



Some novel features of Pythagorean m -polar fuzzy sets with applications

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Abstract

We face many situations in day to day life where multi-polar statistics is offered. The prevailing models like Pythagorean fuzzy sets and m -polar fuzzy sets become inoperable in handling such situation efficiently e.g. if someone wishes to invest his capital in some scheme, he would for sure like to know repeated information about pros and cons of that scheme. Pythagorean m -polar fuzzy sets ($PmFS$ s) serve as the most appropriate model to cope with such situations. The motivation behind this article is to extend the notions of $PmFS$ s coined by Naeem et al. (J Intell Fuzzy Syst 37(6): 8441–8458, 2019) and introduce some new operations and results on $PmFS$ s. Owing to the idea of Pythagorean m -polar fuzzy relation, we render an application in the selection of most appropriate life partner.

Keywords Pythagorean m -polar fuzzy set · (λ, η) -cut · Modal operators · Pythagorean m -polar fuzzy relation · Decision making

Introduction

We bifurcate this section into introductory note and the inspiration behind this article, and the literature review.

Introduction and inspiration

The notion of a set is an embryonic concept in mathematics. The science of logic, reasoning and rationality and set theory are trusted to be the foundation of contemporary mathematics. Writing proofs and dealing with tedious notions of abstract algebra are sublime examples of nexus between the two disciplines. Many results of classical set theory may

be proved with comfort using propositional logic. Indeed these two sciences are naturally mutually dependent regarding the fact that each one of them is delineated in view of the other. Many logical operations are transformed into set theory and contrariwise. In the same way, paradoxes, functions, relations, numbers, modern measure theory and probability theory etc. all are founded on theory of sets. Besides its foundational role, modern set theory owns a massive number of vibrant researchers.

The role of dissociation degree is not taken into account in m -polar fuzzy sets. Intuitionistic fuzzy sets (IFSs) have the hitch that they become of no use while handling the circumstances when addition of affiliation and dissociation degrees goes further than unity. Pythagorean fuzzy sets (PFSs) overcome this shortcoming but are incapable to incorporate multi-polarity like in IFSs. There emerge numerous circumstances where information comprises of multi-polar data. For example, sometimes laboratory technicians need more readings about a suspected patient before giving an accurate diagnosis about a disease. A successful venture capitalist would unquestionably like to secure his capital by gaining frequent statistics about the business where he has been planning to finance. Military operations need repeated information before making an appropriate action. Similarly, before taking a final decision about one's better half, one requires repeated meetings and gossip with the person under consideration. $PmFS$ s offer

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most apposite model for coping with such states of affairs. To cut a long story short, there are many real world situations where other structures lack this issue or that. *PmFSs* have the potential that they don't take off us unaccompanied and single-handedly in such circumstances.

The elementary ambition behind this study is to coin some innovative and expand some existing operations on *PmFSs* presented by Naeem et al. [17]. *PmFSs* provide a strong mathematical structure for dealing with decision making problems including medical analysis, business analysis, military operations, recruitment problems, image processing, artificial intelligence, medicine, forecasting, robotics, economics and trade analysis, robotics, machine learning and a plenty of further day to day problems.

Literature review

In the times of classical set theory, a characteristic function was associated with each member of the set, lacking any other temperate choice for a point in view of belonging-ness to a family. Zadeh [35] coined the idea of fuzzy sets to overcome this drawback. On the words of Zadeh, it may happen that a point be partially a property of a set. Atanassov [4,5], by adding another parameter in fuzzy sets, introduced intuitionistic fuzzy sets (IFSs) by promulgating the constraint that the addition of affiliation and dissociation grades must lie in $[0, 1]$. Later, Yager [32–34] expanded IFSs to Pythagorean fuzzy sets (PFSs). Some remarkable results for PFSs and their practical implementations were studied by Peng et al. [19–23]. Guleria and Bajaj [11] invented matrix form to represent Pythagorean fuzzy soft sets (PFSSs). Naeem et al. [16] introduced some new results and techniques using PFSSs. Rahman et al. [24] presented some basic operations on PFSs. Garg [7–10] explored many techniques for decision making in Pythagorean fuzzy environment. Riaz et al. [26–29] focused on presenting different hybrid structures along with their practical implementations in real world situations.

A generalization of fuzzy sets was proposed in 1994 by Zhang [36] entitled bipolar fuzzy sets. Later, in 2000, Lee [14] brought to canvass an extension of fuzzy sets titled bipolar-valued fuzzy sets and presented two ways of their representation. Consequent upon the discoveries of Zhang and Lee, Chen et al. [6] generalized bipolar fuzzy sets to *m*-polar fuzzy sets. A short time ago, Naeem et al. [17] unveiled Pythagorean *m*-polar fuzzy sets with their practical implementations. Well along, Riaz et al. [25] stretched out the notion to corresponding soft structure by introducing Pythagorean *m*-polar fuzzy soft sets and rendered some fas-

inating applications of the proposed model. Smarandache [31] initiated the idea of neutrosophic sets (NSs) to get by circumstances comprising inexactitude, changeability and indeterminacy. Maji [15] expanded NSs to neutrosophic soft sets (NSSs). Recently, Naeem et al. [18] proposed the notions of fuzzy neutrosophic soft σ -algebra (fns σ -algebra) and fns measure with application in analyzing dynamics factors behind putrefying economy of a country.

The following segment of this article is managed for smooth study as follows: a reflection of some fundamental notions, necessary for further study are presented in Sect. 2. Some novel results on *PmFSs* are presented in Sect. 3. Score and accuracy functions for *PmFSs* along with some chief characteristics and illustrations are presented in Sect. 4. Different forms of (λ, η) -cut and their properties are coined in Sect. 5. Section 6 gives some characteristics relevant to modal operators. Extension principle of *PmFSs* is presented in Sect. 7. Section 8 deals with relations on *PmFSs*. A practical implementation of Pythagorean *m*-polar fuzzy relation, accompanied by an algorithm, in the selection of life partner is rendered in Sect. 9. We finish with concluding remarks and some impending suggestions in Sect. 10.

Preliminaries

In this subdivision, we recollect some ground rules of *PmFSs* with succinctness indispensable to apprehend the advanced part of this article. The multipolarity index i will extend over 1 to m (a positive integer) and \mathfrak{g} will be treated as the part of underlying classical set X , unless expressed something else.

Definition 2.1 [17] A *Pythagorean m-polar fuzzy set* (*PmFS*) \mathfrak{B} is characterized by the mappings $\tau_{\mathfrak{B}}^{(i)}$ (denoting affiliation degrees) and $\sigma_{\mathfrak{B}}^{(i)}$ (meant for dissociation grades) dragging members of X to $[0, 1]$ constrained to obey $0 \leq (\tau_{\mathfrak{B}}^{(i)}(\mathfrak{g}))^2 + (\sigma_{\mathfrak{B}}^{(i)}(\mathfrak{g}))^2 \leq 1$, for all i . The quantity $\varepsilon_{\mathfrak{B}}^{(i)}(\mathfrak{g}) = \sqrt{1 - (\tau_{\mathfrak{B}}^{(i)}(\mathfrak{g}))^2 - (\sigma_{\mathfrak{B}}^{(i)}(\mathfrak{g}))^2}$ is known as *hesitation margin* or *indeterminacy degree* of $\mathfrak{g} \in X$ to \mathfrak{B} . $\varepsilon_{\mathfrak{B}}^{(i)} : X \mapsto [0, 1]$ are mappings expressing lack of knowledge regarding $\mathfrak{g} \in \mathfrak{B}$ or $\mathfrak{g} \notin \mathfrak{B}$. The pair $(\tau^{(i)}, \sigma^{(i)})$ is commonly acknowledged as *Pythagorean fuzzy number* (PFN).

A *PmFS* is usually expressed as

$$\begin{aligned} \beta &= \left\{ \left(\tau_{\beta}^{(1)}(\mathfrak{g}), \sigma_{\beta}^{(1)}(\mathfrak{g}), (\tau_{\beta}^{(2)}(\mathfrak{g}), \sigma_{\beta}^{(2)}(\mathfrak{g})), \dots, (\tau_{\beta}^{(m)}(\mathfrak{g}), \sigma_{\beta}^{(m)}(\mathfrak{g})) \right) \right\} \\ &= \left\{ \frac{\mathfrak{g}}{\left((\tau_{\beta}^{(1)}(\mathfrak{g}), \sigma_{\beta}^{(1)}(\mathfrak{g})), (\tau_{\beta}^{(2)}(\mathfrak{g}), \sigma_{\beta}^{(2)}(\mathfrak{g})), \dots, (\tau_{\beta}^{(m)}(\mathfrak{g}), \sigma_{\beta}^{(m)}(\mathfrak{g})) \right)} \right\} \\ &= \left\{ \frac{\mathfrak{g}}{\left((\tau_{\beta}^{(i)}(\mathfrak{g}), \sigma_{\beta}^{(i)}(\mathfrak{g})) \right)} \right\} \end{aligned}$$

If $|X| = r$, then tabulatory array of β is as in Table 1. The corresponding matrix format is

$$\beta = \begin{pmatrix} (\tau_{\beta}^{(1)}(\mathfrak{g}_1), \sigma_{\beta}^{(1)}(\mathfrak{g}_1)) & (\tau_{\beta}^{(2)}(\mathfrak{g}_1), \sigma_{\beta}^{(2)}(\mathfrak{g}_1)) & \dots & (\tau_{\beta}^{(m)}(\mathfrak{g}_1), \sigma_{\beta}^{(m)}(\mathfrak{g}_1)) \\ (\tau_{\beta}^{(1)}(\mathfrak{g}_2), \sigma_{\beta}^{(1)}(\mathfrak{g}_2)) & (\tau_{\beta}^{(2)}(\mathfrak{g}_2), \sigma_{\beta}^{(2)}(\mathfrak{g}_2)) & \dots & (\tau_{\beta}^{(m)}(\mathfrak{g}_2), \sigma_{\beta}^{(m)}(\mathfrak{g}_2)) \\ \vdots & \vdots & \ddots & \vdots \\ (\tau_{\beta}^{(1)}(\mathfrak{g}_r), \sigma_{\beta}^{(1)}(\mathfrak{g}_r)) & (\tau_{\beta}^{(2)}(\mathfrak{g}_r), \sigma_{\beta}^{(2)}(\mathfrak{g}_r)) & \dots & (\tau_{\beta}^{(m)}(\mathfrak{g}_r), \sigma_{\beta}^{(m)}(\mathfrak{g}_r)) \end{pmatrix}$$

This matrix of size $r \times m$ is titled as *PmF matrix*.

Definition 2.2 [17] β_1 is acknowledged a *PmF subset* of β_2 , written $\beta_1 \sqsubseteq \beta_2$ if $\tau_{\beta_1}^{(i)}(\mathfrak{g}) \leq \tau_{\beta_2}^{(i)}(\mathfrak{g})$ and $\sigma_{\beta_1}^{(i)}(\mathfrak{g}) \geq \sigma_{\beta_2}^{(i)}(\mathfrak{g})$, for all i .

Definition 2.3 [17] A *PmFS* for which $\tau_{\beta}^{(i)}(\mathfrak{g}) = 0$ and $\sigma_{\beta}^{(i)}(\mathfrak{g}) = 1$, for all i , is termed as *null PmFS* and is designated as $\check{\beta}$.

Definition 2.4 [17] A *PmFS* for which $\tau_{\beta}^{(i)}(\mathfrak{g}) = 1$ and $\sigma_{\beta}^{(i)}(\mathfrak{g}) = 0$, for all i , is termed as *absolute PmFS* and is designated as $\check{\check{\beta}}$.

Definition 2.5 [17] The *complement* of a *PmFS*

$$\beta = \left\{ \frac{\mathfrak{g}}{\left((\tau_{\beta}^{(i)}(\mathfrak{g}), \sigma_{\beta}^{(i)}(\mathfrak{g})) \right)} \right\}$$

is given as

$$\beta^c = \left\{ \frac{\mathfrak{g}}{\left((\sigma_{\beta}^{(i)}(\mathfrak{g}), \tau_{\beta}^{(i)}(\mathfrak{g})) \right)} \right\}$$

Definition 2.6 [17] The *union* of β_1 and β_2 is delineated as

$$\beta_1 \sqcup \beta_2 = \left\{ \frac{\mathfrak{g}}{\left(\max(\tau_{\beta_1}^{(i)}(\mathfrak{g}), \tau_{\beta_2}^{(i)}(\mathfrak{g})), \min(\sigma_{\beta_1}^{(i)}(\mathfrak{g}), \sigma_{\beta_2}^{(i)}(\mathfrak{g})) \right)} \right\}$$

Definition 2.7 [17] The *intersection* of β_1 and β_2 is given as

$$\beta_1 \sqcap \beta_2 = \left\{ \frac{\mathfrak{g}}{\left(\min(\tau_{\beta_1}^{(i)}(\mathfrak{g}), \tau_{\beta_2}^{(i)}(\mathfrak{g})), \max(\sigma_{\beta_1}^{(i)}(\mathfrak{g}), \sigma_{\beta_2}^{(i)}(\mathfrak{g})) \right)} \right\}$$

Definition 2.8 [17] The *necessity operator* on β is given as

$$\square\beta = \left\{ \frac{\mathfrak{g}}{\left(\tau_{\beta}^{(i)}(\mathfrak{g}), \sqrt{1 - (\tau_{\beta}^{(i)}(\mathfrak{g}))^2} \right)} \right\}$$

Definition 2.9 [17] The *possibility operator* on β is delineated as

$$\diamond\beta = \left\{ \frac{\mathfrak{g}}{\left(\sqrt{1 - (\sigma_{\beta}^{(i)}(\mathfrak{g}))^2}, \sigma_{\beta}^{(i)}(\mathfrak{g}) \right)} \right\}$$

Table 1 Tabulatory array of β

β				
g_1	$(\tau_{\beta}^{(1)}(g_1), \sigma_{\beta}^{(1)}(g_1))$	$(\tau_{\beta}^{(2)}(g_1), \sigma_{\beta}^{(2)}(g_1))$	\dots	$(\tau_{\beta}^{(m)}(g_1), \sigma_{\beta}^{(m)}(g_1))$
g_2	$(\tau_{\beta}^{(1)}(g_2), \sigma_{\beta}^{(1)}(g_2))$	$(\tau_{\beta}^{(2)}(g_2), \sigma_{\beta}^{(2)}(g_2))$	\dots	$(\tau_{\beta}^{(m)}(g_2), \sigma_{\beta}^{(m)}(g_2))$
\vdots	\vdots	\vdots	\ddots	\vdots
g_r	$(\tau_{\beta}^{(1)}(g_r), \sigma_{\beta}^{(1)}(g_r))$	$(\tau_{\beta}^{(2)}(g_r), \sigma_{\beta}^{(2)}(g_r))$	\dots	$(\tau_{\beta}^{(m)}(g_r), \sigma_{\beta}^{(m)}(g_r))$

Definition 2.10 [17] The *sum* of β_1 and β_2 is defined as

$$\beta_1 \oplus \beta_2 = \left\{ \frac{g}{\left(\sqrt{(\tau_{\beta_1}^{(i)}(g))^2 + (\tau_{\beta_2}^{(i)}(g))^2 - (\tau_{\beta_1}^{(i)}(g)\tau_{\beta_2}^{(i)}(g))^2}, \sigma_{\beta_1}^{(i)}(g)\sigma_{\beta_2}^{(i)}(g) \right)} \right\}$$

Definition 2.11 [17] The *product* of β_1 and β_2 is defined as

$$\beta_1 \otimes \beta_2 = \left\{ \frac{g}{\left(\tau_{\beta_1}^{(i)}(g)\tau_{\beta_2}^{(i)}(g), \sqrt{(\sigma_{\beta_1}^{(i)}(g))^2 + (\sigma_{\beta_2}^{(i)}(g))^2 - (\sigma_{\beta_1}^{(i)}(g)\sigma_{\beta_2}^{(i)}(g))^2} \right)} \right\}$$

Some novel results on PmFSs

We bring together some new upshots on PmFSs in this segment.

Proposition 3.1 Let β be a PmFS. If $\varepsilon_{\beta}^{(i)}(g) \rightarrow 0$ for all i , then

- (1) $\tau_{\beta}^{(i)}(g) \cong \sqrt{\{1 + \sigma_{\beta}^{(i)}(g)\}\{1 - \sigma_{\beta}^{(i)}(g)\}}$.
- (2) $\sigma_{\beta}^{(i)}(g) \cong \sqrt{\{1 + \tau_{\beta}^{(i)}(g)\}\{1 - \tau_{\beta}^{(i)}(g)\}}$.

Proof By definition,

$$\begin{aligned} (\tau_{\beta}^{(i)}(g))^2 + (\sigma_{\beta}^{(i)}(g))^2 + (\varepsilon_{\beta}^{(i)}(g))^2 &= 1 \\ \Rightarrow (\tau_{\beta}^{(i)}(g))^2 + (\sigma_{\beta}^{(i)}(g))^2 &\cong 1 \quad (\because \varepsilon_{\beta}^{(i)}(g) \rightarrow 0) \\ \Rightarrow (\tau_{\beta}^{(i)}(g))^2 &\cong 1 - (\sigma_{\beta}^{(i)}(g))^2 \\ \Rightarrow \tau_{\beta}^{(i)}(g) &\cong \sqrt{1 - (\sigma_{\beta}^{(i)}(g))^2} \\ \text{i.e. } \tau_{\beta}^{(i)}(g) &\cong \sqrt{\{1 + \sigma_{\beta}^{(i)}(g)\}\{1 - \sigma_{\beta}^{(i)}(g)\}} \end{aligned}$$

which proves (1). The proof of (2) may be furnished on the parallel track. \square

Some properties of \oplus , \otimes , \sqcup and \sqcap for PmFSs analogous to crisp sets are presented in following propositions.

Proposition 3.2 If β_1, β_2 and β_3 are three PmFSs, then

- (1) $\beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1$.
- (2) $\beta_1 \otimes \beta_2 = \beta_2 \otimes \beta_1$.
- (3) $\beta_1 \oplus (\beta_2 \oplus \beta_3) = (\beta_1 \oplus \beta_2) \oplus \beta_3$.
- (4) $\beta_1 \otimes (\beta_2 \otimes \beta_3) = (\beta_1 \otimes \beta_2) \otimes \beta_3$.

Proof Let $g \in X$ and $i = 1, 2, \dots, m$.

(1) By definition,

$$\begin{aligned} \mathfrak{B}_1 \oplus \mathfrak{B}_2 &= \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_{\mathfrak{B}_1}^{(i)}(\mathfrak{g}))^2 + (\tau_{\mathfrak{B}_2}^{(i)}(\mathfrak{g}))^2 - (\tau_{\mathfrak{B}_1}^{(i)}(\mathfrak{g})\tau_{\mathfrak{B}_2}^{(i)}(\mathfrak{g}))^2}, \sigma_{\mathfrak{B}_1}^{(i)}(\mathfrak{g})\sigma_{\mathfrak{B}_2}^{(i)}(\mathfrak{g}) \right)} \right\} \\ &= \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_{\mathfrak{B}_2}^{(i)}(\mathfrak{g}))^2 + (\tau_{\mathfrak{B}_1}^{(i)}(\mathfrak{g}))^2 - (\tau_{\mathfrak{B}_2}^{(i)}(\mathfrak{g})\tau_{\mathfrak{B}_1}^{(i)}(\mathfrak{g}))^2}, \sigma_{\mathfrak{B}_2}^{(i)}(\mathfrak{g})\sigma_{\mathfrak{B}_1}^{(i)}(\mathfrak{g}) \right)} \right\} \\ &= \mathfrak{B}_2 \oplus \mathfrak{B}_1 \end{aligned}$$

which proves (1). The proof of (2) is parallel.

(4) For the purpose of brevity, we would write $\tau_{\mathfrak{B}_j}^{(i)}(\mathfrak{g}) = \tau_j^{(i)}$ and $\sigma_{\mathfrak{B}_j}^{(i)}(\mathfrak{g}) = \sigma_j^{(i)}$ onward. Then, by definition, the membership function of $\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \otimes \mathfrak{B}_3)$ is

$$\begin{aligned} \tau_{\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \otimes \mathfrak{B}_3)} &= \tau_1^{(i)} \left(\tau_2^{(i)} \tau_3^{(i)} \right) \\ &= \tau_1^{(i)} \tau_2^{(i)} \tau_3^{(i)} \\ &= \left(\tau_1^{(i)} \tau_2^{(i)} \right) \tau_3^{(i)} \\ &= \tau_{(\mathfrak{B}_1 \otimes \mathfrak{B}_2) \otimes \mathfrak{B}_3} \end{aligned}$$

and the corresponding non-membership function is

$$\begin{aligned} \sigma_{\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \otimes \mathfrak{B}_3)} &= \sqrt{\left(\sigma_1^{(i)} \right)^2 + \left(\sqrt{(\sigma_2^{(i)})^2 + (\sigma_3^{(i)})^2 - (\sigma_2^{(i)}\sigma_3^{(i)})^2} \right)^2 - \left(\sigma_1^{(i)} \right)^2 \left(\sqrt{(\sigma_2^{(i)})^2 + (\sigma_3^{(i)})^2 - (\sigma_2^{(i)}\sigma_3^{(i)})^2} \right)^2} \\ &= \sqrt{\left(\sigma_1^{(i)} \right)^2 + (\sigma_2^{(i)})^2 + (\sigma_3^{(i)})^2 - (\sigma_2^{(i)}\sigma_3^{(i)})^2 - \left(\sigma_1^{(i)} \right)^2 \left((\sigma_2^{(i)})^2 + (\sigma_3^{(i)})^2 - (\sigma_2^{(i)}\sigma_3^{(i)})^2 \right)} \\ &= \sqrt{\left(\sigma_1^{(i)} \right)^2 + (\sigma_2^{(i)})^2 + (\sigma_3^{(i)})^2 - (\sigma_2^{(i)}\sigma_3^{(i)})^2 - (\sigma_1^{(i)}\sigma_2^{(i)})^2 - (\sigma_1^{(i)}\sigma_3^{(i)})^2 + (\sigma_1^{(i)}\sigma_2^{(i)}\sigma_3^{(i)})^2} \\ &= \sqrt{\left(\sqrt{(\sigma_1^{(i)})^2 + (\sigma_2^{(i)})^2 - (\sigma_1^{(i)}\sigma_2^{(i)})^2} \right)^2 + (\sigma_3^{(i)})^2 - \left(\sqrt{(\sigma_1^{(i)})^2 + (\sigma_2^{(i)})^2 - (\sigma_1^{(i)}\sigma_2^{(i)})^2} \right)^2 \cdot \left(\sigma_3^{(i)} \right)^2} \\ &= \sigma_{(\mathfrak{B}_1 \otimes \mathfrak{B}_2) \otimes \mathfrak{B}_3} \end{aligned}$$

Thus, $\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \otimes \mathfrak{B}_3) = (\mathfrak{B}_1 \otimes \mathfrak{B}_2) \otimes \mathfrak{B}_3$.

The proof of (3) is similar. □

Proposition 3.3 Assume that $\mathfrak{B}_1, \mathfrak{B}_2$ and \mathfrak{B}_3 are PmFSs defined over X with $\mathfrak{g} \in X$. Then

- (1) $\mathfrak{B}_1 \oplus (\mathfrak{B}_2 \sqcup \mathfrak{B}_3) = (\mathfrak{B}_1 \oplus \mathfrak{B}_2) \sqcup (\mathfrak{B}_1 \oplus \mathfrak{B}_3)$.
- (2) $\mathfrak{B}_1 \oplus (\mathfrak{B}_2 \sqcap \mathfrak{B}_3) = (\mathfrak{B}_1 \oplus \mathfrak{B}_2) \sqcap (\mathfrak{B}_1 \oplus \mathfrak{B}_3)$.
- (3) $\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \sqcup \mathfrak{B}_3) = (\mathfrak{B}_1 \otimes \mathfrak{B}_2) \sqcup (\mathfrak{B}_1 \otimes \mathfrak{B}_3)$.
- (4) $\mathfrak{B}_1 \otimes (\mathfrak{B}_2 \sqcap \mathfrak{B}_3) = (\mathfrak{B}_1 \otimes \mathfrak{B}_2) \sqcap (\mathfrak{B}_1 \otimes \mathfrak{B}_3)$.

Proof We prove (1) here. The proofs of other parts are similar, so we omit them. By definition,

$$\mathfrak{B}_2 \sqcup \mathfrak{B}_3 = \left\{ \frac{\mathfrak{g}}{\left(\max(\tau_2^{(i)}, \tau_3^{(i)}), \min(\sigma_2^{(i)}, \sigma_3^{(i)}) \right)} \right\}$$

There arise four possibilities:

- (i) $\max(\tau_2^{(i)}, \tau_3^{(i)}) = \tau_2^{(i)}$ and $\min(\sigma_2^{(i)}, \sigma_3^{(i)}) = \sigma_2^{(i)}$;
- (ii) $\max(\tau_2^{(i)}, \tau_3^{(i)}) = \tau_2^{(i)}$ and $\min(\sigma_2^{(i)}, \sigma_3^{(i)}) = \sigma_3^{(i)}$;
- (iii) $\max(\tau_2^{(i)}, \tau_3^{(i)}) = \tau_3^{(i)}$ and $\min(\sigma_2^{(i)}, \sigma_3^{(i)}) = \sigma_2^{(i)}$; and
- (iv) $\max(\tau_2^{(i)}, \tau_3^{(i)}) = \tau_3^{(i)}$ and $\min(\sigma_2^{(i)}, \sigma_3^{(i)}) = \sigma_3^{(i)}$.

We consider only second possibility. The other three cases may be discussed accordingly. Now,

$$\beta_1 \oplus (\beta_2 \sqcup \beta_3) = \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\max(\tau_2^{(i)} \tau_3^{(i)}))^2} - (\tau_1^{(i)} \max(\tau_2^{(i)} \tau_3^{(i)}))^2, \sigma_1^{(i)} \min(\sigma_2^{(i)} \sigma_3^{(i)}) \right)} \right\}$$

$$= \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2, \sigma_1^{(i)} \sigma_3^{(i)} \right)} \right\}$$

and

$$\beta_1 \oplus \beta_2 = \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2, \sigma_1^{(i)} \sigma_2^{(i)} \right)} \right\},$$

$$\beta_1 \oplus \beta_3 = \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\tau_3^{(i)})^2} - (\tau_1^{(i)} \tau_3^{(i)})^2, \sigma_1^{(i)} \sigma_3^{(i)} \right)} \right\}.$$

be three P3FSs defined over $X = \{k, y\}$. Then,

$$\beta_2 \sqcap \beta_3 = \begin{pmatrix} (0.03, 0.31) & (0.52, 0.67) & (0.12, 0.41) \\ (0.38, 0.11) & (0.09, 0.02) & (0.44, 0.77) \end{pmatrix}$$

$$\therefore \beta_1 \otimes (\beta_2 \sqcap \beta_3) = \begin{pmatrix} (0.01, 0.62) & (0.22, 0.89) & (0.10, 0.45) \\ (0.17, 0.12) & (0.03, 0.38) & (0.23, 0.89) \end{pmatrix} \quad (1)$$

So that

$$(\beta_1 \oplus \beta_2) \sqcup (\beta_1 \oplus \beta_3) = \left\{ \frac{\mathfrak{g}}{\left(\max \left(\sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2, \sqrt{(\tau_1^{(i)})^2 + (\tau_3^{(i)})^2} - (\tau_1^{(i)} \tau_3^{(i)})^2 \right), \min(\sigma_1^{(i)} \sigma_2^{(i)}, \sigma_1^{(i)} \sigma_3^{(i)}) \right)} \right\}$$

$$= \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2, \sigma_1^{(i)} \sigma_3^{(i)} \right)} \right\}$$

$$= \beta_1 \oplus (\beta_2 \sqcup \beta_3).$$

This concludes the proof. \square

Example 3.4 Let

$$\beta_1 = \begin{pmatrix} (0.39, 0.56) & (0.43, 0.79) & (0.81, 0.21) \\ (0.45, 0.06) & (0.37, 0.38) & (0.52, 0.69) \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} (0.03, 0.19) & (0.52, 0.67) & (0.12, 0.22) \\ (0.38, 0.10) & (0.09, 0.02) & (0.97, 0.21) \end{pmatrix},$$

and

$$\beta_3 = \begin{pmatrix} (0.82, 0.31) & (0.55, 0.66) & (0.18, 0.41) \\ (0.53, 0.11) & (0.49, 0.01) & (0.44, 0.77) \end{pmatrix}$$

Also,

$$\beta_1 \otimes \beta_2 = \begin{pmatrix} (0.01, 0.58) & (0.22, 0.89) & (0.10, 0.30) \\ (0.17, 0.12) & (0.03, 0.38) & (0.50, 0.71) \end{pmatrix}$$

and

$$\beta_1 \otimes \beta_3 = \begin{pmatrix} (0.32, 0.62) & (0.24, 0.89) & (0.15, 0.45) \\ (0.24, 0.12) & (0.18, 0.38) & (0.23, 0.89) \end{pmatrix}$$

so that

$$(\beta_1 \otimes \beta_2) \sqcap (\beta_1 \otimes \beta_3) = \begin{pmatrix} (0.01, 0.62) & (0.22, 0.89) & (0.10, 0.45) \\ (0.17, 0.12) & (0.03, 0.38) & (0.23, 0.89) \end{pmatrix} \quad (2)$$

Therefore, from (1) & (2): $\beta_1 \otimes (\beta_2 \sqcap \beta_3) = (\beta_1 \otimes \beta_2) \sqcap (\beta_1 \otimes \beta_3)$. Likewise, the other results may be verified.

Proposition 3.5 *Let β_1 and β_2 be two PmFSs. Then,*

- (1) $(\beta_1 \oplus \beta_2)^c = \beta_1^c \otimes \beta_2^c$.
- (2) $(\beta_1 \otimes \beta_2)^c = \beta_1^c \oplus \beta_2^c$.

Proof By definition,

$$\begin{aligned} (\beta_1 \oplus \beta_2)^c &= \left\{ \frac{\mathfrak{g}}{\left(\sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2, \sigma_1^{(i)} \sigma_2^{(i)}\right)} \right\}^c \\ &= \left\{ \frac{\mathfrak{g}}{\left(\sigma_1^{(i)} \sigma_2^{(i)}, \sqrt{(\tau_1^{(i)})^2 + (\tau_2^{(i)})^2} - (\tau_1^{(i)} \tau_2^{(i)})^2\right)} \right\} \\ &= \left\{ \frac{\mathfrak{g}}{(\sigma_1^{(i)}, \tau_1^{(i)})} \right\} \otimes \left\{ \frac{\mathfrak{g}}{(\sigma_2^{(i)}, \tau_2^{(i)})} \right\} \\ &= \left\{ \frac{\mathfrak{g}}{(\tau_1^{(i)}, \sigma_1^{(i)})} \right\}^c \otimes \left\{ \frac{\mathfrak{g}}{(\tau_2^{(i)}, \sigma_2^{(i)})} \right\}^c \\ &= \beta_1^c \otimes \beta_2^c. \end{aligned}$$

This establishes (1). The proof of (2) is similar. □

Example 3.6 Consider β_1 and β_2 given in Example 3.4. We have

$$(\beta_1 \otimes \beta_2)^c = \begin{pmatrix} (0.58, 0.01) & (0.89, 0.22) & (0.30, 0.10) \\ (0.12, 0.17) & (0.38, 0.03) & (0.71, 0.50) \end{pmatrix} \quad (3)$$

Also,

$$\beta_1^c = \begin{pmatrix} (0.56, 0.39) & (0.79, 0.43) & (0.21, 0.81) \\ (0.06, 0.45) & (0.38, 0.37) & (0.69, 0.52) \end{pmatrix}$$

and

$$\begin{aligned} \beta_2^c &= \begin{pmatrix} (0.19, 0.03) & (0.67, 0.52) & (0.22, 0.12) \\ (0.10, 0.38) & (0.02, 0.09) & (0.21, 0.97) \end{pmatrix} \\ \beta_1^c \oplus \beta_2^c &= \begin{pmatrix} (0.58, 0.01) & (0.89, 0.22) & (0.30, 0.10) \\ (0.12, 0.17) & (0.38, 0.03) & (0.71, 0.50) \end{pmatrix} \quad (4) \end{aligned}$$

Hence, from (3) & (4): $(\beta_1 \otimes \beta_2)^c = \beta_1^c \oplus \beta_2^c$. The other result may be verified analogously.

Score and accuracy functions of a PmFN

In this subdivision, we define score and accuracy functions for a Pythagorean m -polar fuzzy number (PmFN) $p = ((\tau_p^{(i)}, \sigma_p^{(i)})_{i=1}^m)$ of a PmFS β .

Definition 4.1 The *score function* of a PmFN $p = ((\tau_p^{(i)}, \sigma_p^{(i)})_{i=1}^m)$ of a PmFS β is defined as

$$s(p) = \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(\mathfrak{g}))^2 - (\sigma_p^{(i)}(\mathfrak{g}))^2 \right\}$$

for all $\mathfrak{g} \in X$. The value of this score function always falls in $[-1, 1]$.

Proposition 4.2 *Let $p = ((\tau_p^{(i)}, \sigma_p^{(i)})_{i=1}^m)$ be a PmFN of a PmFS β defined over X and $\mathfrak{g} \in X$. Then,*

- (1) $s(p) = 1 \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 = m + \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2$.
- (2) $s(p) = 0 \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 = \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2$.
- (3) $s(p) = -1 \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 = \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 - m$.

Proof (1) Let

$$\begin{aligned} s(p) &= 1 \\ \Leftrightarrow \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(\mathfrak{g}))^2 - (\sigma_p^{(i)}(\mathfrak{g}))^2 \right\} &= 1 \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 - \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 &= m \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 &= m + \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 \end{aligned}$$

(2) Let

$$\begin{aligned} s(p) &= 0 \\ \Leftrightarrow \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(\mathfrak{g}))^2 - (\sigma_p^{(i)}(\mathfrak{g}))^2 \right\} &= 0 \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 - \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 &= 0 \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 &= \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 \end{aligned}$$

(3) Let

$$\begin{aligned} s(p) &= -1 \\ \Leftrightarrow \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(\mathfrak{g}))^2 - (\sigma_p^{(i)}(\mathfrak{g}))^2 \right\} &= -1 \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 - \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 &= -m \\ \Leftrightarrow \sum_{i=1}^m (\tau_p^{(i)}(\mathfrak{g}))^2 &= \sum_{i=1}^m (\sigma_p^{(i)}(\mathfrak{g}))^2 - m \end{aligned}$$

Definition 4.3 The *accuracy function* of a PmFN $p = ((\tau_p^{(i)}, \sigma_p^{(i)})_{i=1}^m)$ of a PmFS β is characterized as

$$a(p) = \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(\mathfrak{g}))^2 + (\sigma_p^{(i)}(\mathfrak{g}))^2 \right\}$$

Table 2 Tabular formation of P3FS β

β			
g_1	(0.37, 0.59)	(0.22, 0.54)	(0, 1)
g_2	(0.57, 0.68)	(1, 0)	(1, 0)

Table 3 Values of score and accuracy functions

	β	$s(p)$	$a(p)$
g_1		-0.4848	0.6083
g_2		0.6208	0.9291

for all $g \in X$. The value of this accuracy function always falls in $[0, 1]$.

Proposition 4.4 Let $p = ((\tau_p^{(i)}, \sigma_p^{(i)})_{i=1}^m)$ be a PmFN of a PmFS β defined over X and $g \in X$. Then,

- (1) $a(p) = 0 \Leftrightarrow \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 = m$.
- (2) $a(p) = 1 \Leftrightarrow \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 = 0$.

Proof (1) Let

$$\begin{aligned}
 a(p) &= 0 \\
 \Leftrightarrow \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(g))^2 + (\sigma_p^{(i)}(g))^2 \right\} &= 0 \\
 \Leftrightarrow \sum_{i=1}^m \left\{ 1 - (\varepsilon_p^{(i)}(g))^2 \right\} &= 0 \\
 \Leftrightarrow m - \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 &= 0 \\
 \Leftrightarrow \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 &= m
 \end{aligned}$$

(2) Let

$$\begin{aligned}
 a(p) &= 1 \\
 \Leftrightarrow \frac{1}{m} \sum_{i=1}^m \left\{ (\tau_p^{(i)}(g))^2 + (\sigma_p^{(i)}(g))^2 \right\} &= 1 \\
 \Leftrightarrow \sum_{i=1}^m \left\{ 1 - (\varepsilon_p^{(i)}(g))^2 \right\} &= 1 \\
 \Leftrightarrow m - \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 &= m \\
 \Leftrightarrow \sum_{i=1}^m (\varepsilon_p^{(i)}(g))^2 &= 0
 \end{aligned}$$

□

Example 4.5 Consider a P3FS β defined over the crisp set $X = \{g_1, g_2\}$ given in Table 2.

The values of score and accuracy functions are computed as in Table 3.

Table 4 Tabular formation of P4FS β

β				
g_1	(0.52, 0.47)	(0.23, 0.84)	(0.12, 0.93)	(0.54, 0.31)
g_2	(0.42, 0.11)	(0.36, 0.19)	(0.19, 0.47)	(0.14, 0.08)
g_3	(0.37, 0.32)	(0.41, 0.02)	(0.92, 0.18)	(0.13, 0.26)
g_4	(0.25, 0.11)	(0.54, 0.54)	(0.38, 0.27)	(0.99, 0.12)
g_5	(0.13, 0.60)	(0.12, 0.69)	(0.19, 0.67)	(0.24, 0.70)

(λ, η) -cut of a PmFS

We present different sorts of (λ, η) -cuts of a PmFS in conjunction with a few conspicuous characteristics of these ideas in this fragment.

Definition 5.1 Let β be a PmFS defined over X and $g \in X$. Assume further that $\lambda, \eta \in [0, 1]$.

(1) The strong upper (λ, η) -cut of β is characterized as

$$\beta^{[\lambda, \eta]} = \{g : \tau_\beta^{(i)}(g) \geq \lambda \ \& \ \sigma_\beta^{(i)}(g) \leq \eta\}$$

(2) The weak upper (λ, η) -cut of β is characterized as

$$\beta^{(\lambda, \eta)} = \{g : \tau_\beta^{(i)}(g) > \lambda \ \& \ \sigma_\beta^{(i)}(g) < \eta\}$$

(3) The strong lower (λ, η) -cut of β is characterized as

$$\beta_{[\lambda, \eta]} = \{g : \tau_\beta^{(i)}(g) \leq \lambda \ \& \ \sigma_\beta^{(i)}(g) \geq \eta\}$$

(4) The weak lower (λ, η) -cut of β is characterized as

$$\beta_{(\lambda, \eta)} = \{g : \tau_\beta^{(i)}(g) < \lambda \ \& \ \sigma_\beta^{(i)}(g) > \eta\}$$

for all permissible values of i .

Remark It follows from Definition 5.1 that $\beta^{(\lambda, \eta)} \sqsubseteq \beta^{[\lambda, \eta]}$ and $\beta_{(\lambda, \eta)} \sqsubseteq \beta_{[\lambda, \eta]}$.

Example 5.2 For the P4FS β defined over the crisp set $X = \{g_1, \dots, g_5\}$ given in Table 4, we have

$$\begin{aligned}
 \beta^{[0.25, 0.60]} &= \{g_4\} \\
 \beta^{(0.25, 0.60)} &= \emptyset \\
 \beta_{[0.25, 0.60]} &= \{g_5\} \\
 \beta_{(0.25, 0.60)} &= \emptyset
 \end{aligned}$$

Proposition 5.3 Let \mathcal{B} and \mathcal{Z} be two PmFSs over X . If $\lambda, \lambda_1, \lambda_2, \eta, \eta_1, \eta_2 \in [0, 1]$, then

- (1) $\mathcal{B} \sqsubseteq \mathcal{Z} \Leftrightarrow \mathcal{B}^{[\lambda, \eta]} \sqsubseteq \mathcal{Z}^{[\lambda, \eta]}$.
- (2) $\mathcal{B}^{[\lambda_1, \eta_1]} \sqsubseteq \mathcal{B}^{[\lambda_2, \eta_2]} \Leftrightarrow \lambda_1 \geq \lambda_2 \ \& \ \eta_1 \leq \eta_2$.

Proof (1) Let $\mathcal{B} \sqsubseteq \mathcal{Z}$. Then, for all admissible values of i , $\tau_{\mathcal{B}}^{(i)} \leq \tau_{\mathcal{Z}}^{(i)}$ and $\sigma_{\mathcal{B}}^{(i)} \geq \sigma_{\mathcal{Z}}^{(i)}$. If $\mathfrak{g} \in \mathcal{B} \sqsubseteq \mathcal{Z}$, so $\tau_{\mathcal{Z}}^{(i)} \geq \tau_{\mathcal{B}}^{(i)} \geq \lambda$ and $\sigma_{\mathcal{Z}}^{(i)} \leq \sigma_{\mathcal{B}}^{(i)} \leq \eta$ which in turn yields $\mathcal{B}^{[\lambda, \eta]} \sqsubseteq \mathcal{Z}^{[\lambda, \eta]}$.

The converse follows by reverting the argument.

- (2) Let $\lambda_1 \geq \lambda_2$ and $\eta_1 \leq \eta_2$. Suppose that $\mathfrak{g} \in \mathcal{B}^{[\lambda_1, \eta_1]}$, then $\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}) \geq \lambda_1$ and $\sigma_{\mathcal{B}}^{(i)}(\mathfrak{g}) \leq \eta_1$. Since $\lambda_1 \geq \lambda_2 \ \& \ \eta_1 \leq \eta_2$, so $\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}) \geq \lambda_2$ and $\sigma_{\mathcal{B}}^{(i)}(\mathfrak{g}) \leq \eta_2$. Thus, $\mathcal{B}^{[\lambda_1, \eta_1]} \sqsubseteq \mathcal{B}^{[\lambda_2, \eta_2]}$.

The converse follows directly from definition and properties of inequalities. □

Corollary 5.4 Let \mathcal{B} and \mathcal{Z} be two PmFSs over X . If $\lambda, \lambda_1, \lambda_2, \eta, \eta_1, \eta_2 \in [0, 1]$, then

- (1) $\mathcal{B} \sqsubseteq \mathcal{Z} \Leftrightarrow \mathcal{B}_{[\lambda, \eta]} \sqsubseteq \mathcal{Z}_{[\lambda, \eta]}$.
- (2) $\mathcal{B} \sqsubseteq \mathcal{Z} \Leftrightarrow \mathcal{B}^{(\lambda, \eta)} \sqsubseteq \mathcal{Z}^{(\lambda, \eta)}$.
- (3) $\mathcal{B} \sqsubseteq \mathcal{Z} \Leftrightarrow \mathcal{B}_{(\lambda, \eta)} \sqsubseteq \mathcal{Z}_{(\lambda, \eta)}$.
- (4) $\mathcal{B}_{[\lambda_1, \eta_1]} \sqsubseteq \mathcal{B}_{[\lambda_2, \eta_2]} \Leftrightarrow \lambda_1 \geq \lambda_2 \ \& \ \eta_1 \leq \eta_2$.
- (5) $\mathcal{B}^{(\lambda_1, \eta_1)} \sqsubseteq \mathcal{B}^{(\lambda_2, \eta_2)} \Leftrightarrow \lambda_1 \geq \lambda_2 \ \& \ \eta_1 \leq \eta_2$.
- (6) $\mathcal{B}_{(\lambda_1, \eta_1)} \sqsubseteq \mathcal{B}_{(\lambda_2, \eta_2)} \Leftrightarrow \lambda_1 \geq \lambda_2 \ \& \ \eta_1 \leq \eta_2$.

Proposition 5.5 Let \mathcal{B} and \mathcal{Z} be two PmFSs. If λ, η are real numbers picked from $[0, 1]$, then

- (1) $(\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]} = \mathcal{B}^{[\lambda, \eta]} \sqcup \mathcal{Z}^{[\lambda, \eta]}$.
- (2) $(\mathcal{B} \sqcap \mathcal{Z})^{[\lambda, \eta]} = \mathcal{B}^{[\lambda, \eta]} \sqcap \mathcal{Z}^{[\lambda, \eta]}$.

Proof Since $\mathcal{B} \sqsubseteq \mathcal{B} \sqcup \mathcal{Z}$ and $\mathcal{Z} \sqsubseteq \mathcal{B} \sqcup \mathcal{Z}$, so by Proposition 5.3, $\mathcal{B}^{[\lambda, \eta]} \sqsubseteq (\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]}$ and $\mathcal{Z}^{[\lambda, \eta]} \sqsubseteq (\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]}$. Thus, $\mathcal{B}^{[\lambda, \eta]} \sqcup \mathcal{Z}^{[\lambda, \eta]} \sqsubseteq (\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]}$.

To prove the reverse inclusion, assume that $\mathfrak{g} \in (\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]}$. Then, $\tau_{\mathcal{B} \sqcup \mathcal{Z}}^{(i)} \geq \lambda$ and $\sigma_{\mathcal{B} \sqcup \mathcal{Z}}^{(i)} \leq \eta$, for all i . Now,

$$\begin{aligned} (\mathcal{B} \sqcup \mathcal{Z})^{[\lambda, \eta]} &= \{\mathfrak{g} : \tau_{\mathcal{B} \sqcup \mathcal{Z}}^{(i)}(\mathfrak{g}) \geq \lambda \ \& \ \sigma_{\mathcal{B} \sqcup \mathcal{Z}}^{(i)}(\mathfrak{g}) \leq \eta\} \\ &= \{\mathfrak{g} : \max\{\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}), \tau_{\mathcal{Z}}^{(i)}(\mathfrak{g})\} \\ &\geq \lambda \ \& \ \min\{\sigma_{\mathcal{B}}^{(i)}(\mathfrak{g}), \sigma_{\mathcal{Z}}^{(i)}(\mathfrak{g})\} \leq \eta\} \\ &= \{\mathfrak{g} : \tau_{\mathcal{B}}^{(i)}(\mathfrak{g}) \geq \lambda \ \& \ \sigma_{\mathcal{B}}^{(i)}(\mathfrak{g}) \leq \eta\} \sqcup \{\mathfrak{g} : \tau_{\mathcal{Z}}^{(i)}(\mathfrak{g}) \\ &\geq \lambda \ \& \ \sigma_{\mathcal{Z}}^{(i)}(\mathfrak{g}) \leq \eta\} \\ &\sqsubseteq \mathcal{B}^{[\lambda, \eta]} \sqcup \mathcal{Z}^{[\lambda, \eta]}. \end{aligned}$$

This proves (1). The proof of (2) may be furnished in the same fashion. □

Corollary 5.6 Let \mathcal{B} and \mathcal{Z} be two PmFSs. If λ, η are real numbers picked from $[0, 1]$, then

- (1) $(\mathcal{B} \sqcup \mathcal{Z})^{(\lambda, \eta)} = \mathcal{B}^{(\lambda, \eta)} \sqcup \mathcal{Z}^{(\lambda, \eta)}$.
- (2) $(\mathcal{B} \sqcup \mathcal{Z})_{[\lambda, \eta]} = \mathcal{B}_{[\lambda, \eta]} \sqcup \mathcal{Z}_{[\lambda, \eta]}$.
- (3) $(\mathcal{B} \sqcup \mathcal{Z})_{(\lambda, \eta)} = \mathcal{B}_{(\lambda, \eta)} \sqcup \mathcal{Z}_{(\lambda, \eta)}$.
- (4) $(\mathcal{B} \sqcap \mathcal{Z})^{(\lambda, \eta)} = \mathcal{B}^{(\lambda, \eta)} \sqcap \mathcal{Z}^{(\lambda, \eta)}$.
- (5) $(\mathcal{B} \sqcap \mathcal{Z})_{[\lambda, \eta]} = \mathcal{B}_{[\lambda, \eta]} \sqcap \mathcal{Z}_{[\lambda, \eta]}$.
- (6) $(\mathcal{B} \sqcap \mathcal{Z})_{(\lambda, \eta)} = \mathcal{B}_{(\lambda, \eta)} \sqcap \mathcal{Z}_{(\lambda, \eta)}$.

Remark The results presented in Proposition 5.5 and Corollary 5.6 may be extended to any finite number of PmFSs.

Operations based on necessity and possibility operators

We present some characteristics relevant to modal operators i.e. necessity and possibility operators defined for PmFSs in [17], in this section.

Proposition 6.1 Let \mathcal{B} be a PmFS defined over X , then

- (1) $\square \square \mathcal{B} = \square \mathcal{B}$.
- (2) $\diamond \diamond \mathcal{B} = \diamond \mathcal{B}$.
- (3) $\diamond \square \mathcal{B} = \square \mathcal{B}$.
- (4) $\square \diamond \mathcal{B} = \diamond \mathcal{B}$.
- (5) $(\square \mathcal{B}^c)^c = \diamond \mathcal{B}$.
- (6) $(\diamond \mathcal{B}^c)^c = \square \mathcal{B}$.

Proof Let

$$\mathcal{B} = \left\{ \frac{\mathfrak{g}}{\left(\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}), \sigma_{\mathcal{B}}^{(i)}(\mathfrak{g}) \right)} \right\}$$

be a PmFS defined over X .

- (1) By definition,

$$\begin{aligned} \square \mathcal{B} &= \left\{ \frac{\mathfrak{g}}{\left(\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}), \sqrt{1 - (\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}))^2} \right)} \right\} \\ \therefore \square \square \mathcal{B} &= \left\{ \frac{\mathfrak{g}}{\left(\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}), \sqrt{1 - (\tau_{\mathcal{B}}^{(i)}(\mathfrak{g}))^2} \right)} \right\} \\ &= \square \mathcal{B}. \end{aligned}$$

- (2) Similar to the proof of (1).

(3) By definition,

$$\begin{aligned} \Box B &= \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \\ \therefore \Diamond \Box B &= \left\{ \frac{g}{\left(\sqrt{1 - \left(\sqrt{1 - (\tau_B^{(i)}(g))^2} \right)^2}, \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \\ &= \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \\ &= \Box B. \end{aligned}$$

(4) Similar to the proof of (3).

(5) By definition

$$\begin{aligned} \Box B^c &= \Box \left\{ \frac{g}{\left(\sigma_B^{(i)}(g), \tau_B^{(i)}(g) \right)} \right\} \\ &= \left\{ \frac{g}{\left(\sigma_B^{(i)}(g), \sqrt{1 - (\sigma_B^{(i)}(g))^2} \right)} \right\} \\ \therefore (\Box B^c)^c &= \left\{ \frac{g}{\left(\sqrt{1 - (\sigma_B^{(i)}(g))^2}, \sigma_B^{(i)}(g) \right)} \right\} \\ &= \Diamond B. \end{aligned}$$

(6) Parallel to the proof of (5).

Proposition 6.2 Take B and Z as PmFSSs. Then,

- (1) $\Box(B \sqcup Z) = \Box B \sqcup \Box Z$.
- (2) $\Box(B \sqcap Z) = \Box B \sqcap \Box Z$.
- (3) $\Diamond(B \sqcup Z) = \Diamond B \sqcup \Diamond Z$.
- (4) $\Diamond(B \sqcap Z) = \Diamond B \sqcap \Diamond Z$.

Proof Assume that $\max(\tau_B^{(i)}(g), \tau_Z^{(i)}(g)) = \tau_B^{(i)}(g)$. The other case i.e. $\max(\tau_B^{(i)}(g), \tau_Z^{(i)}(g)) = \tau_Z^{(i)}(g)$ may be discussed in the similar way.

By definition,

$$\begin{aligned} \Box(B \sqcup Z) &= \Box \left\{ \frac{g}{\left(\max(\tau_B^{(i)}(g), \tau_Z^{(i)}(g)), \min(\sigma_B^{(i)}(g), \sigma_Z^{(i)}(g)) \right)} \right\} \\ &= \Box \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \min(\sigma_B^{(i)}(g), \sigma_Z^{(i)}(g)) \right)} \right\} \\ &= \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \end{aligned}$$

and

$$\begin{aligned} \Box B \sqcup \Box Z &= \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \\ \sqcup &\left\{ \frac{g}{\left(\tau_Z^{(i)}(g), \sqrt{1 - (\tau_Z^{(i)}(g))^2} \right)} \right\} \\ &= \left\{ \frac{g}{\left(\tau_B^{(i)}(g), \sqrt{1 - (\tau_B^{(i)}(g))^2} \right)} \right\} \end{aligned}$$

because $\min\left(\sqrt{1 - (\tau_B^{(i)}(g))^2}, \sqrt{1 - (\tau_Z^{(i)}(g))^2}\right) = \sqrt{1 - (\tau_B^{(i)}(g))^2}$.

This proves (1). The proofs of other three assertions may be furnished on the parallel track. \square

Proposition 6.3 Presume that B and Z are PmFSSs. Then,

- (1) $\Box \Box(B \sqcup Z) = \Box(B \sqcup Z)$.
- (2) $\Box \Box(B \sqcap Z) = \Box(B \sqcap Z)$.
- (3) $\Diamond \Diamond(B \sqcup Z) = \Diamond(B \sqcup Z)$.
- (4) $\Diamond \Diamond(B \sqcap Z) = \Diamond(B \sqcap Z)$.
- (5) $\Diamond \Box(B \sqcup Z) = \Box(B \sqcup Z)$.
- (6) $\Diamond \Box(B \sqcap Z) = \Box(B \sqcap Z)$.
- (7) $\Box \Diamond(B \sqcup Z) = \Diamond(B \sqcup Z)$.
- (8) $\Box \Diamond(B \sqcap Z) = \Diamond(B \sqcap Z)$.

Proof Follows from Propositions 6.1 and 6.2. \square

Proposition 6.4 Let B and Z be a PmFSSs defined over X , then

- (1) $(\Box(B \sqcup Z)^c)^c = \Diamond(B \sqcup Z)$.
- (2) $(\Box(B \sqcap Z)^c)^c = \Diamond(B \sqcap Z)$.
- (3) $(\Diamond(B \sqcup Z)^c)^c = \Box(B \sqcup Z)$.
- (4) $(\Diamond(B \sqcap Z)^c)^c = \Box(B \sqcap Z)$.

Proof Takes after from Proposition 6.1. \square

Proposition 6.5 Assume that B and Z are PmFSSs defined over X , then

- (1) $(\Box(B \sqcup Z)^c)^c = \Diamond \Diamond(B \sqcup Z) = \Box \Diamond(B \sqcup Z)$.
- (2) $(\Box(B \sqcap Z)^c)^c = \Diamond \Diamond(B \sqcap Z) = \Box \Diamond(B \sqcap Z)$.
- (3) $\Diamond(\Box(B \sqcup Z)^c)^c = \Box \Box(B \sqcup Z) = \Diamond \Box(B \sqcup Z)$.
- (4) $\Diamond(\Box(B \sqcap Z)^c)^c = \Box \Box(B \sqcap Z) = \Diamond \Box(B \sqcap Z)$.

Proof Follows from Propositions 6.1, 6.3 and 6.4. \square

Extension principle of PmFSs

In this part, we present the extension principle of PmFSs.

Definition 7.1 Let $\mathcal{U} : X \mapsto Y$ be a map connecting non-void crisp sets X and Y . Assume further that β_1 and β_2 are PmFSs defined over X and Y , respectively.

- (1) The *inverse image* of β_2 under \mathcal{U} is a PmFS over X and is characterized as

$$\begin{aligned} \mathcal{U}^{-1}(\beta_2) &= \left\{ \frac{\mathfrak{g}}{\left(\tau_{\mathcal{U}^{-1}(\beta_2)}^{(i)}(\mathfrak{g}), \sigma_{\mathcal{U}^{-1}(\beta_2)}^{(i)}(\mathfrak{g}) \right)} \right\} \\ &= \left\{ \frac{\mathfrak{g}}{\left(\tau_{\beta_2}^{(i)}(\mathcal{U}(\mathfrak{g})), \sigma_{\beta_2}^{(i)}(\mathcal{U}(\mathfrak{g})) \right)} \right\} \end{aligned}$$

- (2) The *image* of β_1 under \mathcal{U} is a PmFS over Y and is characterized as

$$\mathcal{U}(\beta_1) = \left\{ \frac{\mathfrak{g}}{\left(\tau_{\mathcal{U}(\beta_1)}^{(i)}(\mathfrak{g}'), \sigma_{\mathcal{U}(\beta_1)}^{(i)}(\mathfrak{g}') \right)} \right\}$$

where

$$\tau_{\mathcal{U}(\beta_1)}^{(i)}(\mathfrak{g}') = \begin{cases} \bigvee_{\mathfrak{g} \in \mathcal{U}^{-1}(\mathfrak{g}')} \tau_{\beta_1}^{(i)}(\mathfrak{g}), & \text{if } \mathcal{U}^{-1}(\mathfrak{g}') \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma_{\mathcal{U}(\beta_1)}^{(i)}(\mathfrak{g}') = \begin{cases} \bigwedge_{\mathfrak{g} \in \mathcal{U}^{-1}(\mathfrak{g}')} \sigma_{\beta_1}^{(i)}(\mathfrak{g}), & \text{if } \mathcal{U}^{-1}(\mathfrak{g}') \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Proposition 7.2 Let \mathcal{U} be a map driving members of X to Y . If β and \mathcal{Z} are PmFSs defined, respectively, over X and Y , then there exist at least one λ, η in $[0, 1]$ in such a way that

- (1) $\mathcal{U}(\beta^{(\lambda, \eta)}) \subseteq \mathcal{U}(\beta^{[\lambda, \eta]}) \subseteq (\mathcal{U}(\beta))^{[\lambda, \eta]}$.
- (2) $\mathcal{U}^{-1}(\mathcal{Z}^{(\lambda, \eta)}) \subseteq \mathcal{U}^{-1}(\mathcal{Z}^{[\lambda, \eta]}) = (\mathcal{U}^{-1}(\mathcal{Z}))^{[\lambda, \eta]}$.

Proof (1) Since $\beta^{(\lambda, \eta)} \subseteq \beta^{[\lambda, \eta]}$, so $\mathcal{U}(\beta^{(\lambda, \eta)}) \subseteq \mathcal{U}(\beta^{[\lambda, \eta]})$ is trivial. For the next inclusion, assume that $\mathfrak{g}' \in \mathcal{U}(\beta^{[\lambda, \eta]})$. Then, there subsists $\mathfrak{g} \in \beta^{[\lambda, \eta]}$ so that $\mathfrak{g}' = \mathcal{U}(\mathfrak{g})$ and $\tau_{\beta}^{(i)}(\mathfrak{g}) \geq \lambda, \sigma_{\beta}^{(i)}(\mathfrak{g}) \leq \eta$ for all i . Thus, for $\lambda, \eta \in [0, 1]$ and all i , we have

$$\tau_{\beta}^{(i)}(\mathcal{U}^{-1}(\mathfrak{g}')) \geq \lambda \Rightarrow \tau_{\mathcal{U}(\beta)}^{(i)}(\mathfrak{g}') \geq \lambda$$

and

$$\sigma_{\beta}^{(i)}(\mathcal{U}^{-1}(\mathfrak{g}')) \leq \eta \Rightarrow \sigma_{\mathcal{U}(\beta)}^{(i)}(\mathfrak{g}') \leq \eta$$

So, $\mathfrak{g}' \in (\mathcal{U}(\beta))^{[\lambda, \eta]}$ and hence $\mathcal{U}(\beta^{[\lambda, \eta]}) \subseteq (\mathcal{U}(\beta))^{[\lambda, \eta]}$.

- (2) Since $\mathcal{Z}^{(\lambda, \eta)} \subseteq \mathcal{Z}^{[\lambda, \eta]}$, so $\mathcal{U}^{-1}(\mathcal{Z}^{(\lambda, \eta)}) \subseteq \mathcal{U}^{-1}(\mathcal{Z}^{[\lambda, \eta]})$ is trivial.

Now, for each \mathfrak{g}

$$\begin{aligned} \mathfrak{g} \in \mathcal{U}^{-1}(\mathcal{Z}^{[\lambda, \eta]}) &\Leftrightarrow f(\mathfrak{g}) \in \mathcal{Z}^{[\lambda, \eta]} \\ &\Leftrightarrow \tau_{\mathcal{Z}}^{(i)}(\mathcal{U}(\mathfrak{g})) \geq \lambda \ \& \ \sigma_{\mathcal{Z}}^{(i)}(\mathcal{U}(\mathfrak{g})) \leq \eta \end{aligned}$$

Thus,

$$\tau_{\mathcal{U}^{-1}(\mathcal{Z})}^{(i)}(\mathfrak{g}) = \tau_{\mathcal{Z}}^{(i)}(\mathcal{U}(\mathfrak{g})) \geq \lambda$$

and

$$\sigma_{\mathcal{U}^{-1}(\mathcal{Z})}^{(i)}(\mathfrak{g}) = \sigma_{\mathcal{Z}}^{(i)}(\mathcal{U}(\mathfrak{g})) \leq \eta$$

showing that $\mathfrak{g} \in (\mathcal{U}^{-1}(\mathcal{Z}))^{[\lambda, \eta]}$. Thus, $\mathcal{U}^{-1}(\mathcal{Z}^{[\lambda, \eta]}) = (\mathcal{U}^{-1}(\mathcal{Z}))^{[\lambda, \eta]}$. □

Corollary 7.3 Let \mathcal{U} be a map that drags members of X to Y . If β and \mathcal{Z} are PmFSs defined over X and Y in order, then there exist at least one $\lambda, \eta \in [0, 1]$ so that

- (1) $\mathcal{U}(\beta^{(\lambda, \eta)}) \subseteq \mathcal{U}(\beta^{[\lambda, \eta]}) \subseteq (\mathcal{U}(\beta))^{[\lambda, \eta]}$.
- (2) $\mathcal{U}^{-1}(\mathcal{Z}^{(\lambda, \eta)}) \subseteq \mathcal{U}^{-1}(\mathcal{Z}^{[\lambda, \eta]}) = (\mathcal{U}^{-1}(\mathcal{Z}))^{[\lambda, \eta]}$.

Relations on PmFSs

We devote this segment to present some concepts of Pythagorean m -polar fuzzy relation and some its premier characteristics.

Definition 8.1 Take X and Y as non-void classical sets. A *Pythagorean m -polar fuzzy relation (PmFR)* R from X to Y is a PmFS of the Cartesian product $X \times Y$ categorized by the affiliation functions $\tau_R^{(i)} : X \times Y \mapsto [0, 1]^m$ and non-membership functions $\sigma_R^{(i)} : X \times Y \mapsto [0, 1]^m$. We express it as $R(X \leftrightarrow Y)$.

Definition 8.2 Presume that X and Y are non-void classical sets. Assume that β is a PmFS over X . The *max-min-max composition* of $R(X \leftrightarrow Y)$ with β is a PmFS \mathcal{Z} of Y , and is denoted as $\mathcal{Z} = R \circ \beta$. The affiliation and dissociation functions of this composition, in order, are characterized as

$$\tau_{\mathcal{Z}}^{(i)}(\mathfrak{g}') = \max_{\mathfrak{g}} \left(\min \left\{ \tau_{\beta}^{(i)}(\mathfrak{g}), \tau_R^{(i)}(\mathfrak{g}, \mathfrak{g}') \right\} \right)$$

and

$$\sigma_{\mathcal{Z}}^{(i)}(\mathfrak{g}') = \min_{\mathfrak{g}} \left(\max \left\{ \sigma_{\beta}^{(i)}(\mathfrak{g}), \sigma_R^{(i)}(\mathfrak{g}, \mathfrak{g}') \right\} \right)$$

for all $g \in X$ and $g' \in Y$.

This composition may alternatively be computed using

$$R = \tau_R^{(i)}(g') - \sigma_R^{(i)}(g')\varepsilon_R^{(i)}(g')$$

Definition 8.3 Assume that $Q(X \leftrightarrow Y)$ and $R(Y \leftrightarrow Z)$ be PmFRs. The *max-min-max composition* $R \circ Q$ is again a PmFR from X to Z . The affiliation and dissociation functions of this composition, in order, are

$$\tau_{R \circ Q}^{(i)}(g, g'') = \max_{g'} \left(\min \left\{ \tau_Q^{(i)}(g, g'), \tau_R^{(i)}(g', g'') \right\} \right)$$

and

$$\sigma_{R \circ Q}^{(i)}(g, g'') = \min_{g'} \left(\max \left\{ \sigma_Q^{(i)}(g, g'), \sigma_R^{(i)}(g', g'') \right\} \right)$$

for all $g \in X$, $g' \in Y$ and $g'' \in Z$.

This composition may alternatively be computed using

$$R \circ Q = \tau_{R \circ Q}^{(i)}(g, g'') - \sigma_{R \circ Q}^{(i)}(g, g'')\varepsilon_{R \circ Q}^{(i)}(g, g'')$$

Proposition 8.4 Let $R_1(X \leftrightarrow Y)$ and $R_2(Y \leftrightarrow Z)$ be two PmFRs, then

- (1) $(R_1^{-1})^{-1} = R_1$.
- (2) $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$.

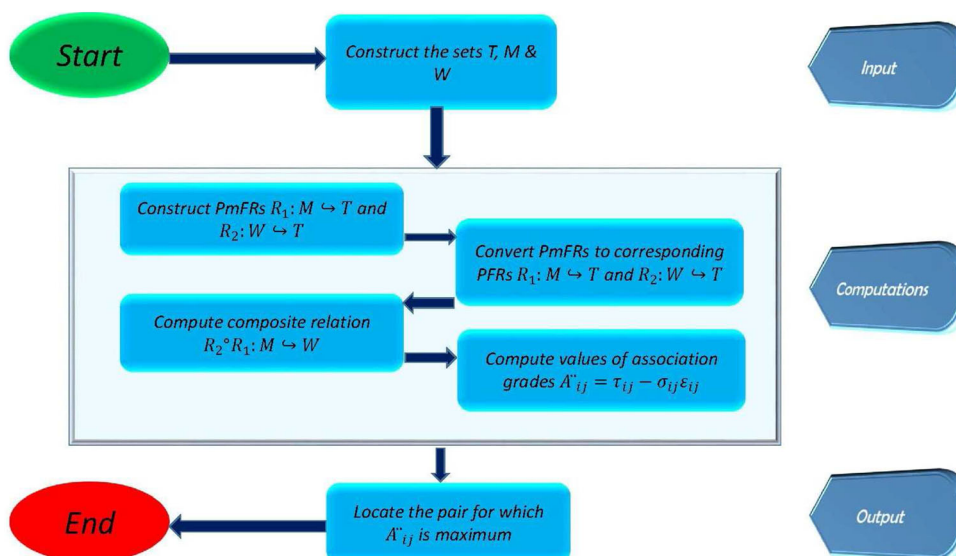
Application of PmFR in choosing life partner

Most of us agree to take a long list of traits that designate our impeccable match, from general traits – cool, sympathetic, curious, exploratory, appreciative – to specific expertise and pastimes – good cook, delight in cricket, civically full of zip, be fond of traveling. But we recognize that we can't detect everything in one person. We hold to reach some compromises by believing that real-world is entirely different from the dreams we see either in closed or open eyes. And then what are the most important things to prioritize if one wants to have a happy and successful relationship? Times of study into relationship consummation and durability points to a number of key wherewithal one may be capable to be familiar with early on:

1. **Trustworthiness:** Trust is the grounds of a successful relationship. Without faith, there is nothing. Indeed, the quality of trustworthiness strengthens a relation.
2. **Clear communication:** It is what leads to developing trust and credibility. Expect for this in a collaborator. Communication is vital—many people have accepted this, because it's honest.
3. **Optimism:** A person who is endowed with ability to see good in things is usually a happy demeanour. The couples who share optimism practice inordinate triumph in problem solving than the pessimists one. Optimism agrees to each individual to donate proficient coping strategies, and higher degrees of cooperation.
4. **Respect:** Respect for others is the key ingredient for a successful and happy life. If a person does not abide by it, he/she is not serious enough for you.
5. **Loves your flaws:** The person you espouse should, undeniably, embellish your fortes, but also escalate your imperfections. It might sound trite, but it's your twist of fate that makes you ... You.
6. **Honesty and integrity:** The epitome partner comprehends the significance of honesty in a faithful relationship. Honesty physiquies trust between people. Dishonesty clouds the other person, betraying their susceptibility and devastating their sense of veracity. Nix conveys a more disparaging impact on a comfortable relationship between two people than untruthfulness and deceit. Even in agonizing circumstances such as unfaithfulness, the flagrant deception involved is every so often equally, if not more, upsetting than the unfaithful act itself. The idyllic partner does its best to sustain a lifetime of totality so that there are no incongruities between words and actions. This works for all layers of communication, both verbal and gestural. Being unblemished and loyal in our most cherished affairs means really knowing ourselves and our commitments. While this can prove hard, it is a struggle worth striving for.



Fig. 1 Flowchart of the algorithm



Before proceeding further to numerical example, we propose an algorithm based upon PmFR as follows:

Algorithm

Input:

1. Collection of elementary data: Construct the set of traits T with the assistance of team of experts. Suppose that the families between whom association level is to be checked are M and W .

Computations:

2. Construct the PmFRs $R_1 : M \leftrightarrow T$ and $R_2 : W \leftrightarrow T$ based on repeated observations, interviews or questionnaires.
3. Convert the PmFRs constructed at step 2 to corresponding PFRs $R_1 : M \leftrightarrow T$ and $R_2 : W \leftrightarrow T$ by taking average values of the observations for each trait.
4. Using max-min-max composition given in Definition 8.3, compute the composite relation $R_2 \circ R_1 : M \leftrightarrow W$.
5. Compute the values of association grades between the members of M and W using $\ddot{A}_{ij} = \tau_{ij} - \sigma_{ij}\epsilon_{ij}$, where $\epsilon_{ij} = \sqrt{1 - \tau_{ij}^2 - \sigma_{ij}^2}$.

Output:

6. Locate the pair for which the value of \ddot{A}_{ij} goes beyond a specific number and declare it as best associated.

Figure 1 portrays this algorithm.

We employ this algorithm on the following example.

Example 9.1 Consider an NGO with the aim to give best match for the people planning to marry. The NGO has a list of 4 men and 4 women registered for possible matrimony. The officials of the NGO had discussion with each of these 8 members thrice at different spans of time and try to get their insight views about different traits thought to be vital

for a happy and successful matrimony. The officials constitute a team of experts to decide the fate of registered persons for possible likelihood. The team prepares PmFSs showing degrees of association $\tau^{(i)}$ and non-association $\sigma^{(i)}$ for a specific trait. Assume that the traits under consideration are

- t = Trustworthiness
- c = Clear communication
- r = Respect & Loves your flaws
- o = Optimism
- h = Honesty & Integrity

whose collection is designated by T . If $M = \{m_1 = \text{Paul}, m_2 = \text{Steve}, m_3 = \text{Kevin}, m_4 = \text{Carl}\}$ is the aggregate of men whereas $W = \{w_1 = \text{Emma}, w_2 = \text{Jane}, w_3 = \text{Angelica}, w_4 = \text{Sussana}\}$ is the assembly of women under study, then PmFR $R_1 : M \leftrightarrow T$ is as in Table 5.

and PmFR $R_2 : T \leftrightarrow W$ is as in Table 6:

For computational ease, we construct the tables of average values given in Tables 5 and 6, which convert PmFR to Pythagorean fuzzy relation (PFR), and represent in Tables 7 and 8 respectively.

Invoking max–min–max composition presented in Definition 8.3, the PFNs for the composite relation $R_2 \circ R_1 : M \leftrightarrow W$ is represented in Table 9.

Now, using $\ddot{A}_{ij} = \tau_{ij} - \sigma_{ij}\epsilon_{ij}$, where $\epsilon_{ij} = \sqrt{1 - \tau_{ij}^2 - \sigma_{ij}^2}$ denotes hesitation margin, the association grades between m_i and w_j are computed in Table 10.

The statistics computed in Table 10 are displayed in Fig. 2.

Assume that the condition for best suitable couple is that the association grade must be greater than 50% i.e. $\ddot{A}_{ij} > 0.5$. Thus, in view of Table 10, it may be concluded that (m_4, w_3) i.e. (Carl, Angelica) is the ideal couple. As long as the other

Table 5 PmFR $R_1 : M \leftrightarrow T$

R_1	t	c	r	o	h
m_1	(0.56, 0.45)	(0.37, 0.29)	(0.67, 0.55)	(0.51, 0.24)	(0.35, 0.25)
	(0.32, 0.26)	(0.31, 0.51)	(0.51, 0.28)	(0.26, 0.29)	(0.46, 0.46)
	(0.76, 0.58)	(0.47, 0.81)	(0.54, 0.79)	(0.82, 0.39)	(0.82, 0.49)
m_2	(0.79, 0.54)	(0.52, 0.47)	(0.98, 0.12)	(0.43, 0.20)	(0.68, 0.51)
	(0.23, 0.48)	(0.74, 0.42)	(0.19, 0.52)	(0.39, 0.01)	(0.59, 0.50)
	(0.57, 0.59)	(0.43, 0.78)	(0.06, 0.53)	(0.81, 0.18)	(0.44, 0.70)
m_3	(0.83, 0.45)	(0.53, 0.31)	(0.51, 0.50)	(0.86, 0.41)	(0.43, 0.34)
	(0.29, 0.32)	(0.43, 0.53)	(0.51, 0.68)	(0.37, 0.11)	(0.56, 0.61)
	(0.48, 0.69)	(0.79, 0.54)	(0.69, 0.52)	(0.72, 0.35)	(0.71, 0.68)
m_4	(0.59, 0.45)	(0.58, 0.39)	(0.56, 0.52)	(0.96, 0.21)	(0.40, 0.31)
	(0.54, 0.51)	(0.34, 0.57)	(0.50, 0.50)	(0.87, 0.10)	(0.51, 0.60)
	(0.64, 0.67)	(0.49, 0.44)	(0.67, 0.43)	(0.65, 0.38)	(0.58, 0.54)

Table 6 PmFR $R_2 : T \leftrightarrow W$

R_2	t	c	r	o	h
w_1	(0.79, 0.58)	(0.46, 0.31)	(0.53, 0.49)	(0.48, 0.53)	(0.52, 0.51)
	(0.68, 0.54)	(0.56, 0.30)	(0.40, 0.63)	(0.12, 0.25)	(0.38, 0.32)
	(0.48, 0.50)	(0.69, 0.70)	(0.75, 0.60)	(0.16, 0.50)	(0.85, 0.49)
w_2	(0.56, 0.54)	(0.79, 0.43)	(0.88, 0.42)	(0.91, 0.37)	(0.38, 0.90)
	(0.48, 0.46)	(0.43, 0.11)	(0.20, 0.18)	(0.21, 0.11)	(0.32, 0.78)
	(0.68, 0.58)	(0.44, 0.54)	(0.43, 0.44)	(0.36, 0.42)	(0.07, 0.12)
w_3	(0.59, 0.54)	(0.86, 0.41)	(0.66, 0.70)	(0.96, 0.21)	(0.95, 0.28)
	(0.79, 0.68)	(0.43, 0.29)	(0.56, 0.59)	(0.82, 0.11)	(0.86, 0.24)
	(0.55, 0.61)	(0.79, 0.51)	(0.48, 0.49)	(0.45, 0.44)	(0.47, 0.39)
w_4	(0.58, 0.54)	(0.47, 0.83)	(0.99, 0.12)	(0.42, 0.31)	(0.55, 0.42)
	(0.48, 0.31)	(0.32, 0.51)	(0.84, 0.16)	(0.41, 0.37)	(0.53, 0.57)
	(0.57, 0.64)	(0.54, 0.55)	(0.58, 0.62)	(0.71, 0.68)	(0.50, 0.49)

Table 7 PFR $R_1 : M \leftrightarrow T$

R_1	t	c	r	o	h
m_1	(0.55, 0.43)	(0.38, 0.54)	(0.57, 0.54)	(0.53, 0.31)	(0.54, 0.40)
m_2	(0.53, 0.54)	(0.56, 0.56)	(0.41, 0.39)	(0.54, 0.13)	(0.57, 0.57)
m_3	(0.53, 0.49)	(0.58, 0.46)	(0.57, 0.57)	(0.65, 0.29)	(0.57, 0.54)
m_4	(0.59, 0.54)	(0.47, 0.47)	(0.58, 0.48)	(0.83, 0.23)	(0.50, 0.48)

Table 8 PFR $R_2 : T \leftrightarrow W$

R_2	t	c	r	o	h
w_1	(0.65, 0.54)	(0.57, 0.44)	(0.56, 0.57)	(0.25, 0.43)	(0.58, 0.44)
w_2	(0.57, 0.53)	(0.55, 0.36)	(0.50, 0.35)	(0.49, 0.30)	(0.26, 0.60)
w_3	(0.64, 0.61)	(0.69, 0.59)	(0.57, 0.59)	(0.74, 0.25)	(0.76, 0.30)
w_4	(0.54, 0.50)	(0.44, 0.63)	(0.80, 0.30)	(0.51, 0.45)	(0.53, 0.49)

Table 9 PFR
 $R_2 \circ R_1 : M \leftrightarrow W$

$R_2 \circ R_1$	w_1	w_2	w_3	w_4
m_1	(0.56, 0.43)	(0.55, 0.31)	(0.57, 0.31)	(0.57, 0.45)
m_2	(0.57, 0.30)	(0.55, 0.30)	(0.57, 0.25)	(0.53, 0.39)
m_3	(0.57, 0.43)	(0.55, 0.30)	(0.58, 0.29)	(0.57, 0.45)
m_4	(0.59, 0.43)	(0.57, 0.30)	(0.74, 0.25)	(0.58, 0.45)

Fig. 2 Association grades between the two genders

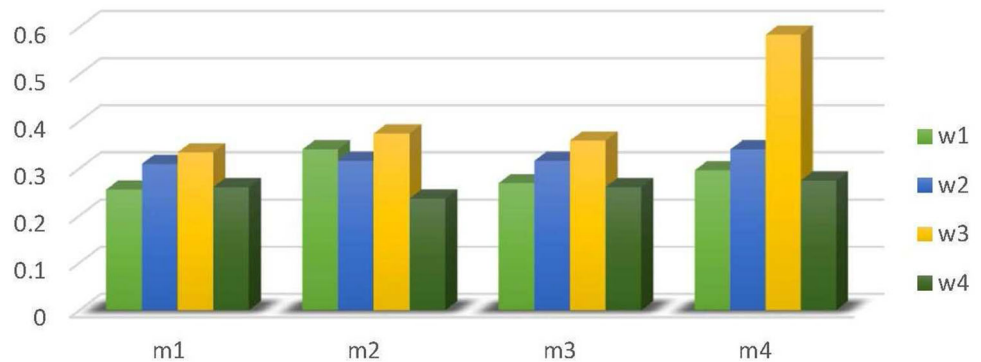


Table 10 Association grades \ddot{A}_{ij} between m_i and w_j

\ddot{A}_{ij}	w_1	w_2	w_3	w_4
m_1	0.25549	0.30960	0.33411	0.26064
m_2	0.34052	0.31617	0.37433	0.23633
m_3	0.26894	0.31617	0.35924	0.26064
m_4	0.29615	0.34052	0.58390	0.27443

Table 11 Values of score function s_{ij} between m_i and w_j

s_{ij}	w_1	w_2	w_3	w_4
m_1	0.12870	0.20640	0.22880	0.12240
m_2	0.23490	0.21250	0.26240	0.12880
m_3	0.14000	0.21250	0.25230	0.12240
m_4	0.16320	0.23490	0.48510	0.13390

members are concerned, they are not suitable for each other for living an ideal matrimony.

For the conformation of the results obtained in Table 10, we compute the score values s_{ij} , using $s_{ij} = \tau_{ij}^2 - \sigma_{ij}^2$, between m_i and w_j represented in that table in the form of PFNs. The results so computed are displayed in Table 11.

It may be observed from Table 11 that the highest value of s_{ij} occurs for (m_4, w_3) , which supports the conclusion drawn through our proposed algorithm. Hence, the proposed algorithm is reliable and yields stable results.

Superiority of proposed algorithm

In this subsection, we discuss how our proposed Algorithm is superior to some existing models.

- In [30], the authors used soft rough Pythagorean m -polar fuzzy sets and Pythagorean m -polar fuzzy soft rough sets for modeling uncertainties. The restrictions on the model presented model are different from the one used in present article.
- In [17], the authors floated the conception of Pythagorean m -polar fuzzy sets. We have extended the operations presented in that paper. Further, the authors have used TOPSIS which suits in the application presented in that paper.
- In [1], the authors have used Pythagorean fuzzy sets as model to deal with uncertainty which does not deal with multi-polarity. The model used in our article deals with multi-polarity as well.
- In [12], the authors have used Pythagorean fuzzy sets as model to deal with uncertainty which does not deal with multi-polarity. The model used in our article deals with multi-polarity as well.
- The model presented in [3], differs from the model proposed in our article in the sense that our model has the edge that it also deals with multi-polar information.
- In [13], the authors have used Pythagorean fuzzy sets to deal with uncertain information. The element of multi-polarity is missing in that model too.
- In [2], the authors used m -polar fuzzy graphs/set which deals with multi-polar information about membership function only. The role of membership function is entirely ignored in that model.

The techniques used in the above mentioned papers either can't handle or become very complex in tackling the problem proposed in application section. For example, in TOPSIS, a

single alternative is to be selected by one or a team of decision experts. In our proposed application, members of both the sets M and W act as decision takers as well as alternatives. Indeed, we tried to find “correlation” amongst members of M and W using uncertain multi-polar information. Hence, if there is no other compatible method, then the question of comparing our proposed technique with any other method diminishes.

Conclusion

The modal operators of necessity and possibility are very nifty in analytic reasoning and logic. The notions of score and accuracy functions are employed in decision taking problems, especially when using aggregation operators. The notions of λ - and (λ, η) -cuts are used in artificial intelligence. $PmFS$ s have the beauty that they amenably handle multi-polar imprecise information with enhanced space for selection of association and dissociation degrees. We established some new results and coined some novel notions relevant to Pythagorean m -polar fuzzy sets including sum \oplus , product \otimes , union \sqcup , intersection \sqcap and complement. We defined score and accuracy functions for $PmFS$ s along with their prime characteristics. Different types of (λ, η) -cuts with their properties have also been brought into light. Operations based on necessity and possibility operators have been discussed with their properties. Extension principle of $PmFS$ s has also been made part of the discussion. We defined Pythagorean m -polar fuzzy relation ($PmFR$). We proposed an algorithm based on $PmFR$ s and rendered a robust decision making application of $PmFR$ in the selection of life partner. The results so obtained are compared with the results obtained using score values and demonstrated that the optimal choice does not differ. The association grades between the two masculinities have been portrayed with the assistance of statistical chart. A number of illustrations, where required, have been included to perceive the notions presented with ease.

The results presented in this paper are also valid for m -polar fuzzy sets, intuitionistic m -polar fuzzy sets and bipolar fuzzy sets. The notions coined have the potential to be extended to q -rung orthopair m -polar fuzzy sets, hesitant fuzzy sets, neutrosophic sets and the soft extensions of all of these structures. Keeping aside the notional and theoretic features, the ideas unveiled have enough potential to be expanded in coping intelligently with day to day life situations including business and trade analysis, robotics, social sciences, life sciences, agricultural sciences, human resource management, pattern recognition, water management, medicine, economics, energy crisis problems, recruit problems and many other fields of practical usage. We expect that this arti-

cle will serve as a helping hand for the active researchers working in this area.

Author contributions The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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