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ORIGINAL ARTICLE

Multi-granulation Pythagorean fuzzy decision-theoretic rough sets based on inclusion measure and their application in incomplete multi-source information systems

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Abstract

Multi-granulation rough sets (MGRSs) and decision-theoretic rough sets (DTRSs) are two important and popular generalizations of classical rough sets. The combination of two generalized rough sets have been investigated by numerous researchers in different extensions of fuzzy settings such as interval-valued fuzzy sets (IVFSs), intuitionistic fuzzy sets (IFSs), bipolar-valued fuzzy sets (BVFSs), etc. Pythagorean fuzzy (PF) set is another extension of fuzzy set, which is more capable in comparison to IFS handle vagueness in real world. However, few studies have focused on the combination of the two rough sets in PF settings. In this study, we combine the two generalized rough sets in PF settings. First, we introduce a type of PF subset (of subset of the given universe) of the PF Set (of the given universe). Then we establish two basic models of multi-granulation PF DTRS (MG-PF-DTRS) of PF subset of the PF set based on PF inclusion measure within the framework of multi-granulation PF approximation space. One model is based on a combination of PF relations (PFRs) and the construction of approximations with respect to the combined PFR. By combining PFRs through intersection and union, respectively, we construct two models. The other model is based on the construction of approximations from PFRs and a combination of the approximations. By using intersection and union to combine the approximations, respectively, we again get two models. As a result, we have total four models. Further for different constraints on parameters, we obtain three kinds of each model of the MG-PF-DTRSs. Then, their principal structure, basic properties and uncertainty measure methods are investigated as well. Second, we give a way to compute PF similarity degrees between two objects and also give a way to compute PF decision-making objects from incomplete multi-source information systems (IMSISs). Then we design an algorithm for decision-making to IMSISs using MG-PFDTRSs and their uncertainty measure methods. Finally, an example about the mutual funds investment is included to show the feasibility and potential of the theoretic results obtained.

Keywords Pythagorean fuzzy set · Pythagorean fuzzy inclusion measure · Multi-granulation Pythagorean fuzzy decision-theoretic rough set · Incomplete multi-source information system

Introduction

In light of Bayesian decision procedure [4], decision theoretic rough set (DTRS) model was proposed by Yao and Wong [35] to analyze the noisy data by considering the tolerance of classification error. Since then the DTRS model has found its applications in various theoretical and practical fields, and it has produced many god results [2,15–17,20,25].

However, the DTRS model cannot deal with numerical data directly. To overcome this disadvantage, researchers used tolerance relations [21], similarity measures [23], domains relations [3], covering [33], inclusion measures [38], fuzzy relations [32,41,42], fuzzy preference relations [26], interval-valued fuzzy preference relations [29], intuitionistic fuzzy relations [19,40], intuitionistic fuzzy inclusion measure [13,13], bipolar-valued fuzzy relations [24,28] in place of equivalence relations.

To handle all types of real data, Yager proposed the concepts of Pythagorean fuzzy set (PFS) [34], which are

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more powerful than intuitionistic fuzzy sets (IFSs) [1] for dealing with the uncertain information in decision-making procedures. For example, if a decision maker gives the membership degree and non-membership degree as 0.8 and 0.6, respectively, then it is only valid for PFS. Fortunately, PFSs generalize the concept of IFSs and the corresponding operational laws, which have been successfully applied to some complex practical decision-making situations, e.g., road-building projects [11], selection of the optimal production strategy [12] and group decision-making problems [8]. Besides, Zhang and Ren [37] investigated Pythagorean fuzzy multigranulation rough set over two universes and its applications in merger and acquisition. Liang et al. [18] gave a method of three-way decisions using ideal TOPSIS solutions on Pythagorean fuzzy informations. Mandal and Ranadive [25] studied decision-theoretic rough sets under Pythagorean fuzzy information.

From the aforementioned literature it is clear that PFSs provide us a novel evaluation format to measure the fuzzy environment, especially when we utilize the positive and negative sides to depict a question [1,19]. Based on the Pythagorean fuzzy environment, we introduce PFSs into multi-granulation rough set (MGRS) and decision-theoretic rough set (DTRS), which are two important and popular extended type of Pawlak's classical rough sets [30] and established multi-granulation Pythagorean fuzzy decision-theoretic rough sets (MG-PF-DTRS) based on Pythagorean fuzzy inclusion measure. First, we introduce a type of Pythagorean fuzzy subset (of subset of the given universe) of the Pythagorean fuzzy Set (of the given universe). Then we establish two basic models of multi-granulation Pythagorean fuzzy decision-theoretic rough set (MG-PF-DTRS) of Pythagorean fuzzy subset of the PFS based on Pythagorean fuzzy inclusion measure within the framework of multi-granulation Pythagorean fuzzy approximation space. One model is based on a combination of PF relations (PFRs) and the construction of approximations with respect to the combined PFR. By combining PFRs through intersection and union, respectively, we construct two models. The other model is based on the construction of approximations from PFRs and a combination of the approximations. By using intersection and union to combine the approximations, respectively, we again get two models. As a result, we have total four models. For different constraints on parameters, we obtain three kinds of each model of the MG-PF-DTRSs. Then, their principal structure, basic properties and uncertainty measure methods are investigated as well. Second, we suggest a decision-making process for incomplete multi-source information systems (IMSISs) using these theoretic results about MG-PF-DTRSs. In this process, we encounter two challenges: (1) How to find the similarity degrees between two objects from IMSISs in the Pythagorean fuzzy settings and (2) how to obtain the Pythagorean fuzzy decision-making objects from IMSISs. To meet out these issues, we first adopt the method of Liu et al. [23] to obtain the unknown value of the given objects with respect to the given attribute. Then we give the method to find the similarity degree between two objects in the Pythagorean fuzzy setting and construct an algorithm for obtaining Pythagorean fuzzy decision-making objects. Having solved these two issues, we design an algorithm for decision-making to IMSISs using MG-PFDTRSs and their uncertainty measure methods. In this algorithm, we address the problem if $X_1, X_2, \ldots, X_r \subseteq U$ (U is the finite universe of discourse) such that $|X_1| = |X_2| = \cdots = |X_r|$, then find the best X_i ($1 \leq i \leq r$), where the elements of X_i are selected randomly. To solve this problem, first we have the IMSIS for the given alternatives with respect to the considered attributes. Second, we compute PFS from IMSIS and PFRs from each sub-information source from IMSIS of U. Then, we derive the PFS and PFRs for each X_i. Third, we obtain the approximation results for each type MG-PF-DTRSs for each X_i. We see that the obtained results are not entirely consistent. For this reason we also suggest several methods of uncertainty measure such as accuracy, approximation degree and approximation quality for four types of MG-PF-DTRSs. Fourth, we obtain best X_i according to the higher accuracy, approximation degree and approximation quality. This is our main objective. In comparison to existing results [13,14], our model has several advantages as listed below:

1. our model can deal with both intuitionistic fuzzy and Pythagorean fuzzy information instead of only intuitionistic fuzzy information;
2. our model can deal with complete and incomplete multi-source information systems instead of only complete multi-source information systems;
3. instead of assuming a fuzzy decision-making object as many researchers do, we give a method to find it.

As far as organization of this paper is concerned we give some necessary concepts of PFSs in "Preliminaries". In "MG-PF-DTRSs based on inclusion measure", we propose a common framework of inclusion measure based on MG-PF-DTRSs and study four types of MG-PF-DTRSs which are constructed using the Pythagorean fuzzy inclusion measure. The uncertainties of the proposed four types of MG-PF-DTRSs are measured in "Uncertainty measures". In "Decision-making to incomplete multi-source information systems using MG-PF-DTRSs", we apply our theoretical results to decision-making in IMSIS. An example about selection of mutual funds is also included in this section to show the feasibility and potential of our proposed decision-making approach. "Conclusions" is the concluding section.

Preliminaries

In this section, we present some basic concepts and terminology used throughout the paper.

Definition 1 (*Peng et al.* [39]) Let U be a universe of discourse. A PFS A in U is given by

$$A = \{\langle x, \mu_{A(U)}(x), \nu_{A(U)}(x)\rangle \mid x \in U\} = \sum_{x\in U} \frac{\langle \mu_A(x), \nu_A(x)\rangle}{x}, \quad (1)$$

where $\mu_A : U \to [0, 1]$ denotes the degree of membership and $\nu_A : U \to [0, 1]$ denotes the degree of nonmembership of the element $x \in U$ for the set A, respectively, with the condition that $0 \le \mu_A^2(x) + \nu_A^2(x) \le 1$. The degree of indeterminacy $\pi_A(x)$ is given by $\sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$. For convenience, Zhang and Xu [39] called $\langle \mu_A(x), \nu_A(x)\rangle$ a Pythagorean fuzzy number (PFN) denoted by $A = \langle \mu_A, \nu_A\rangle$.

If $X \subseteq U$, we define a Pythagorean fuzzy subset of X of the PFS of U in the following way:

$$A(X) = \begin{cases} \langle \mu_A(x), \nu_A(x)\rangle & \text{if } x \in X, \\ \langle 0, 1\rangle & \text{if } x \notin X, \end{cases} \quad (2)$$

which may also denoted by

$$A(X) = \{\langle x, \mu_{A(X)}(x), \nu_{A(X)}(x)\rangle \mid x \in X \subseteq U\} = \sum_{x\in X\subseteq U} \frac{\langle \mu_{A(X)}(x), \nu_{A(X)}(x)\rangle}{x}. \quad (3)$$

In this case the PFN is denoted by $A(X) = \langle \mu_{A(X)}, \nu_{A(X)}\rangle$. The complement of $A(X)$ is denoted by $A(X^c)$ and defined as

$$A(X^c) = \begin{cases} \langle \mu_{A(X}(x), \nu_A(x)\rangle & \text{if } x \notin X. \\ \langle 0, 1\rangle & \text{if } x \in X. \end{cases} \quad (4)$$

For example, let $A = \{\langle x_1, 0.9, 0.3\rangle, \langle x_2, 0.4, 0.7\rangle, \langle x_3, 0.8, 0.4\rangle, \langle x_4, 0.7, 0.2\rangle, \langle x_5, 0.7, 0.6\rangle, \langle x_6, 0.9, 0.2\rangle, \langle x_7, 0.7, 0.5\rangle, \langle x_8, 0.8, 0.2\rangle, \langle x_9, 0.5, 0.7\rangle, \langle x_{10}, 0.7, 0.6\rangle\}$ be a PFS. If $X = \{x_4, x_7, x_8, x_{10}\} \subseteq U$, then $A(X) = \{\langle x_1, 0, 1\rangle, \langle x_2, 0, 1\rangle, \langle x_3, 0, 1\rangle, \langle x_4, 0.7, 0.2\rangle, \langle x_5, 0, 1\rangle, \langle x_6, 0, 1\rangle, \langle x_7, 0.7, 0.5\rangle, \langle x_8, 0.8, 0.2\rangle, \langle x_9, 0, 1\rangle, \langle x_{10}, 0.7, 0.6\rangle\}$ and $A(X^c) = \{\langle x_1, 0.9, 0.3\rangle, \langle x_2, 0.4, 0.7\rangle, \langle x_3, 0.8, 0.4\rangle, \langle x_4, 0, 1\rangle, \langle x_5, 0.7, 0.6\rangle, \langle x_6, 0.9, 0.2\rangle, \langle x_7, 0, 1\rangle, \langle x_8, 0, 1\rangle, \langle x_9, 0.5, 0.7\rangle, \langle x_{10}, 0, 1\rangle\}$.

Throughout this paper by $PFS(U)$ we mean the set of all PFSs defined on U.

Definition 2 (*Peng et al.* [31]) If $A, B \in PFS(U)$, then

1. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in U\}$;
2. $A \subseteq B$ if $\forall x \in U, \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$;
3. $A = B$ iff $\forall x \in U, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$;
4. $\Phi_A = \{\langle x, 1, 0\rangle \mid x \in U\}$;
5. $\emptyset_A = \{\langle x, 0, 1\rangle \mid x \in U\}$;
6. $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in U\}$;
7. $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in U\}$.

Thus it is clear that if $X \subseteq U$, then $A(X), A(X^c) \subseteq A$.

Definition 3 (*Zhang et al.* [36]) A Pythagorean fuzzy relation R on U is a PFS on $U \times U$. That is, R is expressed by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in U \times U\}, \quad (5)$$

where $\mu_R : U \times U \to [0, 1]$ and $\nu_R : U \times U \to [0, 1]$ satisfy $0 \le \mu_R^2(x, y) + \nu_R^2(x, y) \le 1$ for all $(x, y) \in U \times U$.

A Pythagorean fuzzy relation R on U is denoted by $R(U)$ in this paper. If $X \subseteq U$, we define a Pythagorean fuzzy relation $R(U \times X)$ on $U \times X$ as follows:

$$R(U \times X) = \begin{cases} \langle \mu_R(x, y), \nu_A(x, y)\rangle & \text{if } x \in U, y \in X, \\ \langle 0, 1\rangle & \text{if } x \in U, y \notin X. \end{cases} \quad (6)$$

For example, let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and the Pythagorean fuzzy relation on U be as given in Table 1 and $X = \{x_3, x_5\} \subseteq U$; then the Pythagorean fuzzy relation on $U \times X$ will be as given in Table 2.

The Pythagorean fuzzy inclusion measure is also called the Pythagorean fuzzy subsethood measure, which indicates the degree to which one PFS is contained in another PFS. Peng et al. [31] provided a simple definition of the Pythagorean fuzzy inclusion measure as follows:

Definition 4 (*Peng et al.* [31]) Let A, B and C be three PFSs on U. An inclusion measure $I(A, B)$ is a mapping $I : \text{PFS}(U) \times \text{PFS}(U) \to [0, 1]$, possessing the following properties:

1. $0 \le I(A(U), B(U)) \le 1$;
2. $I(A, B) = 1$ iff $A \subseteq B$;
3. $I(A, B) = 0$ iff $A = \Phi_A, B = \emptyset_B$;
4. If $A \subseteq B \subseteq C$, then $I(C, A) \le I(B, A)$ and $I(C, A) \le I(C, B)$.

We provide an inclusion measure on PFS(U) on the basis of the Theorems 3.7(10) and 3.22 presented by Peng et al. [31].

Definition 5 (*Peng et al.* [31]) For $A, B \in \text{PFS}(U)$, an inclusion measure I on PFS(U) can be defined as follows:

Table 1 The Pythagorean fuzzy relation on U

$R(U \times U)$	x_1	x_2	x_3	x_4	x_5
x_1	$\langle 1, 0\rangle$	$\langle 0.5, 0.7\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.5, 0.3\rangle$
x_2	$\langle 0.5, 0.7\rangle$	$\langle 1, 0\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.6, 0.4\rangle$
x_3	$\langle 0.6, 0.7\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 1, 0\rangle$	$\langle 0.7, 0.6\rangle$	$\langle 0.6, 0.5\rangle$
x_4	$\langle 0.4, 0.3\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.7, 0.6\rangle$	$\langle 1, 0\rangle$	$\langle 0.4, 0.3\rangle$
x_5	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 1, 0\rangle$

Table 2 The Pythagorean fuzzy relation on $U \times X$

$R(U \times X)$	x_1	x_2	x_3	x_4	x_5
x_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0, 1\rangle$	$\langle 0.5, 0.3\rangle$
x_2	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0, 1\rangle$	$\langle 0.6, 0.4\rangle$
x_3	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0.6, 0.5\rangle$
x_4	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0.7, 0.6\rangle$	$\langle 0, 1\rangle$	$\langle 0.4, 0.3\rangle$
x_5	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0, 1\rangle$	$\langle 1, 0\rangle$

$$I(A, B) = \frac{\sum_{x\in U}(((\mu_A^2(x) \wedge \mu_B^2(x)) + \nu_A^2(x))}{\sum_{x\in U}(\mu_A^2(x) + (\nu_A^2(x) \vee \nu_B^2(x)))}. \tag{7}$$

The inclusion measure $I(A, B)$ defined in Eq. 5 satisfies the four conditions of the Definition 4.

For example, two PFSs $A = \{\langle x_1, 0.9, 0.3\rangle, \langle x_2, 0.4, 0.7\rangle, \langle x_3, 0.8, 0.4\rangle, \langle x_4, 0.7, 0.2\rangle\}$ and $B = \{\langle x_1, 0.7, 0.6\rangle, \langle x_2, 0.9, 0.2\rangle, \langle x_3, 0.7, 0.5\rangle, \langle x_4, 0.8, 0.2\rangle\}$. Then $I(A, B) = \frac{0.49+0.16+0.49+0.49+0.09+0.49+0.16+0.04}{0.81+0.16+0.64+0.49+0.36+0.49+0.25+0.04} = \frac{2.41}{3.24} = 0.74382716$

MG-PF-DTRSs based on inclusion measure

In this section first we propose and study the models of inclusion measure-based MG-PF-DTRSs, within the framework of multi-granulation Pythagorean fuzzy approximation space.

Definition 6 Let U be a finite universe. For any $X \subseteq U$ and $R_k(U \times X)(1 \le k \le m)$ be m Pythagorean fuzzy relations on $U \times X$. Then, we call $(U, X, R_k(U \times X)(1 \le k \le m))$ a multi-granulation Pythagorean fuzzy approximation space on U.

Definition 7 Let $R_k(U \times X)(1 \le k \le m)$ be m Pythagorean fuzzy relations on $U \times X$, where $X \subseteq U$. For each $x \in U$, two PFSs $[x]_{\cap_{k=1}^m R_k(U\times X)}$ and $[x]_{\cup_{k=1}^m R_k(U\times X)}$ are defined as follows:

$$[x]_{\cap_{k=1}^m R_k(U\times X)} = \left\{\left\langle y, \wedge_{k=1}^m \mu_{R_k(U\times X)}(x, y), \vee_{k=1}^m \nu_{R_k(U\times X)}(x, y)\right\rangle \mid x \in U, y \in X\right\} \tag{8}$$

and

$$[x]_{\cup_{k=1}^m R_k(U\times X)} = \left\{\left\langle y, \vee_{k=1}^m \mu_{R_k(U\times X)}(x, y), \wedge_{k=1}^m \nu_{R_k(U\times X)}(x, y)\right\rangle \mid x \in U, y \in X\right\} \tag{9}$$

for all $y \in X$.

We are now ready to propose a model of four types of MG-PF-DTRSs. This model is shown by Fig. 1. The first two models, called Type-I multi-granulation Pythagorean fuzzy decision-theoretic rough set (Type-I MG-PF-DTRS) and Type-II multi-granulation Pythagorean fuzzy decision-theoretic rough set (Type-II MG-PF-DTRS), are a class based on combination of relation first and then construction of approximations, as demonstrated in the upper half Fig. 1. The next two models, called Type-III multi-granulation Pythagorean fuzzy decision-theoretic rough set (Type-III MG-PF-DTRS) and multi-granulation Pythagorean fuzzy decision-theoretic rough set (Type-IV MG-PF-DTRS), are another class that is used the reverse order, as demonstrated in the lower half Fig. 1.

Now in the following we study four types PF-DTRSs based on the inclusion measure, within the framework of multi-granulation Pythagorean fuzzy approximation space, i.e., four types of MG-PF-DTRSs.

Type-I MG-PF-DTRSs

Definition 8 Let $(U, X, R_k(U\times X)(1 \le k \le m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \le k \le m)$. For any $A(X)$ and $0 \le \beta < \alpha \le 1$, the inclusion measure-based Type-I multi-granulation Pythagorean fuzzy α-lower and β-upper approximations of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$ are defined, respectively, as follows:

$$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^m R_k(U\times X)}(A(X)) = \left\{x \in U : I([x]_{\cap_{k=1}^m R_k(U\times X)}, A(X)) \ge \alpha\right\},$$

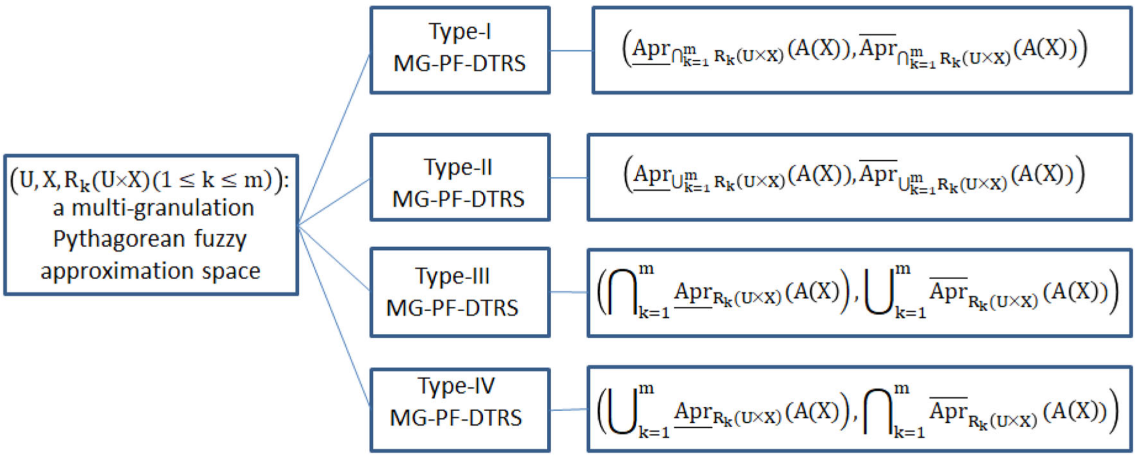

Fig. 1 Models of multi-granulation Pythagorean fuzzy rough sets

$$\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = \left\{x \in U : I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) > \beta\right\}.$$

We call the pair $(\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)), \overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)))$ as inclusion measure based Type-I (α, β)-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$. The positive, negative and boundary regions of $A(X)$ w.r.t. $(U, R_k(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\mathrm{POS}^{\alpha}(A(X)) = \underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)),$$

$$\mathrm{NEG}^{\beta}(A(X)) = (\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)))^{c},$$

$$\mathrm{BND}^{(\alpha,\beta)}(A(X)) = \overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) - \underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)).$$

Remark 1 For any $X = \{y_1, y_2, \ldots, y_n\} \subseteq U$ and for all $x \in U$, we have Eq. (10) from Eq. (7).

$$I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) = \frac{\sum_{i=1}^{n}(((\wedge_{k=1}^{m}\mu^2_{R_k(U\times X)}(x, y_i)) \wedge \mu^2_{A(X)}(y_i)) + (\vee_{k=1}^{m}\nu^2_{R_k(U\times X)}(x, y_i)))}{\sum_{i=1}^{n}((\wedge_{k=1}^{m}\mu^2_{R_k(U\times X)}(x, y_i)) + ((\vee_{k=1}^{m}\nu^2_{R_k(U\times X)}(x, y_i)) \vee \nu^2_{A(X)}(y_i)))}. \tag{10}$$

Remark 2 1. For any $X \subseteq U$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)).$$

2. $\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times U)}(U) = U\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times U)}(U)$.
3. For any $X \subseteq U$ and $0 \leq \beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 \leq 1$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha_2}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \underline{\mathrm{Apr}}^{\alpha_1}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)).$$

and

$$\overline{\mathrm{Apr}}^{\beta_2}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta_1}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)).$$

4. For any $X, Y \subseteq U$ with $A(X) \subseteq A(Y)$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(Y))$$

and

$$\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(Y)).$$

Remark 3 In the Definition 8, the Type-I (α, β)-MG-PF-DTRS is also refereed to as asymmetric MG-PF-DTRS, the aim of which is to approximate Pythagorean fuzzy concepts with high desired level of prediction accuracy. From Definition 8, we can derive the two special kinds of MG-PF-DTRSs are given, respectively, as follows:

1. The Type-I α-MG-PF-DTRS (with $0.5 < \alpha \leq 1$) of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure based Type-I multi-granulation Pythagorean fuzzy α-lower and α-upper approximations as

$$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) \geq \alpha\},$$

$$\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) > 1-\alpha\}.$$

2. The Type-I 0.5-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$ is defined for any $X \subseteq U$ in terms of inclusion measure based Type-I multigranulation Pythagorean fuzzy 0.5-lower and 0.5-upper approximations as

$$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) > 0.5\},$$
$$\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cap_{k=1}^{m} R_k(U\times X)}, A(X)) \ge 0.5\}.$$

Type-II MG-PF-DTRSs

Definition 9 Let $(U, X, R_k(U\times X)(1 \le k \le m))$ be a multigranulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U\times X)(1 \le k \le m)$. For any $A(X)$ and $0 \le \beta < \alpha \le 1$, the inclusion measure-based Type-II multi-granulation Pythagorean fuzzy α-lower and β-upper approximations of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$ are defined, respectively, as follows:

$$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) \ge \alpha\},$$
$$\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) > \beta\}.$$

We call the pair $(\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))), \overline{Apr}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))$ as inclusion measure-based Type-II (α, β)-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$. The positive, negative and boundary regions of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$ are defined, respectively, as follows:

$$\mathrm{POS}^{\alpha}(A(X)) = \underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)),$$
$$\mathrm{NEG}^{\beta}(A(X)) = (\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)))^c,$$
$$\mathrm{BND}^{(\alpha,\beta)}(A(X)) = \overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) - \underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)).$$

Remark 4 For any $X = \{y_1, y_2, \ldots, y_n\} \subseteq U$ and for all $x \in U$, we have Eq. (11) from Eq. (7).

$$I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) = \frac{\sum_{i=1}^{n}(((\vee_{k=1}^{m}\mu^2_{R_k(U\times X)}(x,y_i)) \wedge \mu^2_{A(X)}(y_i)) + (\wedge_{k=1}^{m}\nu^2_{R_k(U\times X)}(x,y_i)))}{\sum_{i=1}^{n}((\vee_{k=1}^{m}\mu^2_{R_k(U\times X)}(x,y_i)) + ((\wedge_{k=1}^{m}\nu^2_{R_k(U\times X)}(x,y_i)) \vee \nu^2_{A(X)}(y_i)))}, \tag{11}$$

Remark 5

1. For any $X \subseteq U$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)).$$

2. $\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times U)}(U) = U = \overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times U)}(U)$.
3. For any $X \subseteq U$ and $0 \le \beta_1 \le \beta_2 \le \alpha_1 \le \alpha_2 \le 1$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha_2}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \underline{\mathrm{Apr}}^{\alpha_1}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))$$

and

$$\overline{\mathrm{Apr}}^{\beta_2}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta_1}_{\cup_{k=1}^{m} R_k}(A(X)).$$

4. For any $X, Y \subseteq U$ with $A(X) \subseteq A(Y)$, we obtain

$$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(Y))$$

and

$$\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) \subseteq \overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(Y)).$$

Remark 6 From the Definition 9, we can derive the two special kinds of MG-PF-RSs, given, respectively, as follows:

1. The Type-II α-MG-PF-DTRS (with $0.5 < \alpha \le 1$) of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \le k \le m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-II multi-granulation Pythagorean fuzzy α-lower and α-upper approximations as

$$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \left\{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) \ge \alpha\right\},$$
$$\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \left\{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) > 1-\alpha\right\}.$$

2. The Type-II 0.5-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-II multi-granulation Pythagorean fuzzy 0.5-lower and 0.5-upper approximations as

$$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) > 0.5\},$$
$$\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x \in U : I([x]_{\cup_{k=1}^{m} R_k(U\times X)}, A(X)) \geq 0.5\}.$$

Type-III MG-PF-DTRSs

Definition 10 Let $(U, X, R_k(U \times X)(1 \leq k \leq m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \leq k \leq m)$. For any $A(X)$ and $0 \leq \beta < \alpha \leq 1$, the inclusion measure-based Type-III multi-granulation Pythagorean fuzzy α-lower and β-upper approximations of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\cap_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)) = \{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq \alpha\},$$
$$\cup_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)) = \{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > \beta\}.$$

We call the pair $(\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)), \cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)))$ as inclusion measure-based Type-III (α, β)-MG-PF-DTRS of A w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$. The positive, negative and boundary regions of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\mathrm{POS}^{\alpha}(A(X)) = \cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)),$$
$$\mathrm{NEG}^{\beta}(A(X)) = (\cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)))^c,$$
$$\mathrm{BND}^{(\alpha,\beta)}(A(X)) = \cup_{k=1}^{m}\overline{Apr}^{\beta}_{R_k(U\times X)}(A(X)) - \cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)).$$

Remark 7 For any $X = \{y_1, y_2, \ldots, y_n\} \subseteq U$ and for all $x \in U$, we have Eqs. (12) and (13) from Eq. (7).

$$\cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) = \bigwedge_{k=1}^{m} \frac{\sum_{i=1}^{n}((\mu^2_{R_k(U\times X)}(x, y_i) \wedge \mu^2_{A(X)}(y_i)) + \nu^2_{R_k(U\times X)}(x, y_i))}{\sum_{i=1}^{n}(\mu^2_{R_k(U\times X)}(x, y_i) + (\nu^2_{R_k(U\times X)}(x, y_i) \vee \nu^2_{A(X)}(y_i))}, \quad (12)$$

$$\cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) = \bigvee_{k=1}^{m} \frac{\sum_{i=1}^{n}((\mu^2_{R_k(U\times X)}(x, y_i) \wedge \mu^2_{A(X)}(y_i)) + \nu^2_{R_k(U\times X)}(x, y_i))}{\sum_{i=1}^{n}(\mu^2_{R_k(U\times X)}(x, y_i) + (\nu^2_{R_k(U\times X)}(x, y_i) \vee \nu^2_{A(X)}(y_i))}. \quad (13)$$

Remark 8
1. For any $X \subseteq U$, we obtain
$$\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)) \subseteq \cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)).$$
2. $\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times U)}(U) = U\ \cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times U)}(U)$.
3. For any $X \subseteq U$ and $0 \leq \beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 \leq 1$, we obtain
$$\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha_2}_{R_k}(A) \subseteq \cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha_1}_{R_k(U\times X)}(A(X))$$
and
$$\cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta_2}_{R_k(U\times X)}(A(X)) \subseteq \cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta_1}_{R_k(U\times X)}(A(X)).$$
4. For any $X, Y \subseteq U$ with $A(X) \subseteq B(X)$, we obtain
$$\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)) \subseteq \cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(Y))$$
and
$$\cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)) \subseteq \cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(Y)).$$

Remark 9 From the Definition 10, we can derive the two special kinds of MG-PF-RSs, are given, respectively, as follows:

1. the Type-III α-MG-PF-DTRS (with $0.5 < \alpha \leq 1$) of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-III multi-granulation Pythagorean fuzzy α-lower and α-upper approximations as

$$\cap_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)) = \{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq \alpha\},$$
$$\cup_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)) = \{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > 1-\alpha\}.$$

2. the Type-III 0.5-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-III multi-granulation Pythagorean fuzzy 0.5-lower and 0.5-upper approximations as

$$\begin{aligned}&\cap_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\\&\quad= \left\{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > 0.5\right\},\\&\cup_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\\&\quad= \left\{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq 0.5\right\}.\end{aligned}$$

Type-IV MG-PF-DTRSs

Definition 11 Let $(U, X, R_k(U \times X)(1 \leq k \leq m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \leq k \leq m)$. For any $A(X)$ and $0 \leq \beta < \alpha \leq 1$, the inclusion measure-based Type-IV multi-granulation Pythagorean fuzzy α-lower and β-upper approximations of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\begin{aligned}&\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\\&\quad= \left\{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq \alpha\right\},\\&\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\\&\quad= \left\{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > \beta\right\}.\end{aligned}$$

We call the pair $(\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)), \cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))$ as inclusion measure-based Type-IV (α, β)-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$. The positive, negative and boundary regions of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\mathrm{POS}^{\alpha}(A(X)) = \cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)),$$

$$\mathrm{NEG}^{\beta}(A(X)) = (\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)))^c,$$

$$\begin{aligned}\mathrm{BND}^{(\alpha,\beta)}(A(X)) = {}&\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\\&- \cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)).\end{aligned}$$

Remark 10 1. $\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times U)}(U) = U \cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times U)}(U)$.

2. For any $X \subseteq U$ and $0 \leq \beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 \leq 1$, we obtain

$$\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha_2}_{R_k(U\times X)}(A(X)) \subseteq \cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha_1}_{R_k(U\times X)}(A(X))$$

and

$$\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta_2}_{R_k(U\times X)}(A(X)) \subseteq \cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta_1}_{R_k(U\times X)}(A(X)).$$

3. For any $X, Y \subseteq U$ with $A(X) \subseteq A(Y)$, we obtain

$$\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X)) \subseteq \cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(Y))$$

and

$$\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X)) \subseteq \cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(Y)).$$

Remark 11 From the Definition 11, we can derive the two special kinds of MG-PF-RSs, are given, respectively, as follows:

1. the Type-IV α-MG-PF-DTRS (with $0.5 < \alpha \leq 1$) of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-IV multi-granulation Pythagorean fuzzy α-lower and α-upper approximations as

$$\begin{aligned}&\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\\&\quad= \{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq \alpha\},\\&\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\\&\quad= \{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > 1 - \alpha\}.\end{aligned}$$

2. the Type-IV 0.5-MG-PF-DTRS of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ is defined for any $X \subseteq U$ in terms of inclusion measure-based Type-IV multi-granulation Pythagorean fuzzy 0.5-lower and 0.5-upper approximations as

$$\begin{aligned}&\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\\&\quad= \{x \in U : \cup_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) > 0.5\},\\&\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\\&\quad= \{x \in U : \cap_{k=1}^{m} I([x]_{R_k(U\times X)}, A(X)) \geq 0.5\}.\end{aligned}$$

Uncertainty measures

In this section, several measures are utilized to calculate the uncertainty of these models which is discussed in previous section. The uncertainty of knowledge is caused by the boundary regions, in the view point of approximations. The larger the boundary area is, the more the uncertainty. The accuracy, roughness and approximation quality are studied in the next.

Definition 12 Let $(U, X, R_k(U \times X)(1 \leq k \leq m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \leq k \leq m)$. For any $A(X)$ and $0 \leq \beta < \alpha \leq 1$, the Type-I MG-PF-DTRS accuracy ρ_{I}, Type-II MG-PF-DTRS accuracy ρ_{II}, Type-III MG-PF-DTRS accuracy ρ_{III} and Type-IV MG-PF-DTRS accuracy ρ_{IV} of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\rho_{\mathrm{I}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{\left|\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X))\right|}, \tag{14}$$

$$\rho_{\mathrm{II}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{\left|\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))\right|}, \tag{15}$$

$$\rho_{\mathrm{III}} = \frac{\left|\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{\left|\cup_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\right|}, \tag{16}$$

$$\rho_{\mathrm{IV}} = \frac{\left|\cup_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{\left|\cap_{k=1}^{m}\overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X))\right|}. \tag{17}$$

Definition 13 Let $(U, X, R_k(U \times X)(1 \leq k \leq m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \leq k \leq m)$. For any $A(X)$ and $0 \leq \beta < \alpha \leq 1$, the Type-I MG-PF-DTRS approximation degree σ_{I}, Type-II MG-PF-DTRS approximation degree σ_{II}, Type-III MG-PF-DTRS approximation degree σ_{III} and Type-IV MG-PF-DTRS approximation degree σ_{IV} of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\sigma_{\mathrm{I}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{|X|}, \tag{18}$$

$$\sigma_{\mathrm{II}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{|X|}, \tag{19}$$

$$\sigma_{\mathrm{III}} = \frac{\left|\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{|X|}, \tag{20}$$

$$\sigma_{\mathrm{IV}} = \frac{\left|\cup_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{|X|}. \tag{21}$$

Definition 14 Let $(U, X, R_k(U \times X)(1 \leq k \leq m))$ be a multi-granulation Pythagorean fuzzy approximation space, for any $X \subseteq U$ with non-empty and finite universe of discourse U and m Pythagorean fuzzy relations $R_k(U \times X)(1 \leq k \leq m)$. For any $A(X)$ and $0 \leq \beta < \alpha \leq 1$, the Type-I MG-PF-DTRS approximation quality ω_{I}, Type-II MG-PF-DTRS approximation quality ω_{II}, Type-III MG-PF-DTRS approximation quality ω_{III} and Type-IV MG-PF-DTRS approximation quality ω_{IV} of $A(X)$ w.r.t. $(U, X, R_k(U \times X)(1 \leq k \leq m))$ are defined, respectively, as follows:

$$\omega_{\mathrm{I}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{|U|}, \tag{22}$$

$$\omega_{\mathrm{II}} = \frac{\left|\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X))\right|}{|U|}, \tag{23}$$

$$\omega_{\mathrm{III}} = \frac{\left|\cap_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{|U|}, \tag{24}$$

$$\omega_{\mathrm{IV}} = \frac{\left|\cup_{k=1}^{m}\underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X))\right|}{|U|}. \tag{25}$$

Decision-making to incomplete multi-source information systems using MG-PF-DTRSs

In this section, based on the MG-PF-DTRSs and their uncertainty measures established in "MG-PF-DTRSs based on inclusion measure" and "Uncertainty measures", we will construct a new method and approach to decision-making with incomplete multi-source information systems. Also, we will present the decision-making algorithm and the general steps for established method in detail.

Incomplete multi-source information systems and the similarity degrees

Definition 15 (*Lin et al.* [22]) A multi-source information system is $\mathrm{MSIS} = \{\mathrm{IS}_l \mid \mathrm{IS}_l = (U, \mathrm{AT}_l, V = \{(V_a)_{a\in \mathrm{AT}_l}\}, f_l\}$, where

1. U is a finite non-empty set of objects, called the universe;
2. AT_l is a non-empty finite set of attributes of each subsystem;
3. $\{V_a\}$ is the domain of the attribute $a \in \mathrm{AT}_l$; and
4. $f_l : U \times \mathrm{AT}_l \mapsto \{(V_a)_{a\in \mathrm{AT}_l}\}$ such that for all $x \in U$ and $a \in \mathrm{AT}_l$, $f(x, a) \in V_a$.

Definition 16 An incomplete multi-source information system (IMSIS) indicates the precise attribute values V_a for some objects are unknown. In this paper, the IMSIS is still denoted without confusion by $\mathrm{IMSIS} = \{\mathrm{IS}_l \mid \mathrm{IS}_l = (U, \mathrm{AT}_l, V, f_l\}$. Here $V = \{(V_a)_{a\in \mathrm{AT}_l}\} \cup \{*\}$, the special symbol "$*$" is used to indicate the unknown value. For instance, if $f(x, a) = *$, the value of object x is unknown on the attribute a.

Liu et al. [23] handle the incomplete single source information system and compute the similarity degree between two objects. Their similarity degree between two objects is fuzzy set. Here, we handle the incomplete multi-source information system and compute the similarity degree between two objects. Our similarity degree between two objects is PFS.

Given an IMSIS $= \{\mathrm{IS}_l \mid \mathrm{IS}_l = (U, \mathrm{AT}_l, V, f_l\}$, suppose the IMSIS contains n objects and t attributes, $U = \{x_1, x_2, \ldots, x_n\}$, $\mathrm{AT}_l \subseteq \{a_1, a_2, \ldots, a_t\}$. For $\forall x, y \in U$, $\forall a_i \in \mathrm{AT}_l$, the relations between (x, a_i) and (y, a_i) can be treated as following four scenarios:

1. Consideration of $(x, a_i) \neq *$ and $(y, a_i) \neq *$, (x, a_i) and (y, a_i) are equality iff $(x, a_i) = (y, a_i)$;
2. Consideration of $(x, a_i) \neq *$ and $(y, a_i) \neq *$, (x, a_i) and (y, a_i) are not the same if $(x, a_i) \neq (y, a_i)$;
3. Consideration of $(x, a_i) = *$ or $(y, a_i) = *$, because of the unknown value "$*$" is treated as "do not care" conditions, it has the probability of $\frac{1}{|V_{a_i}|}$ to equal to one certain value of V_{a_i} (V_{a_i} is a domain of the attribute a_i, $|V_{a_i}|$ denotes the cardinality of a_i).
4. Consideration of $(x, a_i) = (y, a_i) = *$, both of $a_i(x)$ and $a_i(y)$ have the probability of $\frac{1}{|V_{a_i}|}$ to equal to one certain value of V_{a_i}, so the joint probability of $(x, a_i) = (y, a_i)$ is $\frac{1}{|V_{a_i}|^2}$.

With the above discussions, we apply the concepts of PFSs, the similarity degrees between x and y on a_i can be written in the following way:

$$\mu_{\mathrm{Sim}(a_i)}(x, y) = \begin{cases} 1 & (x, a_i) = (y, a_i) \neq *; \\ 0 & (x, a_i) \neq (y, a_i) \wedge (x, a_i) \neq * \wedge (y, a_i) \neq *; \\ \frac{1}{|V_{a_i}|} & (x, a_i) = * \vee (y, a_i) = *; \\ \frac{1}{|V_{a_i}|^2} & (x, a_i) = * \wedge (y, a_i) = * \end{cases} \tag{26}$$

and

$$\nu_{\mathrm{Sim}(a_i)}(x, y) = \begin{cases} 0 & (x, a_i) = (y, a_i) \neq *; \\ 1 & (x, a_i) \neq (y, a_i) \wedge (x, a_i) \neq * \wedge (y, a_i) \neq *; \\ \sqrt{1 - \frac{1}{|V_{a_i}|^2}} & (x, a_i) = * \vee (y, a_i) = *; \\ \sqrt{1 - \frac{1}{|V_{a_i}|^4}} & (x, a_i) = * \wedge (y, a_i) = * \end{cases} \tag{27}$$

Table 3 A car incomplete multi-source information system

U	EC_1		EC_2	
	c_1	c_3	c_2	c_4
x_1	High	Full	Low	Low
x_2	Low	Medium	$*$	Low
x_3	$*$	Compact	$*$	Low
x_4	High	Full	$*$	High
x_5	$*$	Full	$*$	High
x_6	Low	Compact	High	$*$

If $\mathrm{AT}_l = \{b_1, b_1, \ldots, b_s\} \subseteq \{a_1, a_2, \ldots, a_t\}$, then the similarity membership and non-membership degrees between x and y are calculated as follows:

$$\mu_{R_{\mathrm{AT}_l}(U\times U)}(x, y) = \sum_{i=1}^{s} \frac{\mu_{\mathrm{Sim}(b_i)}(x, y)}{s} \tag{28}$$

$$\nu_{R_{\mathrm{AT}_l}(U\times U)}(x, y) = \sum_{i=1}^{s} \frac{\nu_{\mathrm{Sim}(b_i)}(x, y)}{s}. \tag{29}$$

Using Eqs. (28) and (29), from the Definition 3 we have:

$$R_k(U \times U) = \left\{ \left\langle (x, y), \sum_{i=1}^{s} \frac{\mu_{Sim(b_i)}(x, y)}{s}, \sum_{i=1}^{s} \frac{\nu_{Sim(b_i)}(x, y)}{s} \right\rangle \mid x, y \in U \right\}, \tag{30}$$

where $1 \leq k \leq l$.

Example 1 Let us consider an evaluation problem of a car depicted by an IMSIS presented in Table 3. Suppose that $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a set of six cars. Every car in each sub-information (source) system, denoted by EC_1 and EC_2, is described by two attributes. They are $c_1 =$ Price, $c_2 =$ Mileage, $c_3 =$ Size, $c_4 =$ Max-speed, respectively. The domains of the attributes are as follows: $V_{c_1} = \{\mathrm{High}, \mathrm{Low}\}$, $V_{c_2} = \{\mathrm{High}, \mathrm{Low}\}$, $V_{c_3} = \{\mathrm{Full}, \mathrm{Medium}, \mathrm{Compact}\}$, $V_{c_4} = \{\mathrm{High}, \mathrm{Low}\}$.

According to Table 3 and Eqs. (26) and 27) we easily get $\mu_{Sim(c_1)}(x_1, x_2) = 0$, $\mu_{\mathrm{Sim}(c_3)}(x_1, x_2) = 0$, $\nu_{\mathrm{Sim}(c_2)}(x_1, x_2) = 1$ and $\nu_{\mathrm{Sim}(c_4)}(x_1, x_2) = 1$. From Eqs. (28) and (29) we have: $\mu_{R_{\mathrm{EC}_1}(U)}(x_1, x_2) = \frac{0+0}{2} = 0$ and $\nu_{R_{\mathrm{EC}_1}(U)}(x_1, x_2) = \frac{1+1}{2} = 1$. Similarly, we compute $\mu_{R_{\mathrm{EC}_1}(U)}(x_i, x_j)$ and $\nu_{R_{\mathrm{EC}_1}(U)}(x_i, x_j)$ for all $x_i, x_j \in U$ $(i, j = 1, 2, \ldots, 5)$, which is outlined in Table 4.

In the same way, we also compute $\mu_{R_{\mathrm{EC}_2}(U)}$ (x_i, x_j) and $\nu_{R_{\mathrm{EC}}(U)}(x_i, x_j)$ for all $x_i, x_j \in U$ $(i, j = 1, 2, \ldots, 5)$ but here it is not necessary.

Table 4 The computing results of $R_{\mathrm{EC}_1}(U)$

$R_{\mathrm{EC}_1}(U \times U)$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0.2500, 0.9330\rangle$	$\langle 1, 0\rangle$	$\langle 0.7500, 0.4303\rangle$	$\langle 0, 1\rangle$
x_2		$\langle 1, 0\rangle$	$\langle 0.2500, 0.6330\rangle$	$\langle 0, 1\rangle$	$\langle 0.2500, 0.9330\rangle$	$\langle 0.5000, 0.5000\rangle$
x_3			$\langle 1, 0\rangle$	$\langle 0.2500, 0.9330\rangle$	$\langle 0.1250, 0.9841\rangle$	$\langle 0.2500, 0.9330\rangle$
x_4				$\langle 1, 0\rangle$	$\langle 0.7500, 0.4330\rangle$	$\langle 0, 1\rangle$
x_5					$\langle 1, 0\rangle$	$\langle 0.2500, 0.9330\rangle$
x_6						$\langle 1, 0\rangle$

Based on the basic principle of MG-PF-DTRSs, a Pythagorean fuzzy decision-making object is approximated over the multi-granulation Pythagorean fuzzy approximation space. So, we give the approach for compute the Pythagorean fuzzy decision-making object from the IMSIS. Then we give the following algorithm for compute the degrees of membership and non-membership of any alternative with respect to a Pythagorean fuzzy decision-making object from the IMSIS:

Algorithm 1 Computation of the degrees of membership and non-membership of any alternative with respect to a Pythagorean fuzzy decision-making object from the IMSIS.

Step 1: First we consider the reasonable membership degree of the domains of the all attributes.

Step 2: If the membership degree of the domain of the attribute $a_i = \alpha_i$, i.e. $(x, a_i) = \alpha_i$, then we write $\mu_A(x, a_i) = \alpha_i$ and compute $\nu_A(x, a_i) = \sqrt{1 - \alpha_i^2}$. If $(x, a_i) = *$, then we consider $\mu_A(x, a_i) = \frac{1}{|V_{a_i}|}$ and $\nu_A(x, a_i) = \sqrt{1 - \frac{1}{|V_{a_i}|^2}}$.

Step 3: The degrees of membership and non-membership of any $x \in U$ is calculated as follows:

$$\mu_A(x) = \sum_{i=1}^{t} \frac{\mu_{A(U)}(x, a_i)}{t}, \tag{31}$$

$$\nu_A(x) = \sum_{i=1}^{t} \frac{\nu_{A(U)}(x, a_i)}{t}. \tag{32}$$

For clearance we have the following example:

Example 2 (Continued in Example 1) In this example we find the degrees of membership and non-membership of any alternative with respect to a Pythagorean fuzzy decision-making object from a car IMSIS. According to Algorithm 1, we have

Step 1: Let us we consider the reasonable membership degree of the domains of the all attributes, which is shown in Fig. 2.

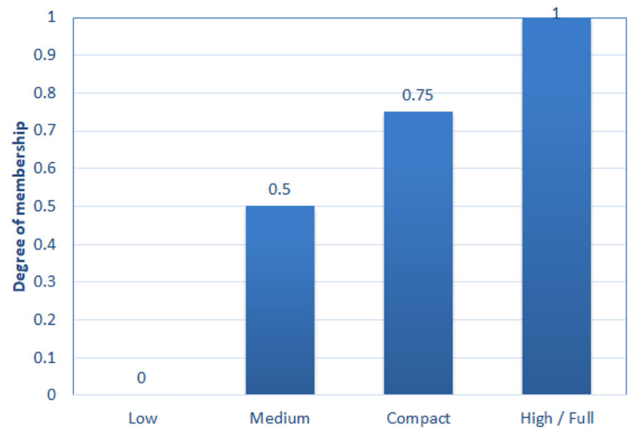

Fig. 2 Membership degree of the domains of the all attributes

Step 2: From Table 3, we get $(x_1, c_1) = \text{High} = 1$, i.e. $\mu_A(x_1, c_1) = 1$, then $\nu_A(x_1, c_1) = 0$. Similarly, $\mu_A(x_1, c_2) = 0$, $\nu_A(x_1, c_2) = 1$, $\mu_A(x_1, c_3) = 1$, $\nu_A(x_1, c_3) = 0$, $\mu_A(x_1, c_4) = 0$, $\nu_A(x_1, c_4) = 1$.

Step 3: Using Eqs. (31) and (32), we have $\mu_A(x_1) = \frac{1+0+1+0}{4} = 0.5$ and $\nu_A(x_1) = \frac{0+1+0+1}{4} = 0.5$. Similarly, we also find $\mu_A(x_i)$ and $\nu_A(x_i)$ $i = \{2, 3, \ldots, 6\}$, which is represented in Eq. (32).

$$A = \frac{\langle 0.5, 0.5\rangle}{x_1} + \frac{\langle 0.25, 0.93\rangle}{x_2} + \frac{\langle 0.44, 0.89\rangle}{x_3} + \frac{\langle 0.88, 0.27\rangle}{x_4} + \frac{\langle 0.75, 0.43\rangle}{x_5} + \frac{\langle 0.65, 0.63\rangle}{x_6}. \tag{33}$$

An algorithm

With the help of the results in "MG-PF-DTRSs based on inclusion measure" and "Uncertainty measures", and the discussion in "Decision-making to incomplete multi-source information systems using MG-PF-DTRSs", we design the algorithm of decision-making based on MG-PF-DTRSs and their uncertainty measure methods, where the information source is multiple and incomplete. The key steps are elaborated as follows:

Step 1: Suppose that a decision making problem the IMSIS is MSIS $= \{\mathrm{IS}_l \mid \mathrm{IS}_l = (U, \mathrm{AT}_l, V = \{(V_a)_{a\in \mathrm{AT}_l}\} \cup \{*\}, f_l\}$. Let us assume that X_1, $X_2, \ldots, X_r$ be the subsets of U, where the elements of X_i $(i = 1, 2, \ldots, r)$ are randomly selected and not repeat any other elements in X_i $(i = 1, 2, \ldots, r)$. To find the best X_i $(i = 1, 2, \ldots, r)$.

Step 2: Computing $R_k(1 \leq k \leq l)(U)$ according to Eq. (30).

Step 3: Constructing the Pythagorean fuzzy decision-making object A according to Algorithm 1.

Step 4: Choose α and β.

Step 5: Computing the inclusion measure based Type-I multi-granulation Pythagorean fuzzy α-lower approximation
$\underline{\mathrm{Apr}}^{\alpha}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X_i))$
and β-upper approximation
$\overline{\mathrm{Apr}}^{\beta}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X_i))$
for each $X_i \subseteq U$, respectively.

Step 6: Computing the inclusion measure-based Type-II multi-granulation Pythagorean fuzzy α-lower approximation
$\underline{\mathrm{Apr}}^{\alpha}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X_i))$
and β-upper approximation
$\overline{\mathrm{Apr}}^{\beta}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X_i))$
for each $X_i \subseteq U$, respectively.

Step 7: Computing the inclusion measure-based Type-III multi-granulation Pythagorean fuzzy α-lower approximation
$\cap_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X_i))$
and β-upper approximation
$\cup_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X_i))$
for each $X_i \subseteq U$, respectively.

Step 8: Computing the inclusion measure based Type-III multi-granulation Pythagorean fuzzy α-lower approximation
$\cup_{k=1}^{m} \underline{\mathrm{Apr}}^{\alpha}_{R_k(U\times X)}(A(X_i))$
and β-upper approximation
$\cap_{k=1}^{m} \overline{\mathrm{Apr}}^{\beta}_{R_k(U\times X)}(A(X_i))$
for each $X_i \subseteq U$, respectively.

Step 9: Computing ρ_{I}, ρ_{II}, ρ_{III} and ρ_{IV} using Eqs. (14)–(17).

Step 10: Computing σ_{I}, σ_{II}, σ_{III} and σ_{IV} using Eqs. (18)–(21).

Step 11: Computing ω_{I}, ω_{II}, ω_{III} and ω_{IV} using Eqs. (22)–(25).

Step 12: Obtain the best X_i according to the higher accuracy, approximation degree and approximation quality for each X_i.

An illustrative example

In this subsection, we apply the proposed algorithm to a real decision making. This example is about quick decision making based on a real investment context, under the MG-PF-DTRSs and their uncertainty measures models, where the information comes from multiple and incomplete.

Problem description

The various types of mutual funds (MFs) of different companies listed in the Growth Enterprise Market board of the India Stock Exchange are a popular investment source to an investor as a long-term investment. However, the sufficient knowledge about the various types of MFs of different companies is always not possible for every investor. Our proposed models are effective for those investors. Suppose an investor plans to invest his/her money in MFs of different companies, with the aim of high returns, while he/she has no sufficient knowledge about all MFs, then He/she chooses initially ten MFs according to the past performances, while he/she invests his/her money intpp the best five MFs out of these ten MFs. For making reasonable five MFs out of ten MFs, we have the following decision analysis.

Decision analysis

We use the algorithm in "An algorithm" of decision analysis based on MG-PF-DTRSs and their uncertainty measure, for decision making.

Suppose an investor initially chooses ten MFs according to the past performances, which is depicted by an IMSIS presented in Table 5. Let $U = \{x_1, x_2, \ldots, x_{10}\}$ be a set of ten MFs. Every MF in each sub-information (source) system, denoted by EC_1, EC_2, and EC_3, is described by attributes. They are c_1 = Sharpe ratio, c_2 = Expence ratio, c_3 = Portfolio concentration ratio, c_4 = Exit load, c_5 = Standard deviation, c_6 = Portfolio turnover ratio, c_7 = Treynor's ratio, c_8 = Beta, c_9 = Fund performance, and c_{10} = Investment philosophy, process and systems followed at the fund house, respectively. The domains of the attributes are: V_{c_1} = {High, Low}, V_{c_2} = {Average, Low}, V_{c_3} = {Moderately low, Moderate}, V_{c_4} = {Good, Fine}, V_{c_5} = {Low, Moderately low, Moderate, Moderately high, high}, V_{c_6} = {Good, Fine, Poor}, V_{c_7} = {Low, Moderately low, Moderate, Moderately high, High}, V_{c_8} = {Good, Fine, Poor}, V_{c_9} = {Low, Moderately low, Moderate, Moderately high, High}, $V_{c_{10}}$ = {Good, Fine, Poor}.

Table 5 Incomplete multi-source information system for ten MFs

U	EC_1				EC_2			EC_3		
	c_1	c_2	c_6	c_7	c_3	c_5	c_8	c_4	c_9	c_{10}
x_1	High	Average	Fine	Moderate	Moderate	Low	Fine	Fine	Moderately high	Fine
x_2	High	$*$	Fine	Moderately high	Moderate	$*$	Fine	Good	High	$*$
x_3	Low	$*$	Fine	$*$	$*$	Moderate	$*$	Good	Moderately low	Good
x_4	$*$	Low	Poor	$*$	$*$	Moderately low	$*$	Fine	$*$	Fine
x_5	High	Average	Good	Low	$*$	High	Good	$*$	High	$*$
x_6	$*$	Low	Fine	$*$	Moderately low	$*$	$*$	Fine	Moderate	$*$
x_7	$*$	$*$	Good	High	$*$	$*$	Fine	$*$	Moderately high	Fine
x_8	Low	$*$	Fine	Moderately high	Moderate	$*$	Good	Fine	$*$	$*$
x_9	$*$	Average	Good	$*$	$*$	Low	Poor	$*$	Low	$*$
x_{10}	Low	Low	Fine	$*$	$*$	Moderate	$*$	Fine	Moderate	Fine

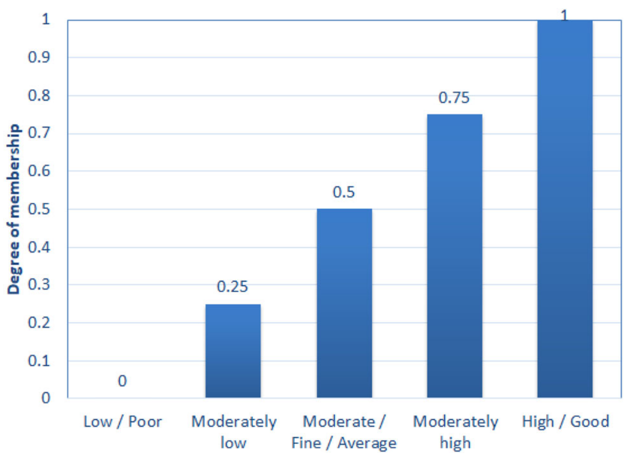

Fig. 3 Membership degree of the domains of the all attributes

For compute A from Table 5, we consider the reasonable membership degree of the domains of the all attributes, which is shown in Fig. 3.

Therefor, we have find $A(U)$ according to Algorithm 1, which is represented in Eq. (34).

$$A = \frac{\langle 0.5250, 0.7724\rangle}{x_1} + \frac{\langle 0.6283, 0.6048\rangle}{x_2} + \frac{\langle 0.4783, 0.7355\rangle}{x_3} + \frac{\langle 0.3483, 0.8469\rangle}{x_4} + \frac{\langle 0.6833, 0.4541\rangle}{x_5} + \frac{\langle 0.3317, 0.9287\rangle}{x_6} + \frac{\langle 0.5950, 0.6837\rangle}{x_7} + \frac{\langle 0.4483, 0.8028\rangle}{x_8} + \frac{\langle 0.3533, 0.8387\rangle}{x_9} + \frac{\langle 0.3533, 0.9119\rangle}{x_{10}} \tag{34}$$

Since the investor invest his/her money to the best five MFs out of ten MFs, but there is no other factors which help the investor. He/She has to randomly select five MFs to make investment that means $|X| = 5$ ($x_i \in X$ is a MF between $x_1, x_2, \ldots, x_{10}$ and not repeat any other MFs in X). Suppose $X = \{x_1, x_2, x_3, x_4, x_5\}$, in order to facilitate understanding of these model, we exhibit a computational process for each type of MG-PF-DTRS model. Then we calculate the Pythagorean fuzzy decision making object $A(X)$ from A using Eq. (2), which is shown in Eq. (35).

$$A(X) = \frac{\langle 0.5250, 0.7724\rangle}{x_1} + \frac{\langle 0.6283, 0.6048\rangle}{x_2} + \frac{\langle 0.4783, 0.7355\rangle}{x_3} + \frac{\langle 0.3483, 0.8469\rangle}{x_4} + \frac{\langle 0.6833, 0.4541\rangle}{x_5} + \frac{\langle 0, 1\rangle}{x_6} + \frac{\langle 0, 1\rangle}{x_7} + \frac{\langle 0, 1\rangle}{x_8} + \frac{\langle 0, 1\rangle}{x_9} + \frac{\langle 0, 1\rangle}{x_{10}}. \tag{35}$$

In the following we have successively obtain for types of MG-PF-DTRSs for $\alpha = 0.9$ and $\beta = 0.8$.

Type-I MG-PF-DTRSs. First, we compute $[x_i]_{\cap_{k=1}^{3} R_k(U\times U)}$ ($i = 1, 2, \ldots, 10$) from Tables 6, 7 and 8 using the Definition 7. Then we obtain $[x_i]_{\cap_{k=1}^{3} R_k(U\times X)}$ ($i = 1, 2, \ldots, 10$) using Eq. (6), which is shown in Table 9.

Using Eq. (10), we compute all inclusion measures $I([x_i]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X))$ ($1 \le i \le 10$) as follows:

$$I([x_1]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8738,$$
$$I([x_2]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.9053,$$
$$I([x_3]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8749,$$
$$I([x_4]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8499,$$
$$I([x_5]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.9265,$$
$$I([x_6]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 1,$$

Table 6 The computing result of $R_1(U)$ from EC_1

$R_1(U)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$< 1, 0 >$	$< 0.6250, 0.4665 >$	$< 0.4250, 0.7115 >$	$< 0.1750, 0.9615 >$	$< 0.5000, 0.5000 >$	$< 0.4250, 0.7115 >$	$< 0.2500, 0.9330 >$	$< 0.3750, 0.7165 >$	$< 0.4250, 0.7115 >$	$< 0.3000, 0.7449 >$
x_2		$< 1, 0 >$	$< 0.3625, 0.7370 >$	$< 0.3000, 0.9280 >$	$< 0.3750, 0.7165 >$	$< 0.5500, 0.6780 >$	$< 0.1875, 0.9586 >$	$< 0.5625, 0.4921 >$	$< 0.3000, 0.9280 >$	$< 0.4250, 0.7115 >$
x_3			$< 1, 0 >$	$< 0.2600, 0.9328 >$	$< 0.1750, 0.9615 >$	$< 0.5100, 0.6828 >$	$< 0.2375, 0.9535 >$	$< 0.6125, 0.4870 >$	$< 0.2600, 0.9328 >$	$< 0.635, 0.4663 >$
x_4				$< 1, 0 >$	$< 0.1750, 0.9615 >$	$< 0.3225, 0.7419 >$	$< 0.2375, 0.9535 >$	$< 0.3000, 0.9280 >$	$< 0.0725, 0.9919 >$	$< 0.385, 0.7163 >$
x_5					$< 1, 0 >$	$< 0.1750, 0.9615 >$	$< 0.5000, 0.6380 >$	$< 0.1250, 0.9665 >$	$< 0.6750, 0.4615 >$	$< 0.0500, 0.9949 >$
x_6						$< 1, 0 >$	$< 0.2375, 0.9535 >$	$< 0.5500, 0.6780 >$	$< 0.0725, 0.9919 >$	$< 0.6350, 0.4663 >$
x_7							$< 1, 0 >$	$< 0.1875, 0.9586 >$	$< 0.4875, 0.7035 >$	$< 0.3000, 0.9280 >$
x_8								$< 1, 0 >$	$< 0.3000, 0.9280 >$	$< 0.6750, 0.4615 >$
x_9									$< 1, 0 >$	$< 0.1350, 0.9663 >$
x_{10}										$< 1, 0 >$

Table 7 The computing result of $R_2(U)$ from EC_2

$R_2(U)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$< 1, 0 >$	$< 0.7333, 0.3266 >$	$< 0.2778, 0.9363 >$	$< 0.2778, 0.9363 >$	$< 0.1667, 0.9553 >$	$< 0.1778, 0.9742 >$	$< 0.5667, 0.6153 >$	$< 0.4000, 0.6599 >$	$< 0.5000, 0.6220 >$	$< 0.2778, 0.9363 >$
x_2		$< 1, 0 >$	$< 0.3444, 0.9295 >$	$< 0.3444, 0.9295 >$	$< 0.2333, 0.9486 >$	$< 0.1244, 0.9807 >$	$< 0.5133, 0.6217 >$	$< 0.3467, 0.6664 >$	$< 0.2333, 0.9486 >$	$< 0.3444, 0.9295 >$
x_3			$< 1, 0 >$	$< 0.1204, 0.9874 >$	$< 0.1944, 0.9704 >$	$< 0.2704, 0.9465 >$	$< 0.2611, 0.9636 >$	$< 0.3444, 0.9295 >$	$< 0.1944, 0.9704 >$	$< 0.4537, 0.6540 >$
x_4				$< 1, 0 >$	$< 0.1204, 0.9874 >$	$< 0.2704, 0.9465 >$	$< 0.2611, 0.9636 >$	$< 0.3444, 0.9295 >$	$< 0.1944, 0.9704 >$	$< 0.1204, 0.9874 >$
x_5					$< 1, 0 >$	$< 0.3444, 0.9295 >$	$< 0.2611, 0.9636 >$	$< 0.5667, 0.6153 >$	$< 0.0833, 0.9894 >$	$< 0.1944, 0.9704 >$
x_6						$< 1, 0 >$	$< 0.2911, 0.9360 >$	$< 0.1244, 0.9807 >$	$< 0.3444, 0.9295 >$	$< 0.2704, 0.9465 >$
x_7							$< 1, 0 >$	$< 0.1800, 0.9551 >$	$< 0.1500, 0.9827 >$	$< 0.2611, 0.9636 >$
x_8								$< 1, 0 >$	$< 0.2333, 0.9486 >$	$< 0.2333, 0.9486 >$
x_9									$< 1, 0 >$	$< 0.1944, 0.9704 >$
x_{10}										$< 1, 0 >$

Table 8 The computing result of $R_3(U)$ from EC_3

$R_3(U)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	< 1, 0 >	< 0.1111, 0.9809 >	< 0, 1 >	< 0.7333, 0.3266 >	< 0.2778, 0.9363 >	< 0.4444, 0.6476 >	< 0.8333, 0.2887 >	< 0.5111, 0.6409 >	< 0.2778, 0.9363 >	< 0.6667, 0.3333 >
x_2		< 1, 0 >	< 0.4444, 0.6476 >	< 0.1778, 0.9742 >	< 0.5370, 0.6199 >	< 0.3704, 0.6646 >	< 0.2778, 0.9363 >	< 0.1037, 0.9912 >	< 0.2704, 0.9465 >	< 0.1111, 0.9809 >
x_3			< 1, 0 >	< 0.0667, 0.9933 >	< 0.2778, 0.9363 >	< 0.1111, 0.9809 >	< 0.1667, 0.9553 >	< 0.1778, 0.9742 >	< 0.2778, 0.9363 >	< 0, 1 >
x_4				< 1, 0 >	< 0.3444, 0.9295 >	< 0.5111, 0.6409 >	< 0.5667, 0.6153 >	< 0.4578, 0.6473 >	< 0.3444, 0.9295 >	< 0.7333, 0.3266 >
x_5					< 1, 0 >	< 0.2037, 0.9533 >	< 0.1944, 0.9704 >	< 0.2704, 0.9465 >	< 0.1204, 0.9874 >	< 0.1944, 0.9704 >
x_6						< 1, 0 >	< 0.2778, 0.9363 >	< 0.4370, 0.6579 >	< 0.2037, 0.9533 >	< 0.7778, 0.3143 >
x_7							< 1, 0 >	< 0.3444, 0.9295 >	< 0.1944, 0.9704 >	< 0.5000, 0.6220 >
x_8								< 1, 0 >	< 0.2704, 0.9465 >	< 0.5111, 0.6409 >
x_9									< 1, 0 >	< 0.2778, 0.9363 >
x_{10}										< 1, 0 >

Table 9 The computing result of $[x_i]_{\cap_{k=1}^{3} R_k(U\times X)}$

$[x_i]_{\cap_{k=1}^{3} R_k(U\times X)}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	< 1, 0 >	< 0.1111, 0.9809 >	< 0, 1 >	< 0.1750, 0.9615 >	< 0.1667, 0.9553 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_2	< 0.1111, 0.9809 >	< 1, 0 >	< 0.3444, 0.9295 >	< 0.1778, 0.9742 >	< 0.2333, 0.9486 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_3	< 0, 1 >	< 0.3444, 0.9295 >	< 1, 0 >	< 0.0667, 0.9933 >	< 0.1750, 0.9704 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_4	< 0.1750, 0.9615 >	< 0.1778, 0.9742 >	< 0.0667, 0.9933 >	< 1, 0 >	< 0.1204, 0.9874 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_5	< 0.1667, 0.9553 >	< 0.2333, 0.9486 >	< 0.1750, 0.9704 >	< 0.1204, 0.9874 >	< 1, 0 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_6	< 0.1778, 0.9742 >	< 0.1244, 0.9807 >	< 0.1111, 0.9809 >	< 0.2704, 0.9465 >	< 0.1750, 0.9615 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_7	< 0.2500, 0.9330 >	< 0.1875, 0.9586 >	< 0.1667, 0.9636 >	< 0.2375, 0.9636 >	< 0.1944, 0.9704 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_8	< 0.3750, 0.7165 >	< 0.1037, 0.9912 >	< 0.1778, 0.9742 >	< 0.3000, 0.9295 >	< 0.1250, 0.9665 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >
x_9	< 0.2778, 0.9363 >	< 0.2333, 0.9486 >	< 0.1944, 0.9704 >	< 0.0725, 0.9919 >	< 0.0833, 0.9894 >	< 0, 0 >	< 0, 1 >	< 0, 1	< 0, 1 >	< 0, 1 >
x_{10}	< 0.2778, 0.9363 >	< 0.1111, 0.9809 >	< 0, 1 >	< 0.1204, 0.9874 >	< 0.0500, 0.9949 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >	< 0, 1 >

$I([x_7]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 1,$
$I([x_8]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.9913,$
$I([x_9]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 1,$
$I([x_{10}]_{\cap_{k=1}^{3} R_k(U\times X)}, A(X)) = 1.$

For $\alpha = 0.9$ and $\beta = 0.8$ we obtain

$$\underline{\mathrm{Apr}}^{0.9}_{\cap_{k=1}^{3} R_k(U\times X)}(A(X)) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$\overline{\mathrm{Apr}}^{0.8}_{\cap_{k=1}^{m} R_k(U\times X)}(A(X)) = U.$$

and

$$\mathrm{POS}^{0.9}(A(X)) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$\mathrm{NEG}^{0.8}(A(X)) = \emptyset,$$
$$\mathrm{BND}^{(0.9,0.8)}(A(X)) = \{x_1\}.$$

Type-II MG-PF-DTRSs First, we compute $[x_i]_{\cup_{k=1}^{3} R_k(U\times U)}$ $(i = 1, 2, \ldots, 10)$ from Tables 6, 7 and 8 using the Definition 7. Then we obtain $[x_i]_{\cup_{k=1}^{3} R_k(U\times X)}$ $(i = 1, 2, \ldots, 10)$ using Eq. (6), which is not display here. Using Eq. (10), we compute all inclusion measures $I([x_i]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X))$ $(1 \le i \le 10)$ as follows:

$I([x_1]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.7209,$
$I([x_2]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8135,$
$I([x_3]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8573,$
$I([x_4]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.7816,$
$I([x_5]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8876,$
$I([x_6]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.9215,$
$I([x_7]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8470,$
$I([x_8]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.8738,$
$I([x_9]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.9777,$
$I([x_{10}]_{\cup_{k=1}^{3} R_k(U\times X)}, A(X)) = 0.7800.$

For $\alpha = 0.9$ and $\beta = 0.8$ we obtain

$$\underline{\mathrm{Apr}}^{0.9}_{\cup_{k=1}^{3} R_k(U\times X)}(A(X)) = \{x_6, x_9\},$$
$$\overline{\mathrm{Apr}}^{0.8}_{\cup_{k=1}^{m} R_k(U\times X)}(A(X)) = \{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

and

$$\mathrm{POS}^{0.9}(A(X)) = \{x_6, x_9\},$$
$$\mathrm{NEG}^{0.8}(A(X)) = \{x_1, x_4\},$$
$$\mathrm{BND}^{(0.9,0.8)}(A(X)) = \{x_2, x_3, x_5, x_7, x_8\}.$$

Type-III MG-PF-DTRSs First we compute $R_1(U \times X)$, $R_2(U \times X)$ and $R_3(U \times X)$ from Tables 6, 7 and 8, and $I([x_i]_{R_1(U\times X)}, A(X))$, $I([x_i]_{R_2(U\times X)}, A(X))$ and $I([x_i]_{R_3(U\times X)}, A(X))$ for $(1 \le i \le 10)$, which are not displayed here. Using Eqs. (12) and (13), we compute all inclusion measures $\cap_{k=1}^{3} I([x_i]_{R_k(U\times X)}, A(X))$ and $\cup_{k=1}^{3} I([x_i]_{R_k(U\times X)}, A(X))$ for $(1 \le i \le 10)$ as follows:

$\cap_{k=1}^{3} I([x_1]_{R_k(U\times X)}, A(X)) = 0.7822,$
$\cap_{k=1}^{3} I([x_2]_{R_k(U\times X)}, A(X)) = 0.8346,$
$\cap_{k=1}^{3} I([x_3]_{R_k(U\times X)}, A(X)) = 0.8581,$
$\cap_{k=1}^{3} I([x_4]_{R_k(U\times X)}, A(X)) = 0.7827,$
$\cap_{k=1}^{3} I([x_5]_{R_k(U\times X)}, A(X)) = 0.8871,$
$\cap_{k=1}^{3} I([x_6]_{R_k(U\times X)}, A(X)) = 0.9328,$
$\cap_{k=1}^{3} I([x_7]_{R_k(U\times X)}, A(X)) = 0.8559,$
$\cap_{k=1}^{3} I([x_8]_{R_k(U\times X)}, A(X)) = 0.9289,$
$\cap_{k=1}^{3} I([x_9]_{R_k(U\times X)}, A(X)) = 0.9785,$
$\cap_{k=1}^{3} I([x_{10}]_{R_k(U\times X)}, A(X)) = 0.8360,$
and $\cup_{k=1}^{3} I([x_1]_{R_k(U\times X)}, A(X)) = 0.8422,$
$\cup_{k=1}^{3} I([x_2]_{R_k(U\times X)}, A(X)) = 0.8877,$
$\cup_{k=1}^{3} I([x_3]_{R_k(U\times X)}, A(X)) = 0.8744,$
$\cup_{k=1}^{3} I([x_4]_{R_k(U\times X)}, A(X)) = 0.8499,$
$\cup_{k=1}^{3} I([x_5]_{R_k(U\times X)}, A(X)) = 0.9266,$
$\cup_{k=1}^{3} I([x_6]_{R_k(U\times X)}, A(X)) = 1,$
$\cup_{k=1}^{3} I([x_7]_{R_k(U\times X)}, A(X)) = 1,$
$\cup_{k=1}^{3} I([x_8]_{R_k(U\times X)}, A(X)) = 0.9281,$
$\cup_{k=1}^{3} I([x_9]_{R_k(U\times X)}, A(X)) = 1,$
$\cup_{k=1}^{3} I([x_{10}]_{R_k(U\times X)}, A(X)) = 0.9883.$

For $\alpha = 0.9$ and $\beta = 0.8$ we obtain

$$\cap_{k=1}^{3} \underline{\mathrm{Apr}}^{0.9}_{R_k(U\times X)}(A(X)) = \{x_6, x_{8,9}\},$$
$$\cup_{k=1}^{3} \overline{\mathrm{Apr}}^{0.8}_{R_k(U\times X)}(A(X)) = U.$$

and

$$\mathrm{POS}^{0.9}(A(X)) = \{x_6, x_8, x_9\},$$
$$\mathrm{NEG}^{0.8}(A(X)) = \emptyset,$$
$$\mathrm{BND}^{(0.9,0.8)}(A(X)) = \{x_1, x_2, x_3, x_4, x_5, x_7, x_{10}\}.$$

Type-IV MG-PF-DTRSs From inclusion measures $\cap_{k=1}^{3} I([x_i]_{R_k(U\times X)}, A(X))$ and $\cup_{k=1}^{3} I([x_i]_{R_k(U\times X)}, A(X))$ for $(1 \le i \le 10)$ we have obtained Type-IV MG-PF-DTRSs for $\alpha = 0.9$ and $\beta = 0.8$ as follows:

$$\cup_{k=1}^{3} \underline{\mathrm{Apr}}^{0.9}_{R_k(U\times X)}(A(X)) = \{x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$\cap_{k=1}^{3} \overline{\mathrm{Apr}}^{0.8}_{R_k(U\times X)}(A(X)) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

Table 10 The results of uncertainty measures

No.	X	Type-I MG-PF-DTRSs			Type-II MG-PF-DTRSs			Type-III MG-PF-DTRSs			Type-IV MG-PF-DTRSs		
		ρ_{I}	σ_{I}	ω_{I}	ρ_{II}	σ_{II}	ω_{II}	ρ_{III}	σ_{III}	ω_{III}	ρ_{IV}	σ_{IV}	ω_{IV}
1	$x_{1,2,3,4,5}$	0.7	1.4	0.7	0.2857	0.4	0.2	0.3	0.6	0.3	0.75	1.2	0.6
2	$x_{1,2,5,6,7}$	0.7	1.4	0.7	0.5	0.6	0.3	0.3	0.6	0.3	0.75	1.2	0.6
3	$x_{1,3,5,7,8}$	0.6	1.2	0.6	0.25	0.4	0.2	0.2	0.4	0.2	0.6667	1.2	0.6
4	$x_{1,3,4,6,7}$	0.5	1	0.5	0.5	0.4	0.2	0.2	0.4	0.2	1	1	0.5
5	$x_{2,4,5,6,7}$	0.7	1.4	0.7	0.5	0.6	0.3	0.4	0.8	0.4	1	1.2	0.6
6	$x_{2,4,5,9,10}$	0.6	1.2	0.6	0.1429	0.2	0.1	0.1	0.2	0.1	0.875	1.4	0.6
7	$x_{2,7,8,9,10}$	0.6	1.2	0.6	0	0	0	0.1	0.2	0.1	0.8571	1.2	0.6
8	$x_{3,4,6,8,9}$	0.5	1	0.5	0.25	0.2	0.1	0.2	0.4	0.2	1	1	0.5
9	$x_{3,5,6,9,10}$	0.6	1.2	0.6	0.2	0.2	0.1	0.1	0.2	0.1	0.8571	1.2	0.6
10	$x_{4,6,7,8,9}$	0.5	1	0.5	0	0	0	0.1	0.2	0.1	1	1	0.5

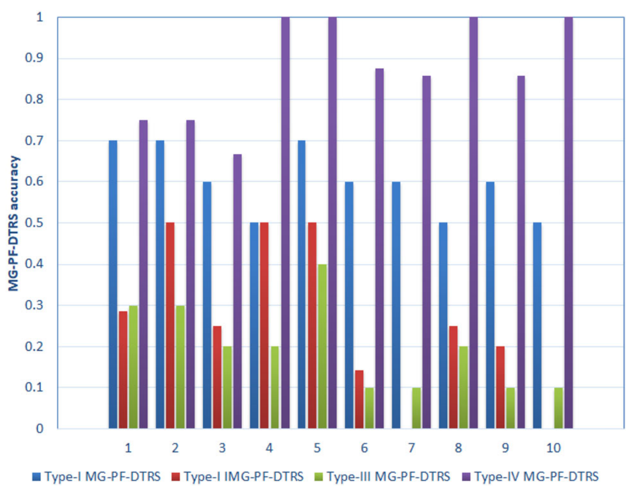

Fig. 4 A comparison of the accuracy

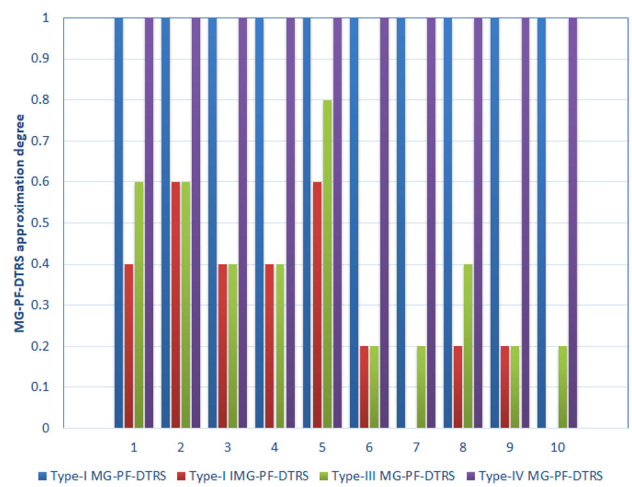

Fig. 5 A comparison of the approximation degree

and

$$\mathrm{POS}^{0.9}(A(X)) = \{x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$\mathrm{NEG}^{0.8}(A(X)) = \{x_1\},$$
$$\mathrm{BND}^{(0.9,0.8)}(A(X)) = \{x_2, x_3\}.$$

It is obvious that these four types of results are not entirely consistent. Then, the uncertainties are not entirely consistent for them and several kinds of uncertainty measure methods are necessary. Consequently, in different fields should select different model according the different requirements in practical application. Based on the calculated approximation sets, we can measure the uncertainty of the alternative MFs to estimate the investment. To evaluate the performance of the proposed uncertainty measure methods, we conduct a series of experiments to calculate these three uncertainty measures. Therefor, in our experiment, we randomly select five MFs from the set U, and uncertainty evolution's for them as shown in Table 10.

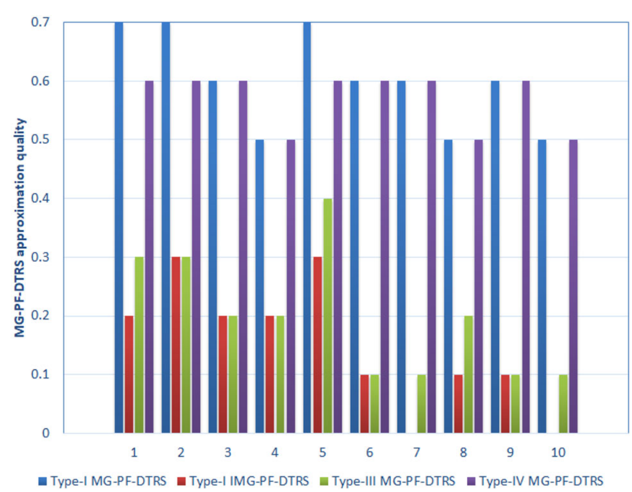

Fig. 6 A comparison of the approximation quality

According to Table 10 and Figs. 4, 5 and 6 we can get that the fifth set $X = \{x_2, x_4, x_5, x_6, x_7\}$ with higher accuracy, approximation degree and approximation quality for each model. That is, x_2, x_4, x_5, x_6 and x_7 are the best five MFs out of ten MFs for the investors to invest. There is no doubt that there are more recommended programs, but its best in the given ten options.

Conclusions

In this paper, we present four types of MG-PF-DTRSs of Pythagorean fuzzy subset (of a subset of the given universe) of the PFS (of the given universe) and study their uncertainty measure methods based on the Pythagorean fuzzy inclusion measure within the framework of multi-granulation Pythagorean fuzzy approximation space. Using this four types of MG-PF-DTRSs and their uncertainty measure methods, we have presented a method for decision-making to IMSIS. In this decision-making method for IMSIS, we have analyzed three issues. (1) How to find the similarity degrees between two objects from IMSISs in the Pythagorean fuzzy settings. (2) How to obtain the Pythagorean fuzzy decision-making objects from IMSISs. (3) The following problem: if $X_1, X_2, \ldots, X_r \subseteq U$ (U is the finite universe of discourse) then find the best X_r, where the elements of X_r are randomly selected and not repeated any other elements in X_r. The studies of this paper are focusing on the basis of the theoretical aspect and the general framework of decision-making process to the IMSIS of the proposed model and method. Therefore, it is recommended that the further improvement of the proposed method to apply more complexity decision-making problems in Garg [5–7,9,10], Mandal and Ranadive [27] and the real-life data be used to test the approach established in this paper.

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Compliance with ethical standards

Conflict of interest Prasenjit Mandal and A. S. Ranadive declare that there is no conflict of interest.

Ethical approval This article does not contain any study performed on humans or animals by the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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