(t_1) = t$ for some $r_1, s_1, t_1 \in N_1$. Now

$$\begin{aligned} \tilde{g}(r-s) &= \tilde{g}(\psi(r_1) - \psi(s_1)) \\ &= \tilde{g}(\psi(r_1 - s_1)) \\ &= \psi^{-1}(\tilde{g})(r_1 - s_1) \\ &\supseteq \psi^{-1}(\tilde{g})(r_1) \cap \psi^{-1}(\tilde{g})(s_1) \\ &= \tilde{g}(\psi(r_1)) \cap \tilde{g}(\psi(s_1)) = \tilde{g}(r) \cap \tilde{g}(s) \\ \gamma(r-s) &= \gamma(\psi(r_1) - \psi(s_1)) \\ &= \gamma(\psi(r_1 - s_1)) \\ &= \psi^{-1}(\gamma)(r_1 - s_1) \\ &\leq \psi^{-1}(\gamma)(r_1) \vee \psi^{-1}(\gamma)(s_1) \\ &= \gamma(\psi(r_1)) \vee \gamma(\psi(s_1)) = \gamma(r) \vee \gamma(s), \end{aligned}$$

and

$$\begin{aligned} \tilde{g}(r+s-r) &= \tilde{g}(\psi(r_1) + \psi(s_1) - \psi(r_1)) \\ &= \tilde{g}(\psi(r_1 + s_1 - r_1)) \\ &= \psi^{-1}(\tilde{g})(r_1 + s_1 - r_1) \\ &\supseteq \psi^{-1}(\tilde{g})(s_1) = \tilde{g}(\psi(s_1)) = \tilde{g}(s) \\ \gamma(r+s-r) &= \gamma(\psi(r_1) + \psi(s_1) - \psi(r_1)) \end{aligned}$$

$$\begin{aligned}
&= \gamma(\psi(r_1 + s_1 - r_1)) \\
&= \psi^{-1}(\gamma)(r_1 + s_1 - r_1) \\
&\leq \psi^{-1}(\gamma)(s_1) = \gamma(\psi(s_1)) = \gamma(s).
\end{aligned}$$

Also

$$\begin{aligned}
\tilde{g}(r(t + s) - rs) &= \tilde{g}(\psi(r_1)(\psi(t_1) + \psi(s_1)) \\
&\quad - \psi(r_1)\psi(s_1)) \\
&= \tilde{g}(\psi(r_1(t_1 + s_1) - r_1s_1)) \\
&= \psi^{-1}(\tilde{g})(r_1(t_1 + s_1) - r_1s_1) \\
&\supseteq \psi^{-1}(\tilde{g})(t_1) = \tilde{g}(\psi(t_1)) = \tilde{g}(t), \\
\gamma(r(t + s) - rs) &= \gamma(\psi(r_1)(\psi(t_1) + \psi(s_1)) \\
&\quad - \psi(r_1)\psi(s_1)) \\
&= \gamma(\psi(r_1(t_1 + s_1) - r_1s_1)) \\
&= \psi^{-1}(\gamma)(r_1(t_1 + s_1) - r_1s_1) \\
&\leq \psi^{-1}(\gamma)(t_1) = \gamma(\psi(t_1)) = \gamma(t).
\end{aligned}$$

If $\tilde{g}_\gamma$ is a hybrid right ideal in N_2 , then

$$\begin{aligned}
\tilde{g}(rs) &= \tilde{g}(\psi(r_1)\psi(s_1)) = \tilde{g}(\psi(r_1s_1)) \\
&= \psi^{-1}(\tilde{g})(r_1s_1) \\
&\supseteq \psi^{-1}(\tilde{g})(r_1) \\
&= \tilde{g}(\psi(r_1)) = \tilde{g}(r_1), \\
\gamma(rs) &= \gamma(\psi(r_1)\psi(s_1)) = \gamma(\psi(r_1s_1)) \\
&= \psi^{-1}(\gamma)(r_1s_1) \\
&\leq \psi^{-1}(\gamma)(r_1) \\
&= \gamma(\psi(r_1)) = \gamma(r_1).
\end{aligned}$$

Therefore we have the following theorem.

Theorem 7 Let $\psi : N_1 \rightarrow N_2$ be a onto homomorphism of near-rings. For every hybrid structure $\tilde{g}_\gamma$ in N_2 over U , if the preimage $\psi^{-1}(\tilde{g}_\gamma)$ of $\tilde{g}_\gamma$ under ψ is a hybrid left (resp. right) ideal of N_1 over U , then $\tilde{g}_\gamma$ is a hybrid left (resp., right) ideal of N_2 over U .

We say that $\tilde{f}_\lambda \in \mathbb{H}(N)$ has the *sup property* if for any subset T of N_1 , there exists $t_0 \in T$ such that $\tilde{f}(t_0) = \bigcup_{t \in T} \tilde{f}(t)$ and $\lambda(t_0) = \bigwedge_{t \in T} \lambda(t)$.

Theorem 8 A near-ring homomorphic image of a hybrid left (resp., right) ideal having the sup property is a hybrid left (resp., right) ideal.

Proof Let $\psi : N_1 \rightarrow N_2$ be a near-ring homomorphism and $\tilde{f}_\lambda$ be a hybrid right ideal of N_1 with the sup property and $\tilde{g}_\mu$ be the image of $\tilde{f}_\lambda$ under ψ .

Given $\psi(r), \psi(t) \in \psi(N_1)$, let $r_0 \in \psi^{-1}(\psi(r)), t_0 \in \psi^{-1}(\psi(t))$ be such that

$$\begin{aligned}
\tilde{f}(r_0) &= \bigcup_{t \in \psi^{-1}(\psi(r))} \tilde{f}(t); \\
\lambda(r_0) &= \bigwedge_{t \in \psi^{-1}(\psi(r))} \lambda(t) \\
\tilde{f}(t_0) &= \bigcup_{t \in \psi^{-1}(\psi(t))} \tilde{f}(t); \\
\lambda(t_0) &= \bigwedge_{t \in \psi^{-1}(\psi(t))} \lambda(t).
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{g}(\psi(r) - \psi(t)) &= \bigcup_{t \in \psi^{-1}(\psi(r) - \psi(t))} \tilde{f}(t) \\
&\supseteq \tilde{f}(r_0 - t_0) \\
&\supseteq \tilde{f}(r_0) \cap \tilde{f}(t_0) \\
&= \left\{ \bigcup_{t \in \psi^{-1}(\psi(r))} \tilde{f}(t) \right\} \cap \left\{ \bigcup_{t \in \psi^{-1}(\psi(t))} \tilde{f}(t) \right\} \\
&= \tilde{g}(\psi(r)) \cap \tilde{g}(\psi(t)), \\
\mu(\psi(r) - \psi(t)) &= \bigwedge_{t \in \psi^{-1}(\psi(r) - \psi(t))} \lambda(t) \\
&\leq \lambda(r_0 - t_0) \leq \lambda(r_0) \vee \lambda(t_0) \\
&= \left\{ \bigwedge_{t \in \psi^{-1}(\psi(r))} \lambda(t) \right\} \vee \left\{ \bigwedge_{t \in \psi^{-1}(\psi(t))} \lambda(t) \right\} \\
&= \mu(\psi(r)) \vee \mu(\psi(t)),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g}(\psi(t) + \psi(r) - \psi(t)) &= \bigcup_{t \in \psi^{-1}(\psi(t) + \psi(r) - \psi(t))} \tilde{f}(t) \\
&\supseteq \tilde{f}(t_0 + r_0 - t_0) \\
&\supseteq \tilde{f}(r_0) \\
&= \bigcup_{t \in \psi^{-1}(\psi(r))} \tilde{f}(t) = \tilde{g}(\psi(r)), \\
\mu(\psi(t) + \psi(r) - \psi(t)) &= \bigwedge_{t \in \psi^{-1}(\psi(t) + \psi(r) - \psi(t))} \lambda(t) \\
&\leq \lambda(\psi(t) + \psi(r) - \psi(t)) \\
&\leq \lambda(r_0) \\
&= \bigwedge_{t \in \psi^{-1}(\psi(r))} \lambda(t) \\
&= \mu(\psi(r)),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(\psi(r)\psi(t)) &= \bigcup_{t \in \psi^{-1}(\psi(r)\psi(t))} \tilde{f}(t) \\
&\supseteq \tilde{f}(r_0 t_0) \\
&\supseteq \tilde{f}(r_0) \\
&= \bigcup_{t \in \psi^{-1}(\psi(r))} \tilde{f}(t) \\
&= \tilde{g}(\psi(r)), \\
\mu(\psi(r)\psi(t)) &= \bigwedge_{t \in \psi^{-1}(\psi(r)\psi(t))} \lambda(t) \\
&\leq \lambda(r_0 t_0) \\
&\leq \lambda(r_0) \\
&= \bigwedge_{t \in \psi^{-1}} (\psi(r)) \\
&= \mu(\psi(r)).
\end{aligned}$$

This proves that $\tilde{g}_\mu$ is a hybrid right ideal in $\psi(N_1)$.

Assume that $\tilde{f}_\lambda$ is hybrid left ideal in N_1 . For $\psi(i) \in \psi(N_1)$, let $i_0 \in \psi^{-1}(\psi(i))$ be such that $\tilde{f}(i_0) = \bigcup_{t \in \psi^{-1}(\psi(i))} \tilde{f}(t)$ and $\lambda(i_0) = \bigwedge_{t \in \psi^{-1}(\psi(i))} \lambda(t)$. Then

$$\begin{aligned}
&\tilde{g}(\psi(r)(\psi(i) + \psi(t)) - \psi(r)\psi(t)) \\
&= \bigcup_{t \in \psi^{-1}(\psi(r)(\psi(i) + \psi(t)) - \psi(r)\psi(t))} \tilde{f}(t) \\
&\supseteq \tilde{f}(r_0(i_0 + t_0) - r_0 t_0) \\
&\supseteq \tilde{f}(i_0) \\
&= \bigcup_{t \in \psi^{-1}(\psi(i))} \tilde{f}(t) \\
&= \tilde{g}(\psi(i)), \\
&\mu(\psi(r)(\psi(i) + \psi(t)) - \psi(r)\psi(t)) \\
&= \bigwedge_{t \in \psi^{-1}(\psi(r)(\psi(i) + \psi(t)) - \psi(r)\psi(t))} \lambda(t) \\
&\leq \lambda(r_0(i_0 + t_0) - r_0 t_0) \\
&\leq \lambda(i_0) \\
&= \bigwedge_{t \in \psi^{-1}(\psi(i))} \lambda(t) \\
&= \mu(\psi(i))
\end{aligned}$$

Hence $\tilde{g}_\mu$ is a hybrid left ideal in $\psi(N_1)$. $\square$

Conclusion

In this paper, we have presented the notion of hybrid ideals in near rings and investigated their properties. Efforts have been made to characterize near-rings in terms of hybrid ideal structures with illustrative examples. The relation of

hybrid intersection and hybrid product of hybrid left (resp., right) ideals in zero-symmetric near-rings has been obtained. For a hybrid left (resp., right) ideal in a near-ring, we have constructed left (resp., right) ideals of near-ring. We have also considered the hybrid image and hybrid preimage of the hybrid left ideal of a near-ring under the near-ring homomorphisms. For future research work, by using the notions and results of this paper, it has been planned to define the concept of hybrid prime (resp., semi) ideals and obtain their various properties and equivalent conditions for a hybrid ideal to be a hybrid prime (resp., semi) ideal in near-rings.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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