Complex & Intelligent Systems (2019) 5:217–228
https://doi.org/10.1007/s40747-019-0105-4

ORIGINAL ARTICLE

# Exponential similarity measures for Pythagorean fuzzy sets and their applications to pattern recognition and decision-making process

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Received: 25 August 2018 / Accepted: 12 April 2019 / Published online: 2 May 2019


**Abstract**

A Pythagorean fuzzy set is one of the successful extensions of the intuitionistic fuzzy set to handle the uncertain and fuzzy information in a more wider way. In this paper, some new exponential similarity measures (SMs) for measuring the similarities between objects are proposed. For it, we used the exponential function for the membership and the non-membership degrees and hence defined some series of the SMs for PFSs. The various desirable properties and their relations are examined. Several counter-intuitive cases are given to show the effectiveness of the proposed measures with the existing SMs. Furthermore, examples to classify the pattern recognition and the decision-making problems are presented and compared with the existing approaches.

**Keywords** Pythagorean fuzzy sets · Similarity measures · Decision-making problems · Pattern recognition · Exponential functions

## Introduction

Decision-making theory is one of the most important theories to trace the finest objects among the set of the feasible ones. In our day-to-day lie situations, we always make a decision to access our decision in such a way that we can get much benefit from them based on our past records. However, due to complex environment these days and insufficient knowledge about the systems due to lack of information or human errors, it is sometimes very difficult to make an optimal decision in a reasonable time. To address the uncertainties in the data, a concept of fuzzy sets (FSs) introduced by Zadeh [1] to handle the uncertain information. In FSs theory, each element is measured with a membership degree (MD) lying between 0 and 1 to represent the partial information of the set. However, FSs does not encounter about the hesitancy between the element of the set. To address it, an extension of the FSs named intuitionistic fuzzy sets (IFSs) [2] by considering a non-membership degree (NMD) $\nu$ of an element along with MD $\mu$, such that they satisfy the linear inequality $\mu + \nu \leq 1$. After their existence, several authors have addressed the decision-making problems (DMPs) under the IFSs' environment. For example, Ye [3] presented a cosine SMs for IFSs. Garg and Kumar [4] presented similarity measures (SMs) for IFSs using the concept of set pair analysis. Hwang et al. [5] defined the SMs based on Jaccard index and applied it to solve the clustering problems. Garg and Kumar [6] defined the exponential-based distance measure for solving the DMPs. However, apart from them, some others kinds of SMs by utilizing the fuzzy information are summarized in [7–16]. In addition, a complete bibliometric analysis of DMPs is summarized in [17,18].

All the above work has been conducted under the IFSs environment which is own restricted to the domain of feasible region $\mu + \nu \leq 1$. Hence, the theory of IFSs is very narrow, and hence, under some special cases, this theory is unable to quantify the analysis. For example, if preference towards the object is given as 0.6 MD and 0.7 as NMD then clearly $0.6 + 0.7 \nleq 1$. Thus, to handle it, a concept of Pythagorean fuzzy sets (PFSs) [19] introduced by expanding the domain of feasible region from $\mu+\nu \leq 1$ to $\mu^2+\nu^2 \leq 1$. It is clearly seen that PFSs expand the region and hence more effective than IFSs. Furthermore, it can easily handle

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the DMPs, where IFSs fail. Therefore, every IFSs is also PFSs. After its existence, several researchers have studied and enhanced the theory of PFSs using aggregation operators (AOs) or the SMs to different fields. For example, Peng and Yang [20] presented some results on PFSs. Beliakov and James [21] defined the averaging AOs for PFSs. Garg [22,23] proposed the weighted averaging and geometric AOs using Einstein t-norm operators for solving DMPs under PFSs environment. Wei [24] defined the interactive averaging AOs for solving the DMPs. Wei and Lu [25] defined the Maclaurin symmetric mean operators to the Pythagorean fuzzy (PF) environment. Ma and Xu [26] proposed the symmetric averaging AOs for the PF information. Garg [27,28] developed exponential and logarithms operations and their based AOs for solving the DMPs under the PFS environment. However, apart from them, several authors [29–31] handled the DMPs under the PFS environment.

The above-stated work is based on the AOs; however, information measures such as SMs, score and accuracy functions, divergence etc., are also useful to solve the DMPs. Under such measures, researchers have also actively participated which can easily be seen through the literature. For example, Zhang and Xu [32] presented the concept of PF numbers (PFNs) and a TOPSIS ("Technique for Order Preference with respect to the Similarity to the Ideal Solution") method to solve the DMPs with PFSs' information. Zeng et al. [33] developed an approach utilizing the AOs and the distance measures for solving the DMPs. Garg [34] defined the correlation coefficients measures for PFSs. Zhang [35] defined an SM-based algorithm to solve the DMPs for PFNs. Wei and Wei [36] define the SMs based on the cosine measures for PFSs. Apart from them, several authors have addressed the extensions of the PFSs such as interval-valued PFSs [42], hesitant PFS [43,44], and linguistic PFS [45] and applied them to solve the various DMPs under the different environments such as health [46] and site selection [47]. Furthermore, some other measures such as an accuracy function [37,38], operations [39], and improved score functions [40,41] are defined for PFS and interval-valued PFS. In the context of DMPs problems, a comparison between two or more objects is an important principle and thus for it, and a concept of SMs is useful.

The existing SMs are based on the Hamming distance which ignore the influences of the MD and NMD independently. Furthermore, to extend the existing measures, in this paper, we introduce some new SMs for PFSs based on the exponential functions defined on both the MDs and NMDs' function. The salient features of these measures are also studied in detail. Furthermore, an algorithm for solving DMPs is addressed in the paper based on the proposed SMs. Finally, numerical examples are taken to illustrate them .

The remaining work is summarized as follows. The basic concepts of PFSs and the SMs are reviewed briefly in "Preliminaries". In "New similarity measures on Pythagorean fuzzy sets", we define some new SMs based on the exponential function for PFSs and studied their properties. Section "Applications of the proposed SMs" deals with the applications of the proposed measures. Finally, a conclusion is given.

## Preliminaries

In this section, we briefly review the basic concepts related to PFS and SM over the set $X$.

**Definition 1** [2] An IFS $A$ in $X$ is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_A, \nu_A : X \to [0, 1]$ be the MD and NMD function, such that $\mu_A + \nu_A \leq 1, \forall x \in X$. For conveniences, Xu [48] denoted this pair as $A = (\mu_A, \nu_A)$.

**Definition 2** [19] A PFS $\mathcal{P}$ is given by

$$\mathcal{P} = \{(x, \mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x)) \mid x \in X\} \tag{2}$$

where $0 \leq \mu_{\mathcal{P}}, \nu_{\mathcal{P}}, \mu_{\mathcal{P}}^2 + \nu_{\mathcal{P}}^2 \leq 1$. A pair of these is written by $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ and called as PFN [32]. Also, the degree of indeterminacy is given as $\pi_{\mathcal{P}} = \sqrt{1 - \mu_{\mathcal{P}}^2 - \nu_{\mathcal{P}}^2}$.

***Note 1*** The collection of all PFSs over $X$ is written as $\Phi(X)$.

**Definition 3** [19,20,27] Let $\mathcal{P} = (\mu, \nu)$, $\mathcal{P}_1 = (\mu_1, \nu_1)$ and $\mathcal{P}_2 = (\mu_2, \nu_2)$ be three PFNs, then we have

(i) $\mathcal{P}^c = (\vartheta, \zeta)$.
(ii) $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if $\mu_1 \leq \mu_2$ and $\nu_1 \geq \nu_2$.
(iii) $\mathcal{P}_1 = \mathcal{P}_2$ if $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{P}_2 \subseteq \mathcal{P}_1$.
(iv) $\mathcal{P}_1 \cap \mathcal{P}_2 = (\min(\mu_1, \mu_2), \max(\nu_1, \nu_2))$.
(v) $\mathcal{P}_1 \cup \mathcal{P}_2 = (\max(\mu_1, \mu_2), \min(\nu_1, \nu_2))$.
(vi) $\mathcal{P}_1 \oplus \mathcal{P}_2 = \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2\right)$.
(vii) $\mathcal{P}_1 \otimes \mathcal{P}_2 = \left(\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2}\right)$.
(viii) $\lambda\mathcal{P}_1 = \left(\sqrt{1 - (1 - \mu_1^2)^\lambda}, \nu_1^\lambda\right), \lambda > 0$.
(ix) $\mathcal{P}_1^\lambda = \left({\mu_1}^\lambda, \sqrt{1 - (1 - \nu_1^2)^\lambda}\right), \lambda > 0$.
(x) $\lambda^{\mathcal{P}} = \begin{cases} \left(\lambda^{\sqrt{1-\zeta^2}}, \sqrt{1 - \lambda^{2\vartheta}}\right) & \text{if } \lambda \in (0, 1) \\ \left((1/\lambda)^{\sqrt{1-\zeta^2}}, \sqrt{1 - (1/\lambda)^{2\vartheta}}\right) & \text{if } \lambda \geq 1 \end{cases}$

**Definition 4** A real-valued function $S : \Phi(X) \times \Phi(X) \to [0, 1]$ is called similarity measure if the following properties are satisfied:

(P1) $0 \leq S(\mathcal{P}, \mathcal{Q}) \leq 1$.
(P2) $S(\mathcal{P}, \mathcal{Q}) = 1 \Leftrightarrow \mathcal{P} = \mathcal{Q}$
(P3) $S(\mathcal{P}, \mathcal{Q}) = S(\mathcal{Q}, \mathcal{P})$
(P4) If $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{R}$ then, $S(\mathcal{P}, \mathcal{R}) \leq S(\mathcal{P}, \mathcal{Q})$ and $S(\mathcal{P}, \mathcal{R}) \leq S(\mathcal{Q}, \mathcal{R})$, where $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Phi(X)$.

**Definition 5** [35] For two PFSs $\mathcal{P}$ and $\mathcal{Q}$ over the finite $X = \{x_1, x_2, \ldots, x_n\}$. Then, the SM-based on distance measure is defined as

$$\mathrm{Sm}(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{n} \omega_i \frac{d(\mathcal{P}_i, \mathcal{Q}_i^C)}{d(\mathcal{P}_i, \mathcal{Q}_i) + d(\mathcal{P}_i, \mathcal{Q}_i^C)}, \tag{3}$$

where $\omega_i > 0$ be the normalized weight vector of $x_i \in X$ and $\mathcal{Q}^C$ is the complement of PFS $\mathcal{Q}$. In addition, $d(\mathcal{P}_i, \mathcal{Q}_i) = \frac{1}{2}\{|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{Q}}^2(x_i)| + |\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{Q}}^2(x_i)| + |\pi_{\mathcal{P}}^2(x_i) - \pi_{\mathcal{Q}}^2(x_i)|\}$ is the distance measure between the PF elements $\mathcal{P}_i$ and $\mathcal{Q}_i$ for all $i = 1, 2, \ldots, n$.

**Definition 6** [36] For two PFSs $\mathcal{P}$ and $\mathcal{Q}$, two cosine SMs between them is defined as

$$\mathrm{PFC}^1(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{n} \omega_i \left( \frac{\mu_{\mathcal{P}}^2(x_i)\mu_{\mathcal{Q}}^2(x_i) + \nu_{\mathcal{P}}^2(x_i)\nu_{\mathcal{Q}}^2(x_i)}{\sqrt{\mu_{\mathcal{P}}^4(x_i) + \mu_{\mathcal{Q}}^4(x_i)}\sqrt{\nu_{\mathcal{P}}^4(x_i) + \nu_{\mathcal{Q}}^4(x_i)}} \right), \tag{4}$$

and

$$\mathrm{PFC}^2(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{n} \omega_i \times \left( \frac{\mu_{\mathcal{P}}^2(x_i)\mu_{\mathcal{Q}}^2(x_i) + \nu_{\mathcal{P}}^2(x_i)\nu_{\mathcal{Q}}^2(x_i) + \pi_{\mathcal{P}}^2(x_i)\pi_{\mathcal{Q}}^2(x_i)}{\sqrt{\mu_{\mathcal{P}}^4(x_i) + \mu_{\mathcal{Q}}^4(x_i) + \pi_{\mathcal{P}}^4(x_i)}\sqrt{\nu_{\mathcal{P}}^4(x_i) + \nu_{\mathcal{Q}}^4(x_i) + \pi_{\mathcal{Q}}^4(x_i)}} \right), \tag{5}$$

where $\omega_i > 0$ is the normalized weight vector of $x_i \in X$.

## New similarity measures on Pythagorean fuzzy sets

This section presents a new SM-based on exponential functions for MDs and NMDs under PFS environment over the finite set $X$.

**Definition 7** For two PFSs $\mathcal{P} = \{\langle x_i, \mu_{\mathcal{P}}(x_i), \nu_{\mathcal{P}}(x_i)\rangle | x_i \in X\}$ and $\mathcal{Q} = \{\langle x_i, \mu_{\mathcal{Q}}(x_i), \nu_{\mathcal{Q}}(x_i)\rangle | x_i \in X\}$, the two exponential functions are defined as

$$\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = e^{-|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{Q}}^2(x_i)|} \tag{6}$$

and

$$\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = e^{-|\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{Q}}^2(x_i)|}. \tag{7}$$

**Theorem 1** *For any two PFSs $\mathcal{P}$ and $\mathcal{Q}$, we have*

(P1) $0 \leq \mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) \leq 1$;
(P2) $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\mu}(\mathcal{Q}, \mathcal{P})$ *and* $\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{Q}, \mathcal{P})$;
(P3) $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = 1$ *if and only if* $\mathcal{P} = \mathcal{Q}$;
(P4) *if* $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{R}$, *then* $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\mu}(\mathcal{Q}, \mathcal{R})\}$ *and* $\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{Q}, \mathcal{R})\}$.

***Proof*** Let $\mathcal{P} = (\mu_{\mathcal{P}}(x_i), \nu_{\mathcal{P}}(x_i))$ and $\mathcal{Q}=(\mu_{\mathcal{Q}}(x_i), \nu_{\mathcal{Q}}(x_i))$ be two PFSs over $X$.

(P1) By definition of PFSs, we have $\mu_{\mathcal{P}}(x_i), \mu_{\mathcal{Q}}(x_i) \leq 1$ and $\mu_{\mathcal{P}}^2(x_i) + \nu_{\mathcal{P}}^2(x_i) \leq 1$ for all $x_i \in X$. Thus, we have

$$-1 \leq -|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{Q}}^2(x_i)| \leq 0 \text{ and}$$
$$-1 \leq -|\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{Q}}^2(x_i)| \leq 0.$$

Hence

$$0 \leq e^{-|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{Q}}^2(x_i)|} \leq 1 \text{ and } 0 \leq e^{-|\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{Q}}^2(x_i)|} \leq 1.$$

Thus, (P1) holds.
(P2) It is obtained from the definition.
(P3) If $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = 1$, then $\mu_{\mathcal{P}}(x_i) = \mu_{\mathcal{Q}}(x_i)$ and $\nu_{\mathcal{P}}(x_i) = \nu_{\mathcal{Q}}(x_i)$ for all $x_i \in X$. It means that $\mathcal{P} = \mathcal{Q}$. On the other hand, if $\mathcal{P} = \mathcal{Q}$, then it is clearly gives that $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = 1$.
(P4) If $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{R}$, then for $x_i \in X$, we have

$$0 \leq \mu_{\mathcal{P}}(x_i) \leq \mu_{\mathcal{Q}}(x_i) \leq \mu_{\mathcal{R}}(x_i) \leq 1$$

and

$$1 \geq \nu_{\mathcal{P}}(x_i) \geq \nu_{\mathcal{Q}}(x_i) \geq \nu_{\mathcal{R}}(x_i) \geq 0.$$

This implies that

$$0 \leq \mu_{\mathcal{P}}^2(x_i) \leq \mu_{\mathcal{Q}}^2(x_i) \leq \mu_{\mathcal{R}}^2(x_i) \leq 1$$

and

$$1 \geq \nu_{\mathcal{P}}^2(x_i) \geq \nu_{\mathcal{Q}}^2(x_i) \geq \nu_{\mathcal{R}}^2(x_i) \geq 0.$$

Hence

$$-|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{R}}^2(x_i)| \leq \min\{-|\mu_{\mathcal{P}}^2(x_i) - \mu_{\mathcal{Q}}^2(x_i)|, -|\mu_{\mathcal{Q}}^2(x_i) - \mu_{\mathcal{R}}^2(x_i)|\}$$

and

$$-|\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{R}}^2(x_i)| \leq \min\{-|\nu_{\mathcal{P}}^2(x_i) - \nu_{\mathcal{Q}}^2(x_i)|, -|\nu_{\mathcal{Q}}^2(x_i) - \nu_{\mathcal{R}}^2(x_i)|\}.$$

It means that

$$\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\mu}(\mathcal{Q}, \mathcal{R})\}$$

and

$$\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{Q}, \mathcal{R})\}.$$

□

Next, based on the two functions defined in Eqs. (6) and (7), we define the weighted SMs for PFSs as below.

**Definition 8** Let $\mathcal{P}, \mathcal{Q}$ be two PFSs defined over $X$ and $\omega_i > 0$ is the weight of the element of $X$ which satisfy $\sum_{i=1}^n \omega_i = 1$. Then, a weighted SMs between them is defined as

$$\mathcal{S}_0(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^n \omega_i \times \mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) \times \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}). \tag{8}$$

**Theorem 2** *The measure defined in Definition 8 is a valid SM for PFSs.*

***Proof*** For two PFSs $\mathcal{P}$ and $\mathcal{Q}$ and from the Theorem 1, we have

(P1) Since $0 \leq \mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) \leq 1$ which implies that $0 \leq \mathcal{S}_0(\mathcal{P}, \mathcal{Q}) \leq \sum_{i=1}^n \omega_i = 1$.
(P2) As $\mathcal{S}_i^{\mu}$ and $\mathcal{S}_i^{\nu}$ are symmetrical for PFSs, so $\mathcal{S}_0$ also have this property.
(P3) As $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = 1$ if and only if $\mathcal{P} = \mathcal{Q}$, so we get $\mathcal{S}_0(\mathcal{P}, \mathcal{Q}) = 1$ if only if $\mathcal{P} = \mathcal{Q}$, because $\sum_{i=1}^n \omega_i = 1$.
(P4) For three PFSs $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ satisfying $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{R}$, we observed from Theorem 1 that $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\mu}(\mathcal{Q}, \mathcal{R})\}$ and $\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{R}) \leq \min \{\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{Q}, \mathcal{R})\}$. Thus, based on it, Eq. (8) becomes $\mathcal{S}_0(\mathcal{P}, \mathcal{R}) \leq \mathcal{S}_0(\mathcal{P}, \mathcal{Q})$ and $\mathcal{S}_0(\mathcal{P}, \mathcal{R}) \leq \mathcal{S}_0(B, C)$. □

Besides this, we can also define some other types of the SMs based on Eqs. (6) and (7), which are summarized in Definitions 9 and 10.

**Definition 9** For two PFSs $\mathcal{P}$ and $\mathcal{Q}$, the weighted average SM of the functions $\mathcal{S}_i^{\mu}$ and $\mathcal{S}_i^{\nu}$ is defined as

$$\mathcal{S}_1(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^n \omega_i \left( \frac{\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) + \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q})}{2} \right), \tag{9}$$

where $\omega_i > 0$ be the normalized weight vector of element of $X$.

**Theorem 3** *The measure given in Definition 9 is a valid SM for PFSs.*

***Proof*** For two PFSs $\mathcal{P}$ and $\mathcal{Q}$ and from the Theorem 1, we have

(P1) Since $0 \leq \mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) \leq 1$, we have

$$0 \leq \mathcal{S}_1(\mathcal{P}, \mathcal{Q}) \leq \sum_{i=1}^n \omega_i = 1.$$

(P2) It can be easily proven, so we omit here.
(P3) As $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}) = \mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}) = 1$ if and only if $\mathcal{P} = \mathcal{Q}$, so by the definition of $\mathcal{S}_1$, we get $\mathcal{S}_1(\mathcal{P}, \mathcal{Q}) = 1$ if only if $\mathcal{P} = \mathcal{Q}$.
(P4) For PFSs $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ such that $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{R}$ then $\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\mu}(\mathcal{Q}, \mathcal{R})\}$ and $\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{R}) \leq \min\{\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}), \mathcal{S}_i^{\nu}(\mathcal{Q}, \mathcal{R})\}$, so that $\mathcal{S}_1(\mathcal{P}, \mathcal{R}) \leq \mathcal{S}_1(\mathcal{P}, \mathcal{Q})$ and $\mathcal{S}_1(\mathcal{P}, \mathcal{R}) \leq \mathcal{S}_1(\mathcal{Q}, \mathcal{R})$.

□

**Definition 10** For two PFSs $\mathcal{P}$ and $\mathcal{Q}$ and using functions $\mathcal{S}_i^{\mu}$ and $\mathcal{S}_i^{\nu}$, a generalized weighted SM $\mathcal{S}_p$ is defined as

$$\mathcal{S}_p(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^n \omega_i \left( \frac{\sqrt[p]{(\mathcal{S}_i^{\mu}(\mathcal{P}, \mathcal{Q}))^p + (\mathcal{S}_i^{\nu}(\mathcal{P}, \mathcal{Q}))^p}}{2} \right) \quad \text{for all } p \in \mathbb{N}^* = \{1, 2, 3, \ldots\}. \tag{10}$$

**Theorem 4** *The function $\mathcal{S}_p$ given in Definition 10 is an SM.*

The proof can be obtained as similar to Theorem 3.

## Applications of the proposed SMs

This section explored the advantages of the proposed SMs in terms of solving pattern recognition problem and DMPs.

### Verification and the comparative analysis

To show the superiority as well as advantages of the proposed measures, we first compare their performance with measures [3,7–16,35,36,49] defined in Table 1 on some common data sets.

The results computed by the proposed SMs ($\mathcal{S}_0$ and $\mathcal{S}_1$) and the existing SMs [3,7–16,35,36,49] are listed in Table 2, which suggests that proposed ones $\mathcal{S}_{BA}$ [16] and $\mathcal{S}_{CC}$ [9] can overcome the drawbacks of the several other existing SMs ($\mathcal{S}_{\mathcal{R}}$ [8], $\mathcal{S}_{HY1}$ [10], $\mathcal{S}_{HY2}$ [10], $\mathcal{S}_{HY3}$ [10], $\mathcal{S}_{HK}$ [11], $\mathcal{S}_{LC}$

**Table 1** Existing similarity measures

| Authors | Similarity measure |
|---|---|
| Li et al. [7] | $\mathcal{S}_L(\mathcal{M},\mathcal{N}) = 1 - \sqrt{\frac{\sum_{i=1}^{n}((\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i))^2 + (\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i))^2)}{2n}}$ |
| Chen [8] | $\mathcal{S}_C(\mathcal{M},\mathcal{N}) = 1 - \frac{\sum_{i=1}^{n}\lvert\mu_{\mathcal{M}}(x_i) - \nu_{\mathcal{M}}(x_i) - (\mu_{\mathcal{N}}(x_i) - \nu_{\mathcal{N}}(x_i))\rvert}{2n}$ |
| Chen and Chang [9] | $\mathcal{S}_{CC}(\mathcal{M},\mathcal{N}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\lvert\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert \times (1 - \frac{\pi_{\mathcal{M}}(x_i) + \pi_{\mathcal{N}}(x_i)}{2}) + \left(\int_0^1 \lvert\mu_{M_{x_i}}(z) - \mu_{N_{x_i}}(z)\rvert d_z\right) \times \left(\frac{\pi_{\mathcal{M}}(x_i) + \pi_{\mathcal{N}}(x_i)}{2}\right)\right)$ |
| | where $\mu_{M_{x_i}}(z) = \begin{cases} 1, & \text{if } z = \mu_{\mathcal{M}}(x_i) = 1 - \nu_{\mathcal{M}}(x_i), \\ \frac{1 - \nu_{\mathcal{M}}(x_i) - z}{1 - \mu_{\mathcal{M}}(x_i) - \nu_{\mathcal{M}}(x_i)}, & \text{if } z \in [\mu_{\mathcal{M}}(x_i), 1 - \nu_{\mathcal{M}}(x_i)], \\ 0, & \text{otherwise.} \end{cases}$ |
| Hung and Yang [10] | $\mathcal{S}_{HY1}(\mathcal{M},\mathcal{N}) = 1 - \frac{\sum_{i=1}^{n}\max(\lvert\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert, \lvert\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i)\rvert)}{n}$, $\mathcal{S}_{HY2}(\mathcal{M},\mathcal{N}) = \frac{e^{\mathcal{S}_{HY1}(\mathcal{M},\mathcal{N})-1} - e^{-1}}{1 - e^{-1}}$ |
| | $\mathcal{S}_{HY3}(\mathcal{M},\mathcal{N}) = \frac{\mathcal{S}_{HY1}(\mathcal{M},\mathcal{N})}{2 - \mathcal{S}_{HY1}(\mathcal{M},\mathcal{N})}$ |
| Hong and Kim [11] | $\mathcal{S}_{HK}(\mathcal{M},\mathcal{N}) = 1 - \frac{\sum_{i=1}^{n}(\lvert\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert + \lvert\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i)\rvert)}{2n}$ |
| Li and Cheng [12] | $\mathcal{S}_{LC}(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}\lvert\psi_M(x_i) - \psi_N(x_i)\rvert^p}{n}}$, |
| | where $\psi_M(x_i) = \frac{\mu_{\mathcal{M}}(x_i) + 1 - \nu_{\mathcal{M}}(x_i)}{2}$, $\psi_N(x_i) = \frac{\mu_{\mathcal{N}}(x_i) + 1 - \nu_{\mathcal{N}}(x_i)}{2}$, and $1 \le p < \infty$. |
| Li and Xu [13] | $\mathcal{S}_{LX}(\mathcal{M},\mathcal{N}) = 1 - \frac{\sum_{i=1}^{n}(\lvert\mu_{\mathcal{M}}(x_i) - \nu_{\mathcal{M}}(x_i) - (\mu_{\mathcal{N}}(x_i) - \nu_{\mathcal{N}}(x_i))\rvert + \lvert\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert + \lvert\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i)\rvert)}{4n}$ |
| Liang and Shi [14] | $\mathcal{S}_{LS1}(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}\lvert\phi_\mu(x_i) + \phi_\nu(x_i)\rvert}{n}}$, $\mathcal{S}_{LS2}(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}\lvert\varphi_\mu(x_i) + \varphi_\nu(x_i)\rvert}{n}}$, |
| | $\mathcal{S}_{LS3}(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\eta_1(x_i) + \eta_2(x_i) + \eta_3(x_i))^p}{3n}}$, |
| | where $\phi_\mu(x_i) = \frac{\lvert\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert}{2}$, $\phi_\nu(x_i) = \frac{\lvert\nu_{\mathcal{M}}(x_i)) - \nu_{\mathcal{N}}(x_i)\rvert}{2}$, $\varphi_\mu(x_i) = \frac{\lvert m_{M1}(x_i) - m_{N1}(x_i)\rvert}{2}$, |
| | $\varphi_\nu(x_i) = \frac{\lvert m_{M2}(x_i) - m_{N2}(x_i)\rvert}{2}$, $m_{M1}(x_i) = \frac{\lvert\mu_M(x_i) + m_M(x_i)\rvert}{2}$, $m_{N1}(x_i) = \frac{\lvert\mu_N(x_i) + m_N(x_i)\rvert}{2}$, |
| | $m_{M2}(x_i) = \frac{\lvert 1 - \nu_M(x_i) + m_M(x_i)\rvert}{2}$, $m_{N2}(x_i) = \frac{\lvert 1 - \nu_N(x_i) + m_N(x_i)\rvert}{2}$, $m_M(x_i) = \frac{\lvert\mu_M(x_i) + 1 - \nu_M(x_i)\rvert}{2}$, |
| | $m_N(x_i) = \frac{\lvert\mu_N(x_i) + 1 - \nu_N(x_i)\rvert}{2}$, $\eta_1(x_i) = \frac{\lvert\mu_M(x_i) - \mu_N(x_i)\rvert + \lvert\nu_M(x_i) - \nu_N(x_i)\rvert}{2}$, |
| | $\eta_2(x_i) = \frac{\lvert(\mu_M(x_i) - \nu_M(x_i)) - (\mu_N(x_i) - \nu_N(x_i))\rvert}{2}$, |
| | $\eta_3(x_i) = \max\left(\frac{\pi_{\mathcal{M}}(x_i)}{2}, \frac{\pi_{\mathcal{N}}(x_i)}{2}\right) - \min\left(\frac{\pi_{\mathcal{M}}(x_i)}{2}, \frac{\pi_{\mathcal{N}}(x_i)}{2}\right)$. |

**Table 1** continued

| Authors | Similarity measure |
|---|---|
| Mitchell [15] | $\mathcal{S}_{\mathcal{M}}(\mathcal{M},\mathcal{N}) = \frac{\rho_\mu(\mathcal{M},\mathcal{N}) + \rho_v(\mathcal{M},\mathcal{N})}{2}$, |
| | where $\rho_\mu(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^n \lvert \mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)\rvert^p}{n}}$,. |
| | $\rho_v(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^n \lvert \nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i)\rvert^p}{n}}$, and $1 \le p < \infty$ |
| Ye [3] | $\mathcal{S}_Y(\mathcal{M},\mathcal{N}) = \frac{1}{n}\sum_{i=1}^n \frac{\mu_{\mathcal{M}}(x_i)\mu_{\mathcal{N}}(x_i) + \nu_{\mathcal{M}}(x_i)\nu_{\mathcal{N}}(x_i)}{\sqrt{\mu^2_{\mathcal{M}}(x_i) + \nu^2_{\mathcal{M}}(x_i)}\sqrt{\mu^2_{\mathcal{N}}(x_i) + \nu^2_{\mathcal{N}}(x_i)}}$ |
| Wei and Wei [36] | $\mathcal{S}_W(\mathcal{M},\mathcal{N}) = \frac{1}{n}\sum_{i=1}^n \frac{\mu^2_{\mathcal{M}}(x_i)\mu^2_{\mathcal{N}}(x_i) + \nu^2_{\mathcal{M}}(x_i)\nu^2_{\mathcal{N}}(x_i)}{\sqrt{\mu^4_{\mathcal{M}}(x_i) + \nu^4_{\mathcal{M}}(x_i)}\sqrt{\mu^4_{\mathcal{N}}(x_i) + \nu^4_{\mathcal{N}}(x_i)}}$ |
| Zhang [35] | $\mathcal{S}_Z(\mathcal{M},\mathcal{N}) = \frac{1}{n}\sum_{i=1}^n \frac{\lvert\mu^2_{\mathcal{M}}(x_i) - \nu^2_{\mathcal{N}}(x_i)\rvert + \lvert\nu^2_{\mathcal{M}}(x_i) - \mu^2_{\mathcal{N}}(x_i)\rvert + \lvert\pi^2_{\mathcal{M}}(x_i) - \pi^2_{\mathcal{N}}(x_i)\rvert}{\lvert\mu^2_{\mathcal{M}}(x_i) - \mu^2_{\mathcal{N}}(x_i)\rvert + \lvert\nu^2_{\mathcal{M}}(x_i) - \nu^2_{\mathcal{N}}(x_i)\rvert}$ |
| | $+\lvert\pi^2_{\mathcal{M}}(x_i) - \pi^2_{\mathcal{N}}(x_i)\rvert + \lvert\mu^2_{\mathcal{M}}(x_i) - \nu^2_{\mathcal{N}}(x_i)\rvert + \lvert\nu^2_{\mathcal{M}}(x_i) - \mu^2_{\mathcal{N}}(x_i)\rvert + \lvert\pi^2_{\mathcal{M}}(x_i) - \pi^2_{\mathcal{N}}(x_i)\rvert$ |
| Peng et al. [49] | $\mathcal{S}_{P1}(\mathcal{M},\mathcal{N}) = 1 - \frac{\sum_{i=1}^n \lvert\mu^2_{\mathcal{M}}(x_i) - \nu^2_{\mathcal{M}}(x_i) - (\mu^2_{\mathcal{N}}(x_i) - \nu^2_{\mathcal{N}}(x_i))\rvert}{2n}$ |
| | $\mathcal{S}_{P2}(\mathcal{M},\mathcal{N}) = \frac{1}{n}\sum_{i=1}^n \frac{(\mu^2_{\mathcal{M}}(x_i) \bigwedge \mu^2_{\mathcal{N}}(x_i)) + (\nu^2_{\mathcal{M}}(x_i) \bigwedge \nu^2_{\mathcal{N}}(x_i))}{(\mu^2_{\mathcal{M}}(x_i) \bigvee \mu^2_{\mathcal{N}}(x_i)) + (\nu^2_{\mathcal{M}}(x_i) \bigvee \nu^2_{\mathcal{N}}(x_i))}$ |
| | $\mathcal{S}_{P3}(\mathcal{M},\mathcal{N}) = \frac{1}{n}\sum_{i=1}^n \frac{(\mu^2_{\mathcal{M}}(x_i) \bigwedge \mu^2_{\mathcal{N}}(x_i)) + (1 - \nu^2_{\mathcal{M}}(x_i)) \bigwedge (1 - \nu^2_{\mathcal{N}}(x_i))}{(\mu^2_{\mathcal{M}}(x_i) \bigvee \mu^2_{\mathcal{N}}(x_i)) + (1 - \nu^2_{\mathcal{M}}(x_i)) \bigvee (1 - \nu^2_{\mathcal{N}}(x_i))}$ |
| Boran and Akay [16] | $\mathcal{S}_{BA}(\mathcal{M},\mathcal{N}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^n \{\lvert t(\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i)) - (\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i))\rvert^p + \lvert t(\nu_{\mathcal{M}}(x_i) - \nu_{\mathcal{N}}(x_i)) - (\mu_{\mathcal{M}}(x_i) - \mu_{\mathcal{N}}(x_i))\rvert^p\}}{2n(t+1)^p}}$ |

**Table 2** Comparison of SMs adopted from [3]

| $\mathcal{M}$<br>$\mathcal{N}$ | Case 1<br>$\{(x, 0.3, 0.3)\}$<br>$\{(x, 0.4, 0.4)\}$ | Case 2<br>$\{(x, 0.3, 0.4)\}$<br>$\{(x, 0.4, 0.3)\}$ | Case 3<br>$\{(x, 1, 0)\}$<br>$\{(x, 0, 0)\}$ | Case 4<br>$\{(x, 0.5, 0.5)\}$<br>$\{(x, 0, 0)\}$ | Case 5<br>$\{(x, 0.4, 0.2)\}$<br>$\{(x, 0.5, 0.3)\}$ | Case 6<br>$\{(x, 0.4, 0.2)\}$<br>$\{(x, 0.5, 0.2)\}$ |
|---|---|---|---|---|---|---|
| $\mathcal{S}_L$ [7] | **0.9** | **0.9** | 0.2929 | 0.5 | **0.9** | 0.9293 |
| $\mathcal{S}_{\mathcal{R}}$ [8] | **1** | 0.9 | 0.5 | **1** | **1** | 0.95 |
| $\mathcal{S}_{CC}$ [9] | 0.9225 | 0.88 | 0.25 | 0.5 | 0.9225 | 0.8913 |
| $\mathcal{S}_{HY1}$ [10] | **0.9** | **0.9** | **0** | 0.5 | **0.9** | **0.9** |
| $\mathcal{S}_{HY2}$ [10] | **0.8495** | **0.8495** | **0** | 0.3775 | **0.8495** | **0.8495** |
| $\mathcal{S}_{HY3}$ [10] | **0.8182** | **0.8182** | **0** | 0.3333 | **0.8182** | **0.8182** |
| $\mathcal{S}_{HK}$ [11] | **0.9** | **0.9** | **0.5** | **0.5** | **0.9** | 0.95 |
| $\mathcal{S}_{LC}$ [12] | **1** | 0.9 | 0.5 | **1** | **1** | 0.95 |
| $\mathcal{S}_{LX}$ [13] | 0.95 | 0.9 | 0.5 | 0.75 | **0.95** | **0.95** |
| $\mathcal{S}_{LS1}$ [14] | **0.9** | **0.9** | **0.5** | **0.5** | **0.9** | 0.95 |
| $\mathcal{S}_{LS2}$ [14] | 0.95 | 0.9 | 0.5 | 0.75 | **0.95** | **0.95** |
| $\mathcal{S}_{LS3}$ [14] | **0.9333** | **0.9333** | 0.5 | 0.6667 | **0.9333** | 0.95 |
| $\mathcal{S}_M$ [15] | **0.9** | **0.9** | **0.5** | **0.5** | **0.9** | 0.95 |
| $\mathcal{S}_Y$ [3] | **1** | 0.96 | N/A | N/A | 0.9971 | 0.9965 |
| $\mathcal{S}_W$ [36] | **1** | 0.8546 | N/A | N/A | **0.9949** | 0.9963 |
| $\mathcal{S}_Z$ [35] | 0.5 | **0** | **0.5** | **0.5** | **0.6** | 0.7 |
| $\mathcal{S}_{P1}$ [49] | **1** | 0.93 | 0.5 | **1** | 0.98 | 0.955 |
| $\mathcal{S}_{P2}$ [49] | **0.5625** | **0.5625** | **0** | **0** | **0.5882** | 0.6897 |
| $\mathcal{S}_{P3}$ [49] | **0.8692** | **0.8692** | 0.5 | 0.6 | **0.8843** | 0.9256 |
| $\mathcal{S}_{BA}$ [16] | 0.967 | 0.9 | 0.5 | 0.8333 | 0.9667 | 0.95 |
| $\mathcal{S}_0$(proposed) | 0.8694 | 0.8694 | 0.3679 | 0.6065 | 0.8694 | 0.9139 |
| $\mathcal{S}_1$(proposed) | 0.9324 | 0.9324 | 0.6839 | 0.7788 | 0.9326 | 0.9570 |

Note: ($p = 1$ in $\mathcal{S}_{\mathcal{M}}, \mathcal{S}_{LC}, \mathcal{S}_{LS1}, \mathcal{S}_{LS2}, \mathcal{S}_{LS3}$, $p = 1, t = 2$ in $\mathcal{S}_{BA}$) "Bold" denotes unreasonable results.
"N/A" denotes that it cannot compute the degree of similarity due to "the division by zero problem"

[12], $\mathcal{S}_{LX}$ [13], $\mathcal{S}_L$ [7], $\mathcal{S}_{LS1}$ [14], $\mathcal{S}_{LS2}$ [14], $\mathcal{S}_{LS3}$ [14], $\mathcal{S}_M$ [15], $\mathcal{S}_Y$ [3], $\mathcal{S}_{P1}$ [49], $\mathcal{S}_{P2}$ [49], $\mathcal{S}_{P3}$ [49], $\mathcal{S}_Z$ [35], and $\mathcal{S}_W$ [35]).

Furthermore, to achieve more advantages of the proposed SMs with the existing measures, we consider another data sets and the results computed by the existing measures [3,7–16,35,36,49] as well as proposed measures ($\mathcal{S}_0, \mathcal{S}_1$) are given in Table 3. It is clearly seen from this table that the proposed SMs overcome the certain drawbacks of the existing measures $\mathcal{S}_{BA}$ [16], $\mathcal{S}_{\mathcal{R}}$ [8], $\mathcal{S}_{HY1}$ [10], $\mathcal{S}_{HY2}$ [10], $\mathcal{S}_{HY3}$ [10], $\mathcal{S}_{HK}$ [11], $\mathcal{S}_{LC}$ [12], $\mathcal{S}_{LX}$ [13], $\mathcal{S}_L$ [7] , $\mathcal{S}_{LS1}$ [14], $\mathcal{S}_{LS2}$ [14], $\mathcal{S}_{LS3}$ [14], $\mathcal{S}_M$ [15], $\mathcal{S}_Y$ [3], $\mathcal{S}_{P1}$ [49], $\mathcal{S}_{P2}$ [49], $\mathcal{S}_Z$ [35], and $\mathcal{S}_W$ [36].

Finally, we further shows that existing measures [3,7–16,35,36,49] also suffer from the shortcoming under some special cases that are listed in Table 4. The computed results by the proposed SM ($\mathcal{S}_0, \mathcal{S}_1$) show the best results as compared to the existing measures $\mathcal{S}_{BA}$ [16], $\mathcal{S}_{\mathcal{R}}$ [8], $\mathcal{S}_{HY1}$ [10], $\mathcal{S}_{HY2}$ [10], $\mathcal{S}_{HY3}$ [10], $\mathcal{S}_{HK}$ [11], $\mathcal{S}_{LC}$ [12], $\mathcal{S}_{LX}$ [13], $\mathcal{S}_L$ [7] , $\mathcal{S}_{LS1}$ [14], $\mathcal{S}_{LS2}$ [14], $\mathcal{S}_{LS3}$ [14], $\mathcal{S}_M$ [15], $\mathcal{S}_Y$ [3], $\mathcal{S}_{P1}$ [49], $\mathcal{S}_{P2}$ [49], $\mathcal{S}_Z$ [35], and $\mathcal{S}_W$ [36].

## Applications related to pattern recognition

***Example 1*** Consider a three known patterns $\mathcal{P}_i (i = 1, 2, 3)$ whose characteristics are represented in terms of PFSs over the feature space $X = \{x_1, x_2, x_3\}$ as follows:

$$\mathcal{P}_1 = \{(x_1, 1, 0), (x_2, 0.8, 0), (x_3, 0.7, 0.1)\};$$
$$\mathcal{P}_2 = \{(x_1, 0.8, 0.1), (x_2, 1, 0), (x_3, 0.9, 0.1)\};$$
$$\mathcal{P}_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0), (x_3, 1, 0)\}.$$

Consider an unknown sample $\mathcal{Q}$ under PFSs and defined as

$$\mathcal{Q} = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}.$$

Our goal is to find out the recognition of the pattern $\mathcal{Q}$ with one of $\mathcal{P}_i$. To achieve it, we choose the arbitrary weight vector $\omega = (0.5, 0.3, 0.2)$ of the elements of $X$, and hence, the measurement values of the SMs along with existing SMs [35, 36] are computed and listed their results in Table 5. From it, we found that the pattern $\mathcal{Q}$ recognizes with $\mathcal{P}_3$ and coincides with the existing measures.

**Table 3** Comparison of SMs adopted from [9]

| | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 |
|---|---|---|---|---|---|---|
| $\mathcal{M}$ | $\{(x, 0.5, 0.5)\}$ | $\{(x, 0.6, 0.4)\}$ | $\{(x, 0, 0.87)\}$ | $\{(x, 0.6, 0.27)\}$ | $\{(x, 0.125, 0.075)\}$ | $\{(x, 0.5, 0.45)\}$ |
| $\mathcal{N}$ | $\{(x, 0, 0)\}$ | $\{(x, 0, 0)\}$ | $\{(x, 0.28, 0.55)\}$ | $\{(x, 0.28, 0.55)\}$ | $\{(x, 0.175, 0.025)\}$ | $\{(x, 0.55, 0.4)\}$ |
| $\mathcal{S}_L$ [7] | 0.5 | 0.4901 | **0.6993** | **0.6993** | **0.95** | **0.95** |
| $\mathcal{S}_{\mathcal{R}}$ [8] | **1** | 0.9 | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{CC}$ [9] | 0.5 | 0.45 | 0.7395 | 0.7055 | 0.9125 | 0.95 |
| $\mathcal{S}_{HY1}$ [10] | 0.5 | 0.4 | **0.68** | **0.68** | **0.95** | **0.95** |
| $\mathcal{S}_{HY2}$ [10] | 0.3775 | 0.2862 | **0.5668** | **0.5668** | **0.9229** | **0.9229** |
| $\mathcal{S}_{HY3}$ [10] | 0.3333 | 0.25 | **0.5152** | **0.5152** | **0.9048** | **0.9048** |
| $\mathcal{S}_{HK}$ [11] | **0.5** | **0.5** | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{LC}$ [12] | **1** | 0.9 | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{LX}$ [13] | 0.75 | 0.7 | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{LS1}$ [14] | **0.5** | **0.5** | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{LS2}$ [14] | **0.75** | **0.75** | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_{LS3}$ [14] | 0.6667 | 0.6333 | **0.7933** | **0.7933** | **0.9667** | **0.9667** |
| $\mathcal{S}_M$ [15] | **0.5** | **0.5** | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_Y$ [3] | N/A | N/A | 0.8912 | 0.7794 | 0.9216 | 0.9946 |
| $\mathcal{S}_W$ [36] | N/A | N/A | 0.968 | 0.438 | 0.9476 | 0.9812 |
| $\mathcal{S}_Z$ [35] | **0.5** | **0.5** | 0.5989 | 0.1696 | 0.625 | 0.6557 |
| $\mathcal{S}_{P1}$ [49] | **1** | 0.9 | 0.7336 | 0.7444 | 0.99 | 0.9525 |
| $\mathcal{S}_{P2}$ [49] | **0** | **0** | 0.3621 | 0.2284 | 0.4483 | 0.8119 |
| $\mathcal{S}_{P3}$ [49] | 0.6 | 0.6176 | 0.3133 | 0.6028 | 0.9806 | 0.9168 |
| $\mathcal{S}_{BA}$ [16] | **0.8333** | **0.8333** | **0.7** | **0.7** | **0.95** | **0.95** |
| $\mathcal{S}_0$(proposed) | 0.6065 | 0.5945 | 0.5870 | 0.5998 | 0.9802 | 0.9094 |
| $\mathcal{S}_1$(proposed) | 0.7788 | 0.7749 | 0.7797 | 0.7747 | 0.9901 | 0.9536 |

Note: ($p = 1$ in $\mathcal{S}_{\mathcal{M}}, \mathcal{S}_{LC}, \mathcal{S}_{LS1}, \mathcal{S}_{LS2}, \mathcal{S}_{LS3}$, $p = 1, t = 2$ in $\mathcal{S}_{BA}$.) "Bold" denotes unreasonable results.
"N/A" denotes that it cannot compute the degree of similarity due to "the division by zero problem"

## Application to the DMPs

This section states the DMP method based on the proposed SMs under PFS environment to determine the finest alternative(s). For it, assume that $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be the set of "$m$" alternatives and $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be the set of "$n$" criteria, whose weight vector is $\omega_j > 0$ with $\sum_{j=1}^{n} \omega_j = 1$. An expert evaluates these alternatives and rates them in terms of PFNs $\gamma_{ij} = (\mu_{ij}, \nu_{ij})$, such that $\mu_{ij}^2 + \nu_{ij}^2 \leq 1$ satisfied. The complete PF decision matrix $D$ is defined as

$$D = \begin{array}{c} \\ \mathcal{P}_1 \\ \mathcal{P}_2 \\ \vdots \\ \mathcal{P}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} \gamma_{11} & \gamma_{12} & \ldots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \ldots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \ldots & \gamma_{mn} \end{pmatrix} \end{array} \tag{11}$$

Then, the following steps are proposed based on the proposed SMs to evaluate them.

Step 1: Determine the weight of each criteria
We determine the weight vector $\omega_{j,k}, (k = 0, 1, 2, \ldots)$ of each criteria $\mathcal{G}_j$ using the following equation:

$$\omega_{j,k} = \frac{(d_j)^k}{\sum_{j=1}^{n} (d_j)^k}, k = 0, 1, 2, \ldots \tag{12}$$

where $d_j = d_{1j} + d_{2j}$ in which $d_{1j} = \max_i \mu_{ij}$, $d_{2j} = \min_i \nu_{ij}$ for all $j = 1, 2, \ldots, n$, such that $\sum_{j=1}^{n} \omega_{j,k} = 1$ for $k = 0, 1, 2, \ldots$.

Step 2: Determine the ideal values
The given criteria are divided into two disjoint sets, namely, the cost $\mathcal{F}_1$ and the benefit $\mathcal{F}_2$. For $\mathcal{F}_1$ criteria, the ideal values are taken as (0,1), while for $\mathcal{F}_2$ criteria, we take (1,0). It is noted here that (1, 0) is the largest value of a PFNs and (0, 1) is the smallest value of a PFN. Therefore, we represent the ideal values for all criteria as $\mathcal{P}_b = (\mathcal{P}_b(1), \mathcal{P}_b(2), \ldots, \mathcal{P}_b(n))$, where $\mathcal{P}_b(j) = (1, 0)$

**Table 4** Comparison of SMs

| | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 |
|---|---|---|---|---|---|---|
| $\mathcal{M}$ | $\{(x, 0.3, 0.7)\}$ | $\{(x, 0.3, 0.7)\}$ | $\{(x, 0.5, 0.5)\}$ | $\{(x, 0.4, 0.6)\}$ | $\{(x, 0.1, 0.5)\}$ | $\{(x, 0.4, 0.2)\}$ |
| $\mathcal{N}$ | $\{(x, 0.4, 0.6)\}$ | $\{(x, 0.2, 0.8)\}$ | $\{(x, 0, 0)\}$ | $\{(x, 0, 0)\}$ | $\{(x, 0.2, 0.3)\}$ | $\{(x, 0.2, 0.3)\}$ |
| $\mathcal{S}_L$ [7] | **0.6863** | **0.6863** | 0.5 | 0.4901 | **0.8419** | **0.8419** |
| $\mathcal{S}_{\mathcal{R}}$ [8] | **0.9** | **0.9** | **1** | 0.9 | **0.85** | **0.85** |
| $\mathcal{S}_{CC}$ [9] | **0.9** | **0.9** | 0.5 | 0.55 | 0.8438 | 0.7685 |
| $\mathcal{S}_{HY1}$ [10] | **0.9** | **0.9** | 0.5 | 0.4 | **0.8** | **0.8** |
| $\mathcal{S}_{HY2}$ [10] | **0.8494** | **0.8494** | 0.3775 | 0.2862 | **0.7132** | **0.7132** |
| $\mathcal{S}_{HY3}$ [10] | **0.8182** | **0.8182** | 0.3333 | 0.25 | **0.6667** | **0.6667** |
| $\mathcal{S}_{HK}$ [11] | **0.9** | **0.9** | **0.5** | **0.5** | **0.85** | **0.85** |
| $\mathcal{S}_{LC}$ [12] | **0.9** | **0.9** | **1** | 0.9 | **0.85** | **0.85** |
| $\mathcal{S}_{LX}$ [13] | **0.9** | **0.9** | 0.75 | 0.7 | **0.85** | **0.85** |
| $\mathcal{S}_{LS1}$ [14] | **0.9** | **0.9** | **0.5** | **0.5** | **0.85** | **0.85** |
| $\mathcal{S}_{LS2}$ [14] | **0.9** | **0.9** | 0.5 | 0.75 | **0.85** | **0.85** |
| $\mathcal{S}_{LS3}$ [14] | **0.95** | **0.95** | 0.6667 | 0.6333 | **0.8833** | **0.8833** |
| $\mathcal{S}_M$ [15] | **0.9** | **0.9** | **0.5** | **0.5** | **0.85** | **0.85** |
| $\mathcal{S}_Y$ [3] | 0.9832 | 0.9873 | N/A | N/A | 0.9249 | 0.8685 |
| $\mathcal{S}_W$ [36] | 0.9721 | 0.9929 | N/A | N/A | 0.9293 | 0.6156 |
| $\mathcal{S}_Z$ [35] | 0.7174 | 0.7857 | **0.5** | **0.5** | 0.5676 | 0.3684 |
| $\mathcal{S}_{P1}$ [49] | **0.9** | **0.9** | **1** | 0.9 | 0.905 | 0.915 |
| $\mathcal{S}_{P2}$ [49] | 0.6923 | 0.726 | **0** | **0** | 0.3448 | 0.32 |
| $\mathcal{S}_{P3}$ [49] | 0.75 | 0.6667 | 0.6 | 0.5517 | 0.8 | 0.8482 |
| $\mathcal{S}_{BA}$ [16] | **0.9** | **0.9** | **0.8333** | **0.8333** | **0.8667** | **0.8667** |
| $\mathcal{S}_0$(proposed) | 0.8187 | 0.8185 | 0.6065 | 0.5945 | 0.8270 | 0.8437 |
| $\mathcal{S}_1$(proposed) | 0.9052 | 0.9060 | 0.7788 | 0.7749 | 0.9113 | 0.9191 |

Note: ($p = 1$ in $\mathcal{S}_{\mathcal{M}}, \mathcal{S}_{LC}, \mathcal{S}_{LS1}, \mathcal{S}_{LS2}, \mathcal{S}_{LS3}$, $p = 1, t = 2$ in $\mathcal{S}_{BA}$) "Bold" denotes unreasonable results.
"N/A" denotes that it cannot compute the degree of similarity due to "the division by zero problem"

**Table 5** Comparison analysis and the ranking order

| Similarity measures | Measurement values of $\mathcal{Q}$ from | | | Ranking order |
|---|---|---|---|---|
| | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | |
| Measure PFC[1] proposed by Wei and Wei [36] | 0.96864 | 0.97113 | 0.98464 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1$ |
| Measure PFC[2] proposed by Wei and Wei [36] | 0.62729 | 0.72373 | 0.92666 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1$ |
| Measure Sm proposed by [35] | 0.64018 | 0.63749 | 0.70645 | $\mathcal{P}_3 \succ \mathcal{P}_1 \succ \mathcal{P}_2$ |
| Measure $\mathcal{S}_0$ proposed in this paper | 0.60585 | 0.63322 | 0.78205 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1$ |
| Measure $\mathcal{S}_1$ proposed in this paper | 0.79017 | 0.80766 | 0.88802 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1$ |
| Measure $\mathcal{S}_2$ proposed in this paper | 0.57250 | 0.58144 | 0.63113 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1$ |

**Table 6** Rating values in terms of PFNs

| | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_3$ | $\mathcal{G}_4$ | $\mathcal{G}_5$ | $\mathcal{G}_6$ |
|---|---|---|---|---|---|---|
| $\mathcal{P}_1$ | (0.2, 0.5) | (0.3, 0.8) | (0.4, 0.9) | (0.3, 0.7) | (0.2, 0.4) | (0.8, 0.4) |
| $\mathcal{P}_2$ | (0.3, 0.5) | (0.8, 0.5) | (0.5, 0.6) | (0.5, 0.6) | (0.4, 0.7) | (0.6, 0.5) |
| $\mathcal{P}_3$ | (0.4, 0.3) | (0.6, 0.8) | (0.6, 0.7) | (0.6, 0.8) | (0.8, 0.6) | (0.7, 0.4) |
| $\mathcal{P}_4$ | (0.4, 0.7) | (0.6, 0.8) | (0.4, 0.7) | (0.7, 0.6) | (0.5, 0.7) | (0.5, 0.8) |
| $\mathcal{P}_5$ | (0.6, 0.8) | (0.4, 0.7) | (0.7, 0.4) | (0.3, 0.4) | (0.7, 0.7) | (0.4, 0.7) |

**Table 7** Comparative study for Example 2

| Approach | Measurement values of the alternatives from $\mathcal{P}_b$ | | | | | Ranking order |
|---|---|---|---|---|---|---|
| | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ | |
| Method by PFC[1] proposed in [36] | 0.45365 | 0.64741 | 0.76863 | 0.42520 | 0.59741 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_5 \succ \mathcal{P}_1 \succ \mathcal{P}_4$ |
| Method by PFC[2] proposed in [36] | 0.35854 | 0.51423 | 0.64996 | 0.39823 | 0.44496 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_5 \succ \mathcal{P}_4 \succ \mathcal{P}_1$ |
| Method by Sm proposed in [35] | 0.45293 | 0.51224 | 0.533656 | 0.38015 | 0.45933 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_5 \succ \mathcal{P}_1 \succ \mathcal{P}_4$ |
| Method by $\mathcal{S}_0$ proposed in the paper | 0.36337 | 0.38339 | 0.4134 | 0.27814 | 0.34522 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1 \succ \mathcal{P}_4 \succ \mathcal{P}_4$ |
| Method by $\mathcal{S}_1$ proposed in the paper | 0.60576 | 0.62673 | 0.6468 | 0.53046 | 0.59481 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1 \succ \mathcal{P}_4 \succ \mathcal{P}_4$ |
| Method by $\mathcal{S}_2$ proposed in the paper | 0.4397 | 0.45107 | 0.46275 | 0.37751 | 0.42857 | $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1 \succ \mathcal{P}_4 \succ \mathcal{P}_4$ |

if $\mathcal{G}_j \in \mathcal{F}_2$ and $\mathcal{P}_b(j) = (0, 1)$ if $\mathcal{G}_j \in \mathcal{F}_1$ for all $j = 1, 2, \ldots, n$.

Step 3: Calculate the SMs of each alternative from its ideal values
Using the proposed SMs, i.e., $\mathcal{S}_0, \mathcal{S}_1$ or $\mathcal{S}_p$, compute the measurement values of each alternative. Based on the rating values and the ideal measures, compute the SMs values using either $\mathcal{S}_0, \mathcal{S}_1$, or $\mathcal{S}_p$ measures.

Step 4: Rank the alternatives
Based on the assessment values of the SMs, rank the given alternatives with the following rules:

$$\mathcal{P}_i \prec \mathcal{P}_p \text{ if and only if } S(\mathcal{P}_i, \mathcal{P}_b) \leq S(\mathcal{P}_p, \mathcal{P}_b)$$
$$\text{for all } i, p = 1, 2, \ldots, m.$$

Here, $S$ represent the SM.

***Example 2*** To demonstrate the above method, we consider an example related to the invest the money in a certain company. For it, a person chooses the five possible companies $\mathcal{P}_i, i = 1, 2, \ldots, 5$ and considered as an alternative. To evaluate these alternatives, a person hires an investment expert which evaluates these companies under the set of six criteria, namely, $\mathcal{G}_1$: "technical ability", $\mathcal{G}_2$: "expected benefit", $\mathcal{G}_3$: "competitive power on the market", $\mathcal{G}_4$: "ability to bear risk", $\mathcal{G}_5$: "management capability", and $\mathcal{G}_6$: "organizational culture". The values of each alternative are listed in Table 6 using PFNs.

Then, the steps of the method are executed as follows:

Step 1: By Eq. (12) with $k = 1$, we can get

$$\omega = (0.13, 0.19, 0.16, 0.16, 0.18, 0.18).$$

Step 2: As $\mathcal{G}_4 \in \mathcal{F}_1$, while others are belongs to $\mathcal{F}_2$, so the ideal values are $\mathcal{P}_b(1) = \mathcal{P}_b(2) = \mathcal{P}_b(3) = \mathcal{P}_b(5) = \mathcal{P}_b(6) = (1, 0)$ and $\mathcal{P}_b(4) = (0, 1)$.

Step 3: Utilize the similarity measure $\mathcal{S}_1$ to compute the measurement values and get $\mathcal{S}_1(\mathcal{P}_1, \mathcal{P}_b) = 0.60576$, $\mathcal{S}_1(\mathcal{P}_2, \mathcal{P}_b) = 0.62673$, $\mathcal{S}_1(\mathcal{P}_2, \mathcal{P}_b) = 0.6468$, $\mathcal{S}_1(\mathcal{P}_4, \mathcal{P}_b) = 0.53046$, and $\mathcal{S}_1(\mathcal{P}_5, \mathcal{P}_b) = 0.59481$.

Step 4: Since the measurement value of $\mathcal{P}_3$ alternative is the highest and hence the best company is $\mathcal{P}_3$. However, the overall ordering is $\mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1 \succ \mathcal{P}_5 \succ \mathcal{P}_4$.

Furthermore, using existing measures [35,36] and the other proposed SMs ($\mathcal{S}_0, \mathcal{S}_p$), we rank the given alternatives in Table 7. This table shows the consistency of the proposed measures as the finest alternative remains the same by all the methods.

## Conclusion

In this paper, we introduce some new SMs between PFSs based on the exponential function of the MDs and NMDs. The desirable combinations and their features are studied in detail. To show the efficiency of the proposed SMs, we give some counter-intuitive examples which shows that existing measures fail under some certain cases, while the proposed one classifies the objects. Later, we solve the pattern recognition as well as DMPs using the proposed SMs. The numerical results are compared with the existing ones to show its consistency. It is revealed from the proposed method that the solution obtained is good compromise than the existing ones and shows it conservative in nature. In the future, we shall expand the proposed measures under the different uncertain and fuzzy environments [50–54].



## References

1. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
2. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

3. Ye J (2011) Cosine similarity measures for intuitionistic fuzzy sets and their applications. Math Comput Model 53(1–2):91–97
4. Garg H, Kumar K (2018) An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making. Soft Comput 22(15):4959–4970
5. Hwang C-M, Yang M-S, Hung W-L (2018) New similarity measures of intuitionistic fuzzy sets based on the jaccard index with its application to clustering. Int J Intell Syst 33(8):1672–1688
6. Garg H, Kumar K (2018) A novel exponential distance and its based TOPSIS method for interval-valued intuitionistic fuzzy sets using connection number of SPA theory. Artif Intell Rev 1–30. https://doi.org/10.1007/s10462-018-9668-5
7. Li YH, Olson DL, Qin Z (2007) Similarity measures between intuitionistic fuzzy (vague) sets: a comparative analysis. Pattern Recognit Lett 28:278–285
8. Chen SM (1997) Similarity measures between vague sets and between elements. IEEE Trans Syst Man Cybern 27(1):153–158
9. Chen SM, Chang CH (2015) A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition. Inf Sci 291:96–114
10. Hung WL, Yang MS (2004) Similarity measures of intuitionistic fuzzy sets based on hausdorff distance. Pattern Recognit Lett 25:1603–1611
11. Hong DH, Kim C (1999) A note on similarity measures between vague sets and between elements. Inf Sci 115:83–96
12. Li DF, Cheng C (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognit Lett 23:221–225
13. Li F, Xu ZY (2001) Measures of similarity between vague sets. J Softw 12(6):922–927
14. Liang Z, Shi P (2003) Similarity measures on intuitionistic fuzzy sets. Pattern Recognit Lett 24:2687–2693
15. Mitchell HB (2003) On the dengfeng chuntian similarity measure and its application to pattern recognition. Pattern Recognit Lett 24:3101–3104
16. Boran FE, Akay D (2014) A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition. Inf Sci 255:45–57
17. Liao H, Mi X, Xu Z, Xu J, Herrera F (2018) Intuitionistic fuzzy analytic network process. IEEE Trans Fuzzy Syst 26(5):2578–2590
18. Liu W, Liao H (2017) A bibliometric analysis of fuzzy decision research during 1970–2015. Int J Fuzzy Syst 19(1):1–14
19. Yager RR (2013) Pythagorean fuzzy subsets, in: Procedings Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, pp 57–61
20. Peng X, Yang Y (2015) Some results for Pythagorean fuzzy sets. Int J Intell Syst 30(11):1133–1160
21. Beliakov G, James S (2014) Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs. In: Fuzzy Systems (FUZZ-IEEE), 2014 IEEE International Conference on, IEEE, pp. 298–305
22. Garg H (2016) A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int J Intell Syst 31(9):886–920
23. Garg H (2018) Generalized Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making. J Exp Theor Artif Intell 30(6):763–794
24. Wei GW (2017) Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making. J Intell Fuzzy Syst 33(4):2119–2132
25. Wei G, Lu M (2018) Pythagorean fuzzy maclaurin symmetric mean operators in multiple attribute decision making. Int J Intell Syst 33(5):1043–1070
26. Ma ZM, Xu ZS (2016) Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems. Int J Intell Syst 31(12):1198–1219
27. Garg H (2018) New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision - making. Int J Intell Syst 33(3):653–683
28. Garg H (2019) New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications. Int J Intell Syst 34(1):82–106
29. Lu M, Wei GW, Alsaadi FE, Hayat T, Alsaedi A (2017) Hesitant Pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. J Intell Fuzzy Syst 33(2):1105–1117
30. Garg H (2019) Hesitant Pythagorean fuzzy maclaurin symmetric mean operators and its applications to multiattribute decision making process. Int J Intell Syst 34(4):601–626
31. Garg H (2017) Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process. Comput Math Org Theory 23(4):546–571
32. Zhang XL, Xu ZS (2014) Extension of TOPSIS to multi-criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29(12):1061–1078
33. Zeng S, Chen J, Li X (2016) A hybrid method for Pythagorean fuzzy multiple-criteria decision making. Int J Inf Technol Decis Mak 15(2):403–422
34. Garg H (2016) A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes. Int J Intell Syst 31(12):1234–1252
35. Zhang XL (2016) A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making. International Journal of Intelligent Systems 31:593–611
36. Wei G, Wei Y (2018) Similarity measures of Pythagorean fuzzy sets based on the cosine function and their applications. Int J Intell Syst 33(3):634–652
37. Garg H (2016) A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. J Intell Fuzzy Syst 31(1):529–540
38. Garg H (2017) A novel improved accuracy function for interval valued Pythagorean fuzzy sets and its applications in decision making process. Int J Intell Syst 31(12):1247–1260
39. Peng X (2019) New operations for interval-valued Pythagorean fuzzy set. Sci Iran 26(2):1049–1076
40. Garg H (2018) A linear programming method based on an improved score function for interval-valued Pythagorean fuzzy numbers and its application to decision-making. Int J Uncert Fuzziness Knowl Based Syst 26(1):67–80
41. Garg H (2017) A new improved score function of an interval-valued Pythagorean fuzzy set based TOPSIS method. Int J Uncert Quantif 7(5):463–474
42. Zhang X (2016) Multicriteria Pythagorean fuzzy decision analysis: a hierarchical QUALIFLEX approach with the closeness index-based ranking. Inf Sci 330:104–124
43. Liang D, Xu Z (2017) The new extension of TOPSIS method for multiple criteria decision making with hesitant pythagorean fuzzy sets. Appl Soft Comput 60:167–179
44. Garg H (2018) Hesitant Pythagorean fuzzy sets and their aggregation operators in multiple attribute decision making. Int J Uncert Quantif 8(3):267–289
45. Garg H (2018) Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. Int J Intell Syst 33(6):1234–1263
46. Ilbahar E, Karaşan A, Cebi S, Kahraman C (2018) A novel approach to risk assessment for occupational health and safety using pythagorean fuzzy AHP and fuzzy inference system. Saf Sci 103:124–136

47. Karasan A, Ilbahar E, Kahraman C (2018) A novel pythagorean fuzzy AHP and its application to landfill site selection problem. Soft Comput 1–16. https://doi.org/10.1007/s00500-018-3649-0
48. Xu ZS (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15:1179–1187
49. Peng X, Yuan H, Yang Y (2017) Pythagorean fuzzy information measures and their applications. Int J Intell Syst 32(10):991–1029
50. Singh S, Garg H (2018) Symmetric triangular interval type-2 intuitionistic fuzzy sets with their applications in multi criteria decision making. Symmetry 10(9):401. https://doi.org/10.3390/sym10090401
51. Rani D, Garg H (2017) Distance measures between the complex intuitionistic fuzzy sets and its applications to the decision-making process. Int J Uncert Quantif 7(5):423–439
52. Rani D, Garg H (2018) Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision-making. Expert Syst 35(6):e12325. https://doi.org/10.1111/exsy.12325
53. Kaur G, Garg H (2018) Generalized cubic intuitionistic fuzzy aggregation operators using t-norm operations and their applications to group decision-making process. Arab J Sci Eng 44(3):2775–2794
54. Garg H, Kumar K (2019) Linguistic interval-valued Atanassov intuitionistic fuzzy sets and their applications to group decision-making problems. IEEE Trans Fuzzy Syst 1–10. https://doi.org/10.1109/TFUZZ.2019.2897961