ORIGINAL ARTICLE

An innovative picture fuzzy distance measure and novel multi-attribute decision-making method

Abdul Haseeb Ganie[1] · Surender Singh[1]

Abstract

Picture fuzzy set (PFS) is a direct generalization of the fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). The concept of PFS is suitable to model the situations that involve more answers of the type yes, no, abstain, and refuse. In this study, we introduce a novel picture fuzzy (PF) distance measure on the basis of direct operation on the functions of membership, non-membership, neutrality, refusal, and the upper bound of the function of membership of two PFSs. We contrast the proposed PF distance measure with the existing PF distance measures and discuss the advantages in the pattern classification problems. The application of fuzzy and non-standard fuzzy models in the real data is very challenging as real data is always found in crisp form. Here, we also derive some conversion formulae to apply proposed method in the real data set. Moreover, we introduce a new multi-attribute decision-making (MADM) method using the proposed PF distance measure. In addition, we justify necessity of the newly proposed MADM method using appropriate counterintuitive examples. Finally, we contrast the performance of the proposed MADM method with the classical MADM methods in the PF environment.

Keywords Picture fuzzy set · Picture fuzzy distance measure · Pattern recognition · TOPSIS · Inferior ratio

Introduction

The comparison of the two distinct objects from various viewpoints is necessary to deal with various real-life problems concerning machine learning, pattern recognition, image processing, decision-making, etc. Depending on the nature of the problem, different researches applied different compatibility/comparison measures. The prominent compatibility/comparison measures are 'similarity measure', 'distance measure', 'correlation measure', etc. The PFS is a non-standard form of fuzziness. As the PFS is a direct extension of FS and IFS, so, it is pertinent to mention the prominent studies regarding the developments and applications of the fuzzy and IFSs.

Zadeh [1] introduced the concept of FSs to study the uncertainty due to lack of complete knowledge. A FS is a group of objects in the universe of discourse with vague, ambiguous, and unsharp boundary. A FS is mainly understood and represented with the help of a membership function assigning a value in the unit interval [0, 1] to each element of the universe of discourse. This assigned value is known as membership degree. The membership degree indicates the extent of belongingness of the element to the FS. If an element has '0' as its membership degree, then the element does not belong to the set, and if it has '1' as membership degree, then the element fully belongs to the set. If the membership degree of an element lies in (0, 1), then the element partially belongs to the FS. Thus, any FS can be determined uniquely with the help of its membership function. To get more insight regarding the development and applications of fuzzy compatibility/comparison measures, we refer to [2–9] and references therein. Because of various perspectives of understanding the vagueness and linguistic imprecision in a system, researchers came up with different mathematical representations. Some important representations among them are intuitionistic FSs (IFSs) [10], type-2 FSs [11–16], Pythagorean FSs [17], q-rung orthopair FSs [18], hesitant FSs [19], etc. These varied expressions are considered as extensions or generalizations of the conventional FS. In the contemporary literature, some prominent

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researches have termed these extensions/generalizations as non-standard FSs. One such extension is the IFS, introduced by Atanassov [10]. Atanassov [10] included the non-membership degree of an element to a FS with the condition that the sum of membership and non-membership degrees should be less or equal to one. Some prominent studies comprising the development and applications of intuitionistic fuzzy (IF)/ Pythagorean fuzzy (PF) compatibility/comparison measures are due to researches [20–29]. Recently, Niu et al. [30] introduced two mentality-parameters and proposed a new method for solving some MADM problems in the interval-valued IF environment. Ejegwa [31] presented the idea of Pythagorean fuzzy relation and demonstrated its application in decision-making. Some exponential Pythagorean fuzzy similarity measures with their applications in decision-making and pattern recognition were put forward by Nguyen and Garg [32]. With the help of probabilistic hesitant FSs and bipartite network projection, Cao et al. [33] introduced a recommendation decision-making algorithm for sharing accommodation.

Although IFSs are more powerful than FSs in expressing uncertain and vague information, they lack an important concept, i.e., degree of neutrality, which has a key role in many situations such as human voting, medical diagnosis, personal selection, etc. In human voting, a person has four options either to vote in favour or to vote against or to abstain or to refuse from voting. In medical diagnosis, the symptoms like temperature and headache may have a null effect on the diseases chest problem and stomach problem. So, to deal with such situations, a new generalization of FSs and IFSs known as PFS was introduced by Cuong and Kreinvoch [34]. In a PFS, each element is specified by the degree of membership, the degree of non-membership, and degree of neutrality together with the condition that the sum of these grades should be less or equal to one. Thereafter, several studies [35–47] investigated the concepts of entropy, similarity, distance, dissimilarity, correlation in the framework of PFSs. The PF cross-entropy with its application in MADM was introduced by Wei [48]. Wei [49] also introduced the TODIM method for solving MADM problems with PF data. Some geometric aggregation operators and some PF operational laws were given by Wang et al. [50]. Wei [51] investigated the application of PF Hamacher aggregation operators in MADM problems. Zhang et al. [52] introduced the evaluation based on distance from the average solution (EDAS) method for solving MADM problems with PF data. Later on, Nhung et al. [53] proposed some new dissimilarity measures involving PF-information and applied them in pattern recognition and multi-criteria decision-making (MCDM). Liu et al. [54] introduced some models of MADM in PF-environment. A PF-divergence measure with its application in multi-criteria decision-making (MCDM) and pattern recognition was proposed by Thao et al. [55]. Some

Dombi aggregation operators in PF-environment with their application in MADM were introduced by Jana et al. [56]. Wei et al. [57] extended the bidirectional projection method to PF-theory for solving a multi-attribute group decision-making (MAGDM) problem. With the help of t-norm and t-conorm, Ashraf et al. [58] introduced a series of weighted geometric operators of PFSs and also demonstrated their application in MAGDM. Wang et al. [59] introduced PF normalized projection-based VIKOR (PFNP-VIKOR) method for risk evaluation. Lin et al. [60] proposed a PF-novel score function, a novel PF-entropy measure, a novel PF-knowledge measure, and PF MULTIMOORA method. Tian et al. [61] suggested a weighted PF power Choquet ordered geometric operator (WPFPCOG) and a weighted PF power Shapley Choquet ordered geometric (WPFPSCOG) operator with their applications in MCDM. For MAGDM problems in PF environment, Zhang et al. [62] introduced an ELECTRE TRI-based outranking approach. To handle emergency decision-making (EDM) problems, a new EDM approach with the help of PFSs was proposed by Ding et al. [63]. Jin et al. [64] introduced Pearson's correlation-based MADM method in PF environment. Zeng et al. [65] introduced an exponential Jensen picture fuzzy divergence measure with its application in MCDM. Joshi [66, 67] investigated some compatibility/comparison measures in the framework of PFSs.

The present work concerns with the study of PF distance measure and its interdisciplinary applications.

The main contributions of this paper can be summarized as follows.

- We introduce a novel PF distance measure that overcomes the drawbacks of the existing PF distance measures.
- We demonstrate the application of the novel PF distance measure in pattern recognition.
- We derive some conversion formulae to generate PF data from the crisp data that enables us to apply our method in real data set.
- We examine the performance of the proposed PF distance measure on the real data set pertaining to the *Iris* plant.
- With the help of the novel PF distance measure, we propose a new method known as the picture fuzzy inferior ratio (PFIR) method for solving MADM problems in the PF environment that is based on the same idea as TOPSIS considering the distance from the positive ideal solution (PIS) as well as from the negative ideal solution (NIS).

The remainder of this paper is organized as follows: we discuss the existing work related to the present study in "Related studies". "Preliminaries" presents some basic concepts related to standard/non-standard fuzzy theory. In "The

picture fuzzy distance measure based on direct operations", a novel PF distance measure, and a weighted novel PF distance measure are proposed and validated. In "Experiments and analysis", we run some experiments on synthetic dataset as well as on real dataset regarding problems of pattern recognition. These numerical experiments facilitate us to contrast the performance of our proposed PF distance measure with various existing PF compatibility measures. In "A new MADM method in a PF system", we discuss the weakness of the traditional TOPSIS method of MADM and present a novel method of MADM. Finally, the section "Conclusion" includes the main findings of this paper and the scope for the future research.

Related work

To compare the features or attributes of two entities in picture fuzzy environment a comparison measure is essential. The functional forms that express the compatibility degree of sets or items are used in automatic classification, physical anthropology, citation analysis, psychology, ecology, pattern recognition, numerical taxonomy, and information retrieval etc. Distance measure, similarity measure and correlation measure are some predominately used indices to compare two feature spaces. These measures largely have similar pragmatic significance. Therefore, to contrast the performance of our newly proposed picture fuzzy distance measure it is essential to mention the existing distance measures, similarity measures and correlation measure concerning the PFSs. We mention the applicability of these measures just after the concerned formulae.

Moreover, we discuss the main drawbacks and limitations of these PF distance measures in Section "Experiments and analysis". First, we mention the prominent existing PF distance measures.

Since, in a PFS G, each element is specified by the degree of membership, the degree of non-membership, and degree of neutrality. Let $m_G(t_k), n_G(t_k)$ and $h_G(t_k)$ represents the membership degree, the non-membership degree, and the neutrality degree, respectively, of the element $t_k \in U$ (Universal set) in the set G such that $0 \le m_G(t_k) + n_G(t_k) + h_G(t_k) \le 1$. In the remaining paper, we shall follow this notation.

PF distance measures $D_{DT1}, D_{DT2}, D_{DT3}, D_{DT4}$ due to Dinh, and Thao [44] are as follows.

$$D_{DT1}(G,H) = \frac{1}{3l} \sum_{k=1}^{l} \left[\begin{array}{l} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \end{array} \right], \qquad (1)$$

$$D_{DT2}(G,H) = \frac{1}{l} \left[\sum_{k=1}^{l} \left(\begin{array}{l} \left(m_G(t_k) - m_H(t_k) \right)^2 \\ + \left(n_G(t_k) - n_H(t_k) \right)^2 \\ + \left(h_G(t_k) - h_H(t_k) \right)^2 \end{array} \right) \right]^{\frac{1}{2}}, \qquad (2)$$

$$D_{DT3}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \max \left(\begin{array}{l} \left| m_G(t_k) - m_H(t_k) \right|, \\ \left| n_G(t_k) - n_H(t_k) \right|, \\ \left| h_G(t_k) - h_H(t_k) \right| \end{array} \right), \qquad (3)$$

$$D_{DT4}(G,H) = \frac{1}{l} \left[\sum_{k=1}^{l} \max \left(\begin{array}{l} \left| m_G(t_k) - m_H(t_k) \right|^2, \\ \left| n_G(t_k) - n_H(t_k) \right|^2, \\ \left| h_G(t_k) - h_H(t_k) \right|^2 \end{array} \right) \right]^{\frac{1}{2}}. \qquad (4)$$

Dinh and Thao [44] have successfully applied the distance measures (1)–(4) in the pattern recognition problem.

PF distance measures $D_{D1}, D_{D2}, D_{D3}, D_{D4}, D_{D5}$ due to Dutta [37] are as follows.

$$D_{D1}(G,H) = \frac{1}{2} \sum_{k=1}^{l} \left[\begin{array}{l} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \\ + \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right], \qquad (5)$$

$$D_{D2}(G,H) = \frac{1}{2l} \sum_{k=1}^{l} \left[\begin{array}{l} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \\ + \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right], \qquad (6)$$

$$D_{D3}(G,H) = \frac{1}{2} \sum_{k=1}^{l} \left[\begin{array}{l} \left(m_G(t_k) - m_H(t_k) \right)^2 \\ + \left(n_G(t_k) - n_H(t_k) \right)^2 \\ + \left(h_G(t_k) - h_H(t_k) \right)^2 \\ + \left(e_G(t_k) - e_H(t_k) \right)^2 \end{array} \right], \qquad (7)$$

$$D_{D4}(G,H) = \frac{1}{2l} \sum_{k=1}^{l} \left[\begin{array}{l} \left(m_G(t_k) - m_H(t_k) \right)^2 \\ + \left(n_G(t_k) - n_H(t_k) \right)^2 \\ + \left(h_G(t_k) - h_H(t_k) \right)^2 \\ + \left(e_G(t_k) - e_H(t_k) \right)^2 \end{array} \right], \qquad (8)$$

$$D_{\text{D5}}(G,H) = \frac{1}{2} \sum_{k=1}^{l} \left[\frac{\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| + \left| n_G(t_k) - n_H(t_k) \right| + \\ \left| h_G(t_k) - h_H(t_k) \right| + \left| e_G(t_k) - e_H(t_k) \right| \end{array}}{\dfrac{\max\left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|, \left| n_G(t_k) - n_H(t_k) \right|, \\ \left| h_G(t_k) - h_H(t_k) \right|, \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right)}{2}} \right] \tag{9}$$

The distance measures (5)–(9) have been applied in the medical diagnosis problem.

Singh et al. [39] defined the following PF distance measures.

$$D_{\text{SMKSS1}}(G,H) = \frac{1}{4l} \sum_{k=1}^{l} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \\ + \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right), \tag{10}$$

$$D_{\text{SMKSS2}}(G,H) = \left[\frac{1}{4l} \sum_{k=1}^{l} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|^2 \\ + \left| n_G(t_k) - n_H(t_k) \right|^2 \\ + \left| h_G(t_k) - h_H(t_k) \right|^2 \\ + \left| e_G(t_k) - e_H(t_k) \right|^2 \end{array} \right) \right]^{\frac{1}{2}}, \tag{11}$$

$$D_{\text{SMKSS3}}(G,H) = \frac{1}{4l} \sum_{k=1}^{l} \max \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|, \\ \left| n_G(t_k) - n_H(t_k) \right|, \\ \left| h_G(t_k) - h_H(t_k) \right|, \\ \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right), \tag{12}$$

$$D_{\text{SMKSS4}}(G,H) = \left[\frac{1}{4l} \sum_{k=1}^{l} \max \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|^2, \\ \left| n_G(t_k) - n_H(t_k) \right|^2, \\ \left| h_G(t_k) - h_H(t_k) \right|^2, \\ \left| e_G(t_k) - e_H(t_k) \right|^2 \end{array} \right) \right]^{\frac{1}{2}}. \tag{13}$$

Singh et al. [39] utilized the distance measures (10)–(13) for the determination the flood disaster risk in the South region of India.

Son [38] defined four PF distance measures D_{S1}, D_{S2}, D_{S3}, D_{S4} as follows.

$$D_{\text{S1}}(G,H) = \frac{\sum_{k=1}^{l} \left(\frac{\Delta m_k + \Delta n_k + \Delta h_k}{3} + \max\left\{ \Delta m_k, \Delta n_k, \Delta h_k \right\} \right)}{\left(\begin{array}{c} \sum_{k=1}^{l} \left(\frac{\Delta m_k + \Delta n_k + \Delta h_k}{3} + \max\left\{ \Delta m_k, \Delta n_k, \Delta h_k \right\} \right) \\ + \max_k \left\{ \Phi_k^G, \Phi_k^H \right\} + \sum_{k=1}^{l} \left| \Phi_k^G - \Phi_k^H \right| + 1 \end{array} \right)}, \tag{14}$$

$$D_{\text{S2}}(G,H) = \frac{\left(\sum_{k=1}^{l} \left(\frac{\Delta m_k^2 + \Delta n_k^2 + \Delta h_k^2}{3} + \max\left\{ \Delta m_k^2, \Delta n_k^2, \Delta h_k^2 \right\} \right) \right)^{\frac{1}{2}}}{\left(\begin{array}{c} \left(\sum_{k=1}^{l} \left(\frac{\Delta m_k^2 + \Delta n_k^2 + \Delta h_k^2}{3} + \max\left\{ \Delta m_k^2, \Delta n_k^2, \Delta h_k^2 \right\} \right) \right)^{\frac{1}{2}} \\ + \left(\max_k \left\{ \Phi_k^G, \Phi_k^H \right\} + \sum_{k=1}^{l} \left| \Phi_k^G - \Phi_k^H \right|^2 \right)^{\frac{1}{2}} + 1 \end{array} \right)}, \tag{15}$$

$$D_{\text{S3}}(G,H) = \frac{\frac{1}{l} \sum_{k=1}^{l} \left(\frac{\Delta m_k + \Delta n_k + \Delta h_k}{3} + \max\left\{ \Delta m_k, \Delta n_k, \Delta h_k \right\} \right)}{\left(\begin{array}{c} \frac{1}{l} \sum_{k=1}^{l} \left(\frac{\Delta m_k + \Delta n_k + \Delta h_k}{3} + \max\left\{ \Delta m_k, \Delta n_k, \Delta h_k \right\} \right) \\ + \max_k \left\{ \Phi_k^G, \Phi_k^H \right\} + \frac{1}{l} \sum_{k=1}^{l} \left| \Phi_k^G - \Phi_k^H \right| + 1 \end{array} \right)}, \tag{16}$$

$$D_{\text{S4}}(G,H) = \frac{\left(\frac{1}{l} \sum_{k=1}^{l} \left(\frac{\Delta m_k^2 + \Delta n_k^2 + \Delta h_k^2}{3} + \max\left\{ \Delta m_k^2, \Delta n_k^2, \Delta h_k^2 \right\} \right) \right)^{\frac{1}{2}}}{\left(\begin{array}{c} \left(\frac{1}{l} \sum_{k=1}^{l} \left(\frac{\Delta m_k^2 + \Delta n_k^2 + \Delta h_k^2}{3} + \max\left\{ \Delta m_k^2, \Delta n_k^2, \Delta h_k^2 \right\} \right) \right)^{\frac{1}{2}} \\ + \left(\max_k \left\{ \Phi_k^G, \Phi_k^H \right\} + \frac{1}{l} \sum_{k=1}^{l} \left| \Phi_k^G - \Phi_k^H \right|^2 \right)^{\frac{1}{2}} + 1 \end{array} \right)}, \tag{17}$$

where $\quad \Delta m_k = \left| m_G(t_k) - m_H(t_k) \right|, k = 1,2,\ldots,l$,

$\Delta n_k = \left| n_G(t_k) - n_H(t_k) \right|, k = 1,2,\ldots,l$,

$\Delta h_k = \left| h_G(t_k) - h_H(t_k) \right|, k = 1,2,\ldots,l$,

$\Phi_k^G = \left| m_G(t_k) + n_G(t_k) + h_G(t_k) \right|, k = 1,2,\ldots,l$,

$\Phi_k^H = \left| m_H(t_k) + n_H(t_k) + h_H(t_k) \right|, k = 1,2,\ldots,l$.

Son [38] has investigated the application of the PF-distance measures (14)–(17) in the clustering analysis.

Bi-parametric PF distance measure D_{KKDK} due to Khan et al. [46] is

$$D_{\mathrm{KKDK}}(G,H)$$

$$= \left[\frac{1}{3l(t+1)^p} \sum_{k=1}^{l} \left(\begin{array}{l} \left| \begin{array}{l} t(m_G(t_k)-m_H(t_k))-(n_G(t_k)-n_H(t_k)) \\ -(h_G(t_k)-h_H(t_k)) \end{array} \right|^p \\ + \left| \begin{array}{l} t(n_G(t_k)-n_H(t_k))-(m_G(t_k)-m_H(t_k)) \\ +(h_G(t_k)-h_H(t_k)) \end{array} \right|^p \\ + \left| \begin{array}{l} t(h_G(t_k)-h_H(t_k))-(m_G(t_k)-m_H(t_k)) \\ +(n_G(t_k)-n_H(t_k)) \end{array} \right|^p \end{array} \right) \right]^{\frac{1}{p}}, \tag{18}$$

where $t = 3, 4, \ldots$ and $p = 1, 2, 3, \ldots$ represent the level of uncertainty and l_p norm, respectively.

Khan et al. [46] demonstrated the application of the PF distance measure (18) in the pattern recognition and medical diagnosis.

Second, we mention the existing PF correlation measures.

PF correlation measures due to Singh [47] are as follows:

$$C_{\mathrm{S1}}(G,H) = \frac{\sum_{k=1}^{l} \left(\begin{array}{l} m_G(t_k)m_H(t_k)+n_G(t_k)n_H(t_k) \\ +h_G(t_k)h_H(t_k) \end{array} \right)}{\left\{ \begin{array}{l} \sqrt{\sum_{k=1}^{l}\left((m_G(t_k))^2+(n_G(t_k))^2+(h_G(t_k))^2\right)} \\ \times\sqrt{\sum_{k=1}^{l}\left((m_H(t_k))^2+(n_H(t_k))^2+(h_H(t_k))^2\right)} \end{array} \right\}}, \tag{19}$$

$$C_{\mathrm{S2}}(G,H) = \frac{\sum_{k=1}^{l} \left(\begin{array}{l} m_G(t_k)m_H(t_k)+n_G(t_k)n_H(t_k) \\ +h_G(t_k)h_H(t_k) \end{array} \right)}{\max \left\{ \begin{array}{l} \sum_{k=1}^{l}\left((m_G(t_k))^2+(n_G(t_k))^2+(h_G(t_k))^2\right), \\ \sum_{k=1}^{l}\left((m_H(t_k))^2+(n_H(t_k))^2+(h_H(t_k))^2\right) \end{array} \right\}}. \tag{20}$$

Singh [47] illustrated the application of PF-correlation coefficients (19)–(20) in bidirectional approximate reasoning and clustering analysis. However, the PF correlation coefficients due to Singh [47] indicate only the degree of correlation and not the nature of correlation.

PF correlation measures due to Ganie et al. [36] are as follows:

$$C_{\mathrm{GSB1}}(G,H)$$

$$= \frac{\frac{1}{l-1}\sum_{k=1}^{l} \left\{ \begin{array}{l} (m_G(t_k)-\overline{m_G})(m_H(t_k)-\overline{m_H}) \\ +(n_G(t_k)-\overline{n_G})(n_H(t_k)-\overline{n_H}) \\ +\left(h_G(t_k)-\overline{h_G}\right)\left(h_H(t_k)-\overline{h_H}\right) \\ +\sigma_k(G)\sigma_k(H) \end{array} \right\}}{\left\{ \begin{array}{l} \sqrt{\frac{1}{l-1}\sum_{k=1}^{l}\left\{ \begin{array}{l} (m_G(t_k)-\overline{m_G})^2+(n_G(t_k)-\overline{n_G})^2 \\ +\left(h_G(t_k)-\overline{h_G}\right)^2+(\sigma_k(G))^2 \end{array} \right\}} \\ \times\sqrt{\frac{1}{l-1}\sum_{k=1}^{l}\left\{ \begin{array}{l} (m_H(t_k)-\overline{m_H})^2+(n_H(t_k)-\overline{n_H})^2 \\ +\left(h_H(t_k)-\overline{h_H}\right)^2+(\sigma_k(H))^2 \end{array} \right\}} \end{array} \right\}}, \tag{21}$$

where $\overline{m_G} = \frac{1}{l}\sum_{k=1}^{l} m_G(t_k), \overline{n_G} = \frac{1}{l}\sum_{k=1}^{l} n_G(t_k),$ $\overline{h_G} = \frac{1}{l}\sum_{k=1}^{l} h_G(t_k), \overline{m_H} = \frac{1}{l}\sum_{k=1}^{l} m_H(t_k),$ $\overline{n_H} = \frac{1}{l}\sum_{k=1}^{l} n_H(t_k), \overline{h_H} = \frac{1}{l}\sum_{k=1}^{l} h_H(t_k),$
$$\sigma_k(G) = \left(\begin{array}{l} (m_G(t_k)-\overline{m_G})-(n_G(t_k)-\overline{n_G}) \\ -\left(h_G(t_k)-\overline{h_G}\right) \end{array} \right) \text{ and}$$
$$\sigma_k(H) = \left(\begin{array}{l} (m_H(t_k)-\overline{m_H})-(n_H(t_k)-\overline{n_H}) \\ -\left(h_H(t_k)-\overline{h_H}\right) \end{array} \right).$$

$$C_{\mathrm{GSB2}}(G,H) = \frac{1}{3}\left(\gamma_1+\gamma_2+\gamma_3\right), \tag{22}$$

where $\gamma_1 = \frac{\sum_{k=1}^{l}(m_G(t_k)-\overline{m_G})(m_H(t_k)-\overline{m_H})}{\sqrt{\sum_{k=1}^{l}(m_G(t_k)-\overline{m_G})^2\times\sum_{k=1}^{l}(m_H(t_k)-\overline{m_H})^2}},$
$\gamma_2 = \frac{\sum_{k=1}^{l}(n_G(t_k)-\overline{n_G})(n_H(t_k)-\overline{n_H})}{\sqrt{\sum_{k=1}^{l}(n_G(t_k)-\overline{n_G})^2\times\sum_{k=1}^{l}(n_H(t_k)-\overline{n_H})^2}}$ and
$\gamma_3 = \frac{\sum_{k=1}^{l}\left(h_G(t_k)-\overline{h_G}\right)\left(h_H(t_k)-\overline{h_H}\right)}{\sqrt{\sum_{k=1}^{l}\left(h_G(t_k)-\overline{h_G}\right)^2\times\sum_{k=1}^{l}\left(h_H(t_k)-\overline{h_H}\right)^2}}.$

The PF correlation coefficients (21)–(22) are capable to compute the extent as well as nature of correlation (positive or negative). Ganie et al. [36] applied these PF correlation coefficients in pattern recognition, medical diagnosis, and clustering analysis. The performance has also been investigated.

Finally, we mention the existing PF similarity measures.

Wei [49] introduced the following PF similarity measures.

$$S_{\mathrm{W1}}(G,H) = \frac{1}{l}\sum_{k=1}^{l} \frac{m_G(t_k)m_H(t_k)+n_G(t_k)n_H(t_k)+h_G(t_k)h_H(t_k)}{\left(\begin{array}{l} \sqrt{(m_G(t_k))^2+(n_G(t_k))^2+(h_G(t_k))^2} \\ \times\sqrt{(m_H(t_k))^2+(n_H(t_k))^2+(h_H(t_k))^2} \end{array} \right)}, \tag{23}$$

$$S_{\mathrm{W2}}(G,H) = \frac{1}{l}\sum_{k=1}^{l} \frac{\left(\begin{array}{l} m_G(t_k)m_H(t_k)+n_G(t_k)n_H(t_k) \\ +h_G(t_k)h_H(t_k)+e_G(t_k)e_H(t_k) \end{array} \right)}{\left(\begin{array}{l} \sqrt{(m_G(t_k))^2+(n_G(t_k))^2+(h_G(t_k))^2+(e_G(t_k))^2} \\ \times\sqrt{(m_H(t_k))^2+(n_H(t_k))^2+(h_H(t_k))^2+(e_H(t_k))^2} \end{array} \right)} \tag{24}$$

$$S_{W3}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cos\left[\frac{\pi}{2} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ \vee \left| n_G(t_k) - n_H(t_k) \right| \\ \vee \left| h_G(t_k) - h_H(t_k) \right| \end{array} \right) \right], \quad (25)$$

$$S_{W4}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cos\left[\frac{\pi}{2} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ \vee \left| n_G(t_k) - n_H(t_k) \right| \\ \vee \left| h_G(t_k) - h_H(t_k) \right| \\ \vee \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right) \right], \quad (26)$$

$$S_{W5}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cos\left[\frac{\pi}{4} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \end{array} \right) \right], \quad (27)$$

$$S_{W6}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cos\left[\frac{\pi}{4} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \\ + \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right) \right], \quad (28)$$

$$S_{W7}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cot\left[\frac{\pi}{4} + \frac{\pi}{4} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ \vee \left| n_G(t_k) - n_H(t_k) \right| \\ \vee \left| h_G(t_k) - h_H(t_k) \right| \end{array} \right) \right], \quad (29)$$

$$S_{W8}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \cot\left[\frac{\pi}{4} + \frac{\pi}{4} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ \vee \left| n_G(t_k) - n_H(t_k) \right| \\ \vee \left| h_G(t_k) - h_H(t_k) \right| \\ \vee \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right) \right]. \quad (30)$$

The PF similarity measures (25)–(30) are based on trigonometric functions. Wei [49] discussed the application of these measures in decision-making. One obvious drawback of trigonometric similarity measures is that these are computationally more expensive.

Wei [43] proposed the PF similarity measure S_{W9} and successfully applied in the mineral field recognition and building material recognition.

$$S_{W9}(G,H) = \frac{1}{l} \sum_{k=1}^{l} \frac{m_G(t_k)m_H(t_k) + n_G(t_k)n_H(t_k) + h_G(t_k)h_H(t_k)}{\max\left(\begin{array}{c} (m_G(t_k))^2 + (n_G(t_k))^2 + (h_G(t_k))^2, \\ (m_H(t_k))^2 + (n_H(t_k))^2 + (h_H(t_k))^2 \end{array} \right)}. \quad (31)$$

Wei and Gao [43] suggested the PF dice similarity measures S_{WG1}, S_{WG2} and illustrated their application in the building material recognition.

$$S_{WG1}(G,H)$$
$$= \frac{1}{l} \sum_{k=1}^{l} \frac{2(m_G(t_k)m_H(t_k) + n_G(t_k)n_H(t_k) + h_G(t_k)h_H(t_k))}{\left(\begin{array}{c} \left((m_G(t_k))^2 + (n_G(t_k))^2 + (h_G(t_k))^2 \right) \\ + \left((m_H(t_k))^2 + (n_H(t_k))^2 + (h_H(t_k))^2 \right) \end{array} \right)}, \quad (32)$$

$$S_{WG2}(G,H)$$
$$= \frac{1}{l} \sum_{k=1}^{l} \frac{2\left(\begin{array}{c} m_G(t_k)m_H(t_k) + n_G(t_k)n_H(t_k) \\ + h_G(t_k)h_H(t_k) + e_G(t_k)e_H(t_k) \end{array} \right)}{\left\{ \begin{array}{c} \left((m_G(t_k))^2 + (n_G(t_k))^2 + (h_G(t_k))^2 + (e_G(t_k))^2 \right) \\ + \left((m_H(t_k))^2 + (n_H(t_k))^2 + (h_H(t_k))^2 + (e_H(t_k))^2 \right) \end{array} \right\}}. \quad (33)$$

Singh et al. [39] derived the PF similarity measures S_{SMKSS1}, S_{SMKSS2}, S_{SMKSS3}.

$$S_{SMKSS1} = 1 - \frac{1}{4l} \sum_{k=1}^{l} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right| \\ + \left| n_G(t_k) - n_H(t_k) \right| \\ + \left| h_G(t_k) - h_H(t_k) \right| \\ + \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right), \quad (34)$$

$$S_{SMKSS2} = 1 - \left[\frac{1}{4l} \sum_{k=1}^{l} \left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|^2 \\ + \left| n_G(t_k) - n_H(t_k) \right|^2 \\ + \left| h_G(t_k) - h_H(t_k) \right|^2 \\ + \left| e_G(t_k) - e_H(t_k) \right|^2 \end{array} \right) \right]^{\frac{1}{2}}, \quad (35)$$

$$S_{SMKSS3} = \frac{1}{4l} \sum_{k=1}^{l} \frac{\min\left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|, \left| n_G(t_k) - n_H(t_k) \right|, \\ \left| h_G(t_k) - h_H(t_k) \right|, \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right)}{\max\left(\begin{array}{c} \left| m_G(t_k) - m_H(t_k) \right|, \left| n_G(t_k) - n_H(t_k) \right|, \\ \left| h_G(t_k) - h_H(t_k) \right|, \left| e_G(t_k) - e_H(t_k) \right| \end{array} \right)}. \quad (36)$$

The PF similarity measure S_{SMKSS3} is computationally more expensive due to min–max operations.

Khan et al. [46] proposed a bi-parametric PF similarity measure

$S_{\text{KKDK}}(G,H)$

$$= 1 - \left[\frac{1}{3l(t+1)^p} \sum_{k=1}^{l} \left(\begin{array}{c} \left| \begin{array}{c} t(m_G(t_k) - m_H(t_k)) - (n_G(t_k) - n_H(t_k)) \\ -(h_G(t_k) - h_H(t_k)) \end{array} \right|^p \\ + \left| \begin{array}{c} t(n_G(t_k) - n_H(t_k)) - (m_G(t_k) - m_H(t_k)) \\ +(h_G(t_k) - h_H(t_k)) \end{array} \right|^p \\ + \left| \begin{array}{c} t(h_G(t_k) - h_H(t_k)) - (m_G(t_k) - m_H(t_k)) \\ +(n_G(t_k) - n_H(t_k)) \end{array} \right|^p \end{array} \right)^{\frac{1}{p}} \right],$$

(37)

where $t = 3, 4, \ldots$ and $p = 1, 2, 3, \ldots$ represent the level of uncertainty and l_p norm, respectively. The applications of this measure have been shown in pattern recognition and medical diagnosis problem.

In the present study, we propose a new PF distance measure and apply it in some real-life problems. Thus, it is essential to reflect upon the necessity and justification of the newly proposed PF distance measure.

The following are the main reasons that motivated us to consider the present study.

- Many of the existing PF distance measures fail to satisfy the axiomatic requirements of being PF distance measures, and others produce counterintuitive results in calculating the distance between different PFSs.
- Most of the existing PF distance measures have been proposed at the "formula" level, and does not satisfy the concerned axiomatic requirements.
- Most of the existing PF distance measures give unreasonable results in the field of pattern recognition.
- For applying a fuzzy or non-standard fuzzy model, the biggest challenge is the non-availability of real data in fuzzy or non-standard fuzzy form. Thus, the relevant formulae for conversion of crisp data to fuzzy/non-standard fuzzy data are essential.
- The TOPSIS method for solving a MADM problem investigates for a compromise solution (alternative) that is closest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). But practically, in many situations, the compromise solution due to TOPSIS is not farthest from NIS, and so the coefficients of ranking mainly consider the closeness to PIS to provide the ranking results (See Examples 4 and 5). Such a situation seems to be inappropriate and hence may deliver false ranking results.

To address the above-mentioned issues, this paper proposes a novel PF distance measure and a new MADM method in the PF settings.

Preliminaries

In this section, we include some basic definitions and operations regarding standard/non-standard fuzzy sets. Throughout this paper, $U = \{t_1, t_2, \ldots, t_l\}$ denotes the universe of discourse and PFS(U) denotes the set of all PFSs on U.

Basic definitions

Definition 1 [1]. A FS G in U is defined as $G = \{t_k, m_G(t_k) | t_k \in U, k = 1, 2, \ldots, l\}$, where $m_G(t_k)$ represents the membership degree of the element $t_k \in U$ in the set G such that $0 \leq m_G(t_k) \leq 1$.

Definition 2 [10]. An IFS G in U is defined as $G = \{t_k, m_G(t_k), n_G(t_k) | t_k \in U, k = 1, 2, \ldots, l\}$, where $m_G(t_k)$ and $n_G(t_k)$ represents the membership degree and non-membership degree, respectively, of the element $t_k \in U$ in the set G such that $0 \leq m_G(t_k) + n_G(t_k) \leq 1$. The pair $m_G(t_k), n_G(t_k)$ is termed as an intuitionistic fuzzy value (IFV).

Definition 3 [34]. A PFS G in U is defined as $G = \{t_k, m_G(t_k), n_G(t_k), h_G(t_k) | t_k \in U, k = 1, 2, \ldots, l\}$, where $m_G(t_k), n_G(t_k)$ and $h_G(t_k)$ represents the membership degree, the non-membership degree, and the neutrality degree, respectively, of the element $t_k \in U$ in the set G such that $0 \leq m_G(t_k) + n_G(t_k) + h_G(t_k) \leq 1$. Also, $e_G(t_k) = 1 - (m_G(t_k) - n_G(t_k) - h_G(t_k)), k = 1, 2, \ldots, l$ is termed as the degree of refusal of the element $t_k \in U$ in the set G. The triad $m_G(t_k), n_G(t_k), h_G(t_k)$ is a picture fuzzy value (PFV).

Definition 4 [35]. For any two PFSs G and H in U, the operations of complement, inclusion, union, and intersection are defined as.

(1) $(G)^c = \left\{ \begin{array}{c} t_k, n_G(t_k), m_G(t_k), h_G(t_k) \\ | t_k \in U, k = 1, 2, \ldots, l \end{array} \right\}.$

(2) $G \subseteq H$, if $m_G(t_k) \leq m_H(t_k), n_G(t_k) \geq n_H(t_k), h_G(t_k) \leq h_H(t_k), \forall t_k \in U.$

$$G \cup H = \left\{ \begin{array}{c} t_k, \max(m_G(t_k), m_H(t_k)), \\ \langle \min(n_G(t_k), n_H(t_k)), \rangle \\ \min(h_G(t_k), h_H(t_k)) \end{array} \middle| t_k \in U \right\}$$

(3)

$$G \cap H = \left\{ \begin{array}{c} t_k, \min(m_G(t_k), m_H(t_k)), \\ \langle \max(n_G(t_k), n_H(t_k)), \rangle \\ \min(h_G(t_k), h_H(t_k)) \end{array} \middle| t_k \in U \right\}$$

(4)

Definition 5 [39]. A function $D(G, H)$ with $G, H \in$ PFS(U) is called a PF distance measure if it satisfies:

(1) $0 \leq D(G,H) \leq 1$.

(2) $D(G,H) = D(H,G)$.

(3) $D(G,H) = 0$ if and only if $G = H$.

(4) If $G \subseteq H \subseteq I$, then $D(G,H) \leq D(G,I)$ and $D(H,I) \leq D(G,I)$.

Now, it is essential to mention some important definitions regarding neutrosophic theory in the sense that a picture fuzzy set is a special case of neutrosophic set and a refined neutrosophic set.

Definition 6 [68] A neutrosophic set A_{NS} in U is defined as $A_{\mathrm{NS}} = \{< t_k, T_A(t_k), I_A(t_k), F_A(t_k) >| t_k \in U, k = 1, 2, \ldots, l\}$, where $T_A(t_k), I_A(t_k), F_A(t_k): U \rightarrow [0, 1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false non-membership, respectively, with $0 \leq T_A(t_k) + I_A(t_k) + F_A(t_k) \leq 3$.

Definition 7 [68] A refined neutrosophic set A_{RNS} in U is defined as $A_{\mathrm{RNS}} = \{< t_k, T_{1A}(t_k), T_{2A}(t_k), \ldots, T_{pA}(t_k); I_{1A}(t_k), I_{2A}(t_k), \ldots, I_{rA}(t_k); F_{1A}(t_k), F_{2A}(t_k), \ldots, F_{sA}(t_k) >| t_k \in U, k = 1, 2, \ldots, l\}$, where all $T_{jA}(t_k), 1 \leq j \leq p, I_{kA}(t_k), 1 \leq m \leq r, F_{lA}(t_k), 1 \leq l \leq s,: U \rightarrow [0, 1]$, and $T_{jA}(t_k)$ represents the jth sub membership degree, $I_{kA}(t_k)$ represents the mth sub-indeterminacy degree, $F_{lA}(x)$ represents the lth sub-non-membership degree, with $p, r, s \geq 1$ integers, where $p + r + s + n \geq 4$, and: $0 \leq \Sigma\, T_{jA}(x) + \Sigma\, I_{kA}(x) + \Sigma\, T_{jA}(x) \leq n$. All neutrosophic sub-components $T_{jA}(x), I_{kA}(x), F_{lA}(x)$ are independent with respect to each other.

In view of the definitions given in this section we can say that a PFS is a particular case of neutrosophic set (NS) [68–70] and refined neutrosophic set (RNS) [68, 70]. The basic difference between a PFS and NS (RNS) is that the components (i.e., membership degrees) in a PFS are dependent whereas the components in a NS (RNS) are independent of each other. Refined neutrosophic set is a generalization of neutrosophic set.

The definitions of NS and RNS seem to inculcate the further generalizations and extensions of the present work.

In the next section, we introduce a new PF distance measure along with its properties.

The picture fuzzy distance measure based on direct operations

Apart from the representation of IFSs given in Definition2, many other possible representations of IFSs are available in the literature. One such interpretation of the IFS $\langle m_G(t_k), n_G(t_k) \rangle$ due to [71] is in terms of the interval $[m_G(t_k), 1 - n_G(t_k)]$ which is the same as the interval-valued fuzzy sets representation of IFSs in which $m_G(t_k)$ and $1 - n_G(t_k)$ are, respectively, the lower and upper bounds

of the degree of membership. Clearly, $[m_G(t_k), 1 - n_G(t_k)]$ is a valid interval as $m_G(t_k) \leq 1 - n_G(t_k)$ holds always for $m_G(t_k) + n_G(t_k) \leq 1$. Continuing this process, we represent the PFS $\langle m_G(t_k), n_G(t_k), h_G(t_k) \rangle$ in the form of an interval $[m_G(t_k), 1 - n_G(t_k) - h_G(t_k)]$, which is a valid interval since $m_G(t_k) + n_G(t_k) + h_G(t_k) \leq 1$. Here $m_G(t_k)$ and $1 - n_G(t_k) - h_G(t_k)$ are the lower and upper bounds of the degree of membership, respectively. Consider the interval representation of two PFSs G and $H \in \mathrm{PFS}(U)$ as $[m_G(t_k), 1 - n_G(t_k) - h_G(t_k)]$, and $[m_H(t_k), 1 - n_H(t_k) - h_H(t_k)]$, then together with lower and upper bounds, the information carried by the PFSs G and H is also determined by the length of intervals. Therefore, we introduce a simple and easy to understand PF distance measure between the PFSs G and H

$$D_{\mathrm{GS}}(G,H) = 1 - \frac{1}{4l} \sum_{k=1}^{l} \begin{bmatrix} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ + 3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ + \sqrt{\begin{array}{c}(1 - m_G(t_k) - n_G(t_k)) \\ \times (1 - m_H(t_k) - n_H(t_k))\end{array}} \\ + \sqrt{\begin{array}{c}(1 - m_G(t_k) - h_G(t_k)) \\ \times (1 - m_H(t_k) - h_H(t_k))\end{array}} \\ + \sqrt{\begin{array}{c}(1 - n_G(t_k) - h_G(t_k)) \\ \times (1 - n_H(t_k) - h_H(t_k))\end{array}} \end{bmatrix}. \tag{38}$$

Here we utilize the concept of consistency in defining this novel PF distance measure. For a PFS $G = \{t_k, m_G(t_k), n_G(t_k), h_G(t_k) | t_k \in U, k = 1, 2, \ldots, l\}$, the membership, non-membership, and neutrality degree can be, respectively, written as $[m_G(t_k), 1 - n_G(t_k) - h_G(t_k)]$, $[n_G(t_k), 1 - m_G(t_k) - h_G(t_k)]$ and $[h_G(t_k), 1 - m_G(t_k) - n_G(t_k)]$. The length of these intervals is $1 - m_G(t_k) - n_G(t_k) - h_G(t_k)$. So, in Eq. (38) $\sqrt{m_G(t_k)m_H(t_k)}$, $\sqrt{n_G(t_k)n_H(t_k)}$, and $\sqrt{h_G(t_k)h_H(t_k)}$, respectively, represent the degree of consistency between the lower bounds of the degree of membership, degree of non-membership, and degree of neutrality. Consequently, $\sqrt{(1 - n_G(t_k) - h_G(t_k))(1 - n_H(t_k) - h_H(t_k))}$, $\sqrt{(1 - m_G(t_k) - h_G(t_k))(1 - m_H(t_k) - h_H(t_k))}$, and $\sqrt{(1 - m_G(t_k) - n_G(t_k))(1 - m_H(t_k) - n_H(t_k))}$ represent the degree of consistency between the upper bounds of the degree of membership, degree of non-membership, and degree of neutrality, respectively. The degree of consistency between interval lengths is described by $\sqrt{e_G(t_k)e_H(t_k)}$.

Now, we show that $D_{\mathrm{GS}}(G,H)$ is a valid PF distance measure.

Theorem 1 $D_{GS}(G, H)$ *is a distance measure between the two PFSs G and H.*

Proof To show that $D_{GS}(G, H)$ is a PF distance measure, we show that it satisfies the four properties given in Definition 5.

(1) For each, $q, r \in [0, +\infty]$, we have $\sqrt{qr} \leq \frac{q+r}{2}$. So, for $0 \leq m_G(t_k) \leq 1$, $0 \leq n_G(t_k) \leq 1$, $0 \leq h_G(t_k) \leq 1$, $0 \leq e_G(t_k) \leq 1$, $0 \leq 1 - m_G(t_k) - n_G(t_k) \leq 1$, $0 \leq 1 - m_G(t_k) - h_G(t_k) \leq 1$ and $0 \leq 1 - n_G(t_k) - h_G(t_k) \leq 1$ we get

$$0 \leq 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} + 3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)}$$
$$+ \sqrt{\left(1 - m_G(t_k) - n_G(t_k)\right)\left(1 - m_H(t_k) - n_H(t_k)\right)} + \sqrt{\left(1 - m_G(t_k) - h_G(t_k)\right)\left(1 - m_H(t_k) - h_H(t_k)\right)}$$
$$+ \sqrt{\left(1 - n_G(t_k) - h_G(t_k)\right)\left(1 - n_H(t_k) - h_H(t_k)\right)}$$

$$\leq \frac{3}{2}\left(m_G(t_k) + m_H(t_k)\right) + \frac{3}{2}\left(n_G(t_k) + n_H(t_k)\right)$$
$$+ \frac{3}{2}\left(h_G(t_k) + h_H(t_k)\right) + \frac{e_G(t_k) + e_H(t_k)}{2}$$
$$+ \frac{\left(1 - m_G(t_k) - n_G(t_k)\right) + \left(1 - m_H(t_k) - n_H(t_k)\right)}{2}$$
$$+ \frac{\left(1 - m_G(t_k) - h_G(t_k)\right) + \left(1 - m_H(t_k) - h_H(t_k)\right)}{2}$$
$$+ \frac{\left(1 - n_G(t_k) - h_G(t_k)\right) + \left(1 - n_H(t_k) - h_H(t_k)\right)}{2}$$
$$= 3 + \frac{m_G(t_k) + n_G(t_k) + h_G(t_k) + e_G(t_k)}{2} + \frac{m_H(t_k) + n_H(t_k) + h_H(t_k) + e_H(t_k)}{2}$$
$$= 3 + \frac{1}{2} + \frac{1}{2} = 4.$$

So, $0 \leq \frac{1}{4l} \sum_{k=1}^{l} \begin{bmatrix} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - n_G(t_k)) \\ \times(1 - m_H(t_k) - n_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - h_G(t_k)) \\ \times(1 - m_H(t_k) - h_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - n_G(t_k) - h_G(t_k)) \\ \times(1 - n_H(t_k) - h_H(t_k))\end{array}} \end{bmatrix} \leq 1$

or $0 \leq 1 - \frac{1}{4l} \sum_{k=1}^{l} \begin{bmatrix} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - n_G(t_k)) \\ \times(1 - m_H(t_k) - n_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - h_G(t_k)) \\ \times(1 - m_H(t_k) - h_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - n_G(t_k) - h_G(t_k)) \\ \times(1 - n_H(t_k) - h_H(t_k))\end{array}} \end{bmatrix} \leq 1.$

Therefore, we have $0 \leq D_{GS}(G, H) \leq 1$.

(2) $D_{GS}(G, H) = D_{GS}(H, G)$ is obvious due to the cumutativeness of the expression of $D_{GS}(G, H)$.

(3) Since $\sqrt{qr}$ achieves its maximum value $\frac{q+r}{2}$ when $q = r$. So, we have

$$D_{GS}(G, H) = 0$$

$$\Leftrightarrow \begin{bmatrix} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{\begin{array}{c}(1-m_G(t_k)-n_G(t_k)) \\ \times(1-m_H(t_k)-n_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1-m_G(t_k)-h_G(t_k)) \\ \times(1-m_H(t_k)-h_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1-n_G(t_k)-h_G(t_k)) \\ \times(1-n_H(t_k)-h_H(t_k))\end{array}} \end{bmatrix} = 4$$

$$\Leftrightarrow m_G(t_k) = m_H(t_k), n_G(t_k) = n_H(t_k), h_G(t_k)$$
$$= h_H(t_k), e_G(t_k) = e_H(t_k), (1-m_G(t_k)-n_G(t_k))$$
$$= (1-m_H(t_k)-n_H(t_k)), (1-m_G(t_k)-h_G(t_k))$$
$$= (1-m_H(t_k)-h_H(t_k)), (1-n_G(t_k)-h_G(t_k))$$
$$= (1-n_H(t_k)-h_H(t_k)) \Leftrightarrow G = H.$$

So, $D_{GS}(G,H) = 0$ if and only if $G = H$.

(4) Let $I \in \text{PFS}(U)$ be another PFS such that $G \subseteq H \subseteq I$. Then we have, $m_G(t_k) \le m_H(t_k) \le m_I(t_k)$, $n_I(t_k) \le n_H(t_k) \le n_G(t_k)$ and $h_G(t_k) \le h_H(t_k) \le h_I(t_k)$.

For $a, b, c \in [0,1], a+b+c \le 1$, we define a fuction f as:

$$f(q,r,s) = 3\sqrt{aq} + 3\sqrt{br} + 3\sqrt{cs}$$
$$+ \sqrt{(1-a-b-c)(1-q-r-s)}$$
$$+ \sqrt{(1-a-b)(1-q-r)}$$
$$+ \sqrt{(1-a-c)(1-q-s)} + \sqrt{(1-b-c)(1-r-s)},$$

where $q, r, s \in [0,1], q+r+s \in [0,1]$.

Now, $\dfrac{\partial f}{\partial q} = \dfrac{3\sqrt{a}}{2\sqrt{q}} - \dfrac{\sqrt{1-a-b-c}}{2\sqrt{1-q-r-s}} - \dfrac{\sqrt{1-a-b}}{2\sqrt{1-q-r}} - \dfrac{\sqrt{1-a-c}}{2\sqrt{1-q-s}}$

$$= \left(\frac{\sqrt{a}}{2\sqrt{q}} + \frac{\sqrt{a}}{2\sqrt{q}} + \frac{\sqrt{a}}{2\sqrt{q}} \right) - \frac{\sqrt{1-a-b-c}}{2\sqrt{1-q-r-s}} - \frac{\sqrt{1-a-b}}{2\sqrt{1-q-r}}$$
$$- \frac{\sqrt{1-a-c}}{2\sqrt{1-q-s}}$$

$$= \left(\frac{\sqrt{a}}{2\sqrt{q}} - \frac{\sqrt{1-a-b-c}}{2\sqrt{1-q-r-s}} \right) + \left(\frac{\sqrt{a}}{2\sqrt{q}} - \frac{\sqrt{1-a-b}}{2\sqrt{1-q-r}} \right)$$
$$+ \left(\frac{\sqrt{a}}{2\sqrt{q}} - \frac{\sqrt{1-a-c}}{2\sqrt{1-q-s}} \right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{a(1-q-r-s)} - \sqrt{q(1-a-b-c)}}{\sqrt{q(1-q-r-s)}} \right)$$
$$+ \frac{1}{2}\left(\frac{\sqrt{a(1-q-r)} - \sqrt{q(1-a-b)}}{\sqrt{q(1-q-r)}} \right)$$
$$+ \frac{1}{2}\left(\frac{\sqrt{a(1-q-s)} - \sqrt{q(1-a-c)}}{\sqrt{q(1-q-s)}} \right)$$

$$= \frac{1}{2}\left(\frac{a(1-q-r-s) - q(1-a-b-c)}{\sqrt{q(1-q-r-s)}\left(\sqrt{a(1-q-r-s)} + \sqrt{q(1-a-b-c)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{a(1-q-r) - q(1-a-b)}{\sqrt{q(1-q-r)}\left(\sqrt{a(1-q-r)} + \sqrt{q(1-a-b)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{a(1-q-s) - q(1-a-c)}{\sqrt{q(1-q-s)}\left(\sqrt{a(1-q-s)} + \sqrt{q(1-a-c)}\right)} \right)$$

$$= \frac{1}{2}\left(\frac{a(1-r-s) - q(1-b-c)}{\sqrt{q(1-q-r-s)}\left(\sqrt{a(1-q-r-s)} + \sqrt{q(1-a-b-c)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{a(1-r) - q(1-b)}{\sqrt{q(1-q-r)}\left(\sqrt{a(1-q-r)} + \sqrt{q(1-a-b)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{a(1-s) - q(1-c)}{\sqrt{q(1-q-s)}\left(\sqrt{a(1-q-s)} + \sqrt{q(1-a-c)}\right)} \right)$$

$$= \frac{1}{2}\left(\frac{(a-q)(1-b-c)}{\sqrt{q(1-q-r-s)}\left(\sqrt{a(1-q-r-s)} + \sqrt{q(1-a-b-c)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{(a-q)(1-b)}{\sqrt{q(1-q-r)}\left(\sqrt{a(1-q-r)} + \sqrt{q(1-a-b)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{(a-q)(1-c)}{\sqrt{q(1-q-s)}\left(\sqrt{a(1-q-s)} + \sqrt{q(1-a-c)}\right)} \right) \text{at } r=b \text{ and } s=c$$

$$\Rightarrow \frac{\partial f}{\partial q} = \frac{1}{2}\left(\frac{(a-q)(1-b-c)}{\sqrt{q(1-q-r-s)}\left(\sqrt{a(1-q-r-s)} + \sqrt{q(1-a-b-c)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{(a-q)(1-b)}{\sqrt{q(1-q-r)}\left(\sqrt{a(1-q-r)} + \sqrt{q(1-a-b)}\right)} \right)$$
$$+ \frac{1}{2}\left(\frac{(a-q)(1-c)}{\sqrt{q(1-q-s)}\left(\sqrt{a(1-q-s)} + \sqrt{q(1-a-c)}\right)} \right).$$

Similarly, we have

$$\frac{\partial f}{\partial r} = \frac{1}{2}\left(\frac{(b-r)(1-a-c)}{\sqrt{r(1-q-r-s)}\left(\sqrt{b(1-q-r-s)}+\sqrt{r(1-a-b-c)}\right)}\right)$$

$$+\frac{1}{2}\left(\frac{(b-r)(1-a)}{\sqrt{r(1-q-r)}\left(\sqrt{b(1-q-r)}+\sqrt{r(1-a-b)}\right)}\right)$$

$$+\frac{1}{2}\left(\frac{(b-r)(1-c)}{\sqrt{r(1-r-s)}\left(\sqrt{b(1-r-s)}+\sqrt{r(1-b-c)}\right)}\right),$$

$$\frac{\partial f}{\partial s} = \frac{1}{2}\left(\frac{(c-s)(1-a-b)}{\sqrt{s(1-q-r-s)}\left(\sqrt{c(1-q-r-s)}+\sqrt{s(1-a-b-c)}\right)}\right)$$

$$+\frac{1}{2}\left(\frac{(c-s)(1-a)}{\sqrt{s(1-q-s)}\left(\sqrt{c(1-q-s)}+\sqrt{s(1-a-c)}\right)}\right)$$

$$+\frac{1}{2}\left(\frac{(c-s)(1-b)}{\sqrt{s(1-r-s)}\left(\sqrt{c(1-r-s)}+\sqrt{s(1-b-c)}\right)}\right).$$

For $a \le q \le 1, b \le 1, c \le 1$, we have $\frac{\partial f}{\partial q} \le 0$, so f is a decreasing function of q for $q \ge a$. For $0 \le q \le a, b \le 1, c \le 1$, we have $\frac{\partial f}{\partial q} \ge 0$, so f is an increasing function of q for $q \le a$. Similarly, for $b \le r \le 1, a \le 1, c \le 1$, we have $\frac{\partial f}{\partial r} \le 0$ and for $0 \le r \le b, a \le 1, c \le 1$, we have $\frac{\partial f}{\partial r} \ge 0$. This means that f is a decreasing function of r when $r \ge b$ and an increasing function of r when $r \le b$. Also, for $c \le s \le 1, a \le 1, b \le 1$, we have $\frac{\partial f}{\partial s} \le 0$ and for $0 \le s \le c, a \le 1, b \le 1$, we have $\frac{\partial f}{\partial s} \ge 0$. This means that f is a decreasing function of s when $s \ge c$ and an increasing function of s when $s \le c$.

Now, for $a = m_G(t_k), b = n_G(t_k), c = h_G(t_k)$ and two triplets $(m_H(t_k), n_H(t_k), h_H(t_k))$, $(m_I(t_k), n_I(t_k), h_I(t_k))$ satisfying $a = m_G(t_k) \le m_H(t_k) \le m_I(t_k), n_I(t_k) \le n_H(t_k) \le n_G(t_k) = b$ and $c = h_G(t_k) \le h_H(t_k) \le h_I(t_k)$, we have

$$f(m_I(t_k), n_I(t_k), h_I(t_k)) \le f(m_H(t_k), n_I(t_k), h_I(t_k))$$

$$\le f(m_H(t_k), n_H(t_k), h_I(t_k))$$

and $f(m_I(t_k), n_I(t_k), h_I(t_k)) \le f(m_I(t_k), n_I(t_k), h_H(t_k))$. $\le f(m_I(t_k), n_H(t_k), h_H(t_k))$.

So,
$$\left[\begin{array}{c} 3\sqrt{m_G(t_k)m_I(t_k)} + 3\sqrt{n_G(t_k)n_I(t_k)} \\ +3\sqrt{h_G(t_k)h_I(t_k)} + \sqrt{e_G(t_k)e_I(t_k)} \\ +\sqrt{(1-m_G(t_k)-n_G(t_k)) \times (1-m_I(t_k)-n_I(t_k))} \\ +\sqrt{(1-m_G(t_k)-h_G(t_k)) \times (1-m_I(t_k)-h_I(t_k))} \\ +\sqrt{(1-n_G(t_k)-h_G(t_k)) \times (1-n_I(t_k)-h_I(t_k))} \end{array}\right],$$

$$\le \left[\begin{array}{c} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{(1-m_G(t_k)-n_G(t_k)) \times (1-m_H(t_k)-n_H(t_k))} \\ +\sqrt{(1-m_G(t_k)-h_G(t_k)) \times (1-m_H(t_k)-h_H(t_k))} \\ +\sqrt{(1-n_G(t_k)-h_G(t_k)) \times (1-n_H(t_k)-h_H(t_k))} \end{array}\right]$$

$$\text{or } 1 - \frac{1}{4l}\sum_{k=1}^{l}\left[\begin{array}{c} 3\sqrt{m_G(t_k)m_I(t_k)} + 3\sqrt{n_G(t_k)n_I(t_k)} \\ +3\sqrt{h_G(t_k)h_I(t_k)} + \sqrt{e_G(t_k)e_I(t_k)} \\ +\sqrt{(1-m_G(t_k)-n_G(t_k)) \times (1-m_I(t_k)-n_I(t_k))} \\ +\sqrt{(1-m_G(t_k)-h_G(t_k)) \times (1-m_I(t_k)-h_I(t_k))} \\ +\sqrt{(1-n_G(t_k)-h_G(t_k)) \times (1-n_I(t_k)-h_I(t_k))} \end{array}\right]$$

$$\ge 1 - \frac{1}{4l}\sum_{k=1}^{l}\left[\begin{array}{c} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{(1-m_G(t_k)-n_G(t_k)) \times (1-m_H(t_k)-n_H(t_k))} \\ +\sqrt{(1-m_G(t_k)-h_G(t_k)) \times (1-m_H(t_k)-h_H(t_k))} \\ +\sqrt{(1-n_G(t_k)-h_G(t_k)) \times (1-n_H(t_k)-h_H(t_k))} \end{array}\right]$$

$\Rightarrow D_{GS}(G, H) \le D_{GS}(G, I)$.

Now, if we let $a = m_I(t_k), b = n_I(t_k), c = h_I(t_k)$ and two triplets $(m_G(t_k), n_G(t_k), h_G(t_k))$, $(m_H(t_k), n_H(t_k), h_H(t_k))$ satisfying

$m_G(t_k) \leq m_H(t_k) \leq m_I(t_k) = a, b = n_I(t_k) \leq n_H(t_k) \leq n_G(t_k)$
and $h_G(t_k) \leq h_H(t_k) \leq h_I(t_k) = c$, we have

$$f\big(m_G(t_k), n_G(t_k), h_G(t_k)\big) \leq f\big(m_H(t_k), n_G(t_k), h_G(t_k)\big)$$
$$\leq f\big(m_H(t_k), n_H(t_k), h_G(t_k)\big)$$

and

$$f\big(m_G(t_k), n_G(t_k), h_G(t_k)\big) \leq f\big(m_G(t_k), n_G(t_k), h_H(t_k)\big)$$

$$\leq f\big(m_G(t_k), n_H(t_k), h_H(t_k)\big).$$

$$\leq \begin{bmatrix} 3\sqrt{m_H(t_k)m_I(t_k)} + 3\sqrt{n_H(t_k)n_I(t_k)} \\ +3\sqrt{h_H(t_k)h_I(t_k)} + \sqrt{e_H(t_k)e_I(t_k)} \\ +\sqrt{\begin{matrix}(1-m_H(t_k)-n_H(t_k))\\ \times(1-m_I(t_k)-n_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-m_H(t_k)-h_H(t_k))\\ \times(1-m_I(t_k)-h_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-n_H(t_k)-h_H(t_k))\\ \times(1-n_I(t_k)-h_I(t_k))\end{matrix}} \end{bmatrix}$$

So, $\begin{bmatrix} 3\sqrt{m_G(t_k)m_I(t_k)} + 3\sqrt{n_G(t_k)n_I(t_k)} \\ +3\sqrt{h_G(t_k)h_I(t_k)} + \sqrt{e_G(t_k)e_I(t_k)} \\ +\sqrt{\begin{matrix}(1-m_G(t_k)-n_G(t_k))\\ \times(1-m_I(t_k)-n_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-m_G(t_k)-h_G(t_k))\\ \times(1-m_I(t_k)-h_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-n_G(t_k)-h_G(t_k))\\ \times(1-n_I(t_k)-h_I(t_k))\end{matrix}} \end{bmatrix}$

or, $1 - \dfrac{1}{4l}\sum\limits_{k=1}^{l} \begin{bmatrix} 3\sqrt{m_G(t_k)m_I(t_k)} + 3\sqrt{n_G(t_k)n_I(t_k)} \\ +3\sqrt{h_G(t_k)h_I(t_k)} + \sqrt{e_G(t_k)e_I(t_k)} \\ +\sqrt{\begin{matrix}(1-m_G(t_k)-n_G(t_k))\\ \times(1-m_I(t_k)-n_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-m_G(t_k)-h_G(t_k))\\ \times(1-m_I(t_k)-h_I(t_k))\end{matrix}} \\ +\sqrt{\begin{matrix}(1-n_G(t_k)-h_G(t_k))\\ \times(1-n_I(t_k)-h_I(t_k))\end{matrix}} \end{bmatrix}$

Table 1 Comparison of PF distance measures

Cases → Distance measures ↓	Case 1 {(0.3, 0.3, 0.2)} {(0.4, 0.4, 0.2)}	Case 2 {(0.3, 0.4, 0.1)} {(0.4, 0.3, 0.1)}	Case 3 {(1.0, 0.0, 0.0)} {(0.0, 0.0, 0.0)}	Case 4 {(0.5, 0.5, 0.0)} {(0.0, 0.0, 0.0)}	Case 5 {(0.3, 0.2, 0.1)} {(0.5, 0.1, 0.1)}	Case 6 {(0.3, 0.2, 0.1)} {(0.5, 0.2, 0.1)}
D_{DT1}	**0.0667**	**0.0667**	**0.3333**	**0.3333**	0.1000	**0.0667**
D_{DT2}	**0.1414**	**0.1414**	1.0000	0.7071	0.2236	0.2000
D_{DT3}	**0.1000**	**0.1000**	1.0000	0.5000	**0.2000**	**0.2000**
D_{DT4}	**0.1000**	**0.1000**	**1.0000**	0.5000	**0.2000**	**0.2000**
D_{D1}	0.2000	0.1000	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{D2}	0.2000	0.1000	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{D3}	**0.1732**	0.1000	1.0000	0.8660	**0.1732**	0.2000
D_{D4}	**0.1732**	0.1000	1.0000	0.8660	**0.1732**	0.2000
D_{D5}	0.2000	0.1000	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{SMKSS1}	0.1000	0.0500	**0.5000**	**0.5000**	**0.1000**	**0.1000**
D_{SMKSS2}	**0.1225**	0.0707	0.7071	0.6124	**0.1225**	0.1414
D_{SMKSS3}	**0.0500**	0.0250	**0.2500**	**0.2500**	**0.0500**	**0.0500**
D_{SMKSS4}	**0.1000**	**0.0500**	**0.5000**	**0.5000**	**0.1000**	**0.1000**
D_{S1}	0.0704	0.0847	0.3077	0.2174	0.1429	0.1176
D_{S2}	0.0562	0.0638	0.3235	0.2110	0.1108	0.1017
D_{S3}	0.0704	0.0847	0.3077	0.2174	0.1429	0.1176
D_{S4}	0.0473	0.0669	0.2779	0.1771	0.1036	0.0902
D_{KKDK}	0.0333	**0.0833**	0.4167	0.1667	0.1250	**0.0833**
D_{GS}(Proposed)	0.0361	0.0065	0.7500	0.6464	0.0205	0.0195

(Counterintuitive cases are in bold type. $t = 3$ and $p = 1$ in D_{KKDK})

Table 2 Comparison of PF distance measures

Cases → Distance measures ↓	Case 1 $\{(0.5, 0.5, 0.0)\}$ $\{(0.0, 0.0, 0.0)\}$	Case 2 $\{(0.6, 0.4, 0.0)\}$ $\{(0.0, 0.0, 0.0)\}$	Case 3 $\{(0.0, 0.87, 0.1)\}$ $\{(0.28, 0.55, 0.1)\}$	Case 4 $\{(0.6, 0.27, 0.1)\}$ $\{(0.28, 0.55, 0.1)\}$
D_{DT1}	**0.3333**	**0.3333**	**0.2000**	**0.2000**
D_{DT2}	0.7071	0.7211	**0.4252**	**0.4252**
D_{DT3}	0.5000	0.6000	**0.3200**	**0.3200**
D_{DT4}	0.5000	0.6000	**0.3200**	**0.3200**
D_{D1}	**1.0000**	**1.0000**	**0.3200**	**0.3200**
D_{D2}	**1.0000**	**1.0000**	**0.3200**	**0.3200**
D_{D3}	0.8660	0.8718	**0.3020**	**0.3020**
D_{D4}	0.8660	0.8718	**0.3020**	**0.3020**
D_{D5}	**1.0000**	**1.0000**	**0.3200**	**0.3200**
D_{SMKSS1}	**0.5000**	**0.5000**	**0.1600**	**0.1600**
D_{SMKSS2}	0.6124	0.6164	**0.2135**	**0.2135**
D_{SMKSS3}	**0.2500**	**0.2500**	**0.0800**	**0.0800**
D_{SMKSS4}	**0.5000**	**0.5000**	**0.1600**	**0.1600**
D_{S1}	0.2174	0.2373	**0.2055**	**0.2055**
D_{S2}	0.2110	0.2322	**0.1662**	**0.1662**
D_{S3}	0.2174	0.2373	**0.2055**	**0.2055**
D_{S4}	0.1771	0.1958	**0.1523**	**0.1523**
D_{KKDK}	0.1667	0.1833	**0.2500**	**0.2500**
D_{GS}(proposed)	0.6464	0.6482	0.1452	0.0547

(Counterintuitive cases are in bold type. $t = 3$ and $p = 1$ in D_{KKDK})

Table 3 Comparison of PF distance measures

Cases → Distance measures ↓	Case 1 $\{(0.3, 0.0, 0.7)\}$ $\{(0.4, 0.0, 0.6)\}$	Case 2 $\{(0.3, 0.0, 0.7)\}$ $\{(0.2, 0.0, 0.8)\}$	Case 3 $\{(0.5, 0.0, 0.5)\}$ $\{(0.0, 0.0, 0.0)\}$	Case 4 $\{(0.4, 0.0, 0.6)\}$ $\{(0.0, 0.0, 0.0)\}$	Case 5 $\{(0.1, 0.4, 0.5)\}$ $\{(0.2, 0.5, 0.3)\}$	Case 6 $\{(0.4, 0.4, 0.2)\}$ $\{(0.2, 0.5, 0.3)\}$
D_{DT1}	**0.0667**	**0.0667**	**0.3333**	**0.3333**	0.1333	0.1333
D_{DT2}	**0.1414**	**0.1414**	0.7071	0.7211	**0.2449**	**0.2449**
D_{DT3}	**0.1000**	**0.1000**	0.5000	0.6000	**0.2000**	**0.2000**
D_{DT4}	**0.1000**	**0.1000**	0.5000	0.6000	**0.2000**	**0.2000**
D_{D1}	**0.1000**	**0.1000**	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{D2}	**0.1000**	**0.1000**	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{D3}	**0.1000**	**0.1000**	0.8660	0.8718	**0.1732**	**0.1732**
D_{D4}	**0.1000**	**0.1000**	0.8660	0.8718	**0.1732**	**0.1732**
D_{D5}	**0.1000**	**0.1000**	**1.0000**	**1.0000**	**0.2000**	**0.2000**
D_{SMKSS1}	**0.0500**	**0.0500**	**0.5000**	**0.5000**	**0.1000**	**0.1000**
D_{SMKSS2}	**0.0707**	**0.0707**	0.6124	0.6164	**0.1225**	**0.1225**
D_{SMKSS3}	**0.0250**	**0.0250**	**0.2500**	**0.2500**	**0.0500**	**0.0500**
D_{SMKSS4}	**0.0500**	**0.0500**	**0.5000**	**0.5000**	**0.1000**	**0.1000**
D_{S1}	**0.0769**	**0.0769**	0.2174	0.2373	**0.1429**	**0.1429**
D_{S2}	**0.0606**	**0.0606**	0.2110	0.2322	**0.1091**	**0.1091**
D_{S3}	**0.0769**	**0.0769**	0.2174	0.2373	**0.1429**	**0.1429**
D_{S4}	**0.0606**	**0.0606**	0.1771	0.1958	**0.1091**	**0.1091**
D_{KKDK}	**0.0833**	**0.0833**	**0.1667**	0.1833	**0.0833**	**0.1667**
D_{GS} (proposed)	0.0055	0.0067	0.6464	0.6482	0.0241	0.0250

(Counterintuitive cases are in bold type. $t = 3$ and $p = 1$ in D_{KKDK}).

$$\geq 1 - \frac{1}{4l} \sum_{k=1}^{l} \begin{bmatrix} 3\sqrt{m_H(t_k)m_I(t_k)} + 3\sqrt{n_H(t_k)n_I(t_k)} \\ +3\sqrt{h_H(t_k)h_I(t_k)} + \sqrt{e_H(t_k)e_I(t_k)} \\ +\sqrt{\begin{array}{c}(1 - m_H(t_k) - n_H(t_k)) \\ \times (1 - m_I(t_k) - n_I(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - m_H(t_k) - h_H(t_k)) \\ \times (1 - m_I(t_k) - h_I(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - n_H(t_k) - h_H(t_k)) \\ \times (1 - n_I(t_k) - h_I(t_k))\end{array}} \end{bmatrix}$$

$$\Rightarrow D_{GS}(H, I) \leq D_{GS}(G, I).$$

Thus, if $G \subseteq H \subseteq I$, then $D_{GS}(G, H) \leq D_{GS}(G, I)$ and $D_{GS}(H, I) \leq D_{GS}(G, I)$.

Hence, $D_{GS}(G, H)$ is a valid PF distance measure.

If we consider the weights w_k of t_k, $k = 1, 2, \ldots, l$, then the weighted distance measure between the PFSs G and H can be defined as

$$D_{WGS}(G, H) = 1 - \frac{1}{4l} \sum_{k=1}^{l} w_k \begin{bmatrix} 3\sqrt{m_G(t_k)m_H(t_k)} + 3\sqrt{n_G(t_k)n_H(t_k)} \\ +3\sqrt{h_G(t_k)h_H(t_k)} + \sqrt{e_G(t_k)e_H(t_k)} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - n_G(t_k)) \\ \times (1 - m_H(t_k) - n_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - m_G(t_k) - h_G(t_k)) \\ \times (1 - m_H(t_k) - h_H(t_k))\end{array}} \\ +\sqrt{\begin{array}{c}(1 - n_G(t_k) - h_G(t_k)) \\ \times (1 - n_H(t_k) - h_H(t_k))\end{array}} \end{bmatrix}$$

Theorem 2 $D_{WGS}(G, H)$ *is a distance measure between the two PFSs G and H.*

In the next section, we discuss the reasonability of the proposed PF distance measure and also discuss the drawbacks of the existing PF distance/correlation/similarity measures with the help of some experiments.

Experiments and analysis

Normally, the reasonability and performance of the distance (similarity) measures between PFSs are mostly confirmed by means of some practical experiments. In this section, two types of experiments are executed to investigate the performance of the newly proposed PF distance measure. First, the numerical experiments with some typically chosen PFSs. Second, we execute numerical experiment concerning the

pattern recognition problems using synthetic data and real data. These experiments enabled us to test the performance and the rationality of the proposed PF distance measure. For numerical experiments, eighteen traditional PF distance measures are chosen to contrast with the proposed PF distance measure. Further, for pattern recognition problems, thirty-seven PF compatibility (similarity/distance/correlation) measures are selected to contrast with the proposed PF distance measure.

Numerical experiments

Usually, the numerical experiments are first used to check the appropriateness of the PF distance (similarity) measures by many researchers, and an effectual distance measure differentiates the distance between PFSs. Three numerical experiments are performed to test the effectiveness of our proposed PF distance measure along with various traditional PF distance measures. In the first numerical experiment, six different cases of PFSs are used, in the second numerical experiment, four different cases of PFSs are used, and in the third numerical experiment, six different cases of PFSs are utilized. The experimental results are listed in Tables 1, 2, 3 and the counter-intuitive results are shown by bold letters. The detailed analysis pertaining to the three numerical experiments is summarized below.

(1) All the existing PF distance measures given in Eqs. (1)–(18) produce unreasonable results in various situations as indicated by bold values in Tables 1, 2, 3.

(2) The PFSs $G = \{(0.3, 0.2, 0.1)\}$, $H = \{(0.5, 0.2, 0.1)\}$ and $I = \{(0.5, 0.1, 0.1)\}$ (Case 5, and Case 6 in Table 1) satisfy $G \subseteq H \subseteq I$ and thus by the fourth axiom of PF distance measure given in Definition 5, we have $D(G, H) \leq D(G, I)$ for any PF distance measure D. But we see from Table 1 that the PF distance measures D_{D3}, D_{D4}, and D_{SMKSS2} given in Eqs. (7), (8) and (11) fail to satisfy this axiom.

Table 4 Calculated values of various PF-compatibility measures regarding Example 1

	(G_1, H)	(G_2, H)	(G_3, H)	Result
D_{DT1}	0.1000	0.1867	0.1933	G_1
D_{DT2}	0.1000	0.1833	0.1929	G_1
D_{DT3}	0.2000	0.2600	0.2600	G_1
D_{DT4}	0.0894	0.1428	0.1456	G_1
D_{D1}	1.0000	1.5000	1.7000	G_1
D_{D2}	0.2000	0.3000	0.3400	G_1
D_{D3}	0.4000	0.6557	0.7071	G_1
D_{D4}	0.1789	0.2933	0.3162	G_1
D_{D5}	0.2000	0.2800	0.3000	G_1
D_{SMKSS1}	0.4000	0.6000	0.6800	G_1
D_{SMKSS2}	0.1265	0.2074	0.2236	G_1
D_{SMKSS3}	0.0500	0.0650	0.0650	G_1
D_{SMKSS4}	0.1000	0.1597	0.1628	G_1
D_{S1}	0.3750	0.5154	0.4755	G_1
D_{S2}	0.3491	0.3951	0.3880	G_1
D_{S3}	0.1250	0.1872	0.1775	G_1
D_{S4}	0.1955	0.2268	0.2232	G_1
D_{KKDK}	0.1117	0.1867	0.2283	G_1
S_{W1}	0.9382	0.7691	0.7609	G_1
S_{W2}	0.9204	0.7813	0.7543	G_1
S_{W3}	0.9511	0.8810	0.8755	G_1
S_{W4}	0.9511	0.8810	0.8755	G_1
S_{W5}	0.9712	0.8703	0.8561	G_1
S_{W6}	0.9511	0.8573	0.8102	G_1
S_{W7}	0.7265	0.6745	0.6781	G_1
S_{W8}	0.7265	0.6745	0.6781	G_1
S_{W9}	0.8191	0.7216	0.6865	G_1
S_{WG1}	0.9263	0.7656	0.7539	G_1
S_{WG2}	0.9124	0.7779	0.7501	G_1
S_{SMKSS1}	0.9000	0.8500	0.8300	G_1
S_{SMKSS2}	0.9840	0.9570	0.9500	G_1
S_{SMKSS3}	0.0000	0.0833	0.0875	G_3
S_{KKDK}	0.8883	0.8133	0.7717	G_1
C_{S1}	0.9168	0.7625	0.7138	G_1
C_{S2}	0.8838	0.7500	0.6739	G_1
C_{GSB1}	0.4929	0.1621	-0.7676	G_1
C_{GSB2}	0.2270	0.1608	-0.6793	G_1
D_{GS}(Proposed)	0.0563	0.0780	0.0958	G_1

($t = 3$ and $p = 1$ in D_{KKDK}, S_{KKDK}).

Table 5 Calculated values of various PF-compatibility measures regarding Example 2

	(G_1, H)	(G_2, H)	(G_3, H)	Result
D_{DT1}	**0.0333**	0.0417	**0.0333**	**Not classified**
D_{DT2}	**0.0500**	0.0559	**0.0500**	**Not classified**
D_{DT3}	**0.0750**	0.0750	**0.0750**	**Not classified**
D_{DT4}	**0.0433**	0.0433	**0.0433**	**Not classified**
D_{D1}	**0.3000**	0.3000	0.4000	**Not classified**
D_{D2}	**0.0750**	0.0750	0.1000	**Not classified**
D_{D3}	**0.1732**	0.1732	0.2236	**Not classified**
D_{D4}	**0.0866**	0.0866	0.1118	**Not classified**
D_{D5}	**0.0750**	0.0750	0.1000	**Not classified**
D_{SMKSS1}	**0.1500**	0.1500	0.2000	**Not classified**
D_{SMKSS2}	**0.0612**	0.0612	0.0791	**Not classified**
D_{SMKSS3}	0.0188	0.0187	0.0250	G_2
D_{SMKSS4}	**0.0433**	0.0433	0.0612	**Not classified**
D_{S1}	0.1646	0.1818	0.1529	G_3
D_{S2}	0.2299	0.2336	0.2282	G_3
D_{S3}	0.0502	0.0545	0.0491	G_3
D_{S4}	0.1303	0.1325	0.1300	G_3
D_{KKDK}	0.0417	0.0521	0.0292	G_3
S_{W1}	0.9906	0.9871	0.9959	G_3
S_{W2}	0.9837	0.9850	0.9726	G_2
S_{W3}	**0.9908**	0.9908	**0.9908**	**Not classified**
S_{W4}	**0.9908**	0.9908	0.9816	**Not classified**
S_{W5}	**0.9954**	0.9931	**0.9954**	**Not classified**
S_{W6}	**0.9908**	0.9908	0.9816	**Not classified**
S_{W7}	**0.8906**	0.8906	**0.8906**	**Not classified**
S_{W8}	**0.8906**	0.8906	0.8587	**Not classified**
S_{W9}	0.9416	0.9156	0.9130	G_1
S_{WG1}	0.9874	0.9828	0.9876	G_3
S_{WG2}	0.9829	0.9831	0.9707	G_2
S_{SMKSS1}	**0.9625**	0.9625	0.9500	**Not classified**
S_{SMKSS2}	**0.9962**	0.9962	0.9938	**Not classified**
S_{SMKSS3}	**N/A**	**N/A**	**N/A**	**Not classified**
S_{KKDK}	0.9583	0.9479	0.9708	G_3
C_{S1}	**0.9846**	0.9846	0.9749	**Not classified**
C_{S2}	**0.9745**	0.9745	0.9505	**Not classified**
C_{GSB1}	0.8896	0.9511	0.9782	G_3
C_{GSB2}	**N/A**	**N/A**	**N/A**	**Not classified**
D_{GS}(proposed)	0.0128	0.0046	0.0192	G_2

(Bold values indicate unreasonable results. "N/A" denotes it cannot calculate the degree of similarity/correlation due to "the division by zero problem". $t = 3$ and $p = 1$ in D_{KKDK}, S_{KKDK}).

(3) The proposed PF distance measure D_{GS} given in Eq. (38) produces reasonable results without any counterintuitive case.

In the next section, we demonstrate the application of the proposed PF distance measure D_{GS} in pattern recognition

and compare the results with the existing PF compatibility measures.

Application in pattern recognition

We discuss the application of the proposed PF distance measure D_{GS} given in the Eq. (38) in pattern recognition problems. In pattern recognition, we are given some known patterns and an unknown pattern. The task is to identify the known pattern with which the resemblance of the unknown pattern is maximum. For this many compatibility measures like distance measure, divergence measure, accuracy measure, similarity measure, correlation measure, dissimilarity measure can be utilized. We, here utilize our proposed PF distance measure D_{GS} given in Eq. (38) for this purpose. In general, a pattern recognition problem in PF environment can be formulated as:

Scenario Let $G_i, i = 1, 2, \ldots, n$ be some known patterns and H be an unknown pattern given in the form of PFSs as

$$G_i = \left\{ \begin{array}{c} \left(t_k, m_{G_i}(t_k), n_{G_i}(t_k), h_{G_i}(t_k)\right) \\ |t_k \in U, k = 1, 2, \ldots, l \end{array} \right\},$$

$i = 1, 2, \ldots, n.$

$$H = \left\{ \begin{array}{c} \left(t_k, m_H(t_k), n_H(t_k), h_H(t_k)\right) \\ |t_k \in U, k = 1, 2, \ldots, l \end{array} \right\}.$$

Aim To find the known pattern $G_i, i = 1, 2, \ldots, n$ with which the resemblance of the unknown pattern H is maximum.

Recognition principle The unknown pattern H can be assigned to one of the known patterns $G_i, i = 1, 2, \ldots, n$ according to the following methods.

Accuracy/correlation/similarity method Let $A(G_i, H) / C(G_i, H) / S(G_i, H), i = 1, 2, \ldots, n$ be the accuracy/correlation/similarity between the known pattern $G_i, i = 1, 2, \ldots, n$ and the unknown pattern H, then H is assigned to G_{i^*}, where $i^* = \arg\max_i A(G_i, H)/C(G_i, H)/S(G_i, H), i = 1, 2, \ldots, n.$

Distance/divergence/dissimilarity method Let $D(G_i, H) / DI(G_i, H) / DS(G_i, H), i = 1, 2, \ldots, n$ be the distance/divergence/dissimilarity between the known pattern $G_i, i = 1, 2, \ldots, n$ and the unknown pattern H, then H is assigned to G_{i^*}, where $i^* = \arg\min_i D(G_i, H)/DI(G_i, H)/DS(G_i, H), i = 1, 2, \ldots, n.$

Now, we solve some pattern recognition problems involving PF information with the help of our proposed PF distance measure given in Eq. (38) and compare the results with some existing PF compatibility measures given in the Eqs. (1)–(37).

Example 1 [36] Consider three known patterns G_1, G_2, G_3, and an unknown pattern H in the form of PFSs as

$$G_1 = \left\{ \begin{array}{c} \left(t_1, 0.4, 0.3, 0.1\right), \left(t_2, 0.5, 0.3, 0.2\right), \\ \left(t_3, 0.4, 0.3, 0.0\right), \left(t_4, 0.7, 0.0, 0.2\right), \\ \left(t_5, 0.6, 0.1, 0.1\right) \end{array} \right\},$$

$$G_2 = \left\{ \begin{array}{c} \left(t_1, 0.7, 0.1, 0.1\right), \left(t_2, 0.2, 0.3, 0.4\right), \\ \left(t_3, 0.2, 0.1, 0.5\right), \left(t_4, 0.1, 0.5, 0.2\right), \\ \left(t_5, 0.3, 0.3, 0.3\right) \end{array} \right\},$$

$$G_3 = \left\{ \begin{array}{c} \left(t_1, 0.1, 0.3, 0.4\right), \left(t_2, 0.4, 0.3, 0.1\right), \\ \left(t_3, 0.3, 0.4, 0.2\right), \left(t_4, 0.2, 0.5, 0.3\right), \\ \left(t_5, 0.5, 0.3, 0.1\right) \end{array} \right\} \text{ and}$$

$$H = \left\{ \begin{array}{c} \left(t_1, 0.6, 0.2, 0.1\right), \left(t_2, 0.3, 0.4, 0.2\right), \\ \left(t_3, 0.4, 0.3, 0.2\right), \left(t_4, 0.7, 0.1, 0.0\right), \\ \left(t_5, 0.4, 0.2, 0.2\right) \end{array} \right\}.$$

The calculated values of PF-correlation/PF distance/ PF-similarity measures between the known patterns $G_i, i = 1, 2, 3$, and the unknown pattern H are summarized in Table 4.

From Table 4, it is clear that all the existing PF compatibility measures except S_{SMKSS3} given in the Eqs. (1)–(37) and our proposed PF distance measure given in the Eq. (38) assign the unknown pattern H to the known

Table 6 Calculated values of various PF-compatibility measures regarding Example 3

	(Setosa, Virginica)	(Versicolor, Virginica)	DoC
D_{DT1}	**0.1235**	0.1497	0.0262
D_{D1}	**0.2662**	0.2908	0.0245
D_{D2}	**0.5160**	0.5392	0.0232
D_{SMKSS1}	**0.1210**	0.1454	0.0244
D_{SMKSS2}	**0.2219**	0.2494	0.0276
D_{SMKSS3}	**0.0534**	0.0633	0099
D_{SMKSS4}	**0.1852**	0.2090	0.0238
D_{S3}	**0.1373**	0.1586	0.0213
D_{S4}	**0.1779**	0.1966	0.0188
D_{KKDK}	**0.0887**	0.1059	0.0172
S_{W1}	**0.0171**	0.0162	0.0008
S_{W2}	**0.0169**	0.0160	0.0008
S_{W3}	**0.0171**	0.0162	0.0008
S_{W4}	**0.0171**	0.0162	0.0008
S_{KKDK}	**0.9113**	0.8941	0.0172
D_{GS}(proposed)	**0.1019**	0.1317	0.0299

(Bold values indicate minimum distance/maximum similarity. $t = 3$ and $p = 1$ in D_{KKDK}, S_{KKDK}).

pattern G_1. This shows that our proposed PFdistance measure is consistent with the existing compatibility measures in PF environment.

Example 2 Consider three known patterns G_1, G_2, G_3, and an unknown pattern H in the form of PFSs as

$$G_1 = \left\{ \begin{array}{l} (t_1, 0.5, 0.0, 0.3), (t_2, 0.7, 0.0, 0.0), \\ (t_3, 0.4, 0.0, 0.5), (t_4, 0.7, 0.0, 0.3) \end{array} \right\},$$

$$G_2 = \left\{ \begin{array}{l} (t_1, 0.5, 0.0, 0.2), (t_2, 0.6, 0.0, 0.1), \\ (t_3, 0.2, 0.0, 0.7), (t_4, 0.7, 0.0, 0.3) \end{array} \right\},$$

$$G_3 = \left\{ \begin{array}{l} (t_1, 0.5, 0.0, 0.4), (t_2, 0.7, 0.0, 0.1), \\ (t_3, 0.4, 0.0, 0.6), (t_4, 0.7, 0.0, 0.2) \end{array} \right\} \text{ and.}$$

$$H = \left\{ \begin{array}{l} (t_1, 0.4, 0.0, 0.3), (t_2, 0.7, 0.0, 0.1), \\ (t_3, 0.3, 0.0, 0.6), (t_4, 0.7, 0.0, 0.3) \end{array} \right\}.$$

The calculated values of PF correlation/PF distance/ PF similarity measures between the known patterns $G_i, i = 1, 2, 3$, and the unknown pattern H are summarized in Table 5.

From Table 5, it is clear that out of thirty-eight PF compatibility measures given in Eqs. (1)–(38), twenty-four fail to classify the unknown pattern H into one of the known patterns $G_i, i = 1, 2, 3$ and the rest fourteen PF compatibility measures including our proposed distance measure D_{GS} classify the unknown pattern H into one of the known patterns $G_i, i = 1, 2, 3$. Thus, we conclude that our proposed PF distance measure D_{GS} is more effective and reasonable than most of the existing PF compatibility measures.

Further, we apply our proposed PF similarity measures on the real data related to the *Iris* plant that has been obtained from the UCI Machine Learning Repository (https://archive.ics.uci.edu/ml/datasets/Iris).

Example 3 Consider the database of the *Iris* plant. In this database, there are 150 samples (Table 12 in the Appendix) that are divided into three categories Setosa, Versicolor, and Virginica. There are four attributes corresponding to each sample namely, Sepal Length (SL), Sepal Width (SW), Petal Length (PL),and Petal Width (PW). We calculate the distance between Virginica and Setosa and between Virginica and Versicolor with the help of our proposed PF distance measure and compare the results with some existing PF compatibility measures.

To assess the performance of the proposed PFdistance measure, we utilize a performance index "Degree of Confidence (DoC)". The more the value of DoC, the more effective is a compatibility measure. The notion of DoC was suggested by Hatzimichailidis et al. [72] in the intuitionistic fuzzy framework. We extend this index in the PF

framework. For the data set under consideration, the DoC is given as follows.

$$\begin{aligned} \text{DoC}(K) = \; & |K(\text{Setosa, Virginica}) - M| \\ & + |K(\text{Versicolor, Virginica}) - M|, \end{aligned} \tag{39}$$

where K is any PF compatibility measure and $M = \min\{K(\text{Setosa, Virginica}), K(\text{Versicolor, Virginica})\}$, if K is a PF distance/dissimilarity measure or $M = \max\{K(\text{Setosa, Virginica}), K(\text{Versicolor, Virginica})\}$, if K is a PF correlation/similarity measure. As the data in the *Iris* database is in crisp form, so we need to convert it into PF form.

We propose the following formulae to convert the crisp data to picture fuzzy data.

The membership $\tilde{m}(t_{ij})$, the non-membership $\tilde{n}(t_{ij})$, and the neutrality $\tilde{h}(t_{ij})$ of the element t_{ij} is calculated as

$$\tilde{m}(t_{ij}) = am(t_{ij}), \; \tilde{n}(t_{ij}) = \left(1 - (m(t_{ij}))^a\right)^{\frac{1}{a}},$$

$$\tilde{h}(t_{ij}) = 1 - \left(1 - m(t_{ij})\right)^a, 0 < a \le 1 \text{ and}$$

$$m(t_{ij}) = 1 - \exp\left(-\frac{t_{ij} - \min(t_{ij})}{\max(t_{ij}) - \min(t_{ij})}\right).$$

Here $\max(t_{ij})$ is the maximum value of an attribute under a particular category and $\min(t_{ij})$ is the minimum value of an attribute under a particular category.

We convert the crisp data given in Table 12 in Appendix into the PF form with the help of the conversion formula.

The PF representation of *Iris setosa* is given in Table 13 (See Appendix) and the PF representation of *Iris versicolor* and *Iris virginica* can be obtained similarly.

Then, we calculate the distance between *Virginica* and *Setosa* and between *Virginica* and *Versicolor* by various PF compatibility measures including our proposed one and also calculate the DoC of each measure as shown in Table 6. From Table 6, we see that *Virginica* has maximum resemblance with *Setosa* as indicated by all the PF compatibility measures. Also, we observe that the *DoC* of our proposed PF distance measure is higher than most of the existing PF distance/similarity measures.

In the next section, we introduce a novel method for solving MADM problems in the PF environment with the help of the proposed PF distance measure D_{GS}.

Table 7 Calculated values regarding Example 4

	$D_{GS}(E_i, E^+)$	$D_{GS}(E_i, E^-)$	γ_i	Ranking
E_1	0.0270	0.0635	0.2984	2
E_2	0.0687	0.0336	0.6717	3
E_3	0.0180	0.0566	0.2414	1

A new MADM method in a picture fuzzy system

In this section, first we point out the weakness of the classical picture fuzzy TOPSIS method. Then, we propose a new TOPSIS like MADM method in the PF settings.

The weakness of classical picture fuzzy TOPSIS

One of the popular and effective methods for solving MADM problems is the technique for order preference by similarity to ideal solution (TOPSIS) introduced originally by Hwang and Yoon [73] and later on extended to the fuzzy environment by Chen [74]. The TOPSIS method is based on the fact that the chosen alternative should be closest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). However, the chosen alternative due to TOPSIS is not actually farthest from NIS as can be seen in the examples given below.

Example 4 Consider the PF-decision matrix with three alternatives $E_i(i = 1, 2, 3)$, and two attributes $A_j(j = 1, 2)$.

$$D_1 = \begin{pmatrix} (0.2, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.4, 0.2, 0.3) & (0.1, 0.3, 0.2) \\ (0.5, 0.2, 0.1) & (0.3, 0.1, 0.4) \end{pmatrix}.$$

Then the picture fuzzy positive ideal solution (PFPIS) E^+ and picture fuzzy negative ideal solution (PFNIS) E^- are given as

$$E^+ = \left\{ \begin{pmatrix} \max(0.2, 0.4, 0.5), \\ \min(0.1, 0.2, 0.2), \\ \min(0.3, 0.3, 0.1) \\ \max(0.5, 0.1, 0.3), \\ \min(0.1, 0.3, 0.1), \\ \min(0.2, 0.2, 0.4) \end{pmatrix} \right\}$$

$$= \{(0.5, 0.1, 0.1), (0.5, 0.1, 0.2)\},$$

$$E^- = \left\{ \begin{pmatrix} \min(0.2, 0.4, 0.5), \\ \max(0.1, 0.2, 0.2), \\ \min(0.3, 0.3, 0.1) \\ \min(0.5, 0.1, 0.3), \\ \max(0.1, 0.3, 0.1), \\ \min(0.2, 0.2, 0.4) \end{pmatrix} \right\}.$$

$$= \{(0.2, 0.2, 0.1), (0.1, 0.3, 0.2)\}.$$

We now calculate the distance of each alternative $E_i(i = 1, 2, 3)$ from the PFPIS E^+, and from PFNIS E^- by using the proposed PF distance measure D_{GS} given in Eq. (38), and then determine the closeness coefficient $\gamma_i = D_{GS}(E_i, E^+)/(D_{GS}(E_i, E^+) + D_{GS}(E_i, E^-))$ of each alternative, and finally rank the alternatives in ascending order of closeness coefficient as shown in Table 7.

From Table 7, we see that the chosen alternative E_3 is closest to PFPIS E^+ but is not farthest from PFNIS E^- because $D_{GS}(E_1, E^-) = 0.0635 > D_{GS}(E_3, E^-) = 0.0566$.

Example 5 Consider the PF-decision matrix with three alternatives $E_i(i = 1, 2, 3)$, and two attributes $A_j(j = 1, 2)$.

$$D_1 = \begin{pmatrix} (0.1, 0.3, 0.6) & (0.4, 0.2, 0.3) \\ (0.7, 0.2, 0.0) & (0.2, 0.0, 0.5) \\ (0.2, 0.1, 0.6) & (0.4, 0.4, 0.2) \end{pmatrix}$$

Then the picture fuzzy positive ideal solution (PFPIS) E^+ and picture fuzzy negative ideal solution (PFNIS) E^- are given as

$$E^+ = \left\{ \begin{pmatrix} \max(0.1, 0.7, 0.2), \\ \min(0.3, 0.2, 0.1), \\ \min(0.6, 0.0, 0.6) \\ \max(0.4, 0.2, 0.4), \\ \min(0.2, 0.0, 0.4), \\ \min(0.3, 0.5, 0.2) \end{pmatrix} \right\}$$

$$= \{(0.7, 0.1, 0.0), (0.4, 0.0, 0.2)\},$$

$$E^- = \left\{ \begin{pmatrix} \min(0.1, 0.7, 0.2), \\ \max(0.3, 0.2, 0.1), \\ \min(0.6, 0.0, 0.6) \\ \min(0.4, 0.2, 0.4), \\ \max(0.2, 0.0, 0.4), \\ \min(0.3, 0.5, 0.2) \end{pmatrix} \right\}$$

$$= \{(0.1, 0.3, 0.0), (0.2, 0.4, 0.2)\}.$$

Table 8 Calculated values regarding Example 5

	$D_{GS}(E_i, E^+)$	$D_{GS}(E_i, E^-)$	γ_i	Ranking
E_1	0.2671	0.1964	0.5763	2
E_2	0.0286	0.1853	0.1338	1
E_3	0.2732	0.1824	0.5996	3

Table 9 Picture fuzzy decision matrix

	A_1	A_2	A_3	A_4
E_1	(0.89, 0.03, 0.08)	(0.42, 0.18, 0.35)	(0.08, 0.02, 0.89)	(0.80, 0.05, 0.11)
E_2	(0.23, 0.11, 0.64)	(0.03, 0.13, 0.82)	(0.73, 0.08, 0.15)	(0.73, 0.14, 0.10)
E_3	(0.52, 0.05, 0.26)	(0.04, 0.10, 0.85)	(0.68, 0.06, 0.26)	(0.43, 0.25, 0.13)
E_4	(0.74, 0.10, 0.16)	(0.02, 0.05, 0.89)	(0.08, 0.84, 0.06)	(0.85, 0.05, 0.09)
E_5	(0.68, 0.21, 0.08)	(0.05, 0.06, 0.87)	(0.13, 0.09, 0.75)	(0.65, 0.02, 0.05)

We now calculate the distance of each alternative $E_i(i = 1, 2, 3)$ from the PFPIS E^+, and from PFNIS E^- by using the proposed PF distance measure D_{GS} given in Eq. (38), and then determine the closeness coefficient $\gamma_i = D_{GS}(E_i, E^+)/(D_{GS}(E_i, E^+) + D_{GS}(E_i, E^-))$ of each alternative, and finally rank the alternatives in ascending order of closeness coefficient as shown in Table 8.

From Table 8, we see that the chosen alternative E_2 is closest to PFPIS E^+ but is not farthest from PFNIS E^- because $D_{GS}(E_1, E^-) = 0.1964 > D_{GS}(E_2, E^-) = 0.1853$.

Thus, from Examples 4 and 5, we see that the best alternative in TOPSIS is not farthest from the NIS. To overcome this major drawback, we introduce a new MADM method in the PF environment known as the picture fuzzy inferior ratio (PFIR) method based on the same concept as in TOPSIS that the chosen alternative should be closest to PFPIS and farthest from PFNIS.

Picture fuzzy inferior ratio method for MADM

Our proposed method establishes a compromise solution (alternative) that is closest to PFPIS and farthest from PFNIS. First of all, we formulate a MADM problem in PF environment as follows:

Scenario We are given a set of n alternatives $E_i(i = 1, 2, \ldots, n)$ and m attributes $A_j(j = 1, 2, \ldots, m)$ along with the attribute weight vector $w = (w_1, w_2, \ldots, w_m), 0 \le w_j \le 1, j = 1, 2, \ldots, m$ and $\sum_{j=1}^m w_j = 1$.

Aim To select the most feasible alternative.

For determining the best alternative, the algorithm is given below.

Algorithm

Step 1 Construct the PF-decision matrix $C = [c_{ij}]_{n \times m}$ in which $c_{ij} = (m_{ij}, n_{ij}, h_{ij})$. is a PFV where m_{ij} is the membership value of alternative E_i under attribute A_j, n_{ij} is the non-membership value of alternative E_i under attribute A_j and h_{ij} is the neutrality value of alternative E_i under attribute A_j.

Step 2 Calculate the normalized PF decision matrix $F = [f_{ij}]_{n \times m}$, where

Table 10 Numerical results related to Example 6

	$D_{GS}(E_i, E^+)$	$D_{GS}(E_i, E^-)$	$\xi(E_i)$	η_i	Ranking
E_1	0.0877	0.1960	0.0000	0.0000	1
E_2	0.1194	0.1499	-0.5976	0.8500	3
E_3	0.1005	0.1277	-0.4946	0.7034	2
E_4	0.1321	0.1709	-0.6349	0.9030	4
E_5	0.1209	0.1325	-0.7031	1.0000	5

Table 11 Ranking of options with the help of various existing methods

Method	Ranking results
PF xross entropy [48]	$E_1 > E_4 > E_2 > E_5 > E_3$
TODIM method [49]	$E_1 > E_4 > E_5 > E_3 > E_2$
EDAS method [52]	$E_1 > E_5 > E_4 > E_3 > E_2$
PF similarity measures [51]	$E_1 > E_4 > E_2 > E_5 > E_3$
PFDWA operator [56]	$E_1 > E_4 > E_5 > E_2 > E_3$
PFDWG operator [56]	$E_1 > E_3 > E_2 > E_4 > E_5$
Generalized PF distance measure [45]	$E_1 > E_4 > E_2 > E_3 > E_5$
VIKOR method [66]	$E_1 > E_4 > E_2 > E_5 > E_3$
PF projection models [74]	$E_1 > E_2 > E_3 > E_4 > E_5$

$$f_{ij} = \begin{cases} c_{ij} = (m_{ij}, n_{ij}, h_{ij}), & \text{for benefit attribute} \\ (c_{ij})^c = (n_{ij}, m_{ij}, h_{ij}), & \text{for cost attribute}. \end{cases}$$

Step 3 Determine the PFPIS E^+ and PFNIS E^-, where $E^+ = \{f_1^+, f_2^+, \ldots, f_m^+\}$ and $E^- = \{f_1^-, f_2^-, \ldots, f_m^-\}$ with $f_j^+ = (\max_i(m_{ij}), \min_i(n_{ij}), \min_i(h_{ij}))$ and $f_j^- = (\min_i(m_{ij}), \max_i(n_{ij}), \min_i(h_{ij}))$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Step 4 Calculate the distance of each alternative $E_i, i = 1, 2, \ldots, n$ from PFPIS E^+, and from PFNIS E^- using our proposed PF distance measure D_{GS} given in Eq. (38) i.e., calculate $D_{GS}(E_i, E^+)$, and $D_{GS}(E_i, E^-)$, $i = 1, 2, \ldots, n$. The smaller $D_{GS}(E_i, E^+)$ is the better E_i is and the greater $D_{GS}(E_i, E^-)$ is the better E_i is.

Step 5 Calculate $D_{GS}(E^+)$, where $D_{GS}(E^+) = \min_{1 \le i \le n}(D_{GS}(E_i, E^+))$, and therefore, the alternative E_i that satisfies $D_{GS}(E^+) = D_{GS}(E_i, E^+)$ is closest to PFPIS.

Step 6 Calculate $D_{\mathrm{GS}}(E^-)$, where $D_{\mathrm{GS}}(E^-) = \max_{1\le i\le n}\left(D_{\mathrm{GS}}\left(E_i, E^-\right)\right)$, and therefore, the alternative E_i that satisfies $D_{\mathrm{GS}}(E^-) = D_{\mathrm{GS}}\left(E_i, E^-\right)$ is farthest from PFNIS.

Step 7. Calculate $\xi(E_i)$, $i = 1, 2, \ldots, n$, for each alternative, where

$$\xi\left(E_i\right) = \frac{D_{\mathrm{GS}}\left(E_i, E^-\right)}{D_{\mathrm{GS}}(E^-)} - \frac{D_{\mathrm{GS}}\left(E_i, E^+\right)}{D_{\mathrm{GS}}(E^+)}.$$

Clearly $\xi(E_i)$ measures the degree to which an alternative G_i, $i = 1, 2, \ldots, n$ is closest to PFPIS and farthest from PFNIS simultaneously. An alternative E_i with $\xi(E_i) = 0$ is the best alternative.

Step 8 Calculate the PFIR η_i for each alternative, where, $\eta_i = \frac{\xi(E_i)}{\min_{1\le i\le n}\xi(E_i)}$.

Step 9 Rank the alternatives in ascending order of values of PFIR η_i.

Proposition 1 $\xi(E_i) \le 0, i = 1, 2, \ldots, n$

Proof Since $D_{\mathrm{GS}}(E^+) = \min_{1\le i\le n}\left(D_{\mathrm{GS}}\left(E_i, E^+\right)\right)$ and $D_{\mathrm{GS}}(E^-) = \max_{1\le i\le n}\left(D_{\mathrm{GS}}\left(E_i, E^-\right)\right)$, so we have.

$D_{\mathrm{GS}}(E^+) \le D\left(E_i, E^+\right)$, and $D_{\mathrm{GS}}(E^-) \ge D_{\mathrm{GS}}\left(E_i, E^-\right), i = 1, 2, \ldots, n$. Therefore, we obtain.
$\frac{D_{\mathrm{GS}}(E_i,E^+)}{D_{\mathrm{GS}}(E^+)} \ge 1$ and $\frac{D_{\mathrm{GS}}(E_i,E^-)}{D_{\mathrm{GS}}(E^-)} \le 1$ and thus $\xi(E_i) = \frac{D_{\mathrm{GS}}(E_i,E^-)}{D_{\mathrm{GS}}(E^-)} - \frac{D_{\mathrm{GS}}(E_i,E^+)}{D_{\mathrm{GS}}(E^+)} \le 1 - 1 = 0.$

Proposition 2 *The* $0 \le \eta_i = \frac{\xi(E_i)}{\min_{1\le i\le n}\xi(E_i)} \le 1, i = 1, 2, \ldots, n$

Proof We have $\xi(E_i) \ge \min_{1\le i\le n}\xi(E_i)$ or $\min_{1\le i\le n}\xi(E_i) \le \xi(E_i) \le 0$ or $1 \ge \frac{\xi(E_i)}{\min_{1\le i\le n}\xi(E_i)} \ge 0$ i.e., $0 \le \eta_i \le 1, i = 1, 2, .., n.$

Now, we apply the proposed PFIR method for solving a MADM problem involving PF data in the following example.

Example 6 [66] A finance company wants to invest its sum of money. The four attributes fixed by the company for choosing the best possible option out of the five possible options (E_1) Mutual Funds (E_2) Health Insurance (E_3) Share Market (E_4) Housing Development Corporations and (E_5) General Insurance are (A_1) Maximum Returns (A_2) Minimum Risk (A_3) Easy withdrawal in case of emergency and (A_4) Transparency. The evaluation of the five possible options E_i, $i = 1, 2, 3, 4, 5$ corresponding to the four attributes A_j, $j = 1, 2, 3, 4$ is provided by the decision experts in the form of PF decision matrix as shown in Table 9.

Since all the attributes are of benefit type, so the normalized PF decision matrix is the same as given in Table 9. With the help of Step 3, we determine the PFPIS E^+ and PFNIS E^- as given below:

$$E^+ = \left\{ \begin{array}{l} (0.89, 0.03, 0.08), (0.42, 0.05, 0.35), \\ (0.73, 0.02, 0.06), (0.85, 0.02, 0.05) \end{array} \right\}$$

$$E^- = \left\{ \begin{array}{l} (0.23, 0.21, 0.08), (0.02, 0.18, 0.35), \\ (0.08, 0.84, 0.06), (0.43, 0.25, 0.05) \end{array} \right\}.$$

Next, using the proposed PF distance measure D_{GS} given in Eq. (38), we calculate the distance of each alternative E_i, $i = 1, 2, 3, 4, 5$ from PFPIS E^+, and from PFNIS E^-. Then we determine $D_{\mathrm{GS}}\left(E^+\right) = \min_{1\le i\le n}\left(D_{\mathrm{GS}}\left(E_i, E^+\right)\right) = 0.0877 = D_{\mathrm{GS}}\left(E_1, E^+\right)$ and $D_{\mathrm{GS}}(E^-) = \max_{1\le i\le n}\left(D_{\mathrm{GS}}\left(E_i, E^-\right)\right) = 0.1960 = D_{\mathrm{GS}}\left(E_1, E^-\right)$. After that, we calculate $\xi(E_i)$, $i = 1, 2, 3, 4, 5$, and η_i, $i = 1, 2, 3, 4, 5$ for each alternative using Step 7 and Step 8, respectively. Finally, in the ascending order of η_i, we rank the alternatives. These calculations are listed in Table 10.

From Table 10, we see that the best option is E_1 and we also observe that E_1 is closest to PFPIS E^+ and simultaneously farthest from PFNIS E^-.

In order to determine the validity and reasonability of the proposed PFIR method, we apply the existing methods [45, 48, 49, 51, 52, 56, 66, 74] for solving the same investment problem given in Example 6, and the results are listed in Table 11.

From Table 11, we observe that all the existing methods [45, 48, 49, 51, 52, 56, 66, 74] indicate that the best option is E_1, the same option indicated by our proposed method. This implies that our proposed method is in agreement with the existing methods in the PF environment.

Conclusion

The proposed PF distance measure is found to be stronger than the existing PF compatibility measures both theoretically and practically. Theoretically, it is valid in view of the metric axioms however some of the existing PF distance measures does not satisfy the metric properties. Practically, the proposed PF distance measure perform better than the existing measures in some counterintuitive situations. We have utilized the proposed PF distance measure for solving the problems related to pattern recognition and compared the results with the existing PF compatibility measures. We observed that in the problems of pattern recognition, the proposed PF distance measure is consistent and in some situations superior to the existing PF compatibility measures. The

application of the proposed PF distance measure has also been investigated in a pattern recognition problem with real data from the UCI machine learning repository. For working with real data (in the crisp form), we have introduced the conversion formula from crisp data to PF data. Also, to assess the performance of the proposed measures a performance index namely "Degree of confidence (DoC)" has been introduced. In view of DoC, our proposed measure is found to outperform the existing PF compatibility measures. We have also presented a new method known as the picture fuzzy inferior ratio (PFIR) method for solving MADM problems in the PF environment. It has been observed that the compromise solution due to the PFIR method is closest to PFPIS and simultaneously farthest from PFNIS. But, in the conventional TOPSIS method this situation fails to occur in many practical problems. We have also seen that the PFIR method is consistent with the existing MADM methods in the PF environment.

In the future, we plan to study:

(1) Generalization of the proposed PF distance measure to spherical fuzzy sets and T-spherical fuzzy sets.
(2) The application of the proposed PF distance measure in clustering analysis, medical diagnosis, image segmentation, etc.

(3) In this paper, we have considered thirty-seven existing compatibility measures concerning picture fuzzy sets and contrasted the performance of proposed measure with them. So, in view of the refernces [68–71] and thirty-eight compatibilty measures considered in this paper, many compatibilty measures can be developed in the framework of neutrosophic sets and refined neutrosophic sets.

Compliance with ethical standards

Conflict of interest Authors declare that there is no conflict ofinterest.

Ethical approval The present article does not contain any studies withhuman participants or animals performed by any of the authors.

Appendix

The *Iris* database obtained from the UCI Machine Learning Repository is given in Tables 12 and the corresponding picture fuzzy representation of the Iris setosa is presented in theTable 13.

Table 12 Iris database

Iris setosa				Iris versicolor				Iris virginica			
SL	SW	PL	PW	SL	SW	PL	PW	SL	SW	PL	PW
5.1	3.5	1.4	0.2	7.0	3.2	4.7	1.4	6.3	3.3	6.0	2.5
4.9	3.0	1.4	0.2	6.4	3.2	4.5	1.5	5.8	2.7	5.1	1.9
4.7	3.2	1.3	0.2	6.9	3.1	4.9	1.5	7.1	3.0	5.9	2.1
4.6	3.1	1.5	0.2	5.5	2.3	4.0	1.3	6.3	2.9	5.6	1.8
5.0	3.6	1.4	0.2	6.5	2.8	4.6	1.5	6.5	3.0	5.8	2.2
5.4	3.9	1.7	0.4	5.7	2.8	4.5	1.3	7.6	3.0	6.6	2.1
4.6	3.4	1.4	0.3	6.3	2.8	4.7	1.6	4.9	2.5	4.5	1.7
5.0	3.4	1.5	0.2	4.9	3.3	3.3	1.0	7.3	2.9	6.3	1.8
4.4	2.9	1.4	0.2	6.6	2.4	4.6	1.3	6.7	2.5	5.8	1.8
4.9	3.1	1.5	0.1	5.2	2.9	3.9	1.4	7.2	3.6	6.1	2.5
5.4	3.7	1.5	0.2	5.0	2.7	3.5	1.0	6.5	3.2	5.1	2.0
4.8	3.4	1.6	0.2	5.9	2.0	4.2	1.5	6.4	2.7	5.3	1.9
4.8	3.0	1.4	0.1	6.0	3.0	4.0	1.0	6.8	3.0	5.5	2.1
4.3	3.0	1.1	0.1	6.1	2.2	4.7	1.4	5.7	2.5	5.0	2.0
5.8	4.0	1.2	0.2	5.6	2.9	3.6	1.3	5.8	2.8	5.1	2.4
5.7	4.4	1.5	0.4	6.7	2.9	4.4	1.4	6.4	3.2	5.3	2.3
5.4	3.9	1.3	0.4	5.6	3.1	4.5	1.5	6.5	3.0	5.5	1.8
5.1	3.5	1.4	0.3	5.8	3.0	4.1	1.0	7.7	3.8	6.7	2.2
5.7	3.8	1.7	0.3	6.2	2.7	4.5	1.5	7.7	2.6	6.9	2.3
5.1	3.8	1.5	0.3	5.6	2.2	3.9	1.1	6.0	2.2	5.0	1.5
5.4	3.4	1.7	0.2	5.9	2.5	4.8	1.8	6.9	3.2	5.7	2.3
5.1	3.7	1.5	0.4	6.1	3.2	4.0	1.3	5.6	2.8	4.9	2.0
4.6	3.6	1.0	0.2	6.3	2.8	4.9	1.5	7.7	2.8	6.7	2.0
5.1	3.3	1.7	0.5	6.1	2.5	4.7	1.2	6.3	2.7	4.9	1.8
4.8	3.4	1.9	0.2	6.4	2.8	4.3	1.3	6.7	3.3	5.7	2.1
5.0	3.0	1.6	0.2	6.6	2.9	4.4	1.4	7.2	3.2	6.0	1.8
5.0	3.4	1.6	0.4	6.8	3.0	4.8	1.4	6.2	2.8	4.8	1.8
5.2	3.5	1.5	0.2	6.7	2.8	5.0	1.7	6.1	3.0	4.9	1.8
5.2	3.4	1.4	0.2	6.0	3.0	4.5	1.5	6.4	2.8	5.6	2.1
4.7	3.2	1.6	0.2	5.7	2.9	3.5	1.0	7.2	3.0	5.8	1.6
4.8	3.1	1.6	0.2	5.5	2.6	3.8	1.1	7.4	2.8	6.1	1.9
5.4	3.4	1.5	0.4	5.5	2.4	3.7	1.0	7.9	3.8	6.4	2.0
5.2	4.1	1.5	0.1	5.8	2.4	3.9	1.2	6.4	2.8	5.6	2.2
5.5	4.2	1.4	0.2	6.0	2.7	5.1	1.6	6.3	2.8	5.1	1.5
4.9	3.1	1.5	0.1	5.4	2.7	4.5	1.5	6.1	2.6	5.6	1.4
5.0	3.2	1.2	0.2	6.0	3.0	4.5	1.6	7.7	3.0	6.1	2.3
5.5	3.5	1.3	0.2	6.7	3.4	4.7	1.5	6.3	3.4	5.6	2.4
4.9	3.1	1.5	0.1	6.3	3.1	4.4	1.3	6.4	3.1	5.5	1.8
4.4	3.0	1.3	0.2	5.6	2.3	4.1	1.3	6.0	3.0	4.8	1.8
5.1	3.4	1.5	0.2	5.5	3.0	4.0	1.3	6.9	3.1	5.4	2.1
5.0	3.5	1.3	0.3	5.5	2.5	4.4	1.2	6.7	3.1	5.6	2.4
4.5	2.3	1.3	0.3	6.1	2.6	4.6	1.4	6.9	3.1	5.1	2.3
4.4	3.2	1.3	0.2	5.8	3.0	4.0	1.2	5.8	2.7	5.1	1.9
5.0	3.5	1.6	0.6	5.0	2.6	3.3	1.0	6.8	3.2	5.9	2.3
5.1	3.8	1.9	0.4	5.6	2.3	4.2	1.3	6.7	3.3	5.7	2.5
4.8	3.0	1.4	0.3	5.7	2.7	4.2	1.2	6.7	3.0	5.2	2.3
5.1	3.8	1.6	0.2	5.7	3.0	4.2	1.3	6.3	2.5	5.0	1.9
4.6	3.2	1.4	0.2	6.2	2.9	4.3	1.3	6.5	3.0	5.2	2.0
5.3	3.7	1.5	0.2	5.1	2.5	3.0	1.1	6.2	3.4	5.4	2.3

Table 12 (continued)

Iris setosa				Iris versicolor				Iris virginica			
SL	SW	PL	PW	SL	SW	PL	PW	SL	SW	PL	PW
5.0	3.3	1.4	0.2	5.7	2.8	4.1	1.3	5.9	3.0	5.1	1.8

Table 13 Picture Fuzzy representation of *Iris setosa*

Iris setosa			
Sepal length (SL)	Sepal width (SW)	Petal length (PL)	Petal width (PW)
(0.2067, 0.1275, 0.2341)	(0.2176, 0.1158, 0.2485)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.1648, 0.1813, 0.1813)	(0.1417, 0.2186, 0.1535)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.1170, 0.2665, 0.1248)	(0.1743, 0.1678, 0.1929)	(0.1417, 0.2186, 0.1535)	(0.0906, 0.3298, 0.0952)
(0.0906, 0.3298, 0.0952)	(0.1584, 0.1911, 0.1734)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.2308, 0.1028, 0.2662)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.2598, 0.0779, 0.3070)	(0.2666, 0.0728, 0.3168)	(0.2703, 0.0701, 0.3222)	(0.2256, 0.1078, 0.2592)
(0.0906, 0.3298, 0.0952)	(0.2039, 0.1306, 0.2304)	(0.1794, 0.1608, 0.1993)	(0.1648, 0.1813, 0.1813)
(0.1865, 0.1516, 0.2081)	(0.2039, 0.1306, 0.2304)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.0322, 0.5566, 0.0328)	(0.1243, 0.2515, 0.1331)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.1648, 0.1813, 0.1813)	(0.1584, 0.1911, 0.1734)	(0.2131, 0.1205, 0.2425)	(0.0000, 1.0000, 0.0000)
(0.2598, 0.0779, 0.3070)	(0.2433, 0.0915, 0.2835)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.1417, 0.2186, 0.1535)	(0.2039, 0.1306, 0.2304)	(0.2433, 0.0915, 0.2835)	(0.0906, 0.3298, 0.0952)
(0.1417, 0.2186, 0.1535)	(0.1417, 0.2186, 0.1535)	(0.1794, 0.1608, 0.1993)	(0.0000, 1.0000, 0.0000)
(0.0000, 1.0000, 0.0000)	(0.1417, 0.2186, 0.1535)	(0.0526, 0.4566, 0.0540)	(0.0000, 1.0000, 0.0000)
(0.3161, 0.0420, 0.3935)	(0.2775, 0.0651, 0.3329)	(0.0996, 0.3065, 0.1052)	(0.0906, 0.328, 0.0952)
(0.3034, 0.0489, 0.3729)	(0.3161, 0.0420, 0.3935)	(0.2131, 0.1205, 0.2425)	(0.2256, 0.1078, 0.2592)
(0.2598, 0.0779, 0.3070)	(0.2666, 0.0728, 0.3168)	(0.1417, 0.2186, 0.1535)	(0.2256, 0.1078, 0.2592)
(0.2067, 0.1275, 0.2341)	(0.2176, 0.1158, 0.2485)	(0.1794, 0.1608, 0.1993)	(0.1648, 0.1813, 0.1813)
(0.3034, 0.0489, 0.3729)	(0.2552, 0.0815, 0.3003)	(0.2703, 0.0701, 0.3222)	(0.1648, 0.1813, 0.1813)
(0.2067, 0.1275, 0.2341)	(0.2552, 0.0815, 0.3003)	(0.2131, 0.1205, 0.2425)	(0.1648, 0.1813, 0.1813)
(0.2598, 0.0779, 0.3070)	(0.2039, 0.1306, 0.2304)	(0.2703, 0.0701, 0.3222)	(0.0906, 0.3298, 0.0952)
(0.2067, 0.1275, 0.2341)	(0.2433, 0.0915, 0.2835)	(0.2131, 0.1205, 0.2425)	(0.2256, 0.1078, 0.2592)
(0.0906, 0.3298, 0.0952)	(0.2308, 0.1028, 0.2662)	(0.0000, 1.0000, 0.0000)	(0.0906, 0.3298, 0.0952)
(0.2067, 0.1275, 0.2341)	(0.1894, 0.1478, 0.2119)	(0.2703, 0.0701, 0.3222)	(0.2753, 0.0665, 0.3297)
(0.1417, 0.2186, 0.1535)	(0.2039, 0.1306, 0.2304)	(0.3161, 0.0420, 0.3935)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.1417, 0.2186, 0.1535)	(0.2433, 0.0915, 0.2835)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.2039, 0.1306, 0.2304)	(0.2433, 0.0915, 0.2835)	(0.2256, 0.1078, 0.2592)
(0.2256, 0.1078, 0.2592)	(0.2176, 0.1158, 0.2485)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.2256, 0.1078, 0.2592)	(0.2039, 0.1306, 0.2304)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.1170, 0.2665, 0.1248)	(0.1743, 0.1678, 0.1929)	(0.2433, 0.0915, 0.2835)	(0.0906, 0.3298, 0.0952)
(0.1417, 0.2186, 0.1535)	(0.1584, 0.1911, 0.1734)	(0.2433, 0.0915, 0.2835)	(0.0906, 0.3298, 0.0952)
(0.2598, 0.0779, 0.3070)	(0.2039, 0.1306, 0.2304)	(0.2131, 0.1205, 0.2425)	(0.2256, 0.1078, 0.2592)
(0.2256, 0.1078, 0.2592)	(0.2878, 0.0582, 0.3486)	(0.2131, 0.1205, 0.2425)	(0.0000, 1.0000, 0.0000)
(0.2753, 0.0665, 0.3297)	(0.2977, 0.0522, 0.3639)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.1648, 0.1813, 0.1813)	(0.1584, 0.1911, 0.1734)	(0.2131, 0.1205, 0.2425)	(0.0000, 1.0000, 0.0000)
(0.0322, 0.5566, 0.0328)	(0.1743, 0.1678, 0.1929)	(0.0996, 0.3065, 0.1052)	(0.0906, 0.3298, 0.0952)
(0.2067, 0.1275, 0.2341)	(0.2176, 0.1158, 0.2485)	(0.1417, 0.2186, 0.1535)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.1584, 0.1911, 0.1734)	(0.2131, 0.1205, 0.2425)	(0.0000, 1.0000, 0.0000)
(0.0624, 0.4182, 0.0645)	(0.1417, 0.2186, 0.1535)	(0.1417, 0.2186, 0.1535)	(0.0906, 0.3298, 0.0952)
(0.0322, 0.5566, 0.0328)	(0.2039, 0.1306, 0.2304)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.2176, 0.1158, 0.2485)	(0.1417, 0.2186, 0.1535)	(0.1648, 0.1813, 0.1813)
(0.0624, 0.4182, 0.0645)	(0.0000, 1.0000, 0.0000)	(0.1417, 0.2186, 0.1535)	(0.1648, 0.1813, 0.1813)

Table 13 (continued)

Iris setosa			
Sepal length (SL)	Sepal width (SW)	Petal length (PL)	Petal width (PW)
(0.0322, 0.5566, 0.0328)	(0.1743, 0.1678, 0.1929)	(0.1417, 0.2186, 0.1535)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.2176, 0.1158, 0.2485)	(0.2433, 0.0915, 0.2835)	(0.3161, 0.0420, 0.3935)
(0.2067, 0.1275, 0.2341)	(0.2552, 0.0815, 0.3003)	(0.3161, 0.0420, 0.3935)	(0.2256, 0.1078, 0.2592)
(0.1417, 0.2186, 0.1535)	(0.1417, 0.2186, 0.1535)	(0.1794, 0.1608, 0.1993)	(0.1648, 0.1813, 0.1813)
(0.2067, 0.1275, 0.2341)	(0.2552, 0.0815, 0.3003)	(0.2433, 0.0915, 0.2835)	(0.0906, 0.3298, 0.0952)
(0.0906, 0.3298, 0.0952)	(0.1743, 0.1678, 0.1929)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)
(0.2433, 0.0915, 0.2835)	(0.2433, 0.0915, 0.2835)	(0.2131, 0.1205, 0.2425)	(0.0906, 0.3298, 0.0952)
(0.1865, 0.1516, 0.2081)	(0.1894, 0.1478, 0.2119)	(0.1794, 0.1608, 0.1993)	(0.0906, 0.3298, 0.0952)

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