A method to multi-attribute decision-making based on interval-valued q-rung dual hesitant linguistic Maclaurin symmetric mean operators

Xue Feng¹ · Xiaopu Shang¹ · Yuan Xu¹ · Jun Wang²

Received: 3 January 2020 / Accepted: 2 April 2020 / Published online: 5 May 2020
© The Author(s) 2020

Abstract

The aim of this paper is to propose a new multi-attribute decision-making (MADM) method to rank all feasible alternatives in complex decision-making scenarios and determine the optimal one. To this end, we first propose the notion of interval-valued q-rung dual hesitant linguistic sets (IVq-RDHLSs) by combining interval-valued q-rung dual hesitant fuzzy (IVq-RDHF) sets with linguistic terms set. The proposed IVq-RDHLSs utilize IVq-RDHF membership and non-membership degrees to assess linguistic terms, so that they can fully express decision-makers' evaluation information. Additionally, some related concepts such as the operational rules, score and accuracy functions, and ranking method of IVq-RDHLSs are presented. Considering the good performance of the classical Maclaurin symmetric mean (MSM) in integrating fuzzy information, we further generalize MSM into IVq-RDHLSs to propose the interval-valued q-rung dual hesitant linguistic MSM operator, the interval-valued q-rung dual hesitant linguistic dual MSM operator, as well as their weighted forms. Afterwards, we study the applications of IVq-RDHLSs and their aggregation operators in decision-making and propose a new MADM method. Some real decision-making problems in daily life are employed to prove the rightness of the proposed method. We also attempt to demonstrate the advantages and superiorities of our proposed method through comparing with some other methods in this paper.

Keywords Multi-attribute decision-making · Interval-valued q-rung dual hesitant fuzzy sets · Interval-valued q-rung dual hesitant linguistic sets · Maclaurin symmetric mean

Introduction

The past years have witnessed the great success of the study on Pythagorean fuzzy sets (PFSs) based multi-attribute decision-making (MADM) methods [1–11]. PFSs were proposed by Prof. Yager [12] to overcome the shortcoming of intuitionistic fuzzy sets (IFSs) [13], which they fail to

describe decision-making situations in which the sum of membership grade (MG) and non-membership grade (NMG) is greater than one. Hence, PFSs have been regarded as an effective tool to comprehensively express decision-makers' (DMs') complex evaluation information. Recently, Yager [14] generalized PFSs by relaxing their constraint and proposed the notion of q-rung orthopair fuzzy sets (q-ROFSs). Due to their high efficiency in expressing fuzzy and complex DMs' assessment information, q-ROFS-based MADM models and methods have been a new research topic in operational research. Recent publications focused on q-ROFSs based decision-making can be roughly divided into three types. The first type is q-rung orthopair fuzzy (q-ROF) aggregation operators (AOs)-based MADM methods. Liu and Wang [15] proposed basic q-ROF operational rules and simple weighted average/geometric AOs, which are the foundation of q-rung orthopair fuzzy AO theory. On the basis of Liu and Wang's [15] pioneering works, some other powerful AOs have been proposed, such as the q-ROF Bonferroni mean [16], q-ROF Heronian mean [17], q-ROF Hamy mean

✉ Xiaopu Shang
sxp@bjtu.edu.cn

Xue Feng
19120604@bjtu.edu.cn

Yuan Xu
xuyuan@bjtu.edu.cn

Jun Wang
wangjun@buct.edu.cn

¹ School of Economics and Management, Beijing Jiaotong University, Beijing, China

² School of Economics and Management, Beijing University of Chemical Technology, Beijing, China

[18], q-ROF Maclaurin symmetric mean (MSM) [19], q-ROF Muirhead mean [20], and q-ROF power MSM [21]. Some other scholars also investigated the existence of heterogeneous interrelationship among q-ROF numbers (q-ROFNs) and proposed some partitioned AOs of q-ROFNs [22–24]. It is noted that the AOs proposed in [15–24] are based on the algebraic t-norm and t-conorm. In addition, some scholars investigated the operation theory of q-ROFNs and proposed novel q-ROF operational laws. For instance, Peng et al. [25] proposed exponential operations and corresponding AOs of q-ROFNs. Xing and her colleagues [26] studied the interaction operations of q-ROFNs, which take the relationship among MGs and NMGs into consideration. Besides, scholars also investigated Hamacher operations [27], Dombi operations [28], and Archimedean operations of q-ROFNs [29], respectively. The second type is information measures of q-ROFSs. Du [30] introduced Minkowski-type distance measures of q-ROFSs. Peng and Liu [31] studied distance measure, similarity measure, entropy, and inclusion measure for q-ROFSs, and invested their application in MADM. Liu et al. [32] discussed the cosine similarity measure and a Euclidean distance measure of q-ROFSs, and investigated their properties and applications. Peng and Dai [33] proposed multi-parametric similarity measure of q-ROFSs and applied it in the assessment of classroom teaching quality. The third type focuses on extensions of classical q-ROFSs and studies their corresponding MADM methods. For example, Liu and Liu [34, 35] employed linguistic terms to represent the MGs and NMGs to propose the linguistics q-ROFSs and investigated their applications in MADM. Li et al. [36] and Wang et al. [37] utilized q-rung orthopair fuzzy MGs and NMGs to portray linguistic variables and proposed the q-rung orthopair linguistic sets (q-ROLSs). Similarly, Wang et al. [38], Xing et al. [39], and Liu et al. [40, 41] proposed the q-rung orthopair uncertain linguistic sets and discussed their applications in MADM, respectively.

Recently, Xu et al. [42, 43] proposed the q-rung dual hesitant fuzzy sets (q-RDHFSs) and interval-valued q-RDHFSs (IVq-RDHFSs) which are powerful to deal with DMs' high hesitancy in giving MGs and NMGs in q-ROFSs context. In addition, Xu et al. [42, 43] introduced the AOs of q-RDHFSs and IVq-RDHFSs to propose novel MADM methods. Numerical examples have proved the effectiveness advantages of these MADM approaches. However, Xu et al.'s [43] method still has shortcomings, owing to its information expression capability. In most practical decision-making situations, it is almost impossible for DMs to comprehensively provide their evaluation values by quantitative methods, due to the high complexity of real decision-making problems. As a matter of fact, more and more scientists and scholars have realized the importance and necessity of depicting DMs' evaluations both quantitatively and qualitatively [44–47]. For example, Liu and Jin [48] generalized intu-

itionistic fuzzy sets to intuitionistic uncertain linguistic sets, which employ intuitionistic fuzzy sets and uncertain linguistic to express DMs' quantitative and qualitative evaluation values, respectively. Similarly, Du et al.'s [49] combined interval-valued PFSs with linguistic terms set (LST) and proposed the interval-valued Pythagorean linguistic sets. In light of this, Xu et al.'s [43] method is flawed as it only considers DMs' quantitative assessment information.

Based on the above analysis, the motivation of this paper is to propose a new MADM method, which overcomes the drawback of Xu et al.'s [43] method. To do so, we first propose the concept of interval-valued q-rung dual hesitant linguistic sets (IVq-RDHLSs) to portray fuzzy decision-making information. The IVq-RDHLS can be regarded as a combination of IVq-ROFS with LTS. The IVq-RDHFLSs are parallel to the interval-valued dual hesitant fuzzy linguistic sets (IVDHFLSs) proposed by Wei et al. [50], but are more powerful and flexible as they have relaxer constraint, which provides DMs more freedom to fully express their evaluation opinions. In addition, IVq-RDHFLSs are also stronger than q-ROLSs as they can effectively deal with DMs' hesitancy and indeterminacy in expressing their decision information. When considering to integrate attribute values in interval-valued q-rung dual hesitant linguistic (IVq-RDHL) MADM problems, the MSM operator has deeply impressed us, due to its flexibility of reflecting the interrelationship among multiple input variables. Because of this characteristic, MSM has been extensively applied to fuse IFs [51, 52], interval-valued IFs [53], hesitant fuzzy sets [54], neutrosophic sets [55]. Therefore, this paper utilizes MSM to aggregate IVq-RDHFL information and proposes a series of AOs of IVq-RDHFL variables (IVq-RDHFLVs). Finally, based on the newly proposed AOs, we investigate MADM problems with IVq-RDHL information and present a new method to select the optimal alternative. The new method is applied to practical MADM problems to verify its validity and effectiveness. We also illustrate the advantages of the new through comparative analysis.

The rest of this paper is organized as follows. In section "Preliminaries", we review the existing basic notions and propose the concept if IVq-RDHLSs. The operations and comparison method are also presented in section "Preliminaries". Section "Aggregation operators for IVq-RDHL information" proposes some AOs of IVq-RDHL information and discusses their properties in detail. Section "A novel method to MADM with IVq-RDHL information" gives the main steps of a novel MADM based on the proposed AOs. Numerical examples are investigated in section "Application of the proposed method in medical equipment selection". Conclusion remarks are provided in "Conclusions".

Preliminaries

In this section, we briefly review the basic notions that will be used in the following parts. In addition, we also propose the concept of IVq-RDHLs as well as their operations and comparison method.

IVq-RDHFSS, IVq-RDHLs, and their related notions

To comprehensively express DMs' fuzzy judgements over alternatives with high hesitancy, Xu et al. [43] introduced the IVq-RDHFSS. The remarkable feature of IVq-RDHFSS is that its MG and NMG are denoted by a set of interval values, satisfying the sum of q th power of MG and q th power of NMG is less than or equal to one ($q \geq 1$). The definition of IVq-RDHFSS is given as follows.

Definition 1 [43] Let X be a fixed set, and then, an interval-valued q -rung dual hesitant fuzzy set (IVq-RDHFSS) D defined on X is expressed as:

$$D = \{ \langle x, h_D(x), g_D(x) \rangle | x \in X \}, \quad (1)$$

where $h_D(x) = \cup_{[r_D^l, r_D^u] \in h_D(x)} \{ [r_D^l, r_D^u] \}$ and $g_D(x) = \cup_{[\eta_D^l, \eta_D^u] \in g_D(x)} \{ [\eta_D^l, \eta_D^u] \}$ are two sets of some interval values in $[0, 1]$, denoting the possible MG and NMG of the element $x \in X$ to the set D , respectively, such that $[r_D^l, r_D^u], [\eta_D^l, \eta_D^u] \subset [0, 1]$, and $0 \leq ((r_D^u)^+)^q + ((\eta_D^u)^+)^q \leq 1, q \geq 1$, where $[r_D^l, r_D^u] \in h_D(x), [\eta_D^l, \eta_D^u] \in g_D(x), (r_D^u)^+ \in h_D^+(x) = \cup_{[r_D^l, r_D^u] \in h_D(x)} \max\{r_D^u\}$, and $(\eta_D^u)^+ \in g_D^+(x) = \cup_{[\eta_D^l, \eta_D^u] \in g_D(x)} \max\{\eta_D^u\}$. For convenience, we call the pair $d(x) = (h_D(x), g_D(x))$ an interval-valued q -rung dual hesitant fuzzy element (IVq-RDHFSE), which can be denoted by $d = (h, g)$ for simplicity. Especially, if $r^l = r^u$ and $\eta^l = \eta^u$, then D reduces to the q -rung dual hesitant fuzzy set [42]; if $q = 1$, then D reduces to interval-valued dual hesitant fuzzy set [56]; if $q = 2$, then D reduces to hesitant interval-valued Pythagorean fuzzy set [57].

Similar to nature language, the LTS is an effective tool proposed by Zadeh [58] to depict DMs' fuzzy evaluations, and, in addition, to more comprehensively and accurately portray DMs' judgements, scholars, and scientists combined LTS with fuzzy set theories to propose new hybrid tools. The typical representatives are the intuitionistic linguistic sets [59], Pythagorean fuzzy linguistic sets [60], hesitant fuzzy linguistic sets [61], q -rung orthopair linguistic sets [36, 37], etc. Similarly, we employ IVq-RDHFSS to denote the possible MGs and NMGs of linguistic terms to propose the

IVq-RDHLs. To this end, we first briefly review the concept of LTS.

Definition 2 [58] Let $S = \{s_i | i = 1, 2, \dots, t\}$ be a linguistic term set (LTS) with odd cardinality. Any label s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [62]: (1) the set is ordered: $s_i > s_j$ if $i > j$; (2) there is the negation operator: $\text{neg}(s_i) = s_j$, such that $i + j = t + 1$. For example, an LTS S can be defined as:

$$S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium}, s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good}\}$$

In the following, we combine IVq-RDHLs with LTS and propose the IVq-RDHLs.

Definition 3 Let X be a fixed set and $\tilde{S}$ be a continuous LTS of $S = \{s_i | i = 1, 2, \dots, t\}$, and then, an interval-valued q -rung dual hesitant linguistic set (IVq-RDHLs) D defined on X is expressed as:

$$D = \{ \langle x, s_{\theta(x)}, h_D(x), g_D(x) \rangle | x \in X \}, \quad (2)$$

where $s_{\theta(x)} \in \tilde{S}$, $h_D(x) = \cup_{[r_D^l, r_D^u] \in h_D(x)} \{ [r_D^l, r_D^u] \}$ and $g_D(x) = \cup_{[\eta_D^l, \eta_D^u] \in g_D(x)} \{ [\eta_D^l, \eta_D^u] \}$ are two sets of interval values in $[0, 1]$, denoting the possible MG and NMG of the element $x \in X$ to the set D , respectively, such that $[r_D^l, r_D^u], [\eta_D^l, \eta_D^u] \in [0, 1]$, and $0 \leq ((r_D^u)^+)^q + ((\eta_D^u)^+)^q \leq 1, q \geq 1$, where $[r_D^l, r_D^u] \in h_D(x), [\eta_D^l, \eta_D^u] \in g_D(x), (r_D^u)^+ \in h_D^+(x) = \cup_{[r_D^l, r_D^u] \in h_D(x)} \max\{r_D^u\}$, and $(\eta_D^u)^+ \in g_D^+(x) = \cup_{[\eta_D^l, \eta_D^u] \in g_D(x)} \max\{\eta_D^u\}$ for all $x \in X$. For convenience, we call $d = \langle s_{\theta(x)}, \{ [r_D^l, r_D^u] \}, \{ [\eta_D^l, \eta_D^u] \} \rangle$ an interval-valued q -rung dual hesitant linguistic variable (IVq-RDHLV), which can be denoted as $d = \langle s_{\theta}, (h, g) \rangle$ for simplicity. In addition, if $r^l = r^u$ and $\eta^l = \eta^u$, then D is a q -rung dual hesitant linguistic set; if $q = 2$, then D is a hesitant interval-valued Pythagorean linguistic set; if $q = 1$ then D is IVDHFLS [50].

The operational laws of IVq-RDHLVs are proposed as follows.

Definition 4 Let $d = \langle s_{\theta}, (h, g) \rangle$, $d_1 = \langle s_{\theta_1}, (h_1, g_1) \rangle$, and $d_2 = \langle s_{\theta_2}, (h_2, g_2) \rangle$ be any three IVq-RDHLVs, and then:

1. $d_1 \oplus d_2 = \langle s_{\theta_1 + \theta_2}, \cup_{r_1 \in h_1, r_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left[\left[\left((r_1^l)^q + (r_2^l)^q - (r_1^l r_2^l)^q \right)^{1/q}, ((r_1^u)^q + (r_2^u)^q - (r_1^u r_2^u)^q)^{1/q} \right] \right\}, \left\{ \left[\eta_1^l \eta_2^l, \eta_1^u \eta_2^u \right] \right\} \right\rangle$;
2. $d_1 \otimes d_2 = \langle s_{\theta_1 \times \theta_2}, \cup_{r_1 \in h_1, r_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left[\left[\left((r_1^l)^q + (r_2^l)^q - (\eta_1^l \eta_2^l)^q \right)^{1/q}, ((r_1^u)^q + (r_2^u)^q - (\eta_1^u \eta_2^u)^q)^{1/q} \right] \right\} \right\}$;
3. $\lambda d = \langle s_{\lambda \theta}, \cup_{r \in h, \eta \in g} \left\{ \left[\left[\left(1 - (1 - (r^l)^q)^\lambda \right)^{1/q}, (1 - (1 - (r^u)^q)^\lambda)^{1/q} \right] \right\}, \left\{ (\eta^l)^\lambda, (\eta^u)^\lambda \right\} \right\}$
4. $d^\lambda = \langle s_{\theta^\lambda}, \cup_{r \in h, \eta \in g} \left\{ \left\{ (r^l)^\lambda, (r^u)^\lambda \right\}, \left[\left[\left(1 - (1 - (\eta^l)^q)^\lambda \right)^{1/q}, (1 - (1 - (\eta^u)^q)^\lambda)^{1/q} \right] \right] \right\} \right\}$.

According to Definition 4, the following theorem can be easily obtained.

Theorem 1 Let $d = \langle s_\theta, (h, g) \rangle$, $d_1 = \langle s_{\theta_1}, (h_1, g_1) \rangle$ and $d_2 = \langle s_{\theta_2}, (h_2, g_2) \rangle$ be any three IVq-RDHLVs, and then:

1. $d_1 \oplus d_2 = d_2 \oplus d_1$;
2. $d_1 \otimes d_2 = d_2 \otimes d_1$;
3. $\lambda(d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2$ ($\lambda_1, \lambda_2 \geq 0$);
4. $\lambda_1 d \oplus \lambda_2 d = (\lambda_1 \oplus \lambda_2) d$ ($\lambda_1, \lambda_2 \geq 0$);
5. $d_1^\lambda \otimes d_2^\lambda = (d_1 \otimes d_2)^\lambda$ ($\lambda \geq 0$);
6. $d_1^\lambda \otimes d_2^\lambda = (d_1 \otimes d_2)^\lambda$ ($\lambda_1, \lambda_2 \geq 0$).

We present the following comparison method to rank any two IVq-RDHLVs.

Definition 5 Let $d = \langle s_\theta, (h, g) \rangle$ be an IVq-RDHLV, and then, the score function of d is defined as:

$$S(d) = \frac{\left(1 + \left(\frac{1}{\#h} \sum_{[r^l, r^u] \in h} r^l\right)^q + \left(\frac{1}{\#h} \sum_{[r^l, r^u] \in h} r^u\right)^q - \left(\frac{1}{\#g} \sum_{[\eta^l, \eta^u] \in g} \eta^l\right)^q - \left(\frac{1}{\#g} \sum_{[\eta^l, \eta^u] \in g} \eta^u\right)^q\right) \times s_\theta}{2}, \quad (3)$$

and the accuracy function of d is defined as:

$$p(d) = \left(\left(\frac{1}{\#h} \sum_{[r^l, r^u] \in h} r^l \right)^q + \left(\frac{1}{\#h} \sum_{[r^l, r^u] \in h} r^u \right)^q - \left(\frac{1}{\#g} \sum_{[\eta^l, \eta^u] \in g} \eta^l \right)^q - \left(\frac{1}{\#g} \sum_{[\eta^l, \eta^u] \in g} \eta^u \right)^q \right) \times s_\theta, \quad (4)$$

where $\#h$ and $\#g$ represent the numbers of interval values in h and g , respectively. Let $d_1 = \langle s_{\theta_1}, (h_1, g_1) \rangle$ and $d_2 = \langle s_{\theta_2}, (h_2, g_2) \rangle$ be any two IVq-RDHLVs, and then:

1. If $S(d_1) > S(d_2)$, then d_1 is superior to d_2 , denoted by $d_1 > d_2$;
2. If $S(d_1) = S(d_2)$, then
if $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2 , denoted by $d_1 = d_2$;
if $p(d_1) > p(d_2)$, then d_1 is superior to d_2 , denoted by $d_1 > d_2$.

Maclaurin symmetric mean

Definition 6 [63] Let α_i ($i = 1, 2, \dots, n$) be a collection of crisp numbers, and $k = 1, 2, \dots, n$. If

$$\text{MSM}^{(k)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \alpha_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \quad (5)$$

and then, $\text{MSM}^{(k)}$ is called the MSM, where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$ and C_n^k is the binomial coefficient.

In addition, Qin and Liu [51] proposed the dual Maclaurin symmetric mean.

Definition 7 [51]. Let α_i ($i = 1, 2, \dots, n$) be a collection of crisp numbers, and $k = 1, 2, \dots, n$. If

$$\text{DMSM}^{(k)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{k} \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k \alpha_{i_j} \right)^{C_n^k} \right), \quad (6)$$

then $\text{DMSM}^{(k)}$ is called dual Maclaurin symmetric mean (DMSM), where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$ and C_n^k is the binomial coefficient.

Aggregation operators for IVq-RDHL information

In this section, we extend MSM and DMSM to IVq-RDHL environment and propose a family of interval-valued q-rung dual hesitant linguistic Maclaurin symmetric mean operators. Moreover, some desirable properties of the proposed AOs are presented and discussed.

$$\text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) = \left(\frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \otimes_{j=1}^k d_{i_j}}{C_n^k} \right)^{1/k}, \quad (7)$$

where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$, and C_n^k is the binomial coefficient.

Based on the operations for IVq-RDHLVs, the following theorem can be obtained.

Theorem 2 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs, and then, the aggregated value by the IVq-RDHLMSM operator is also an IVq-RDHLV and:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) = & \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \right. \\ & \left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \end{aligned} \quad (8)$$

Interval-valued q-rung dual hesitant linguistic Maclaurin symmetric mean operator

Definition 8 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs and $k = 1, 2, \dots, n$, then the interval-valued q-rung dual hesitant linguistic Maclaurin symmetric mean (IVq-RDHLMSM) operator is defined as:

Proof Based on the operations of IVq-RDHLVs introduced in Definition 4, we have:

$$\begin{aligned} \otimes_{j=1}^k d_{i_j} = & \left\langle s \left(\prod_{j=1}^k \theta_{i_j} \right), \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left[\prod_{j=1}^k r_{i_j}^l, \prod_{j=1}^k r_{i_j}^u \right] \right\}, \right. \\ & \left. \left\{ \left[\left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{q}}, \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{q}} \right] \right\} \right\} \end{aligned}$$

and

$$\begin{aligned} \oplus_{1 \leq i_1 < \dots < i_k \leq n} \otimes_{j=1}^k d_{i_j} = & \left\langle s \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j} \right), \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{q}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{q}} \right] \right\}, \right. \\ & \left. \left\{ \left[\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{q}}, \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{q}} \right] \right\} \right\}. \end{aligned}$$

Furthermore:

$$\frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k d_{i_j}}{C_n^k} = \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right), \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\ \left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{q}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{q}} \right] \right\}, \right. \\ \left. \left\{ \left[\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{C_n^k q}}, \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{C_n^k q}} \right] \right\} \right\rangle$$

Hence:

$$\left(\frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k d_{i_j}}{C_n^k} \right)^{\frac{1}{k}} = \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\ \left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \right. \\ \left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\}.$$

In the following, we discuss some desirable properties of the IVq-RDHLMSM operator.

Theorem 3 (Idempotency) Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle$ ($i = 1, 2, \dots, n$) be a collection of IVq-RDHLVs, if all the IVq-RDHLVs are equal, i.e., $d_i = d$ for all i , and d only has one MG and one NMG, then:

$$\text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) = d. \quad (9)$$

Proof Since $d_i = d$ for $i = 1, 2, \dots, n$, we can get:

$$\begin{aligned}
\text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\
&\left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \\
&\left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \\
&= \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \theta_i^k}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\
&\left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (r_i^l)^{kq} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (r_i^u)^{kq} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \\
&\left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (1 - (\eta_i^l)^q)^k \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (1 - (\eta_i^u)^q)^k \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \\
&= \left\langle s \left(\frac{C_n^k \theta^k}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r \in h, \eta \in g} \left\{ \left[\left(1 - \left((1 - (r^l)^{kq})^{\frac{1}{C_n^k}} \right)^{C_n^k} \right)^{\frac{1}{kq}}, \left(1 - \left((1 - (r^u)^{kq})^{\frac{1}{C_n^k}} \right)^{C_n^k} \right)^{\frac{1}{kq}} \right] \right\}, \right. \\
&\left. \left\{ \left[\left(1 - \left(1 - \left((1 - (1 - (\eta^l)^q)^k \right)^{\frac{1}{C_n^k}} \right)^{C_n^k} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \left((1 - (1 - (\eta^u)^q)^k \right)^{\frac{1}{C_n^k}} \right)^{C_n^k} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \\
&= \langle s_\theta, \cup_{r \in h, \eta \in g} \{ [r^l, r^u], [\eta^l, \eta^u] \} \rangle = d.
\end{aligned}$$

Theorem 4 (Monotonicity) Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle$ ($i = 1, 2, \dots, n$) and $d'_i = \langle s_{\theta'_i}, (h'_i, g'_i) \rangle$ ($i = 1, 2, \dots, n$) be two collections of IVq-RDHLVs, if $s_{\theta_i} \geq s_{\theta'_i}$, $r_i \geq r'_i$ and $\eta_i \leq \eta'_i$ holds for all $i = 1, 2, \dots, n$, where $r_i \in h_i, r'_i \in h'_i, \eta_i \in g_i$ and $\eta'_i \in g'_i$, then:

$$\begin{aligned}
\text{IVq-RDHFLMSM}^{(k)}(d_1, d_2, \dots, d_n) &\geq \text{IVq} \\
&- \text{RDHFLMSM}^{(k)}(d'_1, d'_2, \dots, d'_n).
\end{aligned} \tag{10}$$

Proof For easy description, we assume that:

$$\begin{aligned}
\text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\
&\left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \right. \\
&\left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \\
&= \langle s_\theta, \{[g, t]\}, \{[p, q]\} \rangle,
\end{aligned}$$

and

$$\begin{aligned}
\text{IVq-RDHLMSM}^{(k)}(d'_1, d'_2, \dots, d'_n) &= \left\langle s' \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \cup_{r'_{i_j} \in h'_{i_j}, \eta'_{i_j} \in g'_{i_j}} \right. \\
&\left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (r'_{i_j})^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (r'_{i_j})^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\}, \right. \\
&\left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - ((\eta'_{i_j})^l)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - ((\eta'_{i_j})^u)^q) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right] \right\} \right\rangle \\
&= \langle s'_\xi, \{[k, l]\}, \{[m, n]\} \rangle.
\end{aligned}$$

Since $s_{\theta_i} \geq s'_{\theta_i}$ hold for all $i = 1, 2, \dots, n$, it is easy to prove that:

$$s^{\left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k}\right)^{\frac{1}{k}}} \geq s'^{\left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta'_{i_j}}{C_n^k}\right)^{\frac{1}{k}}},$$

i.e., $s_{\theta} \geq s'_{\theta}$.

Since $d_i \geq d'_i$, we can get $r_i \geq (r_i)'$ and $\left(\prod_{j=1}^k r_{i_j}\right)^q \geq \left(\prod_{j=1}^k (r'_{i_j})\right)^q$. Then:

$$1 - \left(\prod_{j=1}^k r_{i_j}\right)^q \leq 1 - \left(\prod_{j=1}^k (r'_{i_j})\right)^q,$$

$$\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}\right)^q\right)^{\frac{1}{C_n^k}} \leq \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (r'_{i_j})\right)^q\right)^{\frac{1}{C_n^k}}$$

and

$$\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}\right)^q\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \geq \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (r'_{i_j})\right)^q\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}.$$

Therefore,

$$\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}\right)^q\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{kq}} \geq \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (r'_{i_j})\right)^q\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{kq}},$$

i.e., $g \geq k$.

Similarly, we also have $t \geq l, p \leq m$ and $q \leq n$.

In the following, we will discuss three situations as follows.

If $g > k$ and $t > l$, we can get $p \leq m$ and $q \leq n$. Then:

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d_{i_1}, d_{i_2}, \dots, d_{i_n}) \\ & > \text{IVq-RDHFLMSM}^{(k)}(d'_{i_1}, d'_{i_2}, \dots, d'_{i_n}). \end{aligned}$$

If $g = k$ and $t = l$, we can get $p < m$ and $q < n$. Then:

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d_{i_1}, d_{i_2}, \dots, d_{i_n}) \\ & > \text{IVq-RDHFLMSM}^{(k)}(d'_{i_1}, d'_{i_2}, \dots, d'_{i_n}) \end{aligned}$$

If $g = k$ and $t = l$, we can get $p = m$ and $q = n$. Then:

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d_{i_1}, d_{i_2}, \dots, d_{i_n}) \\ & = \text{IVq-RDHFLMSM}^{(k)}(d'_{i_1}, d'_{i_2}, \dots, d'_{i_n}). \end{aligned}$$

Therefore, Theorem 4 is kept.

Theorem 5 (Boundedness) Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle$ ($i = 1, 2, \dots, n$) be a collection of IVq-RDHFLVs, if

$$d^+ = \left\langle \max_{i=1}^n s_{\theta_i}, \cup_{r_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \max_{i=1}^n (r_i) \right\}, \left\{ \min_{i=1}^n (\eta_i) \right\} \right\} \right\rangle,$$

and

$$d^- = \left\langle \min_{i=1}^n s_{\theta_i}, \cup_{r_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \min_{i=1}^n (r_i) \right\}, \left\{ \max_{i=1}^n (\eta_i) \right\} \right\} \right\rangle,$$

then

$$d^- \leq \text{IVq-RDHFLMSM}^{(k)}(d_1, d_2, \dots, d_n) \leq d^+. \quad (11)$$

Proof According to Theorem 3 and Theorem 4, we have:

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d_1, d_2, \dots, d_n) \\ & \geq \text{IVq-RDHFLMSM}^{(k)}(d^-, d^-, \dots, d^-) \end{aligned}$$

and

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d^+, d^+, \dots, d^+) \\ & \geq \text{IVq-RDHFLMSM}^{(k)}(d_1, d_2, \dots, d_n). \end{aligned}$$

In addition, both d^- and d^+ only have one MG and NMG. Therefore:

$$\begin{aligned} & \text{IVq-RDHFLMSM}^{(k)}(d^-, d^-, \dots, d^-) = d^- \\ & \text{IVq-RDHFLMSM}^{(k)}(d^+, d^+, \dots, d^+) = d^+. \end{aligned}$$

Therefore, we have $d^- \leq \text{IVq-RDHFLMSM}^{(k)}(d_1, d_2, \dots, d_n) \leq d^+$.

In the following, we discuss some special cases of the proposed IVq-RDHFLMSM operator with respect to parameters k and q .

Case 1 If $k = 1$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic average (IVq-RDHLA) operator, that is:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(1)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{1}{n} \sum_{i=1}^n \theta_i \right), \cup_{r_i \in h_i, \eta_i \in g_i} \left\{ \left[\left(1 - \prod_{i=1}^n \left(1 - (r_i^l)^q \right)^{1/n} \right)^{1/q}, \left(1 - \prod_{i=1}^n \left(1 - (r_i^u)^q \right)^{1/n} \right)^{1/q} \right], \left[\left(\prod_{i=1}^n \eta_i^l \right)^{1/n}, \left(\prod_{i=1}^n \eta_i^u \right)^{1/n} \right] \right\} \right\rangle \\ &= \frac{1}{n} \oplus_{i=1}^n d_i = \text{IVq-RDHLA}(d_1, d_2, \dots, d_n) \end{aligned} \quad (12)$$

Case 2 If $k = 2$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic Bonferroni mean (IVq-RDHLBM) operator; that is:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(2)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{1}{\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n \theta_i \theta_j} \right)^{1/2}, \cup_{r_i \in h_i, r_j \in h_j, \eta_i \in g_i, \eta_j \in g_j} \left\{ \left[\left(1 - \left(\prod_{i,j=1, i \neq j}^n \left(1 - (r_i^l r_j^l)^q \right)^{\frac{1}{n(n-1)}} \right)^{1/2q}, \left(1 - \left(\prod_{i,j=1, i \neq j}^n \left(1 - (r_i^u r_j^u)^q \right)^{\frac{1}{n(n-1)}} \right)^{1/2q} \right] \right\}, \right. \\ &\quad \left. \left\{ \left[\left(1 - \left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (1 - (\eta_i^l)^q)(1 - (\eta_j^l)^q) \right)^{\frac{1}{n(n-1)}} \right)^{1/2} \right)^{1/q}, \left(1 - \left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (1 - (\eta_i^u)^q)(1 - (\eta_j^u)^q) \right)^{\frac{1}{n(n-1)}} \right)^{1/2} \right)^{1/q} \right] \right\} \right\rangle \\ &= \left(\frac{1}{n(n-1)} \oplus_{i,j=1, i \neq j}^n (d_i \otimes d_j) \right)^{1/2} = \text{IVq-RDHLBM}^{1,1}(d_1, d_2, \dots, d_n). \end{aligned} \quad (13)$$

Case 3 If $k = n$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic geometric (IVq-DHLG) operator:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(n)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{1}{\prod_{i=1}^n \theta_i} \right)^{1/n}, \cup_{r_i \in h_i, \eta_i \in g_i} \left\{ \left[\left(\prod_{i=1}^n r_i^l \right)^{1/n}, \left(\prod_{i=1}^n r_i^u \right)^{1/n} \right] \right\}, \right. \\ &\quad \left. \left\{ \left[\left(1 - \prod_{i=1}^n \left(1 - (\eta_i^l)^q \right)^{1/n} \right)^{1/q}, \left(1 - \prod_{i=1}^n \left(1 - (\eta_i^u)^q \right)^{1/n} \right)^{1/q} \right] \right\} \right\rangle \\ &= \frac{1}{n} \otimes_{i=1}^n d_i = \text{IVq-RDHLG}(d_1, d_2, \dots, d_n) \end{aligned} \quad (14)$$

Case 4 If $q = 1$, then the IVq-RDHLMSM operator reduces to interval-valued dual hesitant linguistic Maclaurin symmetric mean (IVDHLMSM) operator; that is:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{1/k}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\ &\quad \left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k r_{i_j}^l \right)^{1/C_n^k} \right)^{1/k}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k r_{i_j}^u \right)^{1/C_n^k} \right)^{1/k} \right] \right\}, \right. \\ &\quad \left. \left\{ \left[1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \eta_{i_j}^l) \right)^{1/C_n^k} \right)^{1/k}, 1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \eta_{i_j}^u) \right)^{1/C_n^k} \right)^{1/k} \right] \right\} \right\rangle \\ &= \text{IVDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) \end{aligned} \quad (15)$$

Case 5 If $q = 2$, then the IVq-RDHLMSM operator reduces to the hesitant interval-valued Pythagorean linguistic Maclaurin symmetric mean (HIVPLMSM) operator; that is:

$$\begin{aligned} \text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \theta_{i_j}}{C_n^k} \right)^{1/k}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\ &\quad \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^l \right)^2 \right)^{1/C_n^k} \right)^{1/2k}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k r_{i_j}^u \right)^2 \right)^{1/C_n^k} \right)^{1/2k} \right] \right\}, \\ &\quad \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^2) \right)^{1/C_n^k} \right)^{1/2}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^2) \right)^{1/C_n^k} \right)^{1/2} \right) \right] \right\} \right\rangle \\ &= \text{HIVPLMSM}^{(k)}(d_1, d_2, \dots, d_n). \end{aligned} \quad (16)$$

Interval-valued q-rung dual hesitant linguistic weighted Maclaurin symmetric mean operator

Definition 9 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHFLVs and $k = 1, 2, \dots, n$. Let $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector, such that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. The interval-valued q-rung dual hesitant linguistic weighted Maclaurin symmetric mean (IVq-RDHLWMSM) operator is defined as:

$$\begin{aligned} \text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) \\ = \left(\frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k (w_{i_j} d_{i_j})}{C_n^k} \right)^{1/k}, \end{aligned} \quad (17)$$

where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$, and C_n^k is the binomial coefficient.

Theorem 6 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs, and then, the aggregated value by the IVq-RDHLWMSM operator is still an IVq-RDHLV and:

$$\begin{aligned} & \text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) \\ &= \left\langle s, \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (w_{i_j} \theta_{i_j})}{C_n^k} \right)^{1/k}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \\ & \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - (r_{i_j}^l)^q)^{w_{i_j}} \right) \right)^{1/C_n^k} \right)^{1/kq} \right. \right. \\ & \left. \left. \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - (r_{i_j}^u)^q)^{w_{i_j}} \right) \right)^{1/C_n^k} \right)^{1/kq} \right] \right\}, \\ & \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^l)^{w_{i_j}q}) \right)^{1/C_n^k} \right)^{1/kq} \right)^{1/q} \right. \right. \\ & \left. \left. \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (\eta_{i_j}^u)^{w_{i_j}q}) \right)^{1/C_n^k} \right)^{1/kq} \right)^{1/q} \right] \right\} \right\rangle \end{aligned} \quad (18)$$

$$\begin{aligned} & \text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) = \left\langle s, \left(\frac{1}{k} \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k \theta_{i_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \right. \\ & \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (r_{i_j}^l)^q)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right. \right. \right. \\ & \left. \left. \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - (r_{i_j}^u)^q)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \right) \right] \right\}, \\ & \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \eta_{i_j}^l \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right. \right. \\ & \left. \left. \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \eta_{i_j}^u \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\} \right\rangle. \end{aligned} \quad (20)$$

The proof of Theorem 6 is similar to that of Theorem 2, which is omitted here. In addition, it is easy to prove that the IVq-RDHLWMSM operator has the properties of monotonicity and boundedness.

Interval-valued q-rung dual hesitant linguistic dual Maclaurin symmetric mean operator

Definition 10 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs, and $k = 1, 2, \dots, n$, and then, the interval-valued q-rung dual hesitant linguistic dual Maclaurin symmetric mean (IVq-RDHLDSM) operator:

$$\begin{aligned} & \text{IVq-RDHLDSM}^{(k)}(d_1, d_2, \dots, d_n) \\ &= \frac{1}{k} \left(\bigotimes_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigoplus_{j=1}^k d_{i_j} \right)^{1/C_n^k} \right), \end{aligned} \quad (19)$$

where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$ and C_n^k is the binomial coefficient.

Theorem 7 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs, and then, the aggregated value by the IVq-RDHLDSM operator is still an IVq-RDHLV and:

The proof of Theorem 7 is similar to that of Theorem 2, which is omitted here. Similar to the IVq-RDHLMSM operator, the IVq-RDHLDSM operator also has the properties of idempotency, monotonicity, and boundedness.

In the followings, we are to investigate the special cases of the IVq-RDHLDSM operator with respect to the parameters k and q .

Case 1 If $k = 1$, the IVq-RDHLDMMSM operator reduces to the IVq-DHLG operator, which is shown as Eq. (14).

Case 2 If $k = 2$, the IVq-RDHLDMMSM operator reduces to the interval-valued q-rung dual hesitant linguistic geometric Bonferroni mean (IVq-RDHLGBM) operator; that is:

Case 5 If $q = 2$, the IVq-RDHLDMMSM operator reduces to the hesitant interval-valued Pythagorean linguistic dual Maclaurin symmetric mean (HIVPLDMMSM) operator; that is:

$$\begin{aligned} \text{IVq-RDHLDMMSM}^{(2)}(d_1, d_2, \dots, d_n) = & \left\langle s^{\frac{1}{2}} \left(\prod_{i,j=1, i \neq j}^n (\theta_i + \theta_j)^{\frac{1}{n(n-1)}} \right)^{\cup_{r_i \in h_i, \eta_i \in g_i, r_j \in h_j, \eta_j \in g_j}}, \right. \\ & \left\{ \left[\left(1 - \left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (1 - (r_i^l)^q)(1 - (r_j^l)^q) \right)^{\frac{1}{n(n-1)}} \right)^{1/2} \right)^{1/q}, \right. \right. \\ & \left. \left(1 - \left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (1 - (r_i^u)^q)(1 - (r_j^u)^q) \right)^{\frac{1}{n(n-1)}} \right)^{1/2} \right)^{1/q} \right] \right\} \\ & \left\{ \left[\left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (\eta_i^l \eta_j^l)^q \right)^{\frac{1}{n(n-1)}} \right)^{1/2q}, \left(1 - \prod_{i,j=1, i \neq j}^n \left(1 - (\eta_i^u \eta_j^u)^q \right)^{\frac{1}{n(n-1)}} \right)^{1/2q} \right] \right\} \Bigg\rangle \\ & = \frac{1}{2} \left(\bigotimes_{i,j=1, i \neq j}^n (d_i \oplus d_j)^{\frac{1}{n(n-1)}} \right) = \text{IVq-RDHLGBM}^{1,1}(d_1, d_2, \dots, d_n). \end{aligned} \quad (21)$$

Case 3 If $k = n$, the IVq-RDHLDMMSM operator reduces to the IVq-RDHLA operator, which is shown as Eq. (12).

Case 4 If $q = 1$, the IVq-RDHLDMMSM operator reduces to the interval-valued dual hesitant linguistic dual Maclaurin symmetric mean (IVDHLDMMSM) operator; that is:

$$\begin{aligned} \text{IVq-RDHLDMMSM}^{(k)}(d_1, d_2, \dots, d_n) = & \left\langle s^{\frac{1}{k}} \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k \theta_{i_j} \right)^{1/C_n^k} \right)^{\cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}}}, \right. \\ & \left\{ \left[1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - r_{i_j}^l) \right)^{1/C_n^k} \right)^{1/k}, 1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - r_{i_j}^u) \right)^{1/C_n^k} \right)^{1/k} \right] \right\}, \\ & \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \eta_{i_j}^l \right)^{1/C_n^k} \right)^{1/k}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \eta_{i_j}^u \right)^{1/C_n^k} \right)^{1/k} \right] \right\} \Bigg\rangle \\ & = \text{IVDHLDMMSM}^{(k)}(d_1, d_2, \dots, d_n) \end{aligned} \quad (22)$$

$$\begin{aligned}
\text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) &= \left\langle s \left(\frac{1}{k} \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k \theta_{i_j} \right)^{1/C_n^k} \right)^{1/C_n^k}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \right. \\
&\quad \left. \left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (r_{i_j}^l)^2 \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right]^{\frac{1}{2}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (r_{i_j}^u)^2 \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right]^{\frac{1}{2}} \right] \right\} \right. \right. \\
&\quad \left. \left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \eta_{i_j}^l \right)^2 \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{2k}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \eta_{i_j}^u \right)^2 \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{2k}} \right] \right\} \right\rangle \\
&= \text{HIVPLDMSM}^{(k)}(d_1, d_2, \dots, d_n).
\end{aligned} \tag{23}$$

Interval-valued q-rung dual hesitant linguistic weighted dual Maclaurin symmetric mean operator

Definition 11 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs and $k = 1, 2, \dots, n$. Let $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector, such that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. The interval-valued q-rung dual hesitant linguistic weighted dual Maclaurin symmetric mean (IVq-RDHLWMSM) operator is defined as:

$$\begin{aligned}
&\text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) \\
&= \frac{1}{k} \left(\bigotimes_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigoplus_{j=1}^k w_{i_j} d_{i_j} \right)^{1/C_n^k} \right),
\end{aligned} \tag{24}$$

where $(i_1, i_2, \dots, i_k)$ traversals all the k -tuple combination of $(1, 2, \dots, n)$, and C_n^k is the binomial coefficient.

Theorem 8 Let $d_i = \langle s_{\theta_i}, (h_i, g_i) \rangle (i = 1, 2, \dots, n)$ be a collection of IVq-RDHLVs, and then, the aggregated value by the IVq-RDHLWMSM operator is still an IVq-RDHLV and:

$$\begin{aligned}
&\text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \dots, d_n) = \left\langle s \left(\frac{1}{k} \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k w_{i_j} \theta_{i_j} \right)^{1/C_n^k} \right)^{1/C_n^k}, \cup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \right. \right. \\
&\quad \left. \left. \left\{ \left[\left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (r_{i_j}^l)^q \right)^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right]^{\frac{1}{q}}, \left(1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (r_{i_j}^u)^q \right)^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right]^{\frac{1}{q}} \right] \right\} \right. \right. \\
&\quad \left. \left. \left\{ \left[\left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (\eta_{i_j}^l)^{w_{i_j}} \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}}, \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (\eta_{i_j}^u)^{w_{i_j}} \right)^q \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{kq}} \right] \right\} \right\rangle.
\end{aligned} \tag{25}$$

The proof of Theorem 8 is similar to that of Theorem 2, which is omitted. In addition, it is easy to prove that the IVq-RDHLWMSM operator has the properties of monotonicity and boundedness.

A novel method to MADM with IVq-RDHL information

In the above sections, we have demonstrated the powerfulness and robustness of the IVq-RDHLVs and their AOs. In this section, we investigate the applications of IVq-RDHLVs and their AOs in MADM. Let us consider a MADM problem in which DMs express their evaluations in the form of IVq-RDHL information. Suppose that there are m alternatives that to be evaluated, which can be denoted as $\{A_1, A_2, \dots, A_m\}$. Let $\{C_1, C_2, \dots, C_n\}$ be a set of attributes, whose weight vector is $w = (w_1, w_2, \dots, w_n)^T$, such that $\sum_{i=1}^n w_i = 1$ and $0 \leq w_i \leq 1$. For the attribute $C_j (j = 1, 2, \dots, n)$

of $A_i (i = 1, 2, \dots, m)$, DMs use an IVq-RDHLV $d_{ij} = \langle s_{\theta_{ij}}, \{ \{ [r_{ij}^l, r_{ij}^u] \}, \{ [\eta_{ij}^l, \eta_{ij}^u] \} \} \rangle$ to denote their evaluation value. Hence, finally, an IVq-RDHL decision matrix can be obtained, which can be denoted as $R = (d_{ij})_{m \times n}$. If we utilize the proposed IVq-RDHL aggregation operators to solve this MADM problem, the main steps are shown as follows.

Step 1. Standardize the original decision matrix according to the following formula:

$$d_{ij} = \begin{cases} \langle s_{\theta_{ij}}, \{ \{ [r_{ij}^l, r_{ij}^u] \}, \{ [\eta_{ij}^l, \eta_{ij}^u] \} \} \rangle & C_i \text{ is benefit type} \\ \langle s_{\theta_{ij}}, \{ \{ [\eta_{ij}^l, \eta_{ij}^u] \}, \{ [r_{ij}^l, r_{ij}^u] \} \} \rangle & C_i \text{ is cost type} \end{cases} \quad (26)$$

Step 2. Calculate the overall evaluation value of each alternative by the IVq-RDHLWMSM operator:

$$d_i = \text{IVq-RDHLWMSM}^{(k)}(d_{i1}, d_{i2}, \dots, d_{in}), \quad (27)$$

or the IVq-RDHLWDMSM operator:

$$d_i = \text{IVq-RDHLWDMSM}^{(k)}(d_{i1}, d_{i2}, \dots, d_{in}). \quad (28)$$

Hence, a set of overall evaluation values are derived.

Step 3. Compute scores of the overall evaluations.

Step 4. Rank the alternatives according to their corresponding scores.

Application of the proposed method in medical equipment selection

The former section introduces a novel MADM method. To better illustrate the procedure of the propose method, we apply it in a real decision-making problem. Furthermore, we also conduct comparative analysis to show the advantages and superiorities of the proposed method.

Example 1 Medical equipment is an important infrastructure for a hospital, which is directly related to the medical service content and service capacity. The choice of medical equipment is influenced by many factors, including the hospitals' budget, whether the fund is sufficient, whether the hospital has the conditions to utilize the equipment, the degree of equipment demand, the technical evaluation of the equipment, the selection of equipment, the maintainability, etc. In this context, medical equipment selection is a problem with high dimension and high complexity. Thus, fuzzy theory is very suitable for the solving the medical equipment

purchasing problem. In this paper, we assume that there are four medical equipment with different brands. Let $A_i (i = 1, 2, 3, 4)$ denote the brand of equipment i . DMs assess each of them under attribute $C_j (j = 1, 2, 3, 4)$, where C_1 represents equipment quality; C_2 represents equipment price; C_3 represents after-sale service; and C_4 represents user evaluation, whose weight vector is $w = (0.25, 0.35, 0.20, 0.20)^T$. Based on the procurement principle, DMs usually express their evaluation information on the medical equipment and mainly focus on 5 levels {unqualified, qualified, medium, good, excellent}. To comprehensively express the evaluation information over the performance of the feasible alternatives under attributes, DMs are requested to evaluate the four alternatives $A_i (i = 1, 2, 3, 4)$ with respect to the four attributes $C_j (j = 1, 2, 3, 4)$ by IVq-RDHLVs and an IVq-RDHL decision matrix $d_{ij} = \langle s_{\theta_{ij}}, \{ \{ [r_{ij}^l, r_{ij}^u] \}, \{ [\eta_{ij}^l, \eta_{ij}^u] \} \} \rangle$ is obtained, which is shown in Table 1.

The decision-making procedure

Step 1. As all attributes are benefit type, the original decision matrix does not need to be standardized.

Step 2. Utilize the IVq-RDHLWMSM operator to aggregate DMs' assessments, so that the overall assessments of alternatives can be derived (suppose that $k = 3, q = 3$). As the aggregation results are so complicated, we omit them here.

Step 3. Compute the scores of the overall assessments and we can obtain:

$$S(d_1) = 0.2621, \quad S(d_2) = 0.2407,$$

$$S(d_3) = 0.3430, \quad S(d_4) = 0.3662.$$

Step 4. Based on the score values, the ranking order of alternatives is derived, i.e., $A_4 > A_3 > A_1 > A_2$. Hence, the optimal choice is equipment is A_4 .

In step 3, if we utilize the IVq-RDHLWDMSM operator to aggregate attribute values (suppose that $k = 3, q = 3$), then the scores of alternatives are:

$$\begin{aligned} S(d_1) &= 0.2786, \quad S(d_2) = 0.2592, \quad S(d_3) \\ &= 0.3695, \quad S(d_4) = 0.3933 \end{aligned}$$

Accordingly, the ranking order is $A_4 > A_3 > A_1 > A_2$ and A_4 is the optimal alternative.

Table 1 Interval-valued q-rung dual hesitant linguistic decision matrix of Example 1

	A ₁	A ₂	A ₃	A ₄
C ₁	$\langle s_4, \{ \{ [0.6, 0.7], [0.7, 0.8] \}, \{ [0.1, 0.2], [0.1, 0.3] \} \} \rangle$	$\langle s_5, \{ \{ [0.6, 0.8], [0.6, 0.7] \}, \{ [0.2, 0.3], [0.1, 0.2] \} \} \rangle$	$\langle s_6, \{ \{ [0.7, 0.8], [0.8, 0.9] \}, \{ [0.1, 0.2], [0.2, 0.3] \} \} \rangle$	$\langle s_5, \{ \{ [0.7, 0.8], [0.8, 0.9] \}, \{ [0.1, 0.2], [0.1, 0.3] \} \} \rangle$
C ₂	$\langle s_4, \{ \{ [0.6, 0.7], [0.5, 0.6] \}, \{ [0.2, 0.3], [0.1, 0.3] \} \} \rangle$	$\langle s_4, \{ \{ [0.5, 0.7], [0.6, 0.7] \}, \{ [0.2, 0.3], [0.3, 0.4] \} \} \rangle$	$\langle s_5, \{ \{ [0.5, 0.6], [0.6, 0.8] \}, \{ [0.2, 0.3], [0.1, 0.3] \} \} \rangle$	$\langle s_5, \{ \{ [0.7, 0.8], [0.6, 0.7] \}, \{ [0.1, 0.3], [0.2, 0.3] \} \} \rangle$
C ₃	$\langle s_3, \{ \{ [0.5, 0.6], [0.7, 0.8] \}, \{ [0.2, 0.3], [0.1, 0.2] \} \} \rangle$	$\langle s_3, \{ \{ [0.7, 0.8], [0.6, 0.8] \}, \{ [0.1, 0.2], [0.3, 0.4] \} \} \rangle$	$\langle s_4, \{ \{ [0.6, 0.8], [0.5, 0.6] \}, \{ [0.1, 0.3], [0.1, 0.2] \} \} \rangle$	$\langle s_4, \{ \{ [0.6, 0.7], [0.5, 0.6] \}, \{ [0.1, 0.2], [0.2, 0.3] \} \} \rangle$
C ₄	$\langle s_5, \{ \{ [0.6, 0.8], [0.6, 0.7] \}, \{ [0.1, 0.3], [0.2, 0.3] \} \} \rangle$	$\langle s_4, \{ \{ [0.7, 0.9], [0.8, 0.9] \}, \{ [0.2, 0.3], [0.2, 0.4] \} \} \rangle$	$\langle s_4, \{ \{ [0.7, 0.8], [0.6, 0.7] \}, \{ [0.1, 0.2], [0.2, 0.3] \} \} \rangle$	$\langle s_5, \{ \{ [0.7, 0.8], [0.8, 0.9] \}, \{ [0.2, 0.3], [0.1, 0.2] \} \} \rangle$

Table 2 Score values and ranking results with different values of k in the IVq-RDHLWMSM operator ($q = 3$)

k	Score values $S(d_i) (i = 1, 2, 3, 4)$	Ranking orders
$k = 1$	$S(d_1) = 0.2874, S(d_2) = 0.2692, S(d_3) = 0.3828, S(d_4) = 0.4057$	$A_4 \succ A_3 \succ A_1 \succ A_2$
$k = 2$	$S(d_1) = 0.2713, S(d_2) = 0.2515, S(d_3) = 0.3584, S(d_4) = 0.3816$	$A_4 \succ A_3 \succ A_1 \succ A_2$
$k = 3$	$S(d_1) = 0.2621, S(d_2) = 0.2407, S(d_3) = 0.3430, S(d_4) = 0.3662$	$A_4 \succ A_3 \succ A_1 \succ A_2$
$k = 4$	$S(d_1) = 0.2513, S(d_2) = 0.2296, S(d_3) = 0.3254, S(d_4) = 0.3484$	$A_4 \succ A_3 \succ A_1 \succ A_2$

Sensitivity analysis

It is noted that our proposed method has two important parameters, i.e., k and q . It is obvious that the two parameters have a significant impact on the final decision results. This section aims to investigate the influence of the parameters on the decision results.

The influence of parameter k on the final results

In this subsection, we assign different values to k in the IVq-RDHLWMSM and IVq-RDHLWDMSM operators and present the decision results in Tables 2 and 3. As we can see from Tables 2 and 3, different score values are obtained with different parameter values of k ; however, the ranking orders are the same, i.e., $A_4 \succ A_3 \succ A_1 \succ A_2$ and A_4 is the optimal alternative. This characteristic demonstrates the robustness of our proposed method. In addition, it is noted that the score value of each alternative derived by the IVq-RDHLWMSM operator becomes smaller with the increase of parameter k . However, in the IVq-RDHLWDMSM operator, the increase of parameter values k leads to the increase of the score value of each alternative. Therefore, the parameter k can be regarded as DMs' attitude to performance of alternatives. If DMs are optimistic about the alternatives, they can select a smaller value of k in the IVq-RDHLWMSM operator, or a larger value of k in the IVq-RDHLWDMSM operator. If DMs are pessimistic about the alternatives, they can choose a larger value of k in the IVq-RDHLWMSM operator and a smaller value of k in the IVq-RDHLWDMSM. If DMs do not have special preference, they choose the value of k can be taken as $k = \lceil k/2 \rceil$, where $\lceil \cdot \rceil$ is the round function and n is the number of attributes.

The influence of parameter q on the final results

To investigate the influence of parameter q on the final decision results, we assign different values of q in the IVq-RDHLWMSM and IVq-RDHLWDMSM operators, and score values of alternatives are shown as Figs. 1 and 2. As

Table 3 Score values and ranking results with different values of k in the IVq-RDHLWMSM operator ($q = 3$)

k	Score values $S(d_i)(i = 1, 2, 3, 4)$	Ranking orders
$k = 1$	$S(d_1) = 0.2513, S(d_2) = 0.2296,$ $S(d_3) = 0.3254, S(d_4) = 0.3484$	$A_4 > A_3 > A_1 > A_2$
$k = 2$	$S(d_1) = 0.2700, S(d_2) = 0.2500,$ $S(d_3) = 0.3557, S(d_4) = 0.3803$	$A_4 > A_3 > A_1 > A_2$
$k = 3$	$S(d_1) = 0.2786, S(d_2) = 0.2592,$ $S(d_3) = 0.3695, S(d_4) = 0.3933$	$A_4 > A_3 > A_1 > A_2$
$k = 4$	$S(d_1) = 0.2874, S(d_2) = 0.2692,$ $S(d_3) = 0.3828, S(d_4) = 0.4057$	$A_4 > A_3 > A_1 > A_2$

seen from Fig. 1, different values of q lead to different score values of alternatives. Moreover, the score values of each alternative become greater with the increase of values of q in the IVq-RDHLWMSM operator. However, the ranking order is always $A_4 > A_3 > A_1 > A_2$ and A_4 is the optimal alternative. Similarly, from Fig. 2, we find out that the increase of the value q in the IVq-RDHLWMSM operator also results in the increase of score value of each alternative. Although the ranking orders are slightly different, the best alternative is always A_4 . In our proposed method, how to select an appropriate value of q is an important issue. Liu [48] and his colleagues proposed a method to determine the value of q properly in q-rung orthopair fuzzy decision context, i.e., the value of q should be taken as the smallest integer that makes $\mu^q + \nu^q \leq 1 (q \geq 1)$, where μ and ν denote the MG and NMG. For instance, a DM employs a q-ROFN (0.75, 0.95) as his/her judgement over a certain alternative. As $0.75^5 + 0.95^5 = 1.0111 > 1$ and $0.75^6 + 0.95^6 = 0.9131 < 1$, the value of q can be taken as 6. In our proposed method, the value of q should be taken as the smallest integer that makes $0 \leq \left((r_D^u)^+\right)^q + \left((\eta_D^u)^+\right)^q \leq 1$. For example, an IVq-RDHLV $\langle s_3, \{[0.1, 0.2], [0.3, 0.4], [0.5, 0.8]\}, \{[0.2, 0.6], [0.7, 0.8]\} \rangle$ is employed to express a DM's evaluation over an alternative. As $0.8^3 + 0.8^3 = 1.024 > 1$ and $0.8^4 + 0.8^4 = 0.8192 < 1$, then the value of q should be taken as 4.

Validation analysis

In this subsection, we utilize our proposed method and some existing MADM methods to solve practical examples and discuss their decision results to demonstrate the validity of our proposed method. These methods involve that presented by Wei et al. [50] based on interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator, and that introduced by Wei [64] based on interval-valued dual hesitant fuzzy uncertain linguistic weighted average (IVDHFULWA) operator.

Example 2 Suppose that there is a panel with three possible logistics outsourcing service providers in the engineering

Fig. 1 Score values of alternatives obtained by the IVq-RDHLWMSM operator when $q \in [1, 10]$ ($k = 3$) **Fig. 2** Score values of alternatives obtained by the IVq-RDHLWMSM operator when $q \in [1, 10]$ ($k = 3$)

material procurement $A_i (i = 1, 2, 3)$ to select with respect to four attributes, i.e., C_1 is the external environment; C_2 is the enterprise management risk; C_3 the logistics business risk; and C_4 is the internal environment risk. The weight vector of the attributes is $w = (0.20, 0.30, 0.30, 0.20)^T$. DMs employed the interval-valued dual hesitant fuzzy linguistic variables (IVDHFLVs) to express their evaluations of alternatives and a normalized decision matrix is listed in Table 4. We utilize our proposed method based on the IVq-RDHLWMSM operator, and Wei et al.'s [50] and Wei's [64] methods to solve Example 2.

It is noted that Wei's [64] method is based on interval-valued dual hesitant fuzzy uncertain linguistic sets (IVDHFULSs). Actually, the interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) is a special case of IVDH-

Table 4 Normalized interval-valued dual hesitant fuzzy linguistic decision matrix of Example 2

	A_1	A_2	A_3
C_1	$\langle s_2, \{\{[0.1, 0.2], [0.2, 0.3]\}, \{[0.3, 0.4]\}\} \rangle$	$\langle s_3, \{\{[0.7, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$	$\langle s_4, \{\{[0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\}\} \rangle$
C_2	$\langle s_1, \{\{[0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\}\} \rangle$	$\langle s_4, \{\{[0.2, 0.3]\}, \{[0.5, 0.6]\}\} \rangle$	$\langle s_1, \{\{[0.7, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$
C_3	$\langle s_4, \{\{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\}\} \rangle$	$\langle s_1, \{\{[0.6, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$	$\langle s_3, \{\{[0.2, 0.5]\}, \{[0.3, 0.4]\}\} \rangle$
C_4	$\langle s_1, \{\{[0.4, 0.7]\}, \{[0.2, 0.3]\}\} \rangle$	$\langle s_4, \{\{[0.3, 0.5]\}, \{[0.3, 0.4]\}\} \rangle$	$\langle s_2, \{\{[0.3, 0.4]\}, \{[0.2, 0.4], [0.4, 0.5]\}\} \rangle$

Table 5 Normalized interval-valued dual hesitant fuzzy uncertain linguistic decision matrix of Example 2

	A_1	A_2	A_3
C_1	$\langle [s_2, s_2], \{\{[0.1, 0.2], [0.2, 0.3]\}, \{[0.3, 0.4]\}\} \rangle$	$\langle [s_3, s_3], \{\{[0.7, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$	$\langle [s_4, s_4], \{\{[0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\}\} \rangle$
C_2	$\langle [s_1, s_1], \{\{[0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\}\} \rangle$	$\langle [s_4, s_4], \{\{[0.2, 0.3]\}, \{[0.5, 0.6]\}\} \rangle$	$\langle [s_1, s_1], \{\{[0.7, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$
C_3	$\langle [s_4, s_4], \{\{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\}\} \rangle$	$\langle [s_1, s_1], \{\{[0.6, 0.8]\}, \{[0.1, 0.2]\}\} \rangle$	$\langle [s_3, s_3], \{\{[0.2, 0.5]\}, \{[0.3, 0.4]\}\} \rangle$
C_4	$\langle [s_1, s_1], \{\{[0.4, 0.7]\}, \{[0.2, 0.3]\}\} \rangle$	$\langle [s_4, s_4], \{\{[0.3, 0.5]\}, \{[0.3, 0.4]\}\} \rangle$	$\langle [s_2, s_2], \{\{[0.3, 0.4]\}, \{[0.2, 0.4], [0.4, 0.5]\}\} \rangle$

Table 6 Decision results of Example 2 by different methods

Method	Score values $S(d_i) (i = 1, 2, 3)$	Ranking order
Wei et al.'s [50] method based on IVDHFLWA operator	$S(d_1) = 3.2517, S(d_2) = 4.0873, S(d_3) = 3.6395$	$A_2 > A_3 > A_1$
Wei's [64] method based on IVDHFULWA operator	$S(d_1) = 6.5034, S(d_2) = 8.1747, S(d_3) = 7.2790$	$A_2 > A_3 > A_1$
The proposed method based on IVq-RDHFLWMSM operator ($q = 1, k = 1$)	$S(d_1) = -0.0512, S(d_2) = -0.0453, S(d_3) = -0.0616$	$A_2 > A_1 > A_3$

FULS, where the upper bound and lower bound of linguistic part are equal. Hence, we can transform an IVDHFLV into an interval-valued dual hesitant fuzzy uncertain linguistic variable (IVDHFULV). For instance, let $\alpha = \langle s_4, \{\{[0.3, 0.4], \{0.5, 0.6]\}\} \rangle$ be an IVDHFLV and we can transform α into an IVDHFULV, i.e. $\alpha = \langle [s_4, s_4], \{\{[0.3, 0.4], \{0.5, 0.6]\}\} \rangle$. Therefore, we can transform Table 4 into Table 5, which is an interval-valued dual hesitant fuzzy uncertain linguistic decision matrix. We list the decision results obtained by our proposed method and those introduced by Wei et al. [50] and Wei [64] in Table 6. As seen from Table 6, although the ranking orders are slightly different, the best alternative is always A_2 , which indicates the feasibility and correctness of our method.

Advantages of the proposed method

In this subsection, we further demonstrate the advantages and superiorities of our proposed method through comparative analysis.

Its powerfulness of expressing DMs' judgements comprehensively

Our proposed method employs IVq-RDHLSs to depict DMs' evaluation values over the performance of alternatives. As we

know, the IVq-RDHLS should satisfy the constraint that the sum of q th power of MG and q th power of NMG is less than or equal to one. In practical decision-making situations, DMs adjust the parameter value q to make DMs' evaluation values to meet this constraint. Hence, our method can comprehensively describe DMs' assessment values in complicated decision-making problems. To better explain this advantage, we provide the following example.

Example 3 In Example 2, DMs employ IVDHFLSs to express their decision preference. The constraint of IVDHFLSs is that the sum of MG and NMG should be not greater than one. Nevertheless, this constraint cannot be always satisfied. For example, we replace the evaluation value of attribute C_1 of A_1 with $\langle s_2, \{\{[0.1, 0.2], [0.2, 0.8]\}, \{[0.3, 0.9]\}\} \rangle$ (The other evaluation values keep unchanged.). If we utilize the Wei et al.'s [50] and Wei's [64] MADM method and our proposed method to solve Example 3, then the decision results are obtained (listed in Table 7).

As we can see from Table 7, the methods proposed by Wei et al. [50] and Wei [64] fail to deal with Example 3, whereas our proposed method can still handle this case and the ranking order is $A_2 > A_3 > A_1$. This is because in the revised example, the evaluation value of the attribute C_1 of alternative A_1 cannot be represented by IVDHFLSs and IVDHFULSs as $0.8 + 0.9 = 1.7 > 1$. Our proposed method

Table 7 Decision results of Example 3 by different methods

Method	Score values $S(d_i)(i = 1, 2, 3)$	Ranking order
Wei et al.'s [50] method based on IVDHFLWA operator	Cannot be calculated	–
Wei's [64] method based on IVDHFULWA operator	Cannot be calculated	–
The proposed method based on IVq-RDHFLWMSM operator ($q = 5, k = 1$)	$S(d_1) = 0.2177, S(d_2) = 0.3206, S(d_3) = 0.2992$	$A_2 \succ A_3 \succ A_1$

Table 8 Decision results of Example 2 by our proposed method based on the IVq-RDHFLWMSM operator with different values of k ($q = 3$)

k	Score values $S(d_i)(i = 1, 2, 3)$	Ranking order
$k = 1$	$S(d_1) = 0.0751, S(d_2) = 0.1479, S(d_3) = 0.1024$	$A_2 \succ A_3 \succ A_1$
$k = 2$	$S(d_1) = 0.0539, S(d_2) = 0.1090, S(d_3) = 0.0667$	$A_2 \succ A_3 \succ A_1$
$k = 3$	$S(d_1) = 0.0444, S(d_2) = 0.0852, S(d_3) = 0.0532$	$A_2 \succ A_3 \succ A_1$
$k = 4$	$S(d_1) = 0.0362, S(d_2) = 0.0694, S(d_3) = 0.0441$	$A_2 \succ A_3 \succ A_1$

can still deal with this case, and if we set $q = 5$, then $0.8^5 + 0.9^5 = 0.9182 < 1$ (The method to appropriately determine the value of q is presented in section “The influence of parameter q on the final results”). Hence, our propose method can deal with more complicated decision information, give DMs more freedom to provide their evaluations, and comprehensively express DMs' judgements.

Its ability of capturing the interrelationship among attributes

In practical decision-making problems, there exists weak or strong interrelationship among attributes. To make the final decision results more reliable, it is necessary to take such interrelationship into consideration when determining the optimal alternative. As we can see, our method and those proposed by Wei et al. [50] and Wei [64] can effectively solve Example 2. In addition, our proposed method is based on the IVq-RDHFLWMSM operator, which has an important parameter k , making the decision-making procedure flexible. If we assign different values to k when dealing with Example 2 by our proposed method, we can obtain the following results (listed in Table 8).

It is noted that the methods proposed by Wei et al. [50] and Wei [64] are based on simple weighted average operator. Hence, the two MADM methods take the evaluation values and the weight information of attributes into account when calculating the overall preference of DMs over alternatives. However, Wei et al.'s [50] and Wei's [64] methods do not consider the interrelationship among attributes. That is

to say, Wei et al.'s [50] and Wei's [64] methods assume that the attributes are independent, which is not always consistent with the reality. Our method based on the IVq-RDHFLWMSM operator can effectively deal with the correlation among attributes. For example, when we set $k = 2$, then the interrelationship between any attributes is reflected. If any three attributes are interacted, we can set $k = 3$. If there is no interrelationship among attributes, we can let $k = 1$. Hence, as in the most practical MADM problems, the attributes are usually dependent, our proposed method is more powerful and suitable than those proposed by Wei et al. [50] and Wei [64].

The ability of portraying DMs' evaluation information both quantitatively and qualitatively

Our proposed method employs IVq-RDHLSs to denote DMs' evaluation information over the performance of alternatives. Basically, the IVq-RDHLS is a hybrid tool that can describe fuzzy decision information from both quantitative and qualitative aspects. When we utilize IVq-RDHLSs to express decision-making information, we first provide DMs a predefined LTS, and DMs can choose proper linguistic terms to express their qualitative decision information. In addition, the IVq-RDHFSs are employed for DMs to express the MG and NMG of a linguistic term, so that their quantitative decision ideas are expressed. Hence, via the LTS and IVq-RDHFSs, we can denote DMs' evaluation information more accurately, so that the final decision results are more reliable. Compared with our proposed MADM method, the decision method proposed by Xu et al. [43] uses the IVq-RDHFSs to denote the DMs' fuzzy judgements, which only concerns the quantitative information and neglects DMs' qualitative evaluation information. In other words, the method of Xu et al. [43] cannot fully express DMs' evaluation values in complicated decision-making context, so that the decision results obtained are insufficient. Hence, compared with Xu et al.'s [43] method, our proposed method is more sufficient to deal with complex decision-making problems and the final ranking orders of alternatives are more reliable. We list the characteristics of all the above-mentioned MADM methods

Table 9 Characteristics of different MADM methods

Method	Whether permits the sum of MD and NMD to be greater than one	Whether considers the interrelationship among multiple attributes	The degree of flexibility of the information process	Whether portrays both DMs' quantitative and qualitative evaluation information
Wei et al.'s [50] method based on IVDHFLWA operator	No	No	Medium	Yes
Wei's [64] method based on IVDHFULWA operator	No	No	Medium	Yes
Xu et al.'s [44] method based on IVq-RDHFWM operator	Yes	Yes	High	No
The proposed method based on IVq-RDHLWMSM operator	Yes	Yes	High	Yes

in Table 9 to better illustrate the advantages and superiorities of our proposed method.

Conclusions

The IVq-RDHFSSs are a good tool in describing fuzzy information. However, the main shortcoming of IVq-RDHFSSs is that they only express DMs' quantitative decision information. To overcome such drawback, we proposed the notion of IVq-RDHLSSs by combining IVq-RDHFSSs with TLS. The IVq-RDHLSSs not only portray DMs' quantitative and qualitative evaluation values, but also effectively deal with DMs' hesitancy in MADM procedures. In addition, due to its laxer constraint, the IVq-RDHLSS also give DMs great freedom to comprehensively express their judgements. To effectively handle MADM under IVq-RDHLSSs, we have proposed the IVq-RDHLMSM, IVq-RDHLWMSM, IVq-RDHLDMSM, and IVq-RDHLWDMSM operators. Besides, we have also introduced a method to determine the optimal alternative based on the IVq-RDHLWMSM and IVq-RDHLWDMSM operators. From the numerical examples, the feasibility and effectiveness of our proposed method have been clearly illustrated. In addition, advantages and superiorities of the new MADM method have been presented through comparative analysis. For future research, we will investigate new AOs of IVq-RDHLs information and propose novel powerful MADM methods.

Acknowledgements This work was supported by National Natural Science Foundation of China (61702023), Humanities and Social Science Foundation of Ministry of Education of China (17YJC870015), Beijing Natural Science Foundation (7192107), and the Beijing Social Science Foundation (19JDGLB022).

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Abdullah L, Goh P (2019) Decision making method based on Pythagorean fuzzy sets and its application to solid waste management. *Complex Intell Syst* 5(2):185–198
2. Rahman K, Abdullah S (2019) Some induced generalized interval-valued Pythagorean fuzzy Einstein geometric aggregation operators and their application to group decision-making. *Comput Appl Math* 38(3):139
3. Rahman K, Abdullah S, Ali A, Amin F (2019) Interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator and their application to group decision making. *Complex Intell Syst* 5(1):41–52
4. Peng X, Selvachandran G (2019) Pythagorean fuzzy set: state of the art and future directions. *Artif Intell Rev* 52(3):1873–1927
5. Khan MSA (2019) The Pythagorean fuzzy Einstein Choquet integral operators and their application in group decision making. *Comput Appl Math* 38(3):128
6. Kumar K, Edalatpanah SA, Jha S, Singh R (2019) A Pythagorean fuzzy approach to the transportation problem. *Complex Intell Syst* 5(2):255–263
7. Mandal P, Ranadive AS (2019) Multi-granulation Pythagorean fuzzy decision-theoretic rough sets based on inclusion measure and their application in incomplete multi-source information systems. *Complex Intell Syst* 5(2):145–163
8. Zhang R, Wang J, Zhu X, Xia M (2017) Some generalized Pythagorean fuzzy Bonferroni mean aggregation operators with

- their application to multi-attribute group decision-making. *Complexity* 2017:5937376
9. Zhu X, Bai K, Wang J, Zhang R, Xing Y (2019) Pythagorean fuzzy interaction power partitioned Bonferroni means with applications to multi-attribute group decision making. *J Intell Fuzzy Syst* 36(4):3423–3438
 10. Yang W, Pang Y (2018) Hesitant interval-valued Pythagorean fuzzy VIKOR method. *Int J Intell Syst* 34(5):754–789
 11. Yang W, Wang C, Liu Y, Sun Y (2019) Hesitant Pythagorean fuzzy interaction aggregation operators and their application in multiple attribute decision-making. *Complex Intell Syst* 5(2):199–216
 12. Yager RR (2013) Pythagorean membership grades in multicriteria decision making. *IEEE T Fuzzy Syst* 22(4):958–965
 13. Atanassov KT (1986) Intuitionistic fuzzy sets. *Fuzzy Sets Syst* 20(1):87–96
 14. Yager RR (2017) Generalized orthopair fuzzy Sets. *IEEE T Fuzzy Syst* 25(5):1222–1230
 15. Liu P, Wang P (2018) Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *Int J Intell Syst* 33(2):259–280
 16. Liu P, Liu J (2018) Some q-rung orthopair fuzzy Bonferroni mean operators and their application to multi-attribute group decision making. *Int J Intell Syst* 33(2):315–347
 17. Wei G, Gao H, Wei Y (2018) Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. *Int J Intell Syst* 33(7):1426–1458
 18. Wang J, Wei G, Lu J, Alsaadi FE, Hayat T, Wei C, Zhang Y (2019) Some q-rung orthopair fuzzy Hamy mean operators in multiple attribute decision-making and their application to enterprise resource planning systems selection. *Int J Intell Syst* 34(10):2429–2458
 19. Wei G, Wei C, Wang J, Gao H, Wei Y (2019) Some q-rung orthopair fuzzy Maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization. *Int J Intell Syst* 34(1):50–81
 20. Wang J, Zhang R, Zhu X, Zhou Z, Shang X, Li W (2019) Some q-rung orthopair fuzzy Muirhead means with their application to multi-attribute group decision making. *J Intell Fuzzy Syst* 36(2):1599–1614
 21. Liu P, Chen S, Wang P (2018) Multiple-attribute group decision-making based on q-rung orthopair fuzzy power Maclaurin symmetric mean operators. *IEEE T Syst Man CY*. <https://doi.org/10.1109/TSMC.2018.2852948>
 22. Bai K, Zhu X, Wang J, Zhang R (2018) Some partitioned Maclaurin symmetric mean based on q-rung orthopair fuzzy information for dealing with multi-attribute group decision making. *Symmetry* 10(9):383
 23. Liu Z, Liu P, Liang X (2018) Multiple attribute decision-making method for dealing with heterogeneous relationship among attributes and unknown attribute weight information under q-rung orthopair fuzzy environment. *Int J Intell Syst* 33(9):1900–1928
 24. Yang W, Pang Y (2019) New q-rung orthopair fuzzy partitioned Bonferroni mean operators and their application in multiple attribute decision making. *Int J Intell Syst* 34(3):439–476
 25. Peng X, Dai J, Garg H (2018) Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function. *Int J Intell Syst* 33(11):2255–2282
 26. Xing Y, Zhang R, Wang J, Bai K, Xue J (2019) A new multi-criteria group decision-making approach based on q-rung orthopair fuzzy interaction Hamy mean operators. *Neural Comput Appl*. <https://doi.org/10.1007/s00521-019-04269-8>
 27. Darko AP, Liang D (2020) Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method. *Eng Appl Artif Intell* 87:103259
 28. Jana C, Muhiuddin G, Pal M (2019) Some Dombi aggregation of Q-rung orthopair fuzzy numbers in multiple-attribute decision making. *Int J Intell Syst* 34(12):3220–3240
 29. Liu P, Wang P (2018) Multiple-attribute decision-making based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers. *IEEE T Fuzzy Syst* 27(5):834–848
 30. Du WS (2018) Minkowski-type distance measures for generalized orthopair fuzzy sets. *Int J Intell Syst* 33(4):802–817
 31. Peng X, Liu L (2019) Information measures for q-rung orthopair fuzzy sets. *Int J Intell Syst* 34(8):1795–1834
 32. Liu D, Chen X, Peng D (2019) Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *Int J Intell Syst* 34(7):1572–1587
 33. Peng X, Dai J (2019) Research on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. *Int J Intell Syst* 34(7):1588–1630
 34. Liu P, Liu W (2019) Multiple-attribute group decision-making based on power Bonferroni operators of linguistic q-rung orthopair fuzzy numbers. *Int J Intell Syst* 34(4):652–689
 35. Liu P, Liu W (2019) Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power Muirhead mean operators based on entropy weight. *Int J Intell Syst* 34(8):1755–1794
 36. Li L, Zhang R, Shang X (2018) Some q-rung orthopair linguistic Heronian mean operators with their application to multi-attribute group decision making. *Arch Control Sci* 28(4):551–583
 37. Wang H, Ju Y, Liu P (2019) Multi-attribute group decision-making methods based on q-rung orthopair fuzzy linguistic sets. *Int J Intell Syst* 34(6):1129–1157
 38. Wang J, Zhang R, Li L, Zhu X, Shang X (2019) A novel approach to multi-attribute group decision making based on q-rung orthopair uncertain linguistic information. *J Intell Fuzzy Syst* 36(6):5565–5581
 39. Xing Y, Zhang R, Zhu X, Bai K (2019) q-Rung orthopair fuzzy uncertain linguistic choquet integral operators and their application to multi-attribute decision making. *J Intell Fuzzy Syst* 37(1):1123–1139
 40. Liu Z, Li L, Li J (2019) q-Rung orthopair uncertain linguistic partitioned Bonferroni mean operators and its application to multiple attribute decision-making method. *Int J Intell Syst* 34(10):2490–2520
 41. Liu Z, Xu H, Yu Y, Li J (2019) Some q-rung orthopair uncertain linguistic aggregation operators and their application to multiple attribute group decision making. *Int J Intell Syst* 34(10):2521–2555
 42. Xu Y, Shang X, Wang J, Wu W, Huang H (2018) Some q-rung dual hesitant fuzzy Heronian mean operators with their application to multiple attribute group decision-making. *Symmetry* 10(10):472
 43. Xu Y, Shang X, Wang J, Zhao H, Zhang R, Bai K (2019) Some interval-valued q-rung dual hesitant fuzzy Muirhead mean operators with their application to multi-attribute decision-making. *IEEE Access* 7:54724–54745
 44. Liu P, Xu H, Geng Y (2020) Normal wiggly hesitant fuzzy linguistic power Hamy mean aggregation operators and their application to multi-attribute decision-making. *Comput Chem Eng* 140:106224
 45. Wang J, Zhang R, Zhu X, Buchmeister B (2018) Some hesitant fuzzy linguistic Muirhead means with their application to multi-attribute group decision-making. *Complexity* 2018:5087851
 46. Liu P, Zhang X (2020) A new hesitant fuzzy linguistic approach for multiple attribute decision making based on Dempster-Shafer evidence theory. *Appl Soft Comput* 86:105897
 47. Cao H, Zhang R, Wang J (2019) Some spherical linguistic Muirhead mean operators with their application to multi-attribute decision making. *J Intell Fuzzy Syst* 37(6):8097–8111

48. Liu P, Jin F (2012) Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. *Inf Sci* 205:58–71
49. Du Y, Hou F, Zafar W, Yu Q, Zhai Y (2017) A novel method for multi-attribute decision making with interval-valued Pythagorean fuzzy linguistic information. *Int J Intell Syst* 32(10):1085–1112
50. Wei G, Lin R, Wang H, Ran LG (2015) Interval-valued dual hesitant fuzzy linguistic arithmetic aggregation operators in multiple attribute decision making. *Int Core J Eng* 1(6):212–222
51. Qin J, Liu X (2014) An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators. *J Intell Fuzzy Syst* 27(5):2177–2190
52. Liu P, Chen S, Wang Y (2020) Multi-attribute group decision making based on intuitionistic fuzzy partitioned Maclaurin symmetric mean operators. *Inf Sci* 512:830–854
53. Liu Z, Teng F, Liu P, Ge Q (2018) Interval-valued intuitionistic fuzzy power Maclaurin symmetric mean aggregation operators and their application to multiple attribute group decision-making. *Int J Uncertain Quan* 8(3):211–232
54. Qin J, Liu X, Pedrycz W (2015) Hesitant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision making. *Int J Intell Syst* 17(4):509–520
55. Liu P, Zhang X (2018) Some Maclaurin symmetric mean operators for single-valued trapezoidal neutrosophic numbers and their applications to group decision making. *Int J Intell Syst* 20(1):45–61
56. Ju Y, Liu X, Yang S (2014) Interval-valued dual hesitant fuzzy aggregation operators and their applications to multiple attribute decision making. *J Intell Fuzzy Syst* 27(3):1203–1218
57. Yang W, Pang Y (2019) Hesitant interval-valued Pythagorean fuzzy VIKOR method. *Int J Intell Syst* 34(5):754–789
58. Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning—Part II. *Inf Sci* 8(4):301–357
59. Wang J, Li J (2009) The multi-criteria group decision making method based on multi-granularity intuitionistic two semantics. *Sci Trans Inf* 33(1):8–9
60. Peng X, Yang Y (2016) Multiple attribute group decision making methods based on Pythagorean fuzzy linguistic set. *Comput Eng Appl* 52(23):50–54
61. Lin R, Zhang F, Wang H, Wei G (2014) Hesitant fuzzy linguistic aggregation operators and their application to multiple attribute decision making. *J Intell Fuzzy Syst* 27(1):49–63
62. Herrera VE, Martinez L, Mata F, Chiclana F (2005) A consensus support system model for group decision-making problems with multigranular linguistic preference relations. *IEEE T Fuzzy Syst* 13(5):44–658
63. Maclaurin C (1729) A second letter to Martin Folkes, Esq concerning the roots of equations, with demonstration of other rules of algebra. *Philos Trans* 36:59–96
64. Wei G (2017) Interval-valued dual hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. *J Intell Fuzzy Syst* 33(3):1881–1893

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.