Complex & Intelligent Systems (2020) 6:401–409
https://doi.org/10.1007/s40747-020-00135-6

ORIGINAL ARTICLE

A study on compact structural soft sets and an application method

Mustafa Burç Kandemir[1] · Damla Yılmaz[2]

Received: 18 September 2019 / Accepted: 25 February 2020 / Published online: 4 April 2020

Abstract
Since the problems of daily life contain a lot of data and obscurity, it has become a necessity to construct new mathematical methods to solve these problems. In this paper, we have established compact-structural soft sets and studied its basic structural properties. Then, we have proposed an application method for decision-making problems using compact-structural soft sets.

Keywords Topology · Compactness · Soft set · Compact structural soft set · Decision-making

Mathematics Subject Classification 54A05 · 54C10 · 54C50 · 54C60 · 54D30 · 90Bxx

Introduction and preliminaries

Humanity uses science in the process of making sense of the universe. Mathematics is the most powerful scientific tool in this process. However, classical mathematical methods are not always easily implemented in this process. Therefore, human beings have developed new mathematical methods and theories. The most important of these are *probability theory* [1], *fuzzy set theory* [2], *rough set theory* [3], *interval mathematics* [4], etc. Over time, scientists have found that these theories have their own difficulties. In 1999, Molodtsov [5] built the *soft set theory* which is a new mathematical tool that overcomes these difficulties and models uncertainties. He described a soft set on a problem universe as a parameterization of some subsets of the problem universe. More formally, let U be an initial universe which is called problem universe, E be a set of parameters, $\mathcal{P}(U)$ be the power set of U, and $A \subseteq E$. Molodtsov [5] defined the soft set in the following manner:

Definition 1 [5] A pair (F, A) is called a *soft set* over U where F is a mapping given by $F : A \to \mathcal{P}(U)$.

✉ Mustafa Burç Kandemir
mbkandemir@mu.edu.tr

Damla Yılmaz
damlayilmaz48000@gmail.com

1 Department of Mathematics, Faculty of Science, Mugla Sıtkı Koçman University, 48000 Muğla, Turkey

2 Department of Mathematics, Institute of Science, Mugla Sıtkı Koçman University, 48000 Muğla, Turkey

All elements in the set $F(a)$ are expressed as elements that provide the a parameter, and $F(a) \in \mathcal{P}(U)$ is called an *a-approximated set* for each $a \in A$ in the soft set (F, A). We denote the family of all soft sets over the universe U via the parameter universe E with $\mathcal{S}(U; E)$

In [6], to better understand a soft set on a problem universe, it is symbolically illustrated as follows:

$$(F, A) = \{a = F(a) \mid a \in A\}.$$

Of course, the symbol $a = F(a)$ expresses the a-approximated set.

Molodtsov has shown in [5] that this theory can be applied to many fields such as analysis, game theory, probability theory, etc. This theory attracted the attention of many scientists and began to study this theory. Set-theoretic operations such as soft subset, soft union, and soft intersection were first defined and studied by [6–8].

Let (F, A) and (G, B) be two soft sets over the initial universe U where $A, B \subseteq E$. If $A \subset B$ and $F(a) \subset G(a)$ for each $a \in A$, it is called that (F, A) is *soft subset* of (G, B), and denoted by $(F, A)\widetilde{\subset}(G, B)$. If $(F, A)\widetilde{\subset}(G, B)$ and $(G, B)\widetilde{\subset}(F, A)$, it is called that (F, A) is *equal* to (G, B). Suppose that $A \cap B \neq \varnothing$. Then, the *soft intersection* of (F, A) and (G, B) is denoted by $(F, A)\widetilde{\cap}(G, B)$, and is defined as $(F, A)\widetilde{\cap}(G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$. The *soft union* of (F, A) and (G, B) over U is the soft set (H, C), denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$, where $C = A \cup B$, and $\forall c \in C$:

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B \\ G(c), & \text{if } c \in B - A \\ F(c) \cup G(c), & \text{if } c \in A \cap B. \end{cases}$$

If $F(a) = \varnothing$ for all $a \in A$. (F, A) is called a *relative null soft set* (with respect to the parameter set A), and denoted by Φ_A. If $F(a) = U$ for all $a \in A$, then (F, A) is called a *relative whole soft set* (with respect to the parameter set A), and denoted by $\mathcal{U}_A$. From this point of view, the relative whole soft set $\mathcal{U}_E$ with respect to the universe set of parameters E is called the *absolute soft set* over $\mathcal{U}$.

Some interesting operations which are called **And** and **Or** operations and different from the known set-theoretic operations were defined by Maji et al. [6]. Let (F, A) and (G, B) be two soft sets over the common universe U. Then, $(F, A)\mathbf{And}(G, B)$ is the soft set (H, C) over U, such that it is defined by $(F, A)\mathbf{And}(G, B) = (H, C)$ where $H((a, b)) = F(a) \cap G(b)$, for all $(a, b) \in C = A \times B$. Similarly, $(F, A)\mathbf{Or}(G, B)$ is a soft set (H, C) over U, such that it is defined by $(F, A)\mathbf{Or}(G, B) = (H, C)$ where $H((a, b)) = F(a) \cup G(b)$, for all $(a, b) \in C = A \times B$.

The *soft complement* of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.

The *Cartesian product* of (F, A) and (G, B) is denoted by $(F, A)\widetilde{\times}(G, B) = (H, A \times B)$ and defined with the mapping $H : A \times B \to \mathcal{P}(U \times U)$, such that $H(a, b) = F(a) \times G(b)$ for each $(a, b) \in A \times B$ [9].

In [10], Min has introduced the concept of similarity between soft sets and investigated some properties. It is called that (F, A) is *similar* to (G, B) (simply $(F, A) \cong (G, B)$) if there exists a bijection function $\phi : A \to B$, such that $F(x) = (G \circ \phi)(x)$ for every $x \in A$, where $(G \circ \phi)(x) = G(\phi(x))$.

Kharal and Ahmad [11], defined the notion of a mapping on soft classes and studied several properties of images and inverse images of soft sets supported by examples and counterexamples. They defined that image and inverse image of a soft set under a soft functions as follows:

Definition 2 [11] Let $\varphi : U_1 \to U_2$ and $\psi : E_1 \to E_2$ be functions. Then, the pair (φ, ψ) is called a soft function from $\mathcal{S}(U_1; E_1)$ to $\mathcal{S}(U_2; E_2)$. The image of each $(F, A) \in \mathcal{S}(U_1; E_1)$ under the soft function (φ, ψ) denoted by $(\varphi, \psi)(F, A) = (\varphi F, \psi(A))$ and defined as, for each $\beta \in \psi(A)$:

$$(\varphi F)(\beta) = \begin{cases} \varphi\left(\bigcup_{\alpha \in \psi^{-1}(\beta) \cap A} F(\alpha)\right), & \psi^{-1}(\beta) \cap A \neq \varnothing \\ \varnothing, & \text{otherwise.} \end{cases}$$

Similarly, the inverse image of each $(G, B) \in \mathcal{S}(U_2; E_2)$ defined as, for each $\alpha \in \psi^{-1}(B)$:

$$(\varphi^{-1}G)(\alpha) = \begin{cases} \varphi^{-1}(G(\psi(\alpha))), & \psi(\alpha) \in B \\ \varnothing, & \text{otherwise,} \end{cases}$$

and denoted by $(\varphi, \psi)^{-1}(G, B) = (\varphi^{-1}G, \psi^{-1}(B))$.

We know that topology which defined as a family of some subsets of a set that satisfy certain conditions is the most important sub-branch of mathematics. More formally, let U be a non-empty set and $\mathcal{P}(U)$ be the power set of U. $\mathcal{T} \subseteq \mathcal{P}(U)$ is called a *topology* on U if the arbitrary union of the elements of $\mathcal{T}$, the finite intersections of the elements of $\mathcal{T}$ are also in $\mathcal{T}$ and $\varnothing, U \in \mathcal{T}$. The pair $(U, \mathcal{T})$ is called a *topological space* if $\mathcal{T}$ is a topology on U. Each element of $\mathcal{T}$ is called an *open set* in this space. If the complement of any subset of U is open, then it is called *closed set*. Therewithal, the concept of compactness is very useful and important topics in topology. The concept of compactness in a topological space can be considered as a generalization of the concept of boundedness and closedness defined on real numbers. The concept of compactness is characterized by the concept of cover. Let $(U, \mathcal{T})$ be a topological space, $\mathcal{C}$ be a family of some subsets of U. If $\bigcup \mathcal{C} = U$, then $\mathcal{C}$ is called a *cover* of U. If $\mathcal{C}$ is finite and $\mathcal{C}$ is a cover of U, then $\mathcal{C}$ is called a *finite cover*. If $\mathcal{C}$ and $\mathcal{C}'$ are covers of U and $\mathcal{C} \subseteq \mathcal{C}'$, then $\mathcal{C}$ is called a *sub-cover* of $\mathcal{C}'$. If all member of $\mathcal{C}$ is open and $\mathcal{C}$ is a cover of U, then it is called *open cover* of U. From this point of view, a topological space $(U, \mathcal{T})$ is called a *compact space* if it has a finite sub-cover of all open covers. Let $X \subseteq U$ and $(U, \mathcal{T})$ be a topological space. X is called a *compact subset* in $(U, \mathcal{T})$ if the sub-space $(X, \mathcal{T}_X)$ is a compact space. We recommend a review of [12] sources for all the topological terms and theorems mentioned here, but not defined and expressed.

In a daily life problem about the future, facts depend on decisions and decisions depend on preferences of the decision-maker. To overcome such problems, decision-making theory, which is a sub-branch of social sciences and many other sciences, is constructed. Decision-making theory provides a rational methodology for making decisions in uncertainties. According to this theory, preferences depend on the tastes of the decision-maker; that is, decisions can vary and are relative. In daily life problems, there are parameters that affect our preferences and so our decisions. For example, in a real-estate problem, when the person choosing a house, there are parameters such as the environment in which the house is located, its cheapness, its cost, the number of rooms, etc., which will affect his or her decision. According to the decision-making theory, the person who buys the house should choose the house that is the most useful and suitable for him or her, and that provides all the parameters he or she cares about at the same time. Many mathematical methods are used in decision-making theory to make decisions more precise. Of course, soft set theory has recently become a frequently used mathematical tool in decision-making theory. Some studies, but not limited to, where soft sets are applied to decision-making problems are given in [13–16]. In addition, Pei and Miao [7] have shown that each soft set is an information system.

Although, theoretically, there is an infinite universe structure, the problems of daily life are solved on a finite universe, especially in science such as engineering, economy, and industry. Therefore, we want to solve our problems in a bounded and closed area. This is directly related to the fact that the problem universe is compact, or a subset of it is compact. Therefore, it is inevitable to express the problems encountered with general topological concepts.

On the basis of all this, in this paper, we establish compact structural soft sets in any topological space and study its properties. We then propose a method for solving decision-making problems using the concept of compact structural soft set. As a result, we will obtain a daily life application of general topological concepts.

Compact structural soft sets

Let $(U, \mathcal{T})$ be a topological space. From now on, we will call the topological space $(U, \mathcal{T})$ as the topological universe U. Now, we can define compact structural soft set over a topological universe as the follows.

Definition 3 Let U be a topological universe and (F, A) be a soft set over U. (F, A) is called *compact structural soft set* (simply *cs-softset*) if $F(e)$ is a compact subset of U for all $e \in A$.

Example 1 Let $\mathbb{R}$ be the usual topological space and $\mathbb{N}$ be the parameters set. Define the mapping $F : \mathbb{N} \to \mathcal{P}(\mathbb{R})$, such that $F(n) = [n-1, n+1]$. Then, $(F, \mathbb{N})$ is a cs-softset over $\mathbb{R}$.

Note that, if the universal set U is a finite set, it is a compact set with respect to each topology defined on it. Moreover, every subset of U is compact. Therefore, a soft set (F, A) defined on a finite topological universe U is a cs-softset.

Now, we discuss the obtained results.

Obviously, if the topological universe U is compact, then the absolute soft set $\mathcal{U}_A$ is cs-softset, and the null soft set Φ_A is cs-softset over U where $A \subseteq E$.

Theorem 1 *Let (F, A) be a cs-softset and $(G, B)\widetilde{\subseteq}(F, A)$ be a soft set over the topological universe U. If $G(b)$ is closed set of U for all $b \in B$, then (G, B) is a cs-softset over U.*

Proof From definition of soft subset and Theorem 26.2 in [12] (i.e., *every closed sub-space of a compact space is compact*), for each $b \in B \subset A$, we have $G(b) \subseteq F(b)$. It follows that $G(b)$ is a compact set. Hence, (G, B) is a cs-softset over U. □

Theorem 2 *Let (F, A) and (G, B) be two cs-softsets over U. Then, $(F, A)\widetilde{\cup}(G, B)$ is a cs-softset over U.*

Proof Since (F, A) and (G, B) are cs-softsets, then $F(a)$ and $G(b)$ are compact for all $a \in A$ and $b \in B$, respectively. Say that $(H, C) = (F, A)\widetilde{\cup}(G, B)$ where $C = A \cup B$ as in definition of soft union of soft sets. There are three situations. If $c \in A - B$, then $H(c) = F(c)$ and $F(c)$ is compact. If $c \in B - A$, then $H(c) = G(c)$ and $G(c)$ is compact. If $c \in A \cap B$, then $H(c) = F(c) \cup G(c)$. Now, let $\mathcal{C}$ be an open cover of $H(c) = F(c) \cup G(c)$ for each $c \in A \cap B$. Then, $\mathcal{C}$ is an open cover of both $F(c)$ and $G(c)$. For each $c \in A \cap B$, $F(c)$ and $G(c)$ are compact sets, so $F(c)$ and $G(c)$ have finite sub-covers $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{C}$, respectively. Their union of $\mathcal{C}_1 \cup \mathcal{C}_2$ is a sub-cover of $\mathcal{C}$ for $F(c) \cup G(c), \forall c \in A \cap B$. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ is finite, then $F(c) \cup G(c)$ is compact for each $c \in A \cap B$. This proves our desire. □

Corollary 1 *The finite union of cs-softsets is a cs-softset.*

Proof It is clear. □

Arbitrary soft unions of cs-softsets may not be cs-softset. Let us see this with an example.

Example 2 Let $\mathbb{R}$ be a topological universe with usual topology. Consider the soft sets $(F_n, \mathbb{N})$ over $\mathbb{R}$, such that $F_n : \mathbb{N} \to \mathcal{P}(\mathbb{R})$ is a mapping for each $n \in \mathbb{N}$ which is defined $F_n(m) = [-(n+m), (n+m)]$ for each $m \in \mathbb{N}$. Therefore, we have soft sets over $\mathbb{R}$ as follows:

$$
\begin{aligned}
(F_0, \mathbb{N}) &= \{0 = \{0\}, 1 = [-1, 1], 2 = [-2, 2], \ldots\}, \\
(F_1, \mathbb{N}) &= \{0 = [-1, 1], 1 = [-2, 2], 2 = [-3, 3] \ldots\}, \\
(F_2, \mathbb{N}) &= \{0 = [-2, 2], 1 = [-3, 3], 2 = [-4, 4] \ldots\}, \\
&\vdots \\
(F_n, \mathbb{N}) &= \{0 = [-n, n], 1 = [-(n+1), (n+1)], \\
&\qquad 2 = [-(n+2), (n+2)] \ldots\}, \\
&\vdots
\end{aligned}
$$

From here, since $\bigcup F_n(m) = \mathbb{R}$ for each $m, n \in \mathbb{N}$, we obtain that:

$$(F, \mathbb{N}) = \widetilde{\bigcup}_{n \in \mathbb{N}} (F_n, \mathbb{N}) = \{0 = \mathbb{R}, 1 = \mathbb{R}, 2 = \mathbb{R}, \ldots\}.$$

$(F, \mathbb{N})$ is not cs-softset over $\mathbb{R}$ because $\mathbb{R}$ is not compact. Hence, arbitrary soft union of cs-softsets may not be cs-softset.

Let (F, A) and (G, B) be two cs-softsets over any topological universe U. The soft intersection of (F, A) and (G, B) may not be cs-softset. For this, we can give the following example.

Example 3 Let $U = \mathbb{R} \cup \{\alpha, \beta\}$ be the universe and:

$$\mathcal{T} = \mathcal{U} \cup \{\{\alpha\} \cup \mathbb{R}, \{\beta\} \cup \mathbb{R}, \{\alpha, \beta\} \cup \mathbb{R}\}$$

be a topology on U where $\mathcal{U}$ is a usual topology on $\mathbb{R}$. Let $E = \{a, b, c\}$ be parameters set and $A = \{a, b\} \subset E$ and $B = \{b, c\} \subset E$. Let us consider the soft set $(F, A) = \{a = \varnothing, b = \{\alpha\} \cup \mathbb{R}\}$ with $F : A \to \mathcal{P}(U)$ and the soft set $(G, B) = \{b = \{\beta\} \cup \mathbb{R}, c = \varnothing\}$ with $G : B \to \mathcal{P}(U)$. Because of from their definitions, (F, A) and (G, B) are cs-softsets, respectively. For $A \cap B = \{b\}$, we obtain that $(F, A)\widetilde{\sqcap}(G, B) = \{b = \mathbb{R}\}$. Since $\mathbb{R}$ is not a compact subset of U, then $(F, A)\widetilde{\sqcap}(G, B)$ is not cs-softset over U.

We can give the following theorem for the soft intersections of two cs-softsets to be cs-softset.

Theorem 3 *Let (F, A) and (G, B) be two cs-softsets over U. If the topological universe U is a Hausdorff space, then $(F, A)\widetilde{\sqcap}(G, B)$ is a cs-softset over U.*

Proof Suppose that $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$ where $C = A \cap B \neq \varnothing$. From definition of soft intersection of soft sets, we have that $H(c) = F(c) \cap G(c)$ for each $c \in C$. Since (F, A) and (G, B) are cs-softsets over U, $F(c)$ and $G(c)$ are compact set in U for each $c \in C = A \cap B$. Since U is Hausdorff universe, then all compact sets are closed from Theorem 26.3 in, and intersection of closed sets is closed. Thus, we have that $F(c) \cap G(c)$ is closed. Hence, it is compact. Consequently, (H, C) is a cs-softset. □

As a direct result of the above theorem:

Corollary 2 *If U is a Hausdorff universe, then the arbitrary intersection of cs-softsets is a cs-softset.*

We obtain following theorem from definition of **And** and **Or** operators and similar to Theorems 2 and 3. Their proofs are similar to proof of Theorems 2 and 3.

Theorem 4 *Let (F, A) and (G, B) be two cs-softsets over U.*

(i) $(F, A)\mathbf{Or}(G, B)$ is a cs-softset.
(ii) If U is a Hausdorff universe, then $(F, A)\mathbf{And}(G, B)$ is a cs-softset.

Theorem 5 *Let (F, A) and (G, B) be two cs-softsets over U. The Cartesian product $(F, A)\widetilde{\times}(G, B)$ is a cs-softset.*

Proof From definition of Cartesian product of soft sets, we have $(F, A)\widetilde{\times}(G, B) = (H, A \times B)$, such that $H(a, b) = F(a) \times G(b)$ for each $(a, b) \in A \times B$. Since (F, A) and (G, B) are cs-softsets, then $F(a)$ is compact for each $a \in A$ and $G(b)$ is compact for each $b \in B$. We have that $F(a) \times G(b)$ is compact set in the topological universe U for each $(a, b) \in A \times B$ from *Tychonoff Theorem* in [12]. Hence, obviously, $(F, A)\widetilde{\times}(G, B)$ is a cs-softset. □

We know that *Tychonoff Theorem* say that "the product of infinitely many compact sets is compact". Therefore, we get the following result which is more general than the above theorem.

Corollary 3 *The Cartesian product of arbitrary number of cs-softets is a cs-softset.*

We obtain the following result from Definition 2.

Theorem 6 *Let U and V be topological universes, $\varphi : U \to V$ be a continuous function $\psi : E \to E'$ be any function. If (F, A) is a cs-softset over U and $b \in \psi(A)$.*

(i) If $\psi^{-1}(b) \cap A = \varnothing$, then $(\varphi, \psi)(F, A)$ is a cs-softset over V.
(ii) If $\psi^{-1}(b) \cap A \neq \varnothing$ and A or U is finite, then $(\varphi, \psi)(F, A)$ is a cs-softset over V.

Proof (i) Say $(\varphi, \psi)(F, A) = (\varphi F, \psi(A))$. We have two cases from Definition 2. Suppose that $\psi^{-1}(b) \cap A = \varnothing$, then we have $(\varphi F)(b) = \varnothing$ for each $b \in \psi(A)$. Since, obviously, $\varnothing$ is compact in V, desired is provided.

(ii) Suppose that U is finite and (F, A) is a cs-soft set over U. Since $\psi^{-1}(b) \cap A \neq \varnothing$, then we have:

$$(\varphi F)(b) = \varphi\left(\bigcup_{a \in \psi^{-1}(b) \cap A} F(a)\right)$$

from Definition 2. Since φ is continuous and $F(a)$s are compact and so their union $\bigcup F(a)$ is compact, then we obtain that $(\varphi F)(b)$ is compact for each $b \in \psi(A)$. Hence, $(\varphi F, \psi(A))$ is a cs-softset over V.

On the other hand, suppose that U is arbitrary and A is finite. Therefore, $\psi(A)$ is finite. For each $b \in \psi(A)$:

$$(\varphi F)(b) = \varphi\left(\bigcup_{a \in \psi^{-1}(b) \cap A} F(a)\right).$$

Since $a \in \psi^{-1}(b) \cap A$ is finite number, it is obtained finite union:

$$\bigcup_{a \in \psi^{-1}(b) \cap A} F(a).$$

Since each $F(a)$ is compact and the union of a finite number of compact subsets of a topological space is also compact, we have $(\varphi F)(b)$ is compact for each $b \in \psi(A)$. Thus, $(\varphi, \psi)(F, A)$ is a cs-softset over V. □

Inverse image of a cs-softset under a soft mapping may not be a cs-softset. To see this, we can give the following example.

Example 4 Let $\mathbb{R}$ be usual topological universe and consider the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = \sin x$. Let the parameter sets $E = E' = \{a, b, c\}$ define the function $\psi : E \to E'$, such that $\psi = \{(a, a), (b, b), (c, c)\}$ and $A = \{a, b\} \subset E'$. Given the soft set over $\mathbb{R}$ such as:

$$(F, A) = \{a = [-1, 1], b = [3, 4]\}.$$

Obviously, (F, A) is a cs-softset.

For each $p \in \psi^{-1}(A) = A \subset E$, we obtain that:

$$\begin{aligned}(\varphi^{-1}F)(a) &= \varphi^{-1}(F(\psi(a))) = \varphi^{-1}(F(a)) \\ &= \varphi^{-1}([-1, 1]) = \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}(\varphi^{-1}F)(b) &= \varphi^{-1}(F(\psi(b))) = \varphi^{-1}(F(b)) \\ &= \varphi^{-1}([3, 4]) = \varnothing.\end{aligned}$$

Therefore, it is obtained that $(\varphi, \psi)^{-1}(F, A) = \{a = \mathbb{R}, b = \varnothing\}$. Thus, $(\varphi, \psi)^{-1}(F, A)$ is not a cs-softset over U.

From here, we have following result.

Theorem 7 *Let U and V be two different topological universes, $\varphi : U \to V$ be a continuous function, and $\psi : E \to E'$ be a function. If U is compact universe, V is Hausdorff universe and (F, A) is a cs-softset over V, then $(\varphi, \psi)^{-1}(F, A)$ is a cs-softset over U.*

Proof From Definition 2, we have $(\varphi, \psi)^{-1}(F, A) = (\varphi^{-1}F, \psi^{-1}A)$ and:

$$(\varphi^{-1}F)(a) = \begin{cases} \varphi^{-1}(F(\psi(a))) & , \psi(a) \in A \\ \varnothing & , \text{otherwise} \end{cases}$$

for each $a \in \psi^{-1}(A)$. If $\psi(a) \notin A$, then $(\varphi^{-1}F)(a) = \varnothing \subset U$ is obviously compact. Suppose that $\psi(a) \in A$. Then, it is obtained that $(\varphi^{-1}F)(a) = \varphi^{-1}(F(\psi(a))) \subset U$. Since $F(\psi(a)) \subset V$ is compact and V is Hausdorff, $F(\psi(a))$ is closed in V from Theorem 26.3 in [12]. Since φ is continuous, $\varphi^{-1}(F(\psi(a)))$ is closed in U. Since U is compact, then $\varphi^{-1}(F(\psi(a)))$ is compact in U from Theorem 26.2 in [12]. Hence, $(\varphi, \psi)^{-1}(F, A)$ is a cs-softset over U. □

In [10], Min gave the concept of similarity of soft sets. Here, we can give the following theorem for the similarity of cs-softset.

Theorem 8 *Let (F, A) and (G, B) be two soft sets over the topological universe U. If (F, A) is similar to (G, B) and (F, A) is cs-softset, then (G, B) is also cs-softset.*

Proof From definition of similarity, the proof of straightforward. □

We call that the topological space $(U, \mathcal{T})$ is *locally compact space* if for all $x \in X$, there exist a compact set C and a neighborhood X of x, such that $x \in X \subset C$. So, C is called a *compact neighborhood* of x in U. Hence, the topological universe is called locally compact if every element of it has a compact neighborhood [12].

We can construct a cs-softset from the definition of locally compactness, obviously. Let define the mapping $F : U \to \mathcal{P}(U)$, such that $F(x)$ is a compact neighborhood of x for each $x \in U$. Obviously, (F, A) is a cs-softset over U. Such cs-softset (F, U) defined on a locally compact universe is called a *compact neighborhood soft set* (briefly *cn-softset*). Obviously, every cn-softset is a cs-softset.

In [17], Kandemir defined the generalized form of similarity of soft sets as follows.

Definition 4 [17] Let E be a set of parameters, U and V be two universes, and (F, A) and (G, B) be soft sets over U and V respectively, where $A, B \subseteq E$. We called that (F, A) is *similar* to (G, B) if there exist bijective functions $\varphi : U \to V$ and $\psi : A \to B$, such that $(\varphi \circ F)(a) = (G \circ \psi)(a)$ for every $a \in A$.

Now, let U and V be two topological universes and $\varphi : U \to V$ be a function. Naturally, the function $\varphi^* : \mathcal{P}(U) \to \mathcal{P}(V)$, such that $\varphi^*(X) = \varphi(X)$, i.e., $\varphi^*(X)$ is a image of X under φ. In this way, using this argument, generalized form of similarity, and the concept of cn-softset, we obtain the following diagram in Fig. 1.

Hereunder,

Theorem 9 *Let U and V be two locally compact topological universes; (F, U) and (G, V) be cn-softsets. If (F, U) is similar to (G, V), then U is homeomorphic to V.*

Proof Since (F, U) is similar to (G, V) and from Fig. 1, there exist a bijection $\varphi : U \to V$, such that $\varphi^* \circ F = G \circ \varphi$ where $\varphi^* : \mathcal{P}(U) \to \mathcal{P}(V)$, such that $\phi^*(X) = \phi(X)$ for $X \in \mathcal{P}(U)$. We need to show that ϕ is open. Because of the cn-softset (F, U), we have an open set $O \subseteq U$, such that $x \in O \subset F(x)$ where $F(x)$ is compact. From here, it is obtained that $\varphi(x) \in \varphi(O) \subset \varphi(F(x))$. From similarity of (F, U) and (G, V), we have:

$$\begin{aligned}\varphi(x) &\in \varphi[O] \subset \varphi[F(x)] = \varphi^*(F(x)) = (\varphi^* \circ F)(x) \\ &= (G \circ \varphi)(x) = G(\varphi(x)).\end{aligned}$$

Since (G, V) is a cn-softset, we obtained that $\varphi(O)$ is open in V. Hence, φ is an open function.

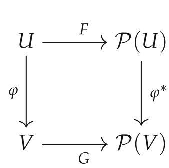

Fig. 1 Diagram of similarity

Now, let us see that φ is continuous. Since (F, A) is similar to (G, V) and from Fig. 1, we have the inverse function of φ, $\varphi^{-1} : V \to U$, such that $(F \circ \varphi^{-1})(y) = ((\varphi^{-1})^* \circ G)(y)$. Suppose that O' is an open set in V. Since (G, V) is a cn-softset, then there is a $y \in V$, such that $y \in O' \subseteq G(y)$. Therefore, we have $\varphi^{-1}(y) \in \varphi^{-1}(O') \subseteq \varphi^{-1}(G(y))$. From similarity of (F, U) and (G, V), we have that:

$$\begin{aligned}\varphi^{-1}(G(y)) &= (\varphi^{-1})^*(G(y)) = ((\varphi^{-1})^* \circ G)(y)\\ &= (F \circ \varphi^{-1})(y) = F(\varphi^{-1}(y)).\end{aligned}$$

Since (F, U) is a cn-softset, $F(\varphi^{-1}(y))$ is a compact neighborhood of $\varphi^{-1}(y)$. Therefore, $\varphi^{-1}(O)$ is an open subset of U. Hence, φ is continuous.

Consequently, φ is a homeomorphism from U to V. □

cn-softset derived from two topological universe that are homeomorphic may not be similar. We can see this with the example below.

Example 5 Let $U = \{a, b, c, d, e\}$ be topological universe with its discrete topology $\mathcal{P}(U)$, and $V = \{1, 2, 3, 4, 5\}$ be other topological universe with its discrete topology $\mathcal{P}(V)$. Define the function $\varphi : U \to V$ such as $\varphi = \{(a, 1), (b, 2), (c, 3), (d, 4), (c, 5)\}$. Obviously, U is homeomorphic to V.

Now, consider cn-softsets:

$$\begin{aligned}&(F, U) = \{a = \{a, b, c\}, b = \{a, b, c\},\\ &c = \{a, b, c\}, d = \{d, e\}, e = \{d, e\}\}\end{aligned}$$

over U and

$$\begin{aligned}&(G, V) = \{1 = \{1, 2, 3\}, 2 = \{2, 3\},\\ &3 = \{3\}, 4 = \{1, 4\}, 5 = V\}\end{aligned}$$

over V. For $b \in U$, we have that $(\varphi^* \circ F)(b) = \varphi^*(F(b)) = \varphi^*(\{a, b, c\}) = \{1, 2, 3\}$ and $(G \circ \varphi)(b) = G(\varphi(b)) = G(2) = \{2, 3\}$. Since $(\varphi^* \circ F)(x) \neq (G \circ \varphi)(x)$ for each $x \in U$, (F, U) is not similar to (G, B).

For a local compact and Hausdorff topological universe U, we can construct a soft set design similar to a cn-softset. Therefore, we can define the set-valued mapping $F^* : U \to \mathcal{P}(U)$, such that $F^*(x)$ is the intersection of compact neighborhoods of x for each $x \in U$, i.e., suppose that $(K_x)_i$ is an arbitrary compact neighborhood of x, then $F^*(x) = \bigcap_{i \in I}(K_x)_i$. Obviously, the soft set (F^*, U) is a cs-softset, and it is called a *minimal compact neighborhood soft set* over U (briefly *mini-cn-softset*).

From this definition, we can give the following theorem for topological universes which are locally compact and Hausdorff.

Theorem 10 *Let U and V be locally compact and Hausdorff topological universes and (F^*, U) and (G^*, V) be mini-cn-softsets derived from these universes. If U is homomorphic to V, then (F^*, U) is similar to (G^*, V).*

Before we prove this theorem, let us give this following lemma whose proof is obvious.

Lemma 11 *If K_x is a compact neighborhood of x, then the homeomorphic image of K_x is a compact neighborhood of image of x under the homeomorphism.*

Now, we can prove Theorem 10.

Proof Since U is homeomorphic to V, then there exists a homeomorphism $\varphi : U \to V$. We know from classical theory that the intersection of an arbitrary number compact set is compact in a Hausdorff topological universe. Using this argument and Lemma 11, we obtain that:

$$\begin{aligned}(\varphi^* \circ F^*)(x) &= \varphi^*\left(\bigcap_{i \in I}(K_x)_i\right)\\ &= \varphi\left[\bigcap_{i \in I}(K_x)_i\right]\\ &= \bigcap_{i \in I}\varphi[(K_x)_i]\\ &= \bigcap_{i \in I}\left(K_{\varphi(x)}\right)_i\\ &= (G^* \circ \varphi)(x).\end{aligned}$$

Thus, From Diagram 1, we have that (F^*, U) is similar to (G^*, V). □

In [18], the author discussed a new perspective on soft topology adhering to Molodtsov's notion. Formal definition is as follows.

Definition 5 [18] Let U be a topological universe and (F, A) be a soft set over U. (F, A) is called a *soft topology* over U if $F(a)$ is a sub-space of U for each $a \in A$. Moreover, $(F, A, \mathcal{T})$ is called a *soft topological space* where $\mathcal{T}$ is a topology on U.

Besides, the author gives the concept of compact soft topological space as follows.

Definition 6 [18] Let $(F, A, \mathcal{T})$ be a soft topological space. It is called that $(F, A, \mathcal{T})$ is a *compact soft topological space* if $F(a)$ is a compact sub-space of U for each $a \in A$.

From Definition 6, we obviously obtain following result.

Theorem 12 *Let U be a topological universe. (F, A) is cs-softset if and only if $(F, A, \mathcal{T})$ is a compact soft topological space.*

An application method of cs-softsets to decision-making problems

Decision-making is usually defined as a process or sequence of activities involving stages of problem recognition, search for information, definition of alternatives, and the selection of an actor of one from two or more alternatives consistent with the ranked preferences. Decision-making theory is a theory of how rational individuals should behave under risk and uncertainty. Mathematical foundations of decision theory have been studied by many scientists until today. We can give [19] as an example.

Although we live in the infinite universe, we have a tendency to solve our problems in finite or limited and closed areas. Because the problems of daily life often do not have an infinite universe perception. Decision-making in a decision-making system is usually done by linear mathematical methods. Of course, there are other mathematical constructs. In this study, we will develop a decision-making method using cs-softset theory on a given topological universe; that is, we give a decision-making technique using general topological concepts.

In developing this method, we need the following well-known theorems that exist in classical theory.

Theorem 13 [max-min (The Extreme Value) Theorem] [12] *Let $(U, \mathcal{T})$ be a topological space, $(\mathbb{R}, \mathcal{U})$ be a usual topological space, and $\varphi : U \to \mathbb{R}$ be a continuous function. If $(U, \mathcal{T})$ is compact, then φ is bounded, and φ has a maximum value and minimum value on U.*

Theorem 14 [12] *Let $(U, \mathcal{T})$ and $(V, \mathcal{T}')$ be topological spaces, $\varphi : U \to V$ be a continuous function, and X be an arbitrary subset of U. The restricted function $\varphi|_X : X \to \mathbb{R}$ is continuous.*

Now, suppose that (F, A) is a cs-softset over a topological universe U and $\varphi : U \to \mathbb{R}$ is a continuous function where $\mathbb{R}$ is the usual topological universe. By Theorem 14, $\varphi|_{F(a)} : F(a) \to \mathbb{R}$ is continuous for each $a \in A$, since $F(a) \subseteq U$ for each $a \in A$. Moreover, since $F(a)$ is compact in U for each $a \in A$, $\varphi|_{F(a)}$ has a maximum value and minimum value by Theorem 13.

Therefore, let us give a decision-making method using the cs-softsets and the arguments above.

As we mentioned before, soft set theory is by nature a mathematical tool that can be easily applied to decision-making problems. Let (F, A) be a soft set over U where $A \subseteq E$. The function F represents *the selector* (or *the decision-maker*), $A \subseteq E$ represents a set of parameters or properties of phenomenon which is selected by selector, and U can be expressed as the universe of the problem that the selector decides. In the theory of decision-making, of course, as we have mentioned above, the universe that affects the choice and the phenomena that we will choose or decide on must be finite. For this reason, suppose that U is finite topological universe, E is finite parameter set, and $A \subseteq E$ and $\varphi : U \to \mathbb{R}$ is continuous where $\mathbb{R}$ is a usual topological universe. The arbitrary soft set (F, A) to be taken over U is naturally a compact structural soft set. Since the compact sets $F(a) \neq \varnothing$ for each $a \in A$ are finite number, then $\bigcap_{a\in A} F(a)$ is compact. This intersection set can be expressed as a set of preferred elements that provides all parameters of interest. By Theorems 13 and 14, the restricted function $\varphi|_{\bigcap_{a\in A} F(a)} : \bigcap_{a\in A} F(a) \to \mathbb{R}$ is continuous and it has a maximum value and minimum value on $\bigcap_{a\in A} F(a)$. Since $\bigcap_{a\in A} F(a)$ is a set of preferred elements that provides all parameters in A, then the maximum value of $\varphi|_{\bigcap_{a\in A} F(a)}$ in $\bigcap_{a\in A} F(a)$ is the element that will choose by decision-maker. In addition to this, we know that the empty set $\varnothing$ is a compact set. If $F(a) = \varnothing$ for any $a \in A$, then we have that $\bigcap_{a\in A} F(a) = \varnothing$. In this case, the restricted function $\varphi|_{\bigcap_{a\in A} F(a)}$ is obtained as an empty function. Therefore, an element to be selected cannot be obtained. At this stage, the selection cannot be made.

We can give this decision-making method in a more formal form using the following algorithms.

Algorithms of decision-making

Let U be a finite topological universe as a problem universe, E be a parameter set, $A \subseteq E$ be a set of interested parameters, and the mapping $\varphi : U \to \mathbb{R}$ is continuous where $\mathbb{R}$ is a usual topological universe. Under these conditions:

Algorithm 1 Define the cs-softset (F, A) according to the selector's preferences.
Algorithm 2 For each $a \in A$, create the preferences set $\bigcap_{a\in A} F(a)$ which provides all parameters at the same time.
Algorithm 3 Find the maximum value of $\varphi|_{\bigcap_{a\in A} F(a)}$ in $\bigcap_{a\in A} F(a)$.

If $\varphi|_{\bigcap_{a\in A} F(a)}$ reaches the maximum value for $x \in \bigcap_{a\in A} F(a)$, then decision-maker (selector) will select x; therefore, when this occurs, it is called that *selection is made* in the (F, A). The cs-softset (F, A) from which the selection is made is called *selection system* (or *stable system*). Otherwise, the system is called *unstable*.

Note that, if there is more than one element $x \in \bigcap_{a\in A} F(a)$ that $\varphi|_{\bigcap_{a\in A} F(a)}$ reaches the same maximum value, the selector can select any of these elements.

Let us see how this decision-making method works with the mythical simple real-estate problem that Molodtsov [5] has given.

Example 6 Let us define the soft set (F, E) as the attractiveness of the houses that Mr. X will buy. For this, let $U = \{h_1, h_2, h_3, h_4\}$ be a set of interested houses, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ be a set of parameters that characterize houses and affect the decision of the decision-maker, such that e_1 represents the expensiveness of the house, e_2 represents the beauty of the house, e_3 represents the house made of wood, e_4 represents the cheapness of the house, e_5 represents that the house is in the green environment, e_6 represents the modernity of the house, e_7 represents that the house is in good repaired, and e_8 represents that the house is in bad repaired. Let $\mathcal{T} = \{\emptyset, U, \{a, c\}, \{b, d\}\}$ be a topology on U, and define the continuous function $\varphi : U \to \mathbb{R}$ such as $\varphi = \{(h_1, 3), (h_2, 1), (h_3, 3), (h_4, 1)\}$ where $\mathbb{R}$ is the usual topological universe. The selection system based on Mr. X's preferences is given as:

$$\begin{aligned}(F, E) = \{&e_1 = \{h_1, h_2\}, e_2 = \{h_1, h_2, h_3\},\\ &e_3 = U, e_4 = \{h_1, h_2, h_4\},\\ &e_5 = U, e_6 = \{h_1, h_2, h_3\},\\ &e_7 = \{h_1, h_2\}, e_8 = \{h_1, h_2, h_3\}\}.\end{aligned}$$

Since U is a finite topological universe, then (F, E) is obviously cs-softset. Hereunder, we have:

$$\bigcap_{e_i \in E} F(e_i) = F(e_1) \cap F(e_2) \cap \cdots \cap F(e_8) = \{h_1, h_2\}.$$

Therefore, we obtained that $\varphi|_{\bigcap F(e_i)}(h_1) = 1$ and $\varphi|_{\bigcap F(e_i)}(h_2) = 3$. Consequently, by algorithms of decision-making, Mr. X will select the house h_2.

In general, of course, the problem space to be decided does not have to be finite. Therefore, we know that if any topological universe is Hausdorff, then the intersection of arbitrary number of compact set in U is compact. Using this argument, we give following theorem for arbitrary topological universe.

Theorem 15 *Let U be a topological universe and $\varphi : U \to \mathbb{R}$, where $\mathbb{R}$ is the usual topological universe. Selection is made in the selection system (F, A) defined on U if and only if U is Hausdorff.*

Proof It is obvious. □

In addition to all these, we can also define another particular soft set given on an arbitrary topological universe.

Definition 7 Let U be a topological universe and (F, A) be a soft set over U. (F, A) is called a *quasi-compact structural soft set* (briefly *quasi-cs-softset*) if there exists an $a \in A$, such that $F(a)$ is compact set in U.

Example 7 Let $\mathbb{R}$ be the usual topological universe and define the soft set $(F, E) = \{a = \{1\}, b = (0, 1), c = [0, 1]\}$ over $\mathbb{R}$ where $E = \{a, b, c\}$. Since $F(a) = \{1\}$ and $F(c) = [0, 1]$ are compact, and but not $F(b)$ in $\mathbb{R}$, so (F, E) is quasi-cs-softset over $\mathbb{R}$.

Clearly, every cs-softset over a topological universe is a quasi-cs-softset.

If we have a quasi-cs-softset over a topological universe, we can define some special parameters in the parameter set.

Definition 8 Let U be a topological universe and (F, A) be a quasi-cs-softset over U. A parameter $a \in A$ is called *prime parameter* if $F(a)$ is a compact subset of U.

From Definition 8, if we throw non-prime parameters from A, we obtain prime parameters set $A_0 \subseteq A$. In this way, we have a reduction of parameters. If we restrict F to A_0, we obtain the set-valued mapping $F_0 : A_0 \to \mathcal{P}(U)$. Therefore, (F_0, A_0) is called *reduced form* of (F, A), and it is obviously cs-softset over U.

Example 8 Consider the quasi-cs-softset (F, A) given in Example 7. The set of prime parameter is $A_0 = \{a, c\} \subset E$ and reduced form of (F, E) is obtained as:

$$(F_0, A_0) = \{a = \{1\}, c = [0, 1]\}.$$

Now, suppose that U be a Hausdorff topological universe and (F, A) is a quasi-cs-softset. Thus, by selecting the prime parameters, the reduced form (F_0, A_0) of (F, A) is obtained. Of course, by its definition, (F_0, A_0) is a cs-softset over U. Since U is a Hausdorff topological universe, then (F_0, A_0) becomes a selection system over U from Theorem 15. By applying the above decision-making algorithms, respectively, it is decided in this system.

Conclusion

Decision-making problems are also frequently encountered in the development of artificial intelligence and operation research. The future and fate of a phenomenon depends on the decisions made on that phenomenon. Mankind has to develop new mathematical tools for solving daily life problems. In this study, we have built a mathematical tool that can be used in decision-making problems, and proposed a method of how to apply it. Of course, this is a very theoretical approach, because our fiction is about using a topological space and general topological concepts. In this manner, We have defined the concept of cs-softset with the notion of a parameterization of compact subsets of a topological universe. Then, we examined some basic theoretical properties. Finally, we proposed an application method for decision-making problems. In this sense, we obtain an application of a very pure field of mathematics to the problems of daily life.

In the future as a continuation of this study, new special soft sets can be identified and their applications explored using extensions and other types of compactness such as paracompactness, metacompactness, semi-compactness, etc.

The authors hope that this article is shed light on to working scientists in these areas.

Acknowledgements The authors thank the anonymous referees and the editors for their constructive comments.

References

1. Laplace PS (1812) Théorie analytique des probabilités, Paris: Courcier, pp 465
2. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
3. Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11:341–356
4. Burkill JC (1924) Functions of intervals. Proc Lond Math Soc 22:275–310
5. Molodtsov D (1999) Soft sets-first results. Comput Math Appl 37:19–31
6. Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45:555–562
7. Pei D, Miao D (2005) From soft sets to information systems. Proc IEEE Int Conf Granul Comput 2:617–621
8. Ali MI, Feng F, Liu XY, Min WK, Shabir M (2009) On some new operations in soft set theory. Comput Math Appl 57:1547–1553
9. Babitha KV, Sunil JJ (2010) Soft set relations and functions. Comput Math Appl 60:1840–1849
10. Min WK (2012) Similarity in soft set theory. Appl Math Lett 25:310–314
11. Kharal A, Ahmad B (2011) Mappings on soft classes. New Math Nat Comput 7:471–481
12. Munkres JR (2000) Topology, 2nd edn. Prentice Hall Inc, Upper Saddle River, NJ
13. Maji PK, Roy A, Biswas R (2002) An applications of soft sets in a decision making problem. Comput Math Appl 44:1077–1083
14. Inthumathi V, Chitra V, Jayasree S (2017) The role of operators on soft sets in decision making problems. Int J Comput Appl Math 12(3):899–910
15. Çetkin V, Aygünoğlu A, Aygün H (2016) A new approach in handling soft decision making problems. J Nonlinear Sci Appl 9:231–239
16. Rose ANM, Awang MI, Hassan H, Zakaria AH, Herawan T, Deris MM (2011) Hybrid Reduction in Soft Set Decision Making. In: Huang DS, Gan Y, Bevilacqua V, Figueroa JC (eds) Advanced Intelligent Computing. ICIC 2011. Lecture Notes in Computer Science, vol 6838. Springer, Berlin, Heidelberg
17. Kandemir MB (2016) Monotonic soft sets and its applications. Ann Fuzzy Math Inform 12(2):295–307
18. Kandemir MB (2018) A new perspective on soft topology. Hittite J Sci Eng 5(2):105–113
19. Barzilai J (2010) Preference function modelling: the mathematical foundations of decision theory. In: Ehrgott M, Figueira J, Greco S (eds) Trends in multiple criteria decision analysis. International series in operations research & management science, vol 142. Springer, Boston