



Group-based generalized q-rung orthopair average aggregation operators and their applications in multi-criteria decision making

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Abstract

The objective of this manuscript is to investigate the concept of generalized q-rung orthopair fuzzy sets ($Gq-ROFSs$) and group generalized q-rung orthopair fuzzy sets ($GGq-ROFSs$) by incorporating the concept of generalized parameter and group generalized parameters in q-rung orthopair fuzzy environment. The main advantage of generalized parameter in q-rung orthopair fuzzy environment is to reduce uncertain errors in the original information to ensure the expert's level of trust and improve the accuracy of final decision. On the base of generalized parameter, some aggregation operators are introduced such as generalized q-rung orthopair fuzzy average aggregation operators and group generalized q-rung orthopair fuzzy average aggregation operators and studied their related properties. Furthermore, a multi-criteria decision-making method technique based on proposed approach is presented. Finally, a numerical example is provided to illustrate the feasibility of the proposed methods and deliver the sensitivity analysis and comparative analysis, which show the superiority of developed approach than existing methods.

Keywords q-rung orthopair fuzzy sets · $Gq-ROFSs$ · $GGq-ROFSs$ · $Gq-ROF$ aggregation operators · $GGq-ROF$ aggregation operators · MCDM

Introduction

In real life, decision making plays a significant role and is commonly used to solve real-world problems. Due to the rapid development of human society in technology and many other fields, decision making is a tedious task for the experts to take an intelligent decision. Decision making is a pre-plan process of selecting the logical choice among several objects. A good decision can change the course of life style. A

decision maker judges the limitations, advantages and characteristics of each alternatives, and then he could reach to the final decision. In recent era, it becomes difficult for an individual expert to cope with all the decision making information. In many situations of real world, multi-experts are needed for decision making. In recent scenario, multi-criteria decision-making (MCDM) methods play a prominent role in modern decision-making environment, in which one of the most important methods is aggregation operators. Aggregation operators collectively aggregate the attributes and then rank the alternatives to get the most suitable choice. To cope with this uncertain and complex environment Atanassov [3] originated the pioneer notion of intuitionistic fuzzy set (IFS) which deals with the uncertain and complex situations in a better way than the dominant concept of Zadeh [38] fuzzy sets. IFS is characterized in two parts that is membership grade and non-membership grade and this concept based on the sum of membership grade and nonmembership grade must belong to $[0, 1]$. Many scholars extended the concept of IFS in several directions in which one of them is aggregation operators which collectively aggregate the information and it is a significant process of decision making. The weighted operators by Yager [30] and ordered weighted aggregation

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operators initiated by Yager and Kacprzyk [35] are used for the fusion of data. Xu [28] originated the concept of IF weighted average ($IFWA$) operator, IF ordered weighted average ($IFOWA$) operator, IF hybrid average ($IFHA$) operator. Feng et al. [5] presented that the existing definition of generalized intuitionistic fuzzy soft sets is clarified and reformulated as a combination of an intuitionistic fuzzy soft set over the universe of discourse and an intuitionistic fuzzy set in the parameter set. Feng et al. [6] presented a number of lexicographic orders by means of several measures such as the membership, non-membership, score, accuracy and expectation score functions. Some equivalent characterizations and illustrative examples are provided, from which the relationships among these lexicographic orders are ascertained. Ali et al. [2] proposed a graphical ranking method based on the uncertainty index and entropy. For the detail and comprehensive analysis of different aggregation operators in the domain of IFSs are given in [7, 11, 18, 29, 36, 37]. To cope the shortcomings of existing decision-making problems, Zhang et al. [39] investigated the concept of J-divergence and evidential reasoning theory under IF environment and for details, see [4, 20, 23]. However, scholars point out a quite few situations where the dominant concept of IFS failed to cope the circumstances. To handle this deficiency, Yager [31] initiated the remarkable notion of Pythagorean fuzzy (PFS) set (PFS), whose influential characteristic consists of square sum of membership grade and non-membership belongs to $[0, 1]$. On the basis of PFS [31], Yager and Abbasov [34] presented the notion of Pythagorean membership grade in PFS and they showed their application in decision making. Since the appearance of PFS many practitioners widely extended this remarkable concept in different directions such as, Yager [32] presented the idea of PFS weighted average ($PFWA$) operator and PFS weighted power average operator to aggregate the information, Peng and Yang [24] presented the detail study of division and subtraction operations in PFS environment, Peng and Yuan [25] investigated the fundamental properties of point aggregation operators and their applications in MCDM, Garg presented the concepts of various aggregation operators in [8–10], Ma and Xu [22] initiated the notion of symmetric PFS weighted averaging and geometric ($SPFWA/G$) operators, Hussain et al. [16] presented rough PFS ideals in semigroups, Joshi [17] presented the combine study of generalized parameter and $PFWA$ operators to construct the concept of generalized $PFWA$ ($GPFWA$), generalized PFS ordered weighted average ($GPFOWA$), and generalized PFS hybrid average ($GPFHA$) operators and Hussain et al. [14] proposed the concept of PFS soft rough set and their desirable properties with detail.

However, in real-life situations, sometimes it is difficult, to take an intelligent decision becomes tedious for the scholars and to point out a quite few situations where the dominant concept of PFS s failed to cope the circumstances. For exam-

ple, if the decision maker/expert assigns membership grade 0.8 and non-membership 0.61, then $(0.8)^2 + (0.61)^2 > 1$. Recently, to cope with this shortcoming Yager [33] initiated the extensive idea of q -rung orthopair fuzzy ($q-ROFS$) set ($q-ROFS$), whose prominent characteristics consists of sum of q th power of membership grade and q th power of non-membership belongs to $[0, 1]$ for $q \geq 1$. So in this case, $(0.8)^q + (0.61)^q < 1$ for $q \geq 3$. It is also observed that the feature of $q-ROFS$ is more stronger than IFS and PFS , so it is clear that $q-ROFS$ is a useful generalization of both IFS and PFS . After the initiation of $q-ROFS$, quite few contributions to this concept are found in the literature. Ali [1] initiated two new approaches such as L-fuzzy sets and the notion of orbit in $q-ROFS$ environment. Hussain et al. [12] presented covering-based $q-ROFS$ rough set hybrid with TOPSIS for multi-attribute decision making. Liu and Wang [21] extended the existing approach of aggregation operators to $q-ROFS$ environment to get the $q-ROFS$ weighted average/geometric ($q-ROFWA/G$) operators and proved their properties. Liu and Liu [19] found the relation between Bonferroni mean operators and $q-ROFS$ numbers to achieved different $q-ROFS$ Bonferroni mean operators. Hussain et al. [15] proposed the concept of $q-ROFS$ soft averaging aggregation operators and presented their desirable properties with detail. The concept of hesitant $q-ROFS$ aggregation operators was proposed by Hussain et al. [13]. Xing et al. [26] studied the point weighted aggregation operators and Xing et al. [27] presented hamy mean operators in $q-ROFS$ environment.

It has been observed that all the considered works are accomplished under $q-ROFS$ environments by assuming that the experts are absolutely familiar with the evaluated objects. But in real life, situations like these are partially fulfilled. For example, in MCDM the provided information by experts are completely based on their own choices and may not lead to the accurate decisions. Therefore, acknowledging the described preferences, it is necessary to justify the initial described preferences from other senior expert/judge. In real cases, there are many circumstances where the initial provided preferences need the support of some other senior experts/judge views. For example, in medical diagnose problem consider a person/patient is suffering from an unknown disease and he visits the hospital for the initial treatment to get his disease diagnosed. He describes his preferences in a set of symptoms regarding his physical condition. These symptoms are completely based on information provided by the patient. If the doctor/physician diagnoses the problem according to the patient's preferences without verifying it from another senior expert/doctor's, then he may not be cured well or may be a cause of causality, because the provided information has not been verified from a senior physician/doctor. Therefore, after acknowledging the patient's described symptoms, it is necessary to certify the described information from senior expert physician/doctors. This is only possible by incorporating the

idea of generalized parameter (in short \mathcal{GP}) to the original information. The primary provided information by the patient is further verified from another expert physician/senior doctor's and he gives his preference in the form of a generalized parameter. The generalized parameter is itself a q -ROFN number (q -ROFN), which reduces the uncertain information and improves the accuracy of the final decision. Without generalized parameter the initial information described by the patient remains in doubt. Similarly, in MCDM process, to get an intelligent decision, the initially provided preferences from other seniors experts/decision makers need to be verified to reduce the complexity and uncertain errors by incorporating the generalized parameter in the initial information to get the accurate decision.

Hence to cope with such situation, the point of views of other senior expert/observer are needed by incorporating the notion of \mathcal{GP} to the original information. In this paper, we introduced the concept generalized q -ROFS (Gq-ROFS) by incorporating generalized parameter to views the expertise of other senior decision makers in q -ROFN environment, which reduce the complexity and uncertainty errors in original information. Then this idea explored to group generalized parameter where the preferences of two or more senior experts/decision makers are analysed in q -ROFN environments to get new concept of group generalized q -ROFSs (GGq-ROFSs). The major advantages of the generalized parameter or group generalized parameter is that to reduced probability of complexities, uncertainties and errors in the original information. The main focus of the present work by the application of MCDM, by utilizing generalized parameter and group generalized parameter. For ranking the alternatives, some aggregation operators are introduced for Gq-ROFSs and GGq-ROFSs. These developed aggregations operators have the ability to adjust the situations in better sequence on the basis of parameterizations character.

The remaining portions of the manuscript are arranged as. Section 2 consists of a brief review of the existing concepts. Section 3 is devoted for the study of q -ROFN ordered weighted average (q -ROFOWA) operator and q -ROFN hybrid average (q -ROFHFA) operator. Section 4 consists of the definition of generalized q -ROFS and the detail study of aggregation operators such as generalized q -ROFN weighted average (Gq-ROFNWA) operator, generalized q -ROFN ordered weighted average (Gq-ROFNOWA) operator and generalized q -ROFN hybrid average (Gq-ROFNHFA) operator. In Sect. 5, the defined aggregation operators in Sect. 4 are extended to group generalized q -ROFN aggregation operators. Section 6 consists of the MCDM process and a decision algorithm based on the proposed concepts. In Sect. 7, the application of the developed method is presented through an illustrative example. The final Sect. 8 present the comparative remarks of the developed method with existing

methods and it has been shown that the developed method is more superior than the existing methods.

Preliminaries

This section consists of a brief discussion of IFS, PFS and q -ROFSs are given which will help in coming sections.

Definition 1 [3] Let \mathcal{X} be a universal set. An IFS \mathcal{J} on \mathcal{X} can be expressed as

$$\mathcal{J} = \{ \langle s, \mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s) \rangle / s \in \mathcal{X} \},$$

where $\mu_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ denotes a membership grade and $\eta_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ denotes a non-membership grade of an object $s \in \mathcal{X}$, to the set \mathcal{J} and it holds that $0 \leq \mu_{\mathcal{J}}(s) + \eta_{\mathcal{J}}(s) \leq 1$.

Definition 2 [31] Consider \mathcal{X} be a universal set. A PFS \mathcal{J} on \mathcal{X} can be expressed as

$$\mathcal{J} = \{ \langle s, \mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s) \rangle / s \in \mathcal{X} \},$$

where $\mu_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ represents a membership grade and $\eta_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ represents a non-membership grade of an object $s \in \mathcal{X}$, to the set \mathcal{J} , respectively, and it holds that $0 \leq (\mu_{\mathcal{J}}(s))^2 + (\eta_{\mathcal{J}}(s))^2 \leq 1$. Furthermore, the degree of hesitancy/nondeterminacy is defined as $\pi_{\mathcal{J}}(s) = \sqrt{1 - (\mu_{\mathcal{J}}(s))^2 - (\eta_{\mathcal{J}}(s))^2}$ for all $s \in \mathcal{X}$.

Definition 3 [33] Let \mathcal{X} be a universal set. A q -ROFS \mathcal{J} on \mathcal{X} can be expressed as

$$\mathcal{J} = \{ \langle s, \mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s) \rangle_q / s \in \mathcal{X} \}$$

where $\mu_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ denotes a membership grade and $\eta_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ denotes a non-membership grade of an object $s \in \mathcal{X}$, to the set \mathcal{J} respectively and it holds that $0 \leq (\mu_{\mathcal{J}}(s))^q + (\eta_{\mathcal{J}}(s))^q \leq 1$ where $q \geq 1$. Further the degree of hesitancy/ nondeterminacy is defined as $\pi_{\mathcal{J}}(s) = \sqrt[q]{1 - (\mu_{\mathcal{J}}(s))^q - (\eta_{\mathcal{J}}(s))^q}$ for $q \geq 1$ and for all $s \in \mathcal{X}$.

For convenience $(\mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s))$ is known to be a q -ROFN number (q -ROFN) and is written as $d = (\mu_d, \eta_d)$. For any three q -ROFNs $d = (\mu_d, \eta_d)$, $d_1 = (\mu_{d_1}, \eta_{d_1})$ and $d_2 = (\mu_{d_2}, \eta_{d_2})$, then the basic operation on them are defined as:

- i: $d_1 \cup d_2 = (\max(\mu_{d_1}, \mu_{d_2}), \min(\eta_{d_1}, \eta_{d_2}))$;
- ii: $d_1 \cap d_2 = (\min(\mu_{d_1}, \mu_{d_2}), \max(\eta_{d_1}, \eta_{d_2}))$;
- iii: $\sim d = (\eta_d, \mu_d)$, where $\sim d$ denotes a complement of d ;

- iv: $d_1 \oplus d_2 = \left(\sqrt[q]{(\mu_{d_1})^q + (\mu_{d_2})^q - \mu_{d_1}^q \mu_{d_2}^q}, \eta_{d_1} \eta_{d_2} \right)$;
- v: $d_1 \otimes d_2 = \left(\mu_{d_1} \mu_{d_2}, \sqrt[q]{(\eta_{d_1})^q + (\eta_{d_2})^q - \eta_{d_1}^q \eta_{d_2}^q} \right)$;
- vi: $\alpha d = \left(\sqrt[q]{1 - (1 - \mu_d^q)^\alpha}, \eta_d^\alpha \right)$;
- vii: $d^\alpha = \left(\mu_d^\alpha, \sqrt[q]{1 - (1 - \eta_d^q)^\alpha} \right)$.

Definition 4 Suppose $d_1 = (\mu_{d_1}, \eta_{d_1})$ and $d_2 = (\mu_{d_2}, \eta_{d_2})$ be two q -ROFNs, then the score function defined by Liu and Wang [21] is given as, $S(d_1) = \mu_{d_1}^q - \eta_{d_1}^q$ and $S(d_2) = \mu_{d_2}^q - \eta_{d_2}^q$. Similarly the accuracy function of d_1 and d_2 are defined as $A(d_1) = \mu_{d_1}^q + \eta_{d_1}^q$ and $A(d_2) = \mu_{d_2}^q + \eta_{d_2}^q$, respectively.

- 1: If $S(d_1) < S(d_2)$, then $d_1 < d_2$ that is d_1 is less than d_2 ;
- 2: If $S(d_1) > S(d_2)$, then $d_1 > d_2$ that is d_1 is greater than d_2 ;

if $S(d_1) = S(d_2)$, then larger the accuracy function batter the orthopair is.

On the bases of above operations, Liu and Wang [21] proved the following properties.

Theorem 1 [21] Let $d = (\mu_d, \eta_d), d_1 = (\mu_{d_1}, \eta_{d_1})$ and $d_2 = (\mu_{d_2}, \eta_{d_2})$ be three q -ROFNs and $\alpha, \alpha_1, \alpha_2 > 0$, then the following are holds:

- i: $d_1 \oplus d_2 = d_2 \oplus d_1$;
- ii: $d_1 \otimes d_2 = d_2 \otimes d_1$;
- iii: $\alpha(d_1 \oplus d_2) = \alpha d_2 \oplus \alpha d_1$;
- iv: $\alpha_1 d \oplus \alpha_2 d = (\alpha_1 + \alpha_2) d$;
- v: $d^{\alpha_1} \otimes d^{\alpha_2} = d^{(\alpha_1 + \alpha_2)}$;
- vi: $d_1^\alpha \otimes d_2^\alpha = (d_1 \otimes d_2)^\alpha$

q-Rung orthopair fuzzy aggregation operator

This section is devoted for a brief discussion of aggregation operators such as q -ROFWA, q -ROFOWA and q -ROFHA operators.

Definition 5 [21] Consider the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ of n q -ROFNs with weight vector $u = (u_1, u_2, \dots, u_n)^T$ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$ (for $\ell = 1, 2, \dots, n$), then the q -ROFWA operator is defined as:

$$q\text{-ROFWA}(d_1, d_2, \dots, d_n) = \left(\sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^n \eta_{d_\ell}^{u_\ell} \right)$$

Example 1 Suppose four q -ROFNs $d_1 = (0.8, 0.7), d_2 = (0.9, 0.5), d_3 = (0.7, 0.9), d_4 = (0.6, 0.3)$ having weight vector $u = \{0.33, 0.15, 0.3, 0.22\}$ for $q = 3$, then

$$\begin{aligned} & \sqrt[3]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^3)^{u_\ell}} = \\ & \sqrt[3]{1 - (1 - 0.8^3)^{0.33} (1 - 0.9^3)^{0.15} (1 - 0.7^3)^{0.3} (1 - 0.6^3)^{0.22}} \\ & = 0.77072 \text{ and} \\ & \sqrt[3]{\prod_{\ell=1}^4 \eta_{d_\ell}^{u_\ell}} = (0.7^{0.33})(0.5^{0.15})(0.9^{0.3})(0.3^{0.22}) \\ & = 0.595617 \end{aligned}$$

Now by Definition 5, we have

$$\begin{aligned} & q\text{-ROFWA}(d_1, d_2, d_3, d_4) \\ & = \left(\sqrt[3]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^3)^{u_\ell}}, \prod_{\ell=1}^4 (\eta_{d_\ell}^{u_\ell}) \right) \\ & = (0.77072, 0.595617) \end{aligned}$$

Definition 6 Let us consider the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$) of n q -ROFNs, then the q -ROFOWA operator is given as;

$$\begin{aligned} & q\text{-ROFOWA}(d_1, d_2, d_3, \dots, d_n) \\ & = \bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \\ & = u_1 \tilde{d}_1 \oplus u_2 \tilde{d}_2 \oplus \dots \oplus u_n \tilde{d}_n. \end{aligned}$$

where $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})$ ($\ell = 1, 2, \dots, n$) indicate the ℓ th largest object of the collection of n q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$.

The aggregation result of Definition 6 through operation rules is described as in Theorem 2.

Theorem 2 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q -ROFNs with weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, then the q -ROFOWA operator is described as:

$$\begin{aligned} & q\text{-ROFOWA}(d_1, d_2, d_3, \dots, d_n) \\ & = \bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \\ & = \left(\sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{\tilde{d}_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^n (\eta_{\tilde{d}_\ell}^{u_\ell}) \right), \end{aligned}$$

where $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})$ indicate the ℓ th largest object of the collection of n q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$).

Example 2 Consider four q -ROFNs $d_1 = (0.7, 0.3), d_2 = (0.8, 0.65), d_3 = (0.9, 0.6), d_4 = (0.88, 0.7)$ having weight vector $u = (0.3, 0.25, 0.1, 0.35)$ for $q = 5$, then to find the score functions of each q -ROFNs, that is

$S(d_1) = 0.7^5 - 0.3^5 = 0.166$, $S(d_2) = 0.8^5 - 0.65^5 = 0.212$, $S(d_3) = 0.9^5 - 0.6^5 = 0.513$, $S(d_4) = 0.88^5 - 0.7^5 = 0.360$.

So $S(d_3) > S(d_4) > S(d_2) > S(d_1)$, this implies that $d_4 > d_3 > d_2 > d_1$. Thus, $\tilde{d}_1 = d_3$, $\tilde{d}_2 = d_4$, $\tilde{d}_3 = d_2$, and $\tilde{d}_4 = d_1$. Further we have

$$\frac{\sqrt[5]{1 - \prod_{\ell=1}^4 (1 - \mu_{\tilde{d}_\ell}^5)^{u_\ell}}}{\sqrt[5]{1 - (1 - 0.9^5)^{0.3} (1 - 0.88^5)^{0.25} (1 - 0.8^5)^{0.1} (1 - 0.7^5)^{0.35}}} = 0.844094$$

$$\prod_{\ell=1}^4 (\eta_{\tilde{d}_\ell}^{u_\ell}) = (0.6^{0.3})(0.7^{0.25})(0.65^{0.1})(0.3^{0.35}) = 0.493178$$

Now by Theorem 2, we have

$$q\text{-ROFOWA}(d_1, d_2, d_3, d_4) = \left(\sqrt[5]{1 - \prod_{\ell=1}^4 (1 - \mu_{\tilde{d}_\ell}^5)^{u_\ell}}, \prod_{\ell=1}^4 (\eta_{\tilde{d}_\ell}^{u_\ell}) \right) = (0.844094, 0.493178)$$

Definition 7 Suppose that $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$), be the collections of q -ROFNs with associated weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$. Then the q -ROFHA operator is given as:

$$q\text{-ROFHA}(d_1, d_2, d_3, \dots, d_n) = \bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell = u_1 \tilde{d}_1 \oplus u_2 \tilde{d}_2 \oplus \dots \oplus u_n \tilde{d}_n$$

where \tilde{d}_ℓ ($\tilde{d}_\ell = \hat{n} u_\ell d_\ell$, for $\ell = 1, 2, \dots, n$) indicate the ℓ th largest object of the collection of q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$), and $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$ be the weight vector of d_ℓ ($\ell = 1, 2, \dots, n$) where $\hat{u}_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n \hat{u}_\ell = 1$ and n indicate the balancing coefficient.

The aggregation result of Definition 7 through operation rules is described as in Theorem 3.

Theorem 3 Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) be the collection of q -ROFNs, then the q -ROFHA operator is described as:

$$q\text{-ROFHA}(d_1, d_2, d_3, \dots, d_n) = \left(\bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \right) = \left(\sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{\tilde{d}_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^n (\eta_{\tilde{d}_\ell}^{u_\ell}) \right),$$

where \tilde{d}_ℓ ($\tilde{d}_\ell = \hat{n} u_\ell d_\ell$, for $\ell = 1, 2, \dots, n$) indicate the ℓ th largest object of the collection of q -ROFNs $d_\ell =$

$(\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) and n indicate the balancing coefficient.

Example 3 Suppose four q -ROFNs $d_1 = (0.5, 0.2)$, $d_2 = (0.83, 0.6)$, $d_3 = (0.95, 0.65)$, $d_4 = (0.9, 0.75)$ having associated weight vector $u = (0.3, 0.2, 0.32, 0.18)$ for $q = 4$, and weight vector $\hat{u} = (0.4, 0.3, 0.1, 0.2)$, then by using operational law

$$n \hat{u}_\ell d_\ell = \left(\sqrt[q]{1 - (1 - \mu_{d_\ell}^q)^{n \hat{u}_\ell}}, \eta_{d_\ell}^{n \hat{u}_\ell} \right)$$

we have

$$4 \hat{u}_1 d_1 = \left(\sqrt[4]{1 - (1 - 0.5^4)^{4 \times 0.4}}, 0.2^{4 \times 0.4} \right) = (0.560, 0.076),$$

$$4 \hat{u}_2 d_2 = \left(\sqrt[4]{1 - (1 - 0.83^4)^{4 \times 0.3}}, 0.6^{4 \times 0.3} \right) = (0.856, 0.542),$$

$$4 \hat{u}_3 d_3 = \left(\sqrt[4]{1 - (1 - 0.95^4)^{4 \times 0.1}}, 0.65^{4 \times 0.1} \right) = (0.837, 0.842), 4 \hat{u}_4 d_4 = \left(\sqrt[4]{1 - (1 - 0.9^4)^{4 \times 0.2}}, 0.75^{4 \times 0.2} \right) = (0.870, 0.794).$$

Now to find their score functions, that is

$$S(4 \hat{u}_1 d_1) = 0.560^4 - 0.076^4 = 0.098, S(4 \hat{u}_2 d_2) = 0.856^4 - 0.542^4 = 0.451, S(4 \hat{u}_3 d_3) = 0.837^4 - 0.842^4 = -0.012, S(4 \hat{u}_4 d_4) = 0.870^4 - 0.794^4 = -0.396.$$

So $S(4 \hat{u}_2 d_2) > S(4 \hat{u}_1 d_1) > S(4 \hat{u}_3 d_3) > S(4 \hat{u}_4 d_4)$, this implies that

$$\tilde{d}_1 = (0.856, 0.542), \tilde{d}_2 = (0.560, 0.076), \tilde{d}_3 = (0.837, 0.842) \text{ and } \tilde{d}_4 = (0.870, 0.794)$$

Further, we have

$$\frac{\sqrt[4]{1 - \prod_{\ell=1}^4 (1 - \mu_{\tilde{d}_\ell}^4)^{u_\ell}}}{\sqrt[4]{1 - (1 - 0.856^4)^{0.3} (1 - 0.560^4)^{0.2} (1 - 0.837^4)^{0.32} (1 - 0.870^4)^{0.18}}} = 0.82465 \text{ and } \prod_{\ell=1}^4 (\eta_{\tilde{d}_\ell}^{u_\ell}) = (0.542^{0.3})(0.076^{0.2})(0.842^{0.32})(0.794^{0.18}) = 0.451263$$

Now by Theorem 3, we have

$$\begin{aligned}
 & q\text{-ROFHA}(d_1, d_2, d_3, d_4) \\
 &= \left(\sqrt[q]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^4)^{u_\ell}}, \prod_{\ell=1}^4 (\eta_{d_\ell}^{u_\ell}) \right) \\
 &= (0.824648, 0.451263)
 \end{aligned}$$

q-Rung orthopair fuzzy averaging aggregation operator under generalized parameter

In this section, first we will define generalized q-ROFS and then we will present the detail study of some average aggregation operators under generalized parameter like as Gq-ROFW A operator, Gq-ROFOWA operator and Gq-ROFHA operator and their properties in detail.

q-Rung orthopair fuzzy information under generalized parameter

Consider a person/patient who is suffering from an unknown disease and he visits the hospital for the initial checkup to diagnose his disease. He describes his preferences in a set of symptoms regarding his physical condition. His preferences are in the form of q-ROFNs that is $\mathcal{X} = \{d_1, d_2, d_3, d_4\}$ where $d_j (j = 1, 2, 3, 4)$ stand for $d_1 =$ high temperature, $d_2 =$ headache, $d_3 =$ cough and $d_4 =$ constipation, respectively. Let the q-ROFS $\mathcal{X} = \{(0.9, 0.6)_{\text{high temperature}}, (0.8, 0.4)_{\text{headache}}, (0.95, 0.5)_{\text{cough}}, (0.7, 0.3)_{\text{constipation}}\}$ denotes the described symptoms of the patient. These symptoms are completely based on the initial information given by the patient. If the expert means doctor/physician diagnoses the patient according to the patient preferences without verifying it from another senior expert/doctor, then he may not be cured well or may be a cause of causality because the provided information has not been verified from a senior physician/doctor. Therefore, after acknowledging the patient’s described symptoms, it is necessary to certify the described information from senior expert physician/doctors. This is only possible by adding the idea of generalized parameter to the original information. The patient information is further verified from another expert physician/senior doctor and he gives their preference with the help of a generalized parameter such as $\tilde{h} = (0.88, 0.65)$, then the new q-ROFS based on generalized parameter is defined as $\mathcal{X} = \{< (0.9, 0.6), (0.8, 0.4), (0.95, 0.5), (0.7, 0.3) >_{(0.88, 0.65)}\}$. The generalized parameter is a q-ROFN which deduct the uncertainty and complexity in original information and improves the accuracy of the final decision. Without generalized parameter the initial information

described by the patient remains in doubt. Thus, the notion of generalized q-ROFS (Gq-ROFS) is defined as.

Definition 8 Suppose \mathcal{X} be a universal set. Then a Gq-ROFS \mathcal{J} of a set \mathcal{X} is defined as

$$\mathcal{J} = \{< s, \mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s) >_q (\mu_{\tilde{h}}, \eta_{\tilde{h}}) / s \in \mathcal{X}\}$$

where the mappings $\mu_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$ and $\eta_{\mathcal{J}} : \mathcal{X} \rightarrow [0, 1]$, represents a membership grade and a non-membership grade of $s \in \mathcal{X}$, to the set \mathcal{J} , respectively, and satisfying that $0 \leq (\mu_{\mathcal{J}}(s))^q + (\eta_{\mathcal{J}}(s))^q \leq 1$ and $\mu_{\tilde{h}}, \eta_{\tilde{h}} \in [0, 1]$ represent the degree of truthfulness and falsity grades of q-ROFS $\{< s, \mu_{\mathcal{J}}(s), \eta_{\mathcal{J}}(s) >_q / s \in \mathcal{X}\}$ such that $0 \leq \mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q \leq 1$ where $q \geq 1$. Further $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ is known to be a generalized parameter. This generalized parameter provide the preference assessment of the another senior decision maker/expert. This generalized parameter is itself a q-ROFN.

The generalized q-rung orthopair fuzzy weighted average operator (Gq-ROFWA)

In this subsection, we will present the study of Gq-ROFWA operator and their properties in detail.

Definition 9 Consider a generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$), then the Gq-ROFWA operator is defined as;

$$\begin{aligned}
 & Gq\text{-ROFWA}(< d_1, d_2, d_3, \dots, d_n >, \tilde{h}) \\
 &= \tilde{h} \otimes q\text{-ROFWA}(d_1, d_2, d_3, \dots, d_n)
 \end{aligned}$$

The aggregation result for q-ROFNs through operation rules is described as in Theorem 4.

Theorem 4 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q-ROFNs with generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ and weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, then the Gq-ROFWA operator is described as:

$$\begin{aligned}
 & Gq\text{-ROFWA}(< d_1, d_2, d_3, \dots, d_n >, \tilde{h}) \\
 &= \tilde{h} \otimes \left(\bigoplus_{\ell=1}^n u_\ell d_\ell \right) \\
 &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right)
 \end{aligned}$$

Proof As we know that

$$d_1 \oplus d_2 = \left(\sqrt[q]{\mu_{d_1}^q + \mu_{d_2}^q - \mu_{d_1}^q \mu_{d_2}^q}, \eta_{d_1} \eta_{d_2} \right)$$

and

$$u_1 d_1 = \left(\sqrt[q]{1 - (1 - \mu_{d_1}^q)^{u_1}}, \eta_{d_1}^{u_1} \right)$$

$$\text{and } u_2 d_2 = \left(\sqrt[q]{1 - (1 - \mu_{d_2}^q)^{u_2}}, \eta_{d_2}^{u_2} \right)$$

We use the mathematical induction to prove this theorem. Now for $n = 2$ we get

$$\begin{aligned} & \text{Gq-ROFWA}(\langle d_1, d_2 \rangle, \tilde{h}) \\ &= \tilde{h} \otimes (u_1 d_1 \oplus u_2 d_2) \\ &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left(\sqrt[q]{1 - (1 - \mu_{d_1}^q)^{u_1} (1 - \mu_{d_2}^q)^{u_2}}, \eta_{d_1}^{u_1} \eta_{d_2}^{u_2} \right) \\ &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^2 (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^2 \eta_{d_\ell}^{u_\ell} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^2 (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^2 (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

this implies for $n = 2$ the result holds. Next suppose that the result true for $n = k$, that is

$$\begin{aligned} & \text{Gq-ROFWA}(\langle d_1, d_2, \dots, d_k \rangle, \tilde{h}) \\ &= \tilde{h} \otimes \left(\bigoplus_{\ell=1}^k u_\ell d_\ell \right) \\ &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^k (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^k \eta_{d_\ell}^{u_\ell} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^k (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^k (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

Now to show that the result hold for $n = k + 1$, then we have

$$\begin{aligned} & \text{Gq-ROFWA}(\langle d_1, d_2, \dots, d_k, d_{k+1} \rangle, \tilde{h}) \\ &= \tilde{h} \otimes \left(u_1 d_1 \oplus u_2 d_2 \oplus \dots \oplus u_k d_k \oplus u_{k+1} d_{k+1} \right) \\ &= \tilde{h} \otimes \{ (u_1 d_1 \oplus u_2 d_2 \oplus \dots \oplus u_k d_k) \oplus u_{k+1} d_{k+1} \} \\ &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left\{ \left(\sqrt[q]{1 - \prod_{\ell=1}^k (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^k (\eta_{d_\ell}^{u_\ell}) \right) \oplus \left(\sqrt[q]{1 - (1 - \mu_{d_{k+1}}^q)^{u_{k+1}}}, \eta_{d_{k+1}}^{u_{k+1}} \right) \right\} \\ &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^k (1 - \mu_{d_\ell}^q)^{u_\ell} \cdot (1 - \mu_{d_{k+1}}^q)^{u_{k+1}}}, \prod_{\ell=1}^k (\eta_{d_\ell}^{u_\ell}) \cdot \eta_{d_{k+1}}^{u_{k+1}} \right) \end{aligned}$$

$$\begin{aligned} &= (\mu_{\tilde{h}}, \eta_{\tilde{h}}) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^{k+1} (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^{k+1} (\eta_{d_\ell}^{u_\ell}) \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^{k+1} (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^{k+1} (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

this implies that n is true for $k + 1$. Hence, under the study of generalized parameter, the given result hold for any number of q-ROFNs. \square

Moreover, in the following, it is shown that the aggregated result achieved from Gq-ROFWA is also a q-ROFNs. Now for any $\ell = 1, 2, 3, \dots, n$ we have $0 \leq \mu_{d_\ell}, \eta_{d_\ell} \leq 1$ with $0 \leq \mu_{d_\ell}^q + \eta_{d_\ell}^q \leq 1$ for $q \geq 1$. This also implies that

$$\begin{aligned} 0 \leq 1 - \mu_{d_\ell}^q \leq 1 &\Rightarrow 0 \leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q) \leq 1 \Rightarrow 0 \\ &\leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \leq 1 \\ \Rightarrow 0 \leq 1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} &\leq 1 \Rightarrow 0 \\ &\leq \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq 1 \end{aligned}$$

As $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ is a generalized parameter where $\mu_{\tilde{h}}, \eta_{\tilde{h}} \in [0, 1]$ with $0 \leq \mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q \leq 1$. Therefore,

$$\begin{aligned} 0 \leq \mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} &\leq 1. \text{ Similarly we can show that} \\ 0 \leq \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} &\leq 1 \end{aligned}$$

Furthermore,

$$\begin{aligned} 0 \leq \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \right)^q & \\ + \left(\sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right)^q & \\ = \mu_{\tilde{h}}^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \right) & \\ + \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \right)^q & \end{aligned}$$

$$\begin{aligned} &= \mu_{\tilde{h}}^q - \mu_{\tilde{h}}^q \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} + \eta_{\tilde{h}}^q \\ &+ \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} - \eta_{\tilde{h}}^q \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \\ &= (\mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q) + \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} - \mu_{\tilde{h}}^q \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \\ &- \eta_{\tilde{h}}^q \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \end{aligned}$$

As $\mu_{d_\ell}^q + \eta_{d_\ell}^q \leq 1 \Rightarrow \eta_{d_\ell}^q \leq 1 - \mu_{d_\ell}^q \Rightarrow -\eta_{d_\ell}^q \geq -(1 - \mu_{d_\ell}^q)$

$$\begin{aligned} &\leq (\mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q) + \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} - \mu_{\tilde{h}}^q \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \\ &- \eta_{\tilde{h}}^q \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \\ &= (\mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q) + \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} - \left(\mu_{\tilde{h}}^q \right. \\ &\left. + \eta_{\tilde{h}}^q \right) \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \\ &= (\mu_{\tilde{h}}^q + \eta_{\tilde{h}}^q) \left(1 - \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \right) + \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} \\ &\leq 1 - \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} + \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell} = 1 \end{aligned}$$

$$\begin{aligned} \text{implies } 0 &\leq \left(\mu_{\tilde{h}}^q \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \right)^q \\ &+ \left(\sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^q)^{u_\ell}} \right)^q \leq 1 \end{aligned}$$

Therefore, it is proved that the aggregated result obtained by Gq-ROFWA operator is also a q-ROFN.

Remark 1 (a) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 1$, then the Gq-ROFWA operator reduces to IFWA operator.

(b) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 2$, then the Gq-ROFWA operator reduces to PFWA operator.

(c) If the value of $q = 2$ is fixed then the Gq-ROFWA operator reduces to GPFWA operator.

Example 4 Suppose the generalized parameter $\tilde{h} = (0.9, 0.6)$ of four q-ROFNs $d_1 = (0.8, 0.7)$, $d_2 = (0.9, 0.5)$, $d_3 = (0.7, 0.9)$, $d_4 = (0.6, 0.3)$ having weight vector $u = \{0.33, 0.15, 0.3, 0.22\}$ for $q = 3$, then

$$\begin{aligned} &\mu_{\tilde{h}} \cdot \sqrt[3]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^3)^{u_\ell}} = \\ &0.9 \times \sqrt[3]{1 - (1 - 0.8^3)^{0.33} (1 - 0.9^3)^{0.15} (1 - 0.7^3)^{0.3} (1 - 0.6^3)^{0.22}} = 0.9 \times \\ &0.77072 = 0.69365 \text{ and} \\ &\sqrt[3]{\eta_{\tilde{h}}^3 + (1 - \eta_{\tilde{h}}^3) \prod_{\ell=1}^4 (\eta_{d_\ell}^3)^{u_\ell}} = \\ &\sqrt[3]{0.6^3 + (1 - 0.6^3)(0.7^{0.33})^3 (0.5^{0.15})^3 (0.9^{0.3})^3 (0.3^{0.22})^3} = \\ &0.72537 \end{aligned}$$

Now by Theorem 4, we have

Gq-ROFWA($\langle d_1, d_2, d_3, d_4 \rangle, \tilde{h}$)

$$\begin{aligned} &= \left(\mu_{\tilde{h}} \cdot \sqrt[3]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^3)^{u_\ell}}, \right. \\ &\left. \sqrt[3]{\eta_{\tilde{h}}^3 + (1 - \eta_{\tilde{h}}^3) \prod_{\ell=1}^4 (\eta_{d_\ell}^3)^{u_\ell}} \right) \\ &= (0.69365, 0.72537) \end{aligned}$$

Theorem 5 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q-ROFNs with generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ and weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, then for Gq-ROFWA operator the following are holes:

i: (Idempotency): If $d_\ell = d$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$Gq-ROFWA(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = \tilde{h} \otimes d.$$

ii: (Boundary condition): If $d_\ell^- = (\min \mu_{\tilde{h} \otimes d_\ell}, \max \eta_{\tilde{h} \otimes d_\ell})$ and $d_\ell^+ = (\max \mu_{\tilde{h} \otimes d_\ell}, \min \eta_{\tilde{h} \otimes d_\ell})$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$d_\ell^- \leq Gq-ROFWA(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \leq d_\ell^+.$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell}^*, \eta_{d_\ell}^*)$ ($\ell = 1, 2, \dots, n$) be the collection of n q-ROFNs such that $\mu_{d_\ell} \leq \mu_{d_\ell}^*$ and $\eta_{d_\ell} \geq \eta_{d_\ell}^*$, then

$$\begin{aligned} &Gq-ROFWA(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \leq \\ &Gq-ROFWA(\langle d_1^*, d_2^*, d_3^*, \dots, d_n^* \rangle, \tilde{h}). \end{aligned}$$

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})$ ($\ell = 1, 2, \dots, n$) be the collection of n q-ROFNs where \tilde{d}_ℓ ($\ell = 1, 2, \dots, n$) is the ℓ th largest object of d_ℓ ($\ell = 1, 2, \dots, n$), then

$$\begin{aligned} &Gq-ROFWA(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \\ &= Gq-ROFWA(\langle \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n \rangle, \tilde{h}). \end{aligned}$$

Proof i: If $d_\ell = d$ (for all $\ell = 1, 2, 3, \dots, n$), then by Theorem 4, we have

$$\begin{aligned} & \text{Gq-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_d^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_d^{u_\ell})^q} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - (1 - \mu_d^q)^{\sum_{\ell=1}^n u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \left(\eta_d^{\sum_{\ell=1}^n u_\ell} \right)^q} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - (1 - \mu_d^q)}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\eta_d)^q} \right) \\ &= \left(\mu_{\tilde{h}} \cdot \mu_d, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\eta_d)^q} \right) \\ &= \tilde{h} \otimes d \end{aligned}$$

ii: Let $d_\ell^- = (\min \mu_{\tilde{h} \otimes d_\ell}, \max \eta_{\tilde{h} \otimes d_\ell})$ and $d_\ell^+ = (\max \mu_{\tilde{h} \otimes d_\ell}, \min \eta_{\tilde{h} \otimes d_\ell})$ where $\min \mu_{\tilde{h} \otimes d_\ell} = \mu_{\tilde{h}}(\min \mu_{d_\ell})$, $\max \mu_{\tilde{h} \otimes d_\ell} = \mu_{\tilde{h}}(\max \mu_{d_\ell})$, $\max \eta_{\tilde{h} \otimes d_\ell} = (\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\max \eta_{d_\ell})^q)^{\frac{1}{q}}$ and $\min \eta_{\tilde{h} \otimes d_\ell} = (\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\min \eta_{d_\ell})^q)^{\frac{1}{q}}$.

Now for each $\ell = 1, 2, 3, \dots, n$, we have $\min \mu_{d_\ell} \leq \mu_{d_\ell} \leq \max \mu_{d_\ell}$ and this also implies that $1 - \max(\mu_{d_\ell}^q) \leq 1 - \mu_{d_\ell}^q \leq 1 - \min(\mu_{d_\ell}^q)$, $\Leftrightarrow \prod_{\ell=1}^n (1 - \max(\mu_{d_\ell}^q))^{u_\ell} \leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \leq \prod_{\ell=1}^n (1 - \min(\mu_{d_\ell}^q))^{u_\ell} \Leftrightarrow (1 - \max(\mu_{d_\ell}^q))^{\sum_{\ell=1}^n u_\ell} \leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \leq (1 - \min(\mu_{d_\ell}^q))^{\sum_{\ell=1}^n u_\ell}$
 $\Leftrightarrow (1 - \max(\mu_{d_\ell}^q)) \leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \leq (1 - \min(\mu_{d_\ell}^q))$
 $\Leftrightarrow 1 - (1 - \min(\mu_{d_\ell}^q)) \leq 1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \leq 1 - (1 - \max(\mu_{d_\ell}^q))$

$$\Leftrightarrow \min(\mu_{d_\ell}) \leq \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq \max(\mu_{d_\ell})$$

As $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ is a generalized parameter where $\mu_{\tilde{h}}, \eta_{\tilde{h}} \in [0, 1]$, then

$$\begin{aligned} \mu_{\tilde{h}}(\min \mu_{d_\ell}) &\leq \mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq \mu_{\tilde{h}}(\max \mu_{d_\ell}) \\ \Leftrightarrow \min \mu_{\tilde{h} \otimes d_\ell} &\leq \mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq \max \mu_{\tilde{h} \otimes d_\ell} \end{aligned} \tag{1}$$

Next for each $\ell = 1, 2, 3, \dots, n$, we have $\min \eta_{d_\ell} \leq \eta_{d_\ell} \leq \max \eta_{d_\ell}$ and this also implies that

$$\begin{aligned} & \prod_{\ell=1}^n ((\min \eta_{d_\ell})^q)^{u_\ell} \\ & \leq \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \leq \prod_{\ell=1}^n ((\max \eta_{d_\ell})^q)^{u_\ell} \\ & \Leftrightarrow ((\min \eta_{d_\ell})^q)^{\sum_{\ell=1}^n u_\ell} \leq \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \leq ((\max \eta_{d_\ell})^q)^{\sum_{\ell=1}^n u_\ell} \\ & \Leftrightarrow (\min \eta_{d_\ell})^q \leq \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \leq (\max \eta_{d_\ell})^q \end{aligned}$$

As $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ is a generalized parameter where $\mu_{\tilde{h}}, \eta_{\tilde{h}} \in [0, 1]$, then

$$\begin{aligned} & (1 - \eta_{\tilde{h}}^q)(\min \eta_{d_\ell})^q \\ & \leq (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \leq (1 - \eta_{\tilde{h}}^q)(\max \eta_{d_\ell})^q \\ & \Leftrightarrow \eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\min \eta_{d_\ell})^q \\ & \leq \eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \leq \eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\max \eta_{d_\ell})^q \\ & \Leftrightarrow \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\min \eta_{d_\ell})^q} \\ & \leq \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \\ & \leq \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q)(\max \eta_{d_\ell})^q} \\ & \Leftrightarrow \min \eta_{\tilde{h} \otimes d_\ell} \\ & \leq \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \leq \max \eta_{\tilde{h} \otimes d_\ell} \end{aligned} \tag{2}$$

If $\text{Gq-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = \delta = (\mu_{\tilde{h} \otimes d_\ell}, \eta_{\tilde{h} \otimes d_\ell})$, then from the analysis of Eqs. (1) and (2), we have

$\min \mu_{\tilde{h} \otimes d_\ell} \leq \mu_{\tilde{h} \otimes d_\ell} \leq \max \mu_{\tilde{h} \otimes d_\ell}$ and $\min \eta_{\tilde{h} \otimes d_\ell} \leq \eta_{\tilde{h} \otimes d_\ell} \leq \max \eta_{\tilde{h} \otimes d_\ell}$. Therefore, by score function, we get

$$d_\ell^- \leq \text{Gq-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \leq d_\ell^+$$

iii: Since $\mu_{d_\ell} \leq \mu_{d_\ell}^*$ and $\eta_{d_\ell} \geq \eta_{d_\ell}^*$ for all ℓ , then this implies

$$\begin{aligned} \mu_{d_\ell} \leq \mu_{d_\ell}^* &\Rightarrow 1 - \mu_{d_\ell}^q \leq 1 - \mu_{d_\ell}^{q*} \\ \Rightarrow \prod_{\ell=1}^n (1 - \mu_{d_\ell}^{q*})^{u_\ell} &\leq \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} \Rightarrow \\ 1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell} &\leq 1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^{q*})^{u_\ell} \end{aligned}$$

$$\Rightarrow \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell^*}^q)^{u_\ell}}$$

As $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ is a generalized parameter where $\mu_{\tilde{h}}, \eta_{\tilde{h}} \in [0, 1]$, then

$$\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}} \leq \mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell^*}^q)^{u_\ell}} \quad (3)$$

Next

$$\begin{aligned} \eta_{d_\ell} \geq \eta_{d_\ell^*} &\Rightarrow \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \geq \prod_{\ell=1}^n (\eta_{d_\ell^*}^{u_\ell})^q \\ &\Rightarrow (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \geq (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell^*}^{u_\ell})^q \\ &\Rightarrow \eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q \geq \eta_{\tilde{h}}^q \\ &\quad + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell^*}^{u_\ell})^q \\ &\Rightarrow \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \\ &\geq \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell^*}^{u_\ell})^q} \end{aligned} \quad (4)$$

Let $Gq-ROFWA(< d_1, d_2, d_3, \dots, d_n >, \tilde{h}) = d$ and $Gq-ROFWA(< d_1^*, d_2^*, d_3^*, \dots, d_n^* >, \tilde{h}) = d^*$. Now by Eqs. (3), (4) and Part **i**: (idempotency), we have

$$\begin{aligned} &(\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}\right) \\ &\quad - \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q\right) \\ &\leq (\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}\right) \\ &\quad - \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q\right) \\ &\Rightarrow S(d^*) \geq S(d) \end{aligned}$$

If $S(d^*) > S(d)$, then

$$Gq-ROFWA(< d_1^*, d_2^*, d_3^*, \dots, d_n^* >, \tilde{h}) > Gq-ROFWA(< d_1, d_2, d_3, \dots, d_n >, \tilde{h})$$

If $S(d^*) = S(d)$, then

$$\begin{aligned} &(\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell^*}^q)^{u_\ell}\right) - \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell^*}^{u_\ell})^q\right) \\ &= (\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}\right) \\ &\quad - \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q\right). \text{ Since } \mu_{d_\ell} \leq \mu_{d_\ell^*} \text{ and } \eta_{d_\ell} \geq \eta_{d_\ell^*} \text{ for all } \ell, \text{ so we have} \\ &\Rightarrow (\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}\right) \\ &= (\mu_{\tilde{h}})^q \cdot \left(1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}\right) \text{ and} \\ &\left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q\right) = \left(\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q\right). \end{aligned}$$

Thus by accuracy function

$$A(d^*) = A(d)$$

Therefore, from the above analysis, we have

$$\begin{aligned} Gq-ROFWA(< d_1, d_2, d_3, \dots, d_n >, \tilde{h}) &\geq \\ Gq-ROFWA(< d_1^*, d_2^*, d_3^*, \dots, d_n^* >, \tilde{h}) &\end{aligned}$$

iv: Proof is straightforward and a trivial case of Theorem 4. □

If it is suppose that the experts have enough knowledge about the evaluated objects and assign value to generalized parameter $\tilde{h} = (1, 0)$. In this case, the proposed $Gq-ROFWA$ operator reduces to $q-ROFWA$ operator. While these types of situations are partially fulfilled in real life. This shortcoming is very carefully tackled by adding the evaluated object/generalized parameter in the initial given preferences.

Proposition 1 (a) *If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (1, 0)$, then the proposed $Gq-ROFWA$ operator reduces to $q-ROFWA$ operator.*

(b) *If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (0, 1)$, then the proposed $Gq-ROFWA$ operator gives the result $(0, 1)$.*

Proof (a) If it is assumed that the experts have enough knowledge about the evaluated objects, that is the generalized parameter $\tilde{h} = (1, 0)$, then in Theorem 4, we have

$$\begin{aligned} &Gq-ROFWA(< d_1, d_2, d_3, \dots, d_n >, \tilde{h}) \\ &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \right. \\ &\quad \left. \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

$$\begin{aligned}
 &= \left(\sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right) \\
 &\text{as } \mu_{\tilde{h}} = 1 \text{ and } \eta_{\tilde{h}} = 0 \\
 &= \left(\sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^n \eta_{d_\ell}^{u_\ell} \right) \\
 &= q\text{-ROFOWA}(d_1, d_2, d_3, \dots, d_n).
 \end{aligned}$$

(b) The proof is similar to the Proof of (a). □

The generalized q-rung orthopair fuzzy ordered weighted average operator (Gq-ROFOWA)

From Gq-ROFOWA, it is clear that in Gq-ROFOWA operators just the q-ROF values are weighed on the basis of generalized parameter, and the Gq-ROFOWA operator weight the ordered positions through scoring the q-ROF values rather than weighting the q-ROF values themselves on the basis of generalized parameter. Therefore, here, we will present the detailed study of Gq-ROFOWA operator and their properties.

Definition 10 Consider a generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$), then the Gq-ROFOWA operator is given as

$$\begin{aligned}
 &Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \\
 &= \tilde{h} \otimes q\text{-ROFOWA}(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n).
 \end{aligned}$$

The aggregation result of Definition 10 through operation rules is described as in Theorem 6.

Theorem 6 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q-ROFNs with generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ and weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, then the Gq-ROFOWA operator is described as:

$$\begin{aligned}
 &Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \\
 &= \tilde{h} \otimes \left(\bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \right) \\
 &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right)
 \end{aligned}$$

where $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})$ indicate the permutation of ℓ th largest object of the collection of n q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$).

Proof The proof is straightforward to Theorem 4. □

Remark 2 (a) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 1$, then the Gq-ROFOWA operator reduces to IFOFWA operator.

(b) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 2$, then the Gq-ROFOWA operator reduces to PFOFWA operator.

(c) If the value of $q = 2$ is fixed then the Gq-ROFOWA operator reduces to GPFOFWA operator.

Example 5 Suppose the generalized parameter $\tilde{h} = (0.95, 0.7)$ of four q-ROFNs $d_1 = (0.7, 0.3)$, $d_2 = (0.8, 0.65)$, $d_3 = (0.9, 0.6)$, $d_4 = (0.88, 0.7)$ having weight vector $u = \{0.3, 0.25, 0.1, 0.35\}$ for $q = 5$, then to find the score functions of each q-ROFNs, that is

$$\begin{aligned}
 &S(d_1) = 0.7^5 - 0.3^5 = 0.166, S(d_2) = 0.8^5 - 0.65^5 = 0.212, \\
 &S(d_3) = 0.9^5 - 0.6^5 = 0.513, S(d_4) = 0.88^5 - 0.7^5 = 0.360.
 \end{aligned}$$

So $S(d_3) > S(d_4) > S(d_2) > S(d_1)$, this implies that $d_4 > d_3 > d_2 > d_1$. Thus $\tilde{d}_1 = d_3, \tilde{d}_2 = d_4, \tilde{d}_3 = d_2$, and $\tilde{d}_4 = d_1$. Further we have

$$\begin{aligned}
 &\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^q)^{u_\ell}} = 0.95 \times \\
 &\sqrt[q]{1 - (1 - 0.9^5)^{0.3} (1 - 0.88^5)^{0.25} (1 - 0.8^5)^{0.1}} = 0.802 \\
 &\sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^4 (\eta_{d_\ell}^{u_\ell})^q} = \\
 &\sqrt[q]{0.7^5 + (1 - 0.7^5)(0.6^{0.3})^5 (0.70^{0.25})^5 (0.65^{0.1})^5 (0.3^{0.35})^5} = 0.720
 \end{aligned}$$

Now by Theorem 6, we have

$$\begin{aligned}
 &Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, d_4 \rangle, \tilde{h}) \\
 &= \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^4 (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^4 (\eta_{d_\ell}^{u_\ell})^q} \right) \\
 &= (0.802, 0.720)
 \end{aligned}$$

Theorem 7 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q-ROFNs with generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ and associated weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, then for Gq-ROFOWA operator the following are holds:

i: (Idempotency): If $\tilde{d}_\ell = \tilde{d}$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = \tilde{h} \otimes \tilde{d}.$$

ii: (Boundary condition): If $\tilde{d}_\ell^- = (\min \mu_{\tilde{h} \otimes \tilde{d}_\ell}, \max \eta_{\tilde{h} \otimes \tilde{d}_\ell})$ and $\tilde{d}_\ell^+ = (\max \mu_{\tilde{h} \otimes \tilde{d}_\ell}, \min \eta_{\tilde{h} \otimes \tilde{d}_\ell})$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$\tilde{d}_\ell^- \leq Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \leq \tilde{d}_\ell^+.$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell^*}, \eta_{d_\ell^*}) (\ell = 1, 2, \dots, n)$ be the collection of n q -ROFNs such that $\mu_{d_\ell} \leq \mu_{d_\ell^*}$ and $\eta_{d_\ell} \geq \eta_{d_\ell^*}$, then

$$Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) \leq Gq\text{-ROFOWA}(\langle d_1^*, d_2^*, d_3^*, \dots, d_n^* \rangle, \tilde{h}).$$

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell}) (\ell = 1, 2, \dots, n)$ be the collection of n q -ROFNs where $\tilde{d}_\ell (\ell = 1, 2, \dots, n)$ is any permutation of $d_\ell (\ell = 1, 2, \dots, n)$, then

$$Gq\text{-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = Gq\text{-ROFOWA}(\langle \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n \rangle, \tilde{h}).$$

Proof Proofs are easy and can be done from Theorem 5. \square

Proposition 2 (a) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (1, 0)$, so in this case the proposed Gq -ROFOWA operator degenerates to q -ROFOWA operator.

(b) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (0, 1)$, so in this case the proposed Gq -ROFOWA operator gives the result $(0, 1)$.

Proof The proofs are similar to the Proposition 1. \square

The generalized q -rung orthopair fuzzy hybrid aggregation (Gq -ROFHA) operator

From the detail discussion of Gq -ROFWA and Gq -ROFOWA operators, it is clear that in Gq -ROFWA operators just the q -ROF values are weighed on the basis of generalized parameter, and similarly in Gq -ROFOWA operator just the ordered positions of the q -ROF values are weighed rather than weighting the q -ROF values themselves on the base of generalized parameter. So, it is clear that the weights denote distinct attributes in both Gq -ROFWA and Gq -ROFOWA operators. However, at the same time both the operators weigh just one of them. To handle this restriction, here we will originate the concept of Gq -ROFHA operator, which weighs both the given values at the same time, that is q -ROF values and its ordered position on the basis of generalized parameter and discuss their properties in detail.

Definition 11 Consider $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$), be the collections of q -ROFNs with associated weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, under generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ and $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)^T$ be the weight vector of $d_\ell (\ell =$

$1, 2, \dots, n)$ where $\tilde{u}_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n \tilde{u}_\ell = 1$. Then, the Gq -ROFHA operator is given as:

$$Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = \tilde{h} \otimes q\text{-ROFHA}(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n).$$

The aggregation result of Definition 11 through operation rules is described as in Theorem 8.

Theorem 8 Consider a generalized parameter $\tilde{h} = (\mu_{\tilde{h}}, \eta_{\tilde{h}})$ for the q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ (for $\ell = 1, 2, \dots, n$), then the Gq -ROFHA operator is described as:

$$Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \tilde{h}) = \tilde{h} \otimes \left(\bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \right) = \left(\mu_{\tilde{h}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{\tilde{d}_\ell}^q)^{u_\ell}}, \sqrt[q]{\eta_{\tilde{h}}^q + (1 - \eta_{\tilde{h}}^q) \prod_{\ell=1}^n (\eta_{\tilde{d}_\ell}^{u_\ell})^q} \right)$$

where $\tilde{d}_\ell (\tilde{d}_\ell = n\tilde{u}_\ell d_\ell, \text{ for } \ell = 1, 2, \dots, n)$ indicate the permutation of ℓ th largest object of the collection of q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$ such that n indicate the balancing coefficient.

Proof Straightforward like Theorem 4. \square

Remark 3 (a) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 1$, so in this case Gq -ROFHA operator degenerates to IFHA operator.

(b) If the generalized parameter $\tilde{h} = (1, 0)$, and $q = 2$, so in this case Gq -ROFHA operator degenerates to PFHA operator. (c) If the value of parameter $q = 2$ is fixed then the Gq -ROFHA operator reduces to GPFA operator.

Example 6 Suppose the generalized parameter $\tilde{h} = (0.8, 0.7)$ of four q -ROFNs $d_1 = (0.5, 0.2)$, $d_2 = (0.83, 0.6)$, $d_3 = (0.95, 0.65)$, $d_4 = (0.9, 0.75)$ having associated weight vector $u = \{0.3, 0.2, 0.32, 0.18\}$ for $q = 4$, and weight vector $\tilde{u} = \{0.4, 0.3, 0.1, 0.2\}$, then using operational law

$$n\tilde{u}_\ell d_\ell = \left(\sqrt[q]{1 - (1 - \mu_{d_\ell}^q)^{n\tilde{u}_\ell}}, \eta_{d_\ell}^{n\tilde{u}_\ell} \right)$$

we have

$$4\tilde{u}_1 d_1 = \left(\sqrt[4]{1 - (1 - 0.5^4)^{4 \times 0.4}}, 0.2^{4 \times 0.4} \right) = (0.560, 0.076),$$

$$4\tilde{u}_2 d_2 = \left(\sqrt[4]{1 - (1 - 0.83^4)^{4 \times 0.3}}, 0.6^{4 \times 0.3} \right) = (0.856, 0.542),$$

$$4\tilde{u}_3 d_3 = \left(\sqrt[4]{1 - (1 - 0.95^4)^{4 \times 0.1}}, 0.65^{4 \times 0.1} \right) = (0.837, 0.842),$$

$$4\check{u}_4d_4 = \left(\sqrt[4]{1 - (1 - 0.9^4)^{4 \times 0.2}}, 0.75^{4 \times 0.2} \right) = (0.870, 0.794).$$

Now to find their score functions, that is

$$\begin{aligned} S(4\check{u}_1d_1) &= 0.560^4 - 0.076^4 = 0.098, \\ S(4\check{u}_2d_2) &= 0.856^4 - 0.542^4 = 0.451, \\ S(4\check{u}_3d_3) &= 0.837^4 - 0.842^4 \\ &= -0.012, \\ S(4\check{u}_4d_4) &= 0.0.870^4 - 0.794^4 = -0.396. \end{aligned}$$

So $S(4\check{u}_2d_2) > S(4\check{u}_1d_1) > S(4\check{u}_3d_3) > S(4\check{u}_4d_4)$, this implies that

$$\begin{aligned} \check{d}_1 &= (0.856, 0.542), \check{d}_2 = (0.560, 0.076), \\ \check{d}_3 &= (0.837, 0.842) \text{ and } \check{d}_4 = (0.870, 0.794) \end{aligned}$$

Furthermore, we have $\mu_{\check{h}} \cdot \sqrt[4]{1 - \prod_{\ell=1}^4 (1 - \mu_{\check{d}_\ell}^4)^{u_\ell}} = 0.8 \times 0.824648 = 0.65972$, and $\sqrt[4]{\eta_{\check{h}}^4 + (1 - \eta_{\check{h}}^4) \prod_{\ell=1}^4 (\eta_{\check{d}_\ell}^{u_\ell})^4} = 0.72192$

Now by Theorem 8, we have

$$\begin{aligned} Gq\text{-ROFHA}(\langle d_1, d_2, d_3, d_4 \rangle, \check{h}) &= \left(\mu_{\check{h}} \cdot \sqrt[4]{1 - \prod_{\ell=1}^4 (1 - \mu_{\check{d}_\ell}^4)^{u_\ell}}, \sqrt[4]{\eta_{\check{h}}^4 + (1 - \eta_{\check{h}}^4) \prod_{\ell=1}^4 (\eta_{\check{d}_\ell}^{u_\ell})^4} \right) \\ &= (0.65972, 0.72192) \end{aligned}$$

Theorem 9 Suppose the collection $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ ($\ell = 1, 2, \dots, n$) of q -ROFNs with associated weight vector $u = (u_1, u_2, \dots, u_n)^T$ of d_ℓ where $u_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n u_\ell = 1$, on the base of generalized parameter $\check{h} = (\mu_{\check{h}}, \eta_{\check{h}})$ and $\check{u} = (\check{u}_1, \check{u}_2, \dots, \check{u}_n)^T$ be the weight vector of d_ℓ ($\ell = 1, 2, \dots, n$) where $\check{u}_\ell \in [0, 1]$ such that $\sum_{\ell=1}^n \check{u}_\ell = 1$ then for Gq -ROFHA operator the following assertions are hole:

i: (Idempotency): If $\check{d}_\ell = \check{d}$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \check{h}) = \check{h} \otimes \check{d}.$$

ii: (Boundary condition): If $\check{d}_\ell^- = (\min \mu_{\check{h} \otimes \check{d}_\ell}, \max \eta_{\check{h} \otimes \check{d}_\ell})$ and $\check{d}_\ell^+ = (\max \mu_{\check{h} \otimes \check{d}_\ell}, \min \eta_{\check{h} \otimes \check{d}_\ell})$ (for all $\ell = 1, 2, 3, \dots, n$), then

$$\check{d}_\ell^- \leq Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \check{h}) \leq \check{d}_\ell^+.$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell^*}, \eta_{d_\ell^*})$ ($\ell = 1, 2, \dots, n$) be two collections of n q -ROFNs in which $\mu_{d_\ell} \leq \mu_{d_\ell^*}$ and $\eta_{d_\ell} \geq \eta_{d_\ell^*}$, then

$$\begin{aligned} Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \check{h}) \\ \leq Gq\text{-ROFHA}(\langle d_1^*, d_2^*, d_3^*, \dots, d_n^* \rangle, \check{h}). \end{aligned}$$

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $\check{d}_\ell = (\mu_{\check{d}_\ell}, \eta_{\check{d}_\ell})$ ($\ell = 1, 2, \dots, n$) be two collection of n q -ROFNs in which \check{d}_ℓ ($\ell = 1, 2, \dots, n$) is the ℓ th largest object of d_ℓ ($\ell = 1, 2, \dots, n$), then

$$\begin{aligned} Gq\text{-ROFHA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \check{h}) \\ = Gq\text{-ROFHA}(\langle \check{d}_1, \check{d}_2, \check{d}_3, \dots, \check{d}_n \rangle, \check{h}). \end{aligned}$$

Proof Proofs are easy and can be done from Theorem 5. \square

Proposition 3 (a) If the priority of the senior decision maker/expert about generalized parameter is taken as $\check{h} = (1, 0)$, then the proposed Gq -ROFHA operator reduces to q -ROFHA operator.

(b) If the priority of the senior decision maker/expert about generalized parameter is taken as $\check{h} = (0, 1)$, so in this case the proposed Gq -ROFHA operator gives the result $(0, 1)$.

Proof Proofs are straightforward. \square

Remark 4 (a) If $\check{u} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the proposed Gq -ROFHA operator reduces to Gq -ROFWA operator.

(b) If $u = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the proposed Gq -ROFHA operator reduces to Gq -ROFOWA operator.

Group generalized parameter based on q-rung orthopair fuzzy average aggregation operator

In this section, we will present the generalized study of the proposed aggregation operators. This analysis will based on two or more expert’s/observer’s opinion in original information by combining the different choice and expertise of the senior decision makers/experts in a more accurate way. Therefore, this can be obtained by introducing a group Gq -ROFWA (GGq-ROFWA) operator, group Gq -ROFOWA (GGq-ROFOWA) operator and group Gq -ROFHA (GGq-ROFHA) operator.

The GGq-ROFWA operator

In this subsection, the idea of generalized q -ROFWA operator is explored to group generalized q -ROFWA operator where the preferences of two or more other senior experts/decision makers are analyze in q -ROF environments.

Definition 12 Consider a group of experts/observers who justify the information under the q -ROF environment. Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the preferences suggested by the senior experts for the q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, then the GG q -ROFWA operators is given as:

$$\text{GG}q\text{-ROFWA}(\langle d_1, d_2, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = q\text{-ROFWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m) \otimes q\text{-ROFWA}(d_1, d_2, \dots, d_n)$$

The aggregation result for q -ROFNs through operation rules is described as in Theorem 10.

Theorem 10 Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the preferences suggested by the senior experts for the q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated weight vector for q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then GG q -ROFWA operator is given as

$$\begin{aligned} &\text{GG}q\text{-ROFWA}(\langle d_1, d_2, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\ &= \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes \left(\bigoplus_{\ell=1}^n u_\ell d_\ell \right) \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q\right) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

Proof We use mathematical induction to prove this theorem. Now for $n = 2$ we get

$$\begin{aligned} &\text{GG}q\text{-ROFWA}(\langle d_1, d_2 \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\ &= \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes (u_1 d_1 \oplus u_2 d_2) \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \otimes (u_1 d_1 \oplus u_2 d_2) \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^2 (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^2 \eta_{d_\ell}^{u_\ell} \right) \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^2 (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^2 (\eta_{d_\ell}^{u_\ell})^q \right) \end{aligned}$$

$$\sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q\right) \prod_{\ell=1}^2 (\eta_{d_\ell}^{u_\ell})^q}$$

this implies for $n = 2$ the result holds.

Next assume that the result holds for $n = K$, that is

$$\text{GG}q\text{-ROFWA}(\langle d_1, d_2, \dots, d_K \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes \left(\bigoplus_{\ell=1}^K u_\ell d_\ell \right)$$

$$\begin{aligned} &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^K (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^K \eta_{d_\ell}^{u_\ell} \right) \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^K (1 - \mu_{d_\ell}^q)^{u_\ell}}, \sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q\right) \prod_{\ell=1}^K (\eta_{d_\ell}^{u_\ell})^q} \right) \end{aligned}$$

Now to show that the result hold for $n = K + 1$, then we have

$$\begin{aligned} &\text{GG}q\text{-ROFWA}(\langle d_1, d_2, \dots, d_K, d_{K+1} \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\ &= \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes (u_1 d_1 \oplus u_2 d_2 \oplus \dots \oplus u_K d_K \oplus u_{K+1} d_{K+1}) \\ &= \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes \{(u_1 d_1 \oplus u_2 d_2 \oplus \dots \oplus u_K d_K) \oplus u_{K+1} d_{K+1}\} \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \otimes \left\{ \left(\sqrt[q]{1 - \prod_{\ell=1}^K (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^K (\eta_{d_\ell}^{u_\ell}) \right) \oplus \left(\sqrt[q]{1 - (1 - \mu_{d_{K+1}}^q)^{u_{K+1}}}, \eta_{d_{K+1}}^{u_{K+1}} \right) \right\} \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^K (1 - \mu_{d_\ell}^q)^{u_\ell}} \cdot (1 - \mu_{d_{K+1}}^q)^{u_{K+1}}, \prod_{\ell=1}^K (\eta_{d_\ell}^{u_\ell}) \cdot \eta_{d_{K+1}}^{u_{K+1}} \right) \end{aligned}$$

$$\begin{aligned}
 &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}}, \prod_{k=1}^m \eta_{\tilde{h}_k}^{\hat{u}_k} \right) \\
 &\otimes \left(\sqrt[q]{1 - \prod_{\ell=1}^{K+1} (1 - \mu_{d_\ell}^q)^{u_\ell}}, \prod_{\ell=1}^{K+1} \eta_{d_\ell}^{u_\ell} \right) \\
 &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^{K+1} (1 - \mu_{d_\ell}^q)^{u_\ell}}, \right. \\
 &\quad \left. \sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q\right) \prod_{\ell=1}^{K+1} (\eta_{d_\ell}^{u_\ell})^q} \right) \text{ this implies that } n \text{ is}
 \end{aligned}$$

true for $K + 1$. Therefore, the given result is holds for any number of q-ROFNs on the based of expert preferences. \square

Moreover, the aggregated result achieved from GGq-ROFWA is also a q-ROFNs.

Remark 5 (a) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 1$, so in this case the GGq-ROFWA operator degenerates to IFWA operator.

(b) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 2$, then the GGq-ROFWA operator degenerates to PFWA operator. (c) If the assign value $q = 2$ is fixed then the GGq-ROFWA operator reduces to GGPFWA operator.

Theorem 11 Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k})(k = 1, 2, \dots, m)$ be the preferences suggested by the senior experts for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})(\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated weight vector for q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then for GGq-ROFWA operator the following assertions are hold:

i: (Idempotency): If $d_\ell = d$ (for all $\ell = 1, 2, 3, \dots, n$), and $\tilde{h}_k = \tilde{h}$ (for all $k = 1, 2, \dots, m$) then

$$\begin{aligned}
 &GGq\text{-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \\
 &(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = \tilde{h} \otimes d.
 \end{aligned}$$

ii: (Boundary condition): If $d_\ell^- = (\min \mu_{\tilde{h}_k \otimes d_\ell}, \max \eta_{\tilde{h}_k \otimes d_\ell})$ and $d_\ell^+ = (\max \mu_{\tilde{h}_k \otimes d_\ell}, \min \eta_{\tilde{h}_k \otimes d_\ell})(\forall \ell = 1, 2, 3, \dots, n)$ ($\forall k = 1, 2, 3, \dots, m$), then

$$\begin{aligned}
 &d_\ell^- \leq GGq\text{-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \\
 &(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq d_\ell^+.
 \end{aligned}$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell^*}, \eta_{d_\ell^*})(\ell = 1, 2, \dots, n)$ be two collection of n q-ROFNs such that $\mu_{d_\ell} \leq \mu_{d_\ell^*}$ and $\eta_{d_\ell} \geq \eta_{d_\ell^*}$, then $GGq\text{-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq GGq\text{-ROFWA}(\langle d_1^*, d_2^*, d_3^*, \dots, d_n^* \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$.

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})(\ell = 1, 2, \dots, n)$ be two collection of n q-ROFNs where $\tilde{d}_\ell(\ell = 1, 2, \dots, n)$ is the ℓ th largest object of d_ℓ , then

$$\begin{aligned}
 &GGq\text{-ROFWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \\
 &(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = GGq\text{-ROFWA} \\
 &(\langle \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)).
 \end{aligned}$$

Proof Straightforward. \square

Proposition 4 (a) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h}_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), then the proposed GGq-ROFWA operator reduces to q-ROFWA operator.

(b) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h}_k = (0, 1)$ (for all $k = 1, 2, \dots, m$), then the proposed GGq-ROFWA operator presents the same result $(0, 1)$.

Proof Proofs are straightforward. \square

The GGq-ROFOWA operator

From GGq-ROFWA, it is clear that in GGq-ROFWA operators just the q-ROF values are weighed on the basis of group generalized parameter, while the GGq-ROFOWA operator weights the ordered positions after scoring the q-ROF values rather than weighting the q-ROF values themselves on the base of group generalized parameter. Therefore, here, we will present the detail study of GGq-ROFOWA operator and their properties.

Definition 13 Consider a group of experts/observers who justify the information under the q-ROF environment. Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k})(k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})(\ell = 1, 2, \dots, n)$, then the GGq-ROFOWA operators is given as:

$$\begin{aligned}
 &GGq\text{-ROFOWA}(\langle d_1, d_2, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\
 &= q\text{-ROFOWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m) \otimes \\
 &= q\text{-ROFOWA}(d_1, d_2, \dots, d_n)
 \end{aligned}$$

The aggregation result for q-ROFNs through operation rules is described as in Theorem 12.

Theorem 12 Suppose that $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k})(k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})(\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated

weight vector for q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then GGq-ROFOWA operator is given as:

$$\begin{aligned} & \text{GGq-ROFOWA}(\langle d_1, d_2, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\ &= \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes \left(\bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \right). \\ &= \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{\tilde{d}_\ell}^q)^{u_\ell}}, \right. \\ & \left. \sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^q)^{\hat{u}_k} + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^q)^{\hat{u}_k}\right) \prod_{\ell=1}^n (\eta_{\tilde{d}_\ell}^q)^{u_\ell}} \right) \end{aligned}$$

where $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell})$ indicate the ℓ th largest object of n q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$.

Proof Proof is easy and directly follows from Theorem 10. \square

Remark 6 (a) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 1$, then the GGq-ROFOWA operator reduces to IFWA operator.

(b) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 2$, then the GGq-ROFOWA operator reduces to PFWA operator.

(c) If the value of parameter $q = 2$ is fixed then the GGq-ROFOWA operator reduces to GGPFOWA operator.

Theorem 13 Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated weight vector for q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then the GGq-ROFOWA operator has the properties:

i: (Idempotency): If $\tilde{d}_\ell = \tilde{d}$ (for all $\ell = 1, 2, 3, \dots, n$), and $\tilde{h}_k = \tilde{h}$ (for all $k = 1, 2, \dots, m$) then

$$\begin{aligned} & \text{GGq-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \\ & (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \\ &= \tilde{h} \otimes \tilde{d}. \end{aligned}$$

ii: (Boundary condition): If $\tilde{d}_\ell^- = \left(\min \mu_{\tilde{h}_k \otimes \tilde{d}_\ell}, \max \eta_{\tilde{h}_k \otimes \tilde{d}_\ell} \right)$ and $\tilde{d}_\ell^+ = \left(\max \mu_{\tilde{h}_k \otimes \tilde{d}_\ell}, \min \eta_{\tilde{h}_k \otimes \tilde{d}_\ell} \right)$ ($\forall \ell = 1, 2, 3, \dots, n$) ($\forall k = 1, 2, 3, \dots, m$), then

$$\begin{aligned} & \tilde{d}_\ell^- \leq \text{GGq-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, \\ & (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq \tilde{d}_\ell^+. \end{aligned}$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell^*}, \eta_{d_\ell^*}) (\ell = 1, 2, \dots, n)$ be two collection of n q -ROFNs such that $\mu_{d_\ell} \leq \mu_{d_\ell^*}$ and $\eta_{d_\ell} \geq \eta_{d_\ell^*}$, then $\text{GGq-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq \text{GGq-ROFOWA}(\langle d_1^*, d_2^*, d_3^*, \dots, d_n^* \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$.

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $\tilde{d}_\ell = (\mu_{\tilde{d}_\ell}, \eta_{\tilde{d}_\ell}) (\ell = 1, 2, \dots, n)$ be two collection of n q -ROFNs where $\tilde{d}_\ell (\ell = 1, 2, \dots, n)$ is represents the ℓ th largest object of d_ℓ , then $\text{GGq-ROFOWA}(\langle d_1, d_2, d_3, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = \text{GGq-ROFOWA}(\langle \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$.

Proof Proofs is straightforward. \square

Proposition 5 (a) If the priorities/preferences of the senior decision makers/experts about generalized parameters are taken as $\tilde{h}_k = (1, 0) (k = 1, 2, \dots, m)$, then the proposed GGq-ROFOWA operator reduces to q -ROFOWA operator.

(b) If the priorities/preferences of the senior decision makers/experts about generalized parameters are taken as $\tilde{h}_k = (0, 1) (k = 1, 2, \dots, m)$, then the proposed GGq-ROFOWA operator presents the same result $(0, 1)$.

Proof Straightforward. \square

The GGq-ROFHA operator

From the above discussion, it is concluded that the GGq-ROFOWA operator just weights the q -ROF values on the basis of group generalized parameter, and similarly the GGq-ROFOWA operator just weights the ordered positions after scoring the q -ROF values rather than weighing the q -ROF values themselves on the basis of group generalized parameter. Therefore, it is clear that the weights denote distinct attributes in both GGq-ROFOWA and GGq-ROFOWA operators. However, at the same time both the operators weighs just one of them. To handle this restriction, here we will originate the study of GGq-ROFHA operator, which weighs both the given values at the same time, that is q -ROF values and its ordered position on the basis of group generalized parameter.

Definition 14 Consider a group of experts/observers who justify the information under the q -ROF environment. Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q -ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, then the GGq-ROFHA operator is given as:

$$\text{GGq-ROFHA}(\langle d_1, d_2, \dots, d_n \rangle, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$$

$$= q\text{-ROFHA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m) \otimes q\text{-ROFHA}(d_1, d_2, \dots, d_n)$$

The aggregation result for q-ROFNs through operation rules is described as in Theorem 14.

Theorem 14 Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)^T$ with $\sum_{\ell=1}^n \tilde{u}_\ell = 1$ where $\tilde{u}_\ell \in [0, 1]$, be the weight vector and $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated weight vector of q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then GGq-ROFHA operator is given as;

$$GGq\text{-ROFHA}(< d_1, d_2, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = \left(\bigoplus_{k=1}^m \hat{u}_k \tilde{h}_k \right) \otimes \left(\bigoplus_{\ell=1}^n u_\ell \tilde{d}_\ell \right) = \left(\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{\tilde{d}_\ell}^q)^{u_\ell}}, \sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k})^{\hat{u}_k} + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k})^{\hat{u}_k}\right) \prod_{\ell=1}^n (\eta_{\tilde{d}_\ell})^q} \right)$$

where $\tilde{d}_\ell (\tilde{d}_\ell = n\tilde{u}_\ell d_\ell, \text{ for } \ell = 1, 2, \dots, n)$ indicate the permutation which is the ℓ th largest object of the collection of q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$ and n indicate the balancing coefficient.

Remark 7 (a) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 1$, then the GGq-ROFHA operator reduces to IFHA operator.

(b) If the generalized parameter $p_k = (1, 0)$ (for all $k = 1, 2, \dots, m$), and $q = 2$, then the GGq-ROFHA operator reduces to PFHA operator.

(c) If the value of parameter $q = 2$ is fixed then the GGq-ROFHA operator reduces to GGPFHA operator.

Theorem 15 Let $\tilde{h}_k = (\mu_{\tilde{h}_k}, \eta_{\tilde{h}_k}) (k = 1, 2, \dots, m)$ be the priorities/preferences suggested by the senior experts for the q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$, having weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$ with $\sum_{k=1}^m \hat{u}_k = 1$ where $\hat{u}_k \in [0, 1]$. Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)^T$ with $\sum_{\ell=1}^n \tilde{u}_\ell = 1$ where $\tilde{u}_\ell \in [0, 1]$, be the weight vector and $u = (u_1, u_2, \dots, u_n)^T$ with $\sum_{\ell=1}^n u_\ell = 1$ where $u_\ell \in [0, 1]$, be the associated weight vector of q-ROFNs $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$, then for GGq-ROFHA operator the following conditions are hold:

i: (Idempotency): If $d_\ell = \tilde{d}$ (for all $\ell = 1, 2, 3, \dots, n$), and $\tilde{h}_k = \tilde{h}$ (for all $k = 1, 2, \dots, m$) then

$$GGq\text{-ROFHA}(< d_1, d_2, d_3, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = \tilde{h} \otimes \tilde{d}.$$

ii: (Boundary condition): If $\tilde{d}_\ell^- = \left(\min \mu_{\tilde{h}_k \otimes \tilde{d}_\ell}, \max \eta_{\tilde{h}_k \otimes \tilde{d}_\ell} \right)$ and $\tilde{d}_\ell^+ = \left(\max \mu_{\tilde{h}_k \otimes \tilde{d}_\ell}, \min \eta_{\tilde{h}_k \otimes \tilde{d}_\ell} \right)$ ($\forall \ell = 1, 2, 3, \dots, n$) ($\forall k = 1, 2, 3, \dots, m$), then

$$\tilde{d}_\ell^- \leq GGq\text{-ROFHA}(< d_1, d_2, d_3, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq \tilde{d}_\ell^+.$$

iii: (Monotonicity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell})$ and $d_\ell^* = (\mu_{d_\ell^*}, \eta_{d_\ell^*}) (\ell = 1, 2, \dots, n)$ be two collection of n q-ROFNs such that $\mu_{d_\ell} \leq \mu_{d_\ell^*}$ and $\eta_{d_\ell} \geq \eta_{d_\ell^*}$, then $GGq\text{-ROFHA}(< d_1, d_2, d_3, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) \leq GGq\text{-ROFHA}(< d_1^*, d_2^*, d_3^*, \dots, d_n^* >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$.

iv: (Commutativity): Suppose $d_\ell = (\mu_{d_\ell}, \eta_{d_\ell}) (\ell = 1, 2, \dots, n)$ be two collection of n q-ROFNs where $\tilde{d}_\ell (\ell = 1, 2, \dots, n)$ is any permutation of d_ℓ , then $GGq\text{-ROFHA}(< d_1, d_2, d_3, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m)) = GGq\text{-ROFHA}(< \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$.

Proof Proofs are straightforward. □

Proposition 6 (a) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (1, 0)$, so in this case the proposed GGq-ROFHA operator degenerates to q-ROFHA operator.

(b) If the priority of the senior decision maker/expert about generalized parameter is taken as $\tilde{h} = (0, 1)$, then the proposed GGq-ROFHA operator gives the result $(0, 1)$.

Remark 8 (a) If $\tilde{u} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, so in this case the improved GGq-ROFHA operator degenerates to GGq-ROFHA operator.

(b) If $u = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, so in this case the proposed GGq-ROFHA operator degenerates to GGq-ROFHA operator.

An approach to MCDM and its application based on the generalization parameter

In this section, the technique of MCDM is constructed on the concept of GGq-ROF information under generalized parameter. The general concept and steps of construction of the developed approach are given below.

MCDM approach

To demonstrate the MCDM approach on the basis of developed approach, let $\mathcal{X} = \{d_1, d_2, \dots, d_m\}$ be a certain discrete set of alternatives and $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ be the corresponding set of criteria having weight vector $u = \{u_1, u_2, \dots, u_n\}$ where $u_i \in [0, 1]$ with $\sum_{i=1}^n u_i = 1$, on the base of generalized parameter. A team of experts is called to for the judgement of each alternative $d_i (i = 1, 2, \dots, m)$ to their corresponding criteria $c_j (j = 1, 2, \dots, n)$. The experts give their assessment details in the form of q-ROFNs denoted by $d_{ij} = (\mu_{d_{ij}}, \eta_{d_{ij}})$ where in points of view of experts $\mu_{d_{ij}}$ represents the membership and $\eta_{d_{ij}}$ represents the non-membership grades to which alternative d_{ij} satisfies the the criteria c_j having the condition that $0 \leq (\mu_{d_{ij}})^q + (\eta_{d_{ij}})^q \leq 1$ for $q \geq 1$.

To certify the collected information in more accurate manner, the group $\{\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_k\}$ of other senior experts are constituted which provide their preferences/ priorities for each alternative in the form of q-ROFNs represented by $\tilde{h}_l = (\mu_{\tilde{h}_l}, \eta_{\tilde{h}_l}) (l = 1, 2, \dots, k)$, with weight vector $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_k)^T$ such that $\hat{u}_l \in [0, 1]$ with $\sum_{l=1}^k \hat{u}_l = 1$.

Algorithm

The algorithm for the developed operator consists of the following steps.

step 1 From the above analysis collect the decision makers/expert's information provided for each alternative to their corresponding criteria and then construct a decision matrix

$$[\mathcal{X}]_{m \times n} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{pmatrix},$$

step 2 Collect the priorities/preferences of the senior's group experts against each alternatives under the generalized parameter and from these information construct the generalized parameter matrix

$$[\mathcal{Y}]_{m \times k} = \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1k} \\ \tilde{h}_{21} & \tilde{h}_{22} & \cdots & \tilde{h}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{m1} & \tilde{h}_{m2} & \cdots & \tilde{h}_{mk} \end{pmatrix},$$

step 3 Append the matrices obtained from steps 1: and 2: to get the new matrix $[\mathcal{Z}]_{m \times (n+k)}$ which shows the priorities/preferences of expert's for each alternatives d_{ij}

corresponding to criteria c_j on the bases of generalized parameter \tilde{h}_l .

$$[\mathcal{Z}]_{m \times (n+k)} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1(n+k)} \\ d_{21} & d_{22} & \cdots & d_{2(n+k)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{m(n+k)} \end{pmatrix},$$

step 4 Utilized the developed GGq-ROFWA operator to get the overall aggregated result from matrix $[\mathcal{Z}]_{m \times (n+k)}$ for alternative d_i ,

step 5 Determine the score of each aggregated result in step 4: and rank the results in a specific order to get the best alternative.

Numerical illustration

This section is devoted for the presentation of an illustrating example to demonstrate the validity and effectiveness of the developed approach with q-ROF information.

Consider Pneumonia is a common disease and initially having four basic symptoms such as *chest pain*, *fever*, *cough* and *fatigue*. The disease of Pneumonia can be treated with four medicines. These medicines have various therapeutic effects on these four symptoms. Now for treating this disease, a doctor advise needs to analyze the best and worst therapeutic effects of these four medicines. Let $\mathcal{X} = \{d_1, d_2, d_3, d_4\}$ denotes the four medicines (alternatives) and $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ represents the fours symptoms (criteria). Furthermore, considering the significance degree of the four symptoms, the doctor provide the weight vector $u = (u_1 = 0.3, u_2 = 0.25, u_3 = 0.18, u_4 = 0.27)^T$ of the criteria set. The doctor presents their evaluation for each medicine (alternative) to their corresponding symptom (criteria) in the form of q-ROFNs which is given in Table 1. To justify the collected information in more accurate manner, consider a group of senior doctors/experts $\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}$ provide their priority/preferences with weight vector $\hat{u} = (\hat{u}_1 = 0.35, \hat{u}_2 = 0.32, \hat{u}_3 = 0.33)^T$ such that $\hat{u}_i \in [0, 1]$ with $\sum_{i=1}^3 \hat{u}_i = 1$. A group of senior doctors/experts provide their preferences report in the form of q-ROFNs which is given in Table 2. The steps-wise algorithm of the presented approach for MCDM is given as

step 1 The collected information of decision maker/expert evaluation $[\mathcal{X}]_{m \times n} = (d_{ij})_{m \times n} = (\mu_{d_{ij}}, \eta_{d_{ij}})_{m \times n}$ for each alternative to their corresponding criteria are given in Table 1.

step 2 The preferences/priorities of the group of other senior decision makers/experts evaluation against each alternatives under the generalized parameter

Table 1 q -ROF expert’s evaluation matrix $[\mathcal{X}]_{m \times n}$

	c_1	c_2	c_3	c_4
d_1	(0.9, 0.2)	(0.8, 0.3)	(0.95, 0.15)	(0.4, 0.2)
d_2	(0.8, 0.5)	(0.7, 0.2)	(0.93, 0.35)	(0.6, 0.1)
d_3	(0.9, 0.3)	(0.6, 0.8)	(0.85, 0.52)	(0.7, 0.5)
d_4	(0.7, 0.2)	(0.9, 0.4)	(0.78, 0.62)	(0.5, 0.3)

Table 2 Other senior expert’s evaluation matrix $[\mathcal{Y}]_{m \times k}$ based on q -ROF gen-eralized Parameter

	\tilde{h}_1	\tilde{h}_2	\tilde{h}_3
d_1	(0.91, 0.23)	(0.88, 0.38)	(0.9, 0.2)
d_2	(0.85, 0.5)	(0.86, 0.35)	(0.7, 0.3)
d_3	(0.9, 0.3)	(0.65, 0.58)	(0.6, 0.4)
d_4	(0.76, 0.25)	(0.91, 0.42)	(0.5, 0.6)

matrix $[\mathcal{Y}]_{m \times k} = [\tilde{h}_{il}]_{m \times k} = (\mu_{\tilde{h}_{il}}, \eta_{\tilde{h}_{il}})_{m \times k}$ is given in Table 2.

step 3 Combine the matrices of steps 1: and 2: to get the new matrix $[\mathcal{Z}]_{m \times (n+k)}$ which shows the priorities/preferences of experts for each alternatives d_{ij} corresponding to criteria c_j on the bases of generalized parameter \tilde{h}_l .

step 4 Utilized the developed GGq-ROFWA operator to get the over all aggregated result from matrix $[\mathcal{Z}]_{m \times (n+k)}$ row wise for each alternative d_j .

Now the aggregated result for d_1 is as, for $q = 3$;

$$\xi_1 = GGq-ROFWA(< d_1, d_2, \dots, d_n >, (\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m))$$

$$\xi_1 = \left(\frac{\sqrt[q]{1 - \prod_{k=1}^m (1 - \mu_{\tilde{h}_k}^q)^{\hat{u}_k}} \cdot \sqrt[q]{1 - \prod_{\ell=1}^n (1 - \mu_{d_\ell}^q)^{u_\ell}}}{\sqrt[q]{\prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q + \left(1 - \prod_{k=1}^m (\eta_{\tilde{h}_k}^{\hat{u}_k})^q\right) \prod_{\ell=1}^n (\eta_{d_\ell}^{u_\ell})^q}} \right)$$

$$= (0.89792 \times 0.8477, \sqrt[3]{1.7156 \times 10^{-2} + 0.98284 \times 9.2831 \times 10^{-3}})$$

$$\xi_1 = (0.76117, 0.29731)$$

Similarly we can find the others;

$$\xi_2 = (0.64677, 0.40586), \xi_3 = (0.62707, 0.55885), \xi_4 = (0.61011, 0.45385).$$

step 5 Determine the score of each aggregated result in step 4: and rank the results in descending order to get the best alternative.

$$S(\xi_1) = 0.41473, S(\xi_2) = 0.20370, S(\xi_3) = 0.072038, S(\xi_4) = 0.13362$$

Hence from the score values we get the ranking result as;

$d_1 \geq d_2 \geq d_4 \geq d_3$. Therefore, from overall calculation it is clear that, the best medicine (alternative) against the given symptom (criteria) is d_1 .

Comparative analysis

From the above analysis, it is clear that the best alternative to the corresponding criteria is d_1 . If a single senior expert is recommended rather than a group of senior experts, which provide his preference/priority for the mention information, then the following results are concluded,

1: If the expert \tilde{h}_1 is recommended for the consideration of mentioned information, then the score results are given as,

$$S(\xi_1) = 0.43771, S(\xi_2) = 0.16556, S(\xi_3) = 0.24166, S(\xi_4) = 0.14959.$$

This implies that $d_1 \geq d_3 \geq d_2 \geq d_4$.

2: If the expert \tilde{h}_2 is recommended for the consideration of mentioned information, then the score results are given as,

$$S(\xi_1) = 0.35148, S(\xi_2) = 0.25734, S(\xi_3) = -0.14418, S(\xi_4) = 0.23582.$$

This implies that $d_1 \geq d_2 \geq d_4 \geq d_3$.

3: Similarly if the expert \tilde{h}_3 is recommended for the consideration of mentioned information, then the score results are given as,

$$S(\xi_1) = 0.42686, S(\xi_2) = 0.12847, S(\xi_3) = -0.058680, S(\xi_4) = -0.18629.$$

This implies that $d_1 \geq d_2 \geq d_3 \geq d_4$.

The ranking results for the single expert for the same alternatives to their corresponding criteria is different but the best alternative remain same, which represents the importance of expert preferences, knowledge, consciousness and expertise on their preference values.

Moreover, by comparing the superiorities and advantages of the developed approach with existing methods in the literature using the same example and ignoring the group generalized parameter matrix $[\mathcal{Y}]_{m \times k}$. These methods including intuitionistic fuzzy weighted averaging (IFWA) operator presented by Xu [28] and Li [18], Pythagorean fuzzy weighted averaging (PFWA) operator presented by Yagger [32], Ma and Xu [22], symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator presented by Ma and Xu [22], group generalized parameter Pythagorean fuzzy weighted averaging (GGPFWA) operator initiated by Joshi [17], q-ROFWA

Table 3 GGq-ROF expert’s priority/preferences matrix $[\mathcal{Z}]_{m \times (n+k)}$

	c_1	c_2	c_3	c_4	\tilde{h}_1	\tilde{h}_2	\tilde{h}_3
d_1	(0.9, 0.2)	(0.8, 0.3)	(0.95, 0.15)	(0.4, 0.2)	(0.91, 0.23)	(0.88, 0.38)	(0.9, 0.2)
d_2	(0.8, 0.5)	(0.7, 0.2)	(0.93, 0.35)	(0.6, 0.1)	(0.85, 0.5)	(0.86, 0.35)	(0.7, 0.3)
d_3	(0.9, 0.3)	(0.6, 0.8)	(0.85, 0.52)	(0.7, 0.5)	(0.9, 0.3)	(0.65, 0.58)	(0.6, 0.4)
d_4	(0.7, 0.2)	(0.9, 0.4)	(0.78, 0.62)	(0.5, 0.3)	(0.76, 0.25)	(0.91, 0.42)	(0.5, 0.6)

Table 4 Comparative analysis of distinct aggregation operators

Operators	Score values				Ranking
	ξ_1	ξ_2	ξ_3	ξ_4	
IFWA ^{5,9}	Inaccessible				×
PFWA ¹⁵	0.75078	0.71123	0.60872	0.66425	$d_1 \geq d_2 \geq d_4 \geq d_3$
PFWA ²⁰	0.66052	0.55713	0.40420	0.47255	$d_1 \geq d_2 \geq d_4 \geq d_3$
SPFWA ²⁰	0.53681	0.46357	0.28674	0.3795	$d_1 \geq d_2 \geq d_4 \geq d_3$
GGPFWA ²¹	0.46035	0.21881	0.018209	0.11228	$d_1 \geq d_2 \geq d_4 \geq d_3$
q-ROFWA ²⁴	0.59987	0.47911	0.40693	0.41913	$d_1 \geq d_2 \geq d_4 \geq d_3$
GGq-ROFWA	0.41473	0.20370	0.072038	0.13362	$d_1 \geq d_2 \geq d_4 \geq d_3$

Table 5 Comparative analysis of distinct aggregation operators on $[\mathcal{Z}]_{m \times (n+k)}$

Operators	Score values				Ranking
	ξ_1	ξ_2	ξ_3	ξ_4	
IFWA ^{5,9}	Inaccessible				×
PFWA ^{15,20}	Inaccessible				times
SPFWA ²⁰	Inaccessible				×
GGPFWA ²¹	0.46035	0.21881	0.018209	0.11228	$d_1 \geq d_2 \geq d_4 \geq d_3$
q-ROFWA ²⁴	Inaccessible				×
GGq-ROFWA	0.41473	0.20370	0.072038	0.13362	$d_1 \geq d_2 \geq d_4 \geq d_3$

operator presented by Liu and Wang [21]. The ranking results of these aggregation operators are given in Table 4.

From the analysis the Table 4 it is clear that only IFWA operator by Xu [28] and Li [18] is inaccessible to rank the MCDM problem because it cannot tackle the assessment value satisfy $\mu_d + \eta_d > 1$. The ranking result for the rest of MCDM methods remain same and the best optimal value is d_1 . But the methods proposed by Yager [32], Ma and Xu [22] and Joshi [17] have also some restriction and they cannot handle the assessment value satisfy $(\mu_d)^2 + (\eta_d)^2 > 1$. For example, if we make a minor change in Table 4 that is, by replace the alternatives d_{11}, d_{24}, d_{32} and d_{44} by (0.9, 0.85). Then the methods presented by Yager [32], Ma and Xu [22] and Joshi [17] are also fail. However, the method presented in Liu and Wang [21] and the method developed in this paper still deal the situations by adjusting the value of q .

If we consider GGq-ROF expert’s priority/preferences matrix $[\mathcal{Z}]_{m \times (n+k)}$ as given in Table 3, and utilize the methods proposed by Yager [32], Ma and Xu [22], Joshi [17] and Liu and Wang [21] on matrix $[\mathcal{Z}]_{m \times (n+k)}$, as their ranking result is shown in Table 5.

From Table 5, it is observed that the methods proposed by Yager [32], Ma and Xu [22] and Liu and Wang [21] are inaccessible to provide the ranking results and the method presented by Joshi [17] and our developed method is still working and produces the same result.

However, the method presented by Joshi [17] has some limitations and it cannot handle the assessment value satisfactorily $(\mu_d)^2 + (\eta_d)^2 > 1$. For example, if we make a minor change in Table 5 that is, by replace just a single alternatives by (0.9, 0.8). Then, the method presented by Joshi [17] fails to handle the situation. However, the method developed in this paper is still works by adjusting the value of q . Thus IFWA operator by Xu [28] and Li [18], PFWA operator by Yager [32], Ma and Xu [22], GGPFWA operator by Joshi [17], and q-ROFWA operator Liu and Wang [21] are the special cases of the developed aggregation operators as shown in Remark 1, and Proposition 1. Finally from the above analysis and comparison, this fact is observed that the method proposed in this paper is more effective, powerful and superior to solve the MCDM problems than the existing methods.

Conclusion

It has been observed, that in real-life situation provided information of a single expert are completely based on his own priority and may not lead to the accurate decisions. Therefore, acknowledging the initial preferences, it is necessary to justify the initially described preferences from other senior experts/judges to ensure the expert's level of trust and improve the accuracy of the final decision. This is only possible by adding the idea of generalized parameter to the original information. In this paper, the concept Gq-ROFSs is introduced by incorporating generalized parameter to the original information to the views of other senior decision makers or to the expertise of other senior decision makers in q-ROF environment. Then this idea explored to the group generalized parameter in which the preferences of two or more other senior experts/decision makers are analyzed in q-ROF environments. Different aggregation operators are presented on the basis of generalized parameter. Then, the defined aggregation operators are extended to GGq-ROF aggregation operators. These developed aggregations operators have the ability to adjust the situations in a better sequence on the basis of parameterization character. The major advantages of the developed concept is to reduce the probability of complexities, uncertainties and errors in the original information. The main focus of the developed work is on MCDM application by using the proposed approach. Finally, through comparative remark, it has been shown that the developed method is superior to the existing methods.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflicts of interest regarding the publication of this paper.

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