A Six Parameters Beta Distribution with Application for Modeling Waiting Time of Muslim Early Morning Prayer

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Received: 21 February 2020 / Revised: 19 April 2020 / Accepted: 24 April 2020 / Published online: 18 May 2020
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Abstract

Beta distribution is a well-known and widely used distribution for modeling and analyzing lifetime data, due to its interesting characteristics. In this paper, a six parameters beta distribution is introduced as a generalization of the two (standard) and the four parameters beta distributions. This distribution is closed under scaling and exponentiation, and has reflection symmetry property, has some well-known distributions as special cases, such as, the two and four parameters beta, generalized modification of the Kumaraswamy, generalized beta of the first kind, the power function, Kumaraswamy power function, Minimax, exponentiated Pareto, and the generalized uniform distributions. Its moments about the origin, moment generating function, incomplete moments, mean deviations, are derived. The maximum likelihood estimation method is used for estimating its parameters and applied to estimate the parameters of the six different simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes. Finally, two real life data sets, represent the waiting period of Muslim worshipers from the time of entering the mosque till the actual time of starting Alfajir pray in two different mosques, were used to illustrate the usefulness and the flexibility of this distribution, as well as, presents better fitting than the other gamma, exponential, the four parameters beta, and the generalized beta of the first kind distributions

Keywords Beta distribution · Maximum likelihood estimator · Moments · Simulation study · Applications

Mathematics Subject Classification 60E05 · 62E15 · 65C05

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1 Introduction

Due to its interesting characteristics, the beta distribution is one of the well-known continuous distribution, that has a wide range of application in various filed, such as reliability applications and production quality control. It has a flexible shape, that reflects a wide range of natural and empirical phenomena in nature and reality that can be modelling with this distribution. Its domain, the interval from zero to one, add another interesting characteristic to this distribution by allowing it to consider as a probability distribution of probabilities, such as fraction of time, measurements whose values (or relative values) all lie between zero and one, or the random behavior of percentages and fractions, especially, in the cases when we have no idea about the probability, and therefore, it can be used to represents all probabilities. Another area that used beta distribution for representing possible values of probabilities or a distribution of the probabilities is the Bayesian studies, as being the prior distribution, that is widely used. In fact, it is one of the three common distributions, with the rectangular/uniform and normal distributions, that are employed to represents within the framework Bayesian analysis of continuous variables, Sheskin [1, p. 397]. Data mining methods and techniques need to use information about the prior probability knowledge, hence the beta distribution is representing a candidate for such situations, see Shi [2], and Olson and Shi [3] for further details. For an intensive reference of the beta distribution see Johnson et al. [4, p. 210–275].

The probability density function (pdf) of the four parameters beta distribution, Johnson et al. [5, p. 210], is given by;

$$f(t) = \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{B(\alpha, \beta)(b-a)^{\alpha+\beta-1}}, \quad a < t < b, \quad (1)$$

where, the parameters α, β, a and b satisfy that $\alpha > 0, \beta > 0, a$ and b are real number such that $a < b$, $B(\alpha, \beta)$ is the beta function, Abramowitz and Stegun [6, p. 258], defined by;

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (2)$$

and $\Gamma(\alpha)$, the gamma function, Abramowitz and Stegun [6, p. 255], defined by;

$$\Gamma(\alpha) = \int_0^1 t^{\alpha-1} e^{-t} dt \quad (3)$$

The common widely used form of beta distribution in the literature, is the pdf given by;

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad (4)$$

This two parameters form is called sometimes, the standard beta distribution, which is obtained from (1) by making the transformation; $x = \frac{t-a}{b-a}$.

One direction of the research employing the beta distribution is the generalization of the form given by (4), in order to be even more flexible and cover a lot of shapes.

Armoro and Bayarri [7] introduced the Gauss hypergeometric distribution, with parameters p, q, r and δ , as a generalization to the beta distribution when they studied a Bayesian queuing theory problem, with the following pdf;

$$f(x) = \frac{x^{p-1}(1-x)^{q-1}(1+\delta x)^{-r}}{B(p, q)F_{(2,1)}(r, p; p+q; -\delta)}, \quad 0 < x < 1, \quad (5)$$

where $p > 0, q > 0, -\infty < r < \infty, \delta > -1$, and $F_{(2,1)}$ is the generalized hypergeometric function defined for non-negative integers n and m by;

$$F_{(n,m)}(a_1, \dots, a_n; b_1, \dots, b_m; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_n)_k}{(b_1)_k \dots (b_m)_k} \frac{z^k}{k!} \quad (6)$$

and $(a)_k$ is defined by;

$$(a)_k = \begin{cases} 0 & \text{if } k = 0 \\ a(a+1) \dots (a+k-1), & k = 1, 2, 3, \dots \end{cases} \quad (7)$$

Gordy [8] introduced the confluent hypergeometric distribution, with parameters p, q and s , with pdf given by;

$$f(x) = \frac{x^{p-1}(1-x)^{q-1}\exp(-sx)}{B(p, q)F_{(1,1)}(p, p+q, -s)}, \quad 0 < x < 1,$$

where $p > 0, q > 0$ and $-\infty < s < \infty$.

Pathan et al. [9] introduced a five parameters distribution as a generalization beta distribution, called it generalized beta distribution, with pdf given by;

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}(1-\sigma x)^{\rho-1}\exp(-\gamma x)}{B(\alpha, \beta)\Phi_1(\alpha, \rho; \alpha+\beta; \sigma, -\gamma)}, \quad 0 < x < 1,$$

where the parameters $\alpha, \beta, \sigma, \rho$ and γ satisfy that $\alpha > 0, \beta > 0, 0 \leq \sigma < 1, \rho$ and γ are real numbers and $\Phi_1(\cdot)$ is the Humbert's confluent hypergeometric function given in Srivastava and Manocha [10, p. 58, Eq. (36)], and derive expressions for its distribution function moments.

Ng et al. [11] study the properties and evaluate the prediction level of a 6 parameters generalized beta distribution model with pdf given by;

$$f(x) = \frac{\frac{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}{\Gamma(\gamma+\rho)\Gamma(\gamma+\rho-\alpha-\beta)}(1-z)^{\sigma}x^{\gamma-1}(1-x)^{\rho-1}(1-\sigma x)^{\rho-1}(1-zx)^{-\sigma}F_{(2,1)}(\alpha, \beta; \gamma; x)}{F_{(3,2)}\left(\rho, \sigma; \gamma+\rho-\alpha-\beta; \gamma+\rho-\alpha, \gamma+\rho-\beta; \frac{z}{z-1}\right)B(\gamma, \rho)} \quad 0 < x < 1, \quad (8)$$

where the parameters $\alpha, \beta, \gamma, \sigma, z$ and ρ satisfy that $\alpha > 0, \beta > 0, \gamma > 0, \sigma > 0, z < 0.5$ and $\rho > \alpha + \beta - \gamma$.

Although Ng et al. [11], who provided a nice literature review for the beta family, showed interesting advances of this distribution with pdf given by (8) for fitting many different types of data, as well as that of Armero and Bayarri [7], Gordy [8] and Pathan et al. [9], but it is not easy to work with empirically. Finally, Gómez-Déniz and Sarabia [12] introduced a generalization of the standard beta distribution with bounded support, and study some of its basic properties, the behavior of its maximum likelihood estimators through simulation and derive its multivariate version.

The rest of the paper is organized as follows. Section 2 defines the six parameters beta distribution (SPBD). Section 3 gives some properties of this distribution, these properties are; boundaries and some limits of the pdf of SPBD, series expansion of its pdf, its mode, quantile function, reliability function, hazard function, special cases of SPBD, some transformation of the SPBD, its scaling, exponentiation and reflection symmetry properties, generation of its random variates, its order statistics distribution, moments about the origin, mean and variance, moment generating function, harmonic mean, incomplete moments, mean deviations, probability weighted moments, Renyi entropy, and Lorenz and Bonferroni curves. Section 4 introduces estimation of its parameters using the method of maximum likelihood estimation (MLE). Section 5 gives six miscellaneous simulation study of the SPBD to check the performance of the MLE. Section 6 uses the SPBD and other nested and related distributions to fit two different real-life data. Finally, Sect. 7 ends with conclusions.

2 The Six Parameters Beta Distribution

Let $0 < a, b, \alpha, \beta, A, B < \infty$, such that $A < B$, and define the function f by:

$$f(x; a, b, \alpha, \beta, A, B) = \begin{cases} \frac{bx^{b-1}}{a^b(B-A)^{\alpha+\beta-1}B(\alpha, \beta)} \left[\left(\frac{x}{a} \right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a} \right)^b \right]^{\beta-1}, & aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}} \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

where $B(\alpha, \beta)$ is the beta function defined by (2). We will write $f(x)$ instead of $f(x; a, b, \alpha, \beta, A, B)$ for simplicity. We have the following proposition;

Proposition 1 *The function f defined by (9) is a pdf with its cumulative distribution function (CDF) F given by;*

$$F_X(x; a, b, c, \alpha, \beta) = \begin{cases} 0, & x \leq a\alpha^{\frac{1}{b}} \\ \frac{1}{B(\alpha, \beta)} B\left(\frac{\left(\frac{x}{a}\right)^b - A}{B-A}; \alpha, \beta\right), & aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}} \\ 1, & x \geq a\beta^{\frac{1}{b}} \end{cases} \quad (10)$$

where $B(z; \alpha, \beta)$ is the incomplete beta function, Abramowitz and Stegun [6, p. 263], defined by;

$$B(z; \alpha, \beta) = \int_0^z t^{\alpha-1} (1-t)^{\beta-1} dt. \quad (11)$$

Proof Since $0 < a, b, \alpha, \beta, A, B < \infty$, and $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$, then $A < \left(\frac{x}{a}\right)^b < B$, hence $\left(\frac{x}{a}\right)^b - A > 0$ and also $B - \left(\frac{x}{a}\right)^b > 0$, implying that f given in (9) is non-negative. Now;

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) dx &= \int_{aA^{\frac{1}{b}}}^{aB^{\frac{1}{b}}} \frac{bx^{b-1}}{a^b(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \left[\left(\frac{x}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a}\right)^b \right]^{\beta-1} dx \\ &= \frac{b}{a^b(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \int_{aA^{\frac{1}{b}}}^{aB^{\frac{1}{b}}} x^{b-1} \left[\left(\frac{x}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a}\right)^b \right]^{\beta-1} dx \end{aligned}$$

Let $\left(\frac{x}{a}\right)^b - A = (B-A)t$, then $x = a[(B-A)t + A]^{\frac{1}{b}}$, and $dx = \frac{a(B-A)}{b} [(B-A)t + A]^{\frac{1}{b}-1} dt$, then;

$$\begin{aligned} \int_{aA^{\frac{1}{b}}}^{aB^{\frac{1}{b}}} x^{b-1} \left[\left(\frac{x}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a}\right)^b \right]^{\beta-1} dx &= \frac{a^b}{b} (B-A)^{\alpha+\beta-1} \int_0^1 t^{\alpha-1} e^{-t} dt \\ &= \frac{a^b}{b} (B-A)^{\alpha+\beta-1} B(\alpha, \beta) \end{aligned}$$

Hence, $\int_{-\infty}^{+\infty} f(x) dx = 1$. It follows that, for any x such that, $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$;

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_{aA^{\frac{1}{b}}}^x \frac{bt^{b-1}}{a^b(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \left[\left(\frac{t}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{t}{a}\right)^b \right]^{\beta-1} dt \end{aligned} \quad (12)$$

Now by using the transformation $z = \frac{\left(\frac{t}{a}\right)^b - A}{B-A}$, (12) reduces to;

$$= \frac{1}{B(\alpha, \beta)} \int_0^{\left(\frac{x}{a}\right)^{\frac{1}{b}} - A} z^{\alpha-1} e^{-z} dz$$

from which we get (10).

We note that the F_X can be written, for $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$, in the form;

$$F_X(x) = I\left(\frac{\left(\frac{x}{a}\right)^{\frac{1}{b}} - A}{B - A}; \alpha, \beta\right) \quad (13)$$

where $I(z; \alpha, \beta)$ is the regularized incomplete beta function, Abramowitz and Stegun [6, p. 263], defined by;

$$I(z; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^z t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{B(z; \alpha, \beta)}{B(\alpha, \beta)} \quad (14)$$

□

Definition of the SPBD The rv X is said to have a SPBD with parameters a, b, α, β, A and B written as $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, if its pdf is given by (9), or equivalently, its CDF is given by (10) or (13).

Figure 1 shows some plots of the pdf of the SPBD for some of its parameter's values, indicating that this distribution has a lot of different flexible shapes.

3 Some Characteristics of the SPBD

3.1 Boundaries and Some Limits of the pdf

Let us study the behavior of the pdf of the $\text{SPBD}(a, b, \alpha, \beta, A, B)$ at certain points. At the boundary's points, we have from (9) for $0 < \beta < \infty$, that;

$$f\left(aA^{\frac{1}{b}}\right) = \begin{cases} \infty, & 0 < \alpha < 1 \\ \frac{\beta b A^{1-\frac{1}{b}}}{a(B-A)}, & \alpha = 1 \\ 0, & \alpha > 1 \end{cases}$$

Therefore;

Fig. 1 Different pdf plots of the SPBD models

$$\begin{aligned}\lim_{a \rightarrow 0^+} f\left(aA^{\frac{1}{b}}\right) &= \lim_{b \rightarrow \infty} f\left(aA^{\frac{1}{b}}\right) = \lim_{\beta \rightarrow \infty} f\left(aA^{\frac{1}{b}}\right) = \infty, \\ \lim_{a \rightarrow \infty} f\left(aA^{\frac{1}{b}}\right) &= \lim_{b \rightarrow 0^+} f\left(aA^{\frac{1}{b}}\right) = \lim_{\beta \rightarrow 0^+} f\left(aA^{\frac{1}{b}}\right) = \lim_{B \rightarrow \infty} f\left(aA^{\frac{1}{b}}\right) = 0, \\ \lim_{A \rightarrow 0^+} f\left(aA^{\frac{1}{b}}\right) &= 0 \quad \text{if } b \neq 1, \quad \text{and} \quad \lim_{A \rightarrow 0^+} f\left(aA^{\frac{1}{b}}\right) = \frac{\beta b}{aB} \quad \text{if } b = 1\end{aligned}$$

Similarly;

$$f\left(aA^{\frac{1}{b}}\right) = \begin{cases} \infty, & 0 < \beta < 1 \\ \frac{abB^{1-\frac{1}{b}}}{a(B-A)}, & \beta = 1 \\ 0, & \beta > 1 \end{cases}$$

Therefore;

$$\begin{aligned}\lim_{a \rightarrow 0^+} f\left(aB^{\frac{1}{b}}\right) &= \lim_{b \rightarrow \infty} f\left(aB^{\frac{1}{b}}\right) = \lim_{\alpha \rightarrow \infty} f\left(aB^{\frac{1}{b}}\right) = \lim_{\beta \rightarrow 0^+} f\left(aB^{\frac{1}{b}}\right) = \infty, \\ \lim_{a \rightarrow \infty} f\left(aB^{\frac{1}{b}}\right) &= \lim_{b \rightarrow 0^+} f\left(aB^{\frac{1}{b}}\right) = \lim_{\alpha \rightarrow 0^+} f\left(aB^{\frac{1}{b}}\right) = \lim_{\beta \rightarrow \infty} f\left(aB^{\frac{1}{b}}\right) = 0, \\ \text{and} \quad \lim_{A \rightarrow 0^+} f\left(aB^{\frac{1}{b}}\right) &= \frac{ab}{a} B^{-\frac{1}{b}}\end{aligned}$$

3.2 Series Expansion

Proposition 2 *The function f given by (9) can be written in the following expansion series.*

$$f(x; a, b, \alpha, \beta, A, B) = \frac{bB^{\beta-1}}{(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C(i, j; a, b, \alpha, \beta, A, B) x^{b(\alpha+j-i)-1} \quad (15)$$

where

$$C(i, j; a, b, \alpha, \beta, A, B) = (-1)^{i+j} \binom{\alpha-1}{i} \binom{\beta-1}{j} \frac{A^i}{a^{b(\alpha+j-i)} B^j}$$

Proof Since $0 < a$ and $aA^{\frac{1}{b}} < x$, then $A < \left(\frac{x}{a}\right)^b$, that is $\frac{A}{\left(\frac{x}{a}\right)^b} < 1$, we can write

$$\left[\left(\frac{x}{a}\right)^b - A\right]^{\alpha-1} = \left(\frac{x}{a}\right)^{b(\alpha-1)} \left[1 - \frac{A}{\left(\frac{x}{a}\right)^b}\right]^{\alpha-1}$$

Therefore, using the binomial series expansion, Abramowitz and Stegun [6, p. 14], we can write;

$$\left[\left(\frac{x}{a} \right)^b - A \right]^{\alpha-1} = \left(\frac{x}{a} \right)^{b(\alpha-1)} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \left[\frac{A}{\left(\frac{x}{a} \right)^b} \right]^i \quad (16)$$

Similarly, we have that;

$$\left[B - \left(\frac{x}{a} \right)^b \right]^{\beta-1} = B^{\beta-1} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \left[\frac{\left(\frac{x}{a} \right)^b}{B} \right]^j \quad (17)$$

Hence, using (16) and (17) into the function f given by (9) we get (15). $\square$

3.3 The Mode

For $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$, we can see that the pdf of the SPBD satisfies the following;

$$\frac{\partial}{\partial x} f(x) = \left\{ \frac{\frac{b}{a}(\alpha-1)\left(\frac{x}{a}\right)^{b-1}}{\left(\frac{x}{a}\right)^b - A} - \frac{b(\beta-1)\left(\frac{x}{a}\right)^{b-1}}{a\left[B - \left(\frac{x}{a}\right)^b\right]} + \frac{b-1}{x} \right\} f(x) \quad (18)$$

Therefore, $\frac{\partial}{\partial x} f(x) = 0$, is equivalent to either $f(x) = 0$, which is discussed in Sect. 3.1 above, or

$$\frac{\frac{b}{a}(\alpha-1)\left(\frac{x}{a}\right)^{b-1}}{\left(\frac{x}{a}\right)^b - A} - \frac{b(\beta-1)\left(\frac{x}{a}\right)^{b-1}}{a\left[B - \left(\frac{x}{a}\right)^b\right]} + \frac{b-1}{x} = 0 \quad (19)$$

Multiplying (19) by $ax\left[\left(\frac{x}{a}\right)^b - A\right]\left[B - \left(\frac{x}{a}\right)^b\right]$, and setting $y = \left(\frac{x}{a}\right)^b$, it reduces to;

$$c_1 y^2 + c_2 y + c_3 = 0 \quad (20)$$

where

$$\begin{aligned} c_1 &= b(\alpha + \beta - 1) - 1 \\ c_2 &= (1 - b\alpha)B + b(1 - b\beta)A \\ c_3 &= (b-1)AB \end{aligned}$$

Let discuss the real roots of (20), according to the following cases.

Case 1 If $\alpha + \beta \neq 1$, $b = \frac{1}{\alpha + \beta - 1}$, and $(1 - b\alpha)B \neq b(1 - b\beta)A$, that is when $c_1 = 0$ and $c_2 \neq 0$, then (20) has a single root given by;

$$y = \frac{-(b-1)AB}{(1-b\alpha)B + b(1-b\beta)A}$$

Hence; the root in term of x is given by;

$$x_1 = a \left[\frac{(1-b)AB}{(1-b\alpha)B + b(1-b\beta)A} \right]^{\frac{1}{b}}$$

Case 2 If $b \neq \frac{1}{\alpha + \beta - 1}$, that is $c_1 \neq 0$, then the real roots of (20) in terms of x , that is when $c_2^2 - 4c_1c_3 \geq 0$, are given by;

$$x_2 = a \left[\frac{(b\alpha - 1)B + b(b\beta - 1)A - \sqrt{((1 - b\alpha)B + b(1 - b\beta)A)^2 - 4(b-1)AB[b(\alpha + \beta - 1) - 1]}}{2b(\alpha + \beta - 1) - 2} \right]^{\frac{1}{b}}$$

$$x_3 = a \left[\frac{(b\alpha - 1)B + b(b\beta - 1)A + \sqrt{((1 - b\alpha)B + b(1 - b\beta)A)^2 - 4(b-1)AB[b(\alpha + \beta - 1) - 1]}}{2b(\alpha + \beta - 1) - 2} \right]^{\frac{1}{b}}$$

Since $\frac{\partial^2}{\partial x^2} f(x_i)$, for $i = 1, 2$, and 3 is not easy to be evaluated, an empirical evaluation has to be studied to see at which point x_i we have a local maximum in order to determined the mode of the SPBD.

3.4 Quantile Function

Let $0 < p < 1$, then the quantile function of the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, Q , is defined by;

$$Q(u) = \inf\{x \in \mathbb{R}; p \leq F(x)\}$$

can be found using (13), to be;

$$Q(p) = a \left[A + (B - A)I^{-1}(p; \alpha, \beta) \right]^{\frac{1}{b}} \quad (21)$$

where I^{-1} is the inverse of regularized incomplete beta function.

In particular, the median of X , $Med(X)$; is given by;

$$Med(X) = a \left[A + (B - A)I^{-1}(0.5; \alpha, \beta) \right]^{\frac{1}{b}}$$

Table 1 represents parameters values and domain ranges of the some selected SPBD data sets, which has different shapes and domain range, that will use for our simulation study in Sect. 5, as well as, will be used for computing of certain

Table 1 Parameters values of the some selected SPBD data sets

Data set	Parameters						Domain Range	
	a	b	α	β	A	B	Minimum	Maximum
1	1.8	2.3	1.4	3.9	0	1	0	1.8
2	1.5	3.1	0.93	2.65	0.015	1.1	0.387020171	1.546834102
3	1.5	5.75	0.43	5.65	0.01	2	0.673387689	1.69217121
4	2	1.3	1.6	3.8	0.5	1.8	1.17346046	3.143355214
5	2	1.2	2.3	1.8	0.5	1.8	1.122462048	3.264052108
6	2	0.45	2.15	0.65	0.4	1.2	0.261047095	2.999081861

Fig. 2 Plots of the quantile function of the six selected SPBD data sets

statistics of SPBD later in this section, while Fig. 2 represents the plots of the quantile functions of these SPBD data sets.

3.5 Reliability Function

The reliability (survival) function of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ using (13), is given by;

$$R(x) = 1 - F(x) = 1 - I \left(\left(\frac{x}{a} \right)^b - A, \alpha, \beta \right), \quad aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}, \quad (22)$$

3.6 Hazard Function

The hazard function, $h(x)$, of the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, using (9) and (13), is given for $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$, by;

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\frac{bx^{b-1}}{a^b(B-A)^{a+\beta-1}B(\alpha, \beta)} \left[\left(\frac{x}{a} \right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a} \right)^b \right]^{\beta-1}}{1 - I\left(\frac{\left(\frac{x}{a} \right)^b - A}{B-A}; \alpha, \beta \right)}$$

3.7 Special Cases of SPBD

1. The SPBD(1, 1, p, q, a, b) is the 4 parameters Beta distribution, Johnson et al. [4, p. 210], with pdf;

$$f(x) = \frac{1}{B(p, q)} \frac{(x-a)^{p-1}(b-x)^{q-1}}{(b-a)^{p+q-1}}, \quad a < x < b$$

2. The SPBD($a, b, 1, c, \alpha, \beta$) is the generalized modification of the Kumaraswamy distribution, Alshkaki [13], with pdf;

$$f(x) = \frac{bc}{a} (\beta - \alpha)^{-c} \left(\frac{x}{a} \right)^{b-1} \left[\beta - \left(\frac{x}{a} \right)^b \right]^{c-1}, \quad a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$$

3. The SPBD(1, $a, 1, b, 0, 1$) is the Kumaraswamy distribution, Kumaraswamy [14], with pdf;

$$f(x) = abx^{a-1} [1 - x^a]^{b-1}, \quad 0 < x < 1$$

4. The SPBD($a, b, p, q, 0, 1$) is the generalized beta of the first kind distribution, McDonald [15], with pdf;

$$f(x) = \frac{|a|}{b^{ap} B(p, q)} x^{ap-1} \left[1 - \left(\frac{x}{a} \right)^b \right]^{q-1}, \quad 0 < x < 1$$

5. The SPBD(1, 1, 1, 1, 0, 1) is the standard uniform distribution with pdf;

$$f(x) = 1, \quad 0 < x < 1$$

6. The SPBD(1, 2, 1, 1, 0, 1) is the triangular distribution with pdf;

$$f(x) = 2x, \quad 0 < x < 1$$

7. The SPBD(1, 1, 1, δ, α, β) is the power function distribution with pdf;

$$f(x) = \frac{\delta}{(\beta - \alpha)} \left[\frac{\beta - x}{\beta - \alpha} \right]^{\delta-1}, \quad \alpha < x < \beta$$

8. The SPBD($\lambda, a\theta, 1, b, 0, 1$) is the Kumaraswamy power function distribution, Abdul-Moniem [16], with pdf;

$$f(x) = \frac{ab\theta}{\lambda} \left(\frac{x}{\lambda}\right)^{a\theta-1} \left[1 - \left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1}, \quad 0 < x < \lambda$$

9. The SPBD($1, \beta, 1, \gamma, 0, 1$) is the Minimax distribution, McDonald [15], with pdf;

$$f(x) = \beta\gamma x^{\beta-1} [1 - x^\beta]^{\gamma-1}, \quad 0 < x < 1$$

10. The SPBD($c, -\alpha, 1, \theta, 0, 1$) is the exponentiated Pareto distribution, Gupta et al. [17], with pdf;

$$f(x) = \theta\alpha c^\alpha x^{-(\alpha+1)} \left[1 - \left(\frac{x}{c}\right)^{-\alpha}\right]^{\theta-1}, \quad 0 < c < x$$

11. The SPBD($\alpha, \beta, 1, 1, 0, 1$) is the generalized uniform distribution, Tiwari et al. [18], with pdf;

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1}, \quad 0 < x < \alpha$$

We may note that the Kumaraswamy (Case 4), standard uniform (Case 5), triangular (Case 6), Kumaraswamy power function (Case 8), minimax (Case 9), Pareto (Case 10), and the generalized uniform (Case 11) distribution are all special cases of the generalized beta of the first kind distribution (Case 4).

3.8 Transformations

Lemma 2

1. Let the rv U has the standard uniform distribution, $U(0, 1)$, and the rv X defined by $X = a[A + (B - A)I^{-1}(U, \alpha, \beta)]^{\frac{1}{\beta}}$, then $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$.
2. Let the rv $X \sim \text{SPBD}(1, 1, 1, 1, 0, 1)$, then the rv Y defined by $Y = \left(\frac{1-X}{X}\right)^{\frac{1}{\delta}} \cdot e^{-\frac{\gamma}{\delta}}$ has a log-logistic distribution with parameters δ and γ , Johnson et al. [4, p. 151], with CDF given by;

$$F_Y(y) = 1 - [1 + y^\delta e^\gamma]^{-1}, \quad y \geq 0$$

3. Let the rv $X \sim \text{SPBD}(a, b, 1, c, A, B)$, then the rv Y defined by $Y = B - \left(\frac{x}{a}\right)^b$ has the generalized uniform distribution, Tiwari et al. [18], with CDF given by;

$$F_Y(y) = \left[\frac{y}{B-A}\right]^c, \quad 0 \leq y \leq B-A$$

4. Let the rv $X \sim \text{SPBD}(a, b, 1, c, A, B)$ then the rv Y defined by $Y = (B-A)^{-1} \left[B - \left(\frac{x}{a}\right)^b\right]$ has the beta distribution with parameters 1 and c , with CDF given by;

$$F_Y(y) = y^c, \quad 0 \leq y \leq 1$$

5. Let the rv $X \sim \text{SPBD}(1, b, 1, 1, 0, 1)$, then the rv Y defined by $Y = \theta - b^2 \log(X)$ has an exponential distribution with parameters θ and b , Johnson et al. [5, p. 494], with CDF given by;

$$F_Y(y) = 1 - e^{-\left(\frac{y-\theta}{b}\right)}, \quad x > \theta$$

6. Let the rv $X \sim \text{SPBD}(1, b, 1, 1, 0, 1)$, then the rv Y defined by $Y = \mu - \beta \log[-\log(X^b)]$ has a Gumbel (generalized extreme value type-I) distribution with parameters μ and β , Forbes et al. [19, p. 98], with CDF given by;

$$F_Y(y) = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}}$$

7. Let the rv $X \sim \text{SPBD}(1, b, 1, 1, 0, 1)$, then the rv Y defined by $Y = \alpha + \beta \log\left(\frac{X^b}{1-X^b}\right)$ has a logistic distribution with parameters α and β , Johnson et al. [4, p. 115], with CDF given by;

$$F_Y(y) = \left[1 + e^{-\left(\frac{y-\alpha}{\beta}\right)}\right]^{-1}$$

8. Let the rv $X \sim \text{SPBD}(1, b, 1, 1, 0, 1)$, then the rv Y defined by $Y = \frac{k}{X}$, where k is a positive constant, has a Pareto distribution with parameters k and b , Johnson et al. [4, p. 574], with CDF given by;

$$F_Y(y) = 1 - \left(\frac{k}{y}\right)^b$$

9. Let the rv $X \sim \text{SPBD}(1, b, 1, 1, 0, 1)$, then the rv Y defined by $Y = \xi + \alpha [-\log(X^b)]^{\frac{1}{\theta}}$ has a Weibull distribution with parameters ξ , α , and θ , Johnson et al. [4, p. 629], with its CDF given by;

$$F_Y(y) = 1 - e^{-\left(\frac{y-\xi}{\alpha}\right)^{\theta}}$$

Proof For case (1), we have;

$$\begin{aligned} F_X(x) &= P(X \leq x) = P\left(a[A + (B-A)I^{-1}(U; \alpha, \beta)]^{\frac{1}{b}} \leq x\right) \\ &= P\left(I^{-1}(U; \alpha, \beta) \leq \frac{\left(\frac{x}{a}\right)^b - A}{B-A}\right) = P\left(U \leq I\left(\frac{\left(\frac{x}{a}\right)^b - A}{B-A}; \alpha, \beta\right)\right) = I\left(\frac{\left(\frac{x}{a}\right)^b - A}{B-A}; \alpha, \beta\right) \end{aligned}$$

Therefore, the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$.

Proof of cases (2) through (9) can be shown on the same lines as the proof of (1).

We may note that the SPBDs stated in Cases 2, 5, 6, 8, and 9 are all special cases of the generalized beta of the first kind distribution (see Case 4 of Sect. 3.7). $\square$

3.9 Scaling Property

Proposition 3 (The SPBD is closed under scaling) *Let the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ and let the rv $Y = cX$, where $0 < c < \infty$, then $Y \sim \text{SPBD}(ca, b, \alpha, \beta, A, B)$.*

$$F_Y(y) = P(Y \leq y) = P(cX \leq y) = P\left(X \leq \frac{y}{c}\right) = F_X\left(\frac{y}{c}; a, b, \alpha, \beta, A, B\right)$$

Proof

Therefore, using (13), we have that;

$$F_Y(y) = I\left(\frac{\left(\frac{y}{c}\right)^b - A}{B - A}; \alpha, \beta\right) = I\left(\frac{\left(\frac{y}{ca}\right)^b - A}{B - A}; \alpha, \beta\right) = F_X(y; ca, b, \alpha, \beta, A, B)$$

On the same lines as the proof of Proposition 3, we can prove the following Propositions 4 and 5. $\square$

3.10 Exponentiation Property

Proposition 4 (The SPBD is closed under exponentiation) *Let the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ and let the rv $Y = X^c$, where $0 < c < \infty$, then $Y \sim \text{SPBD}\left(a^c, \frac{b}{c}, \alpha, \beta, A, B\right)$.*

3.11 Reflection Symmetry Property

Proposition 5 *Let the rv $X \sim \text{SPBD}(a, 1, \alpha, \beta, A, B)$ and let the rv $Y = a(B + A) - X$, then $Y \sim \text{SPBD}(a, 1, \beta, \alpha, A, B)$.*

3.12 Generate SPBD Random Variates

Using result Lemma 2(1), we can generate $\text{SPBD}(a, b, \alpha, \beta, A, B)$ random variates as follows;

1. Generate $u \sim U(0, 1)$.
2. Compute $y = I^{-1}(U, \alpha, \beta)$.
3. Set $x = a[A + (B - A)y]^{\frac{1}{b}}$.

3.13 Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from $\text{SPBD}(a, b, \alpha, \beta, A, B)$, with pdf f and CDF F , and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be their order statistics, then for $i = 1, 2, 3, \dots, n$, the pdf of i -th order statistics $X_{i:n}$, $f_{i:n}(x)$, is given by;

$$f_{i:n}(x) = \begin{cases} \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i}, & a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}} \\ 0 & \text{otherwise,} \end{cases}$$

Hence, for $a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$, and using the fact that;

$$[1 - F(x)]^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j [F(x)]^j$$

we have that;

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j [F(x)]^j \\ &= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \left[I \left(\frac{\left(\frac{x}{a}\right)^b - A}{B - A}; \alpha, \beta \right) \right]^{i+j-1} \end{aligned}$$

Then the pdf of the rv $X_{i:n}$, $f_{i:n}$, can be written as;

$$f_{i:n}(x) = f(x) \sum_{j=0}^{i-1} A(i, j; n) \left[I \left(\frac{\left(\frac{x}{a}\right)^b - A}{B - A}; \alpha, \beta \right) \right]^{i+j-1} \quad (23)$$

where

$$A(i, j; n) = (-1)^j \frac{n!}{(i-1)!(n-i-j)!j!}.$$

3.14 Moments about the Origin

Let $k = 1, 2, 3, \dots$, then the moment of the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ of order k about zero is given by;

$$E(X^k) = \int_{-\infty}^{+\infty} x^k f(x) dx = \int_{aA^{\frac{1}{b}}}^{aB^{\frac{1}{b}}} \frac{bx^{k+p-1}}{a^b(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \left[\left(\frac{x}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a}\right)^b \right]^{\beta-1} dx$$

and by using the transformation $y = \left(\frac{x}{a}\right)^b$, it reduces to;

$$E(X^k) = \frac{a^k}{(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \int_A^B y^{\frac{k}{b}} (y-A)^{\alpha-1} (B-y)^{\beta-1} dy$$

from which we have that;

$$E(X^k) = a^k A^{\frac{k}{b}} \Gamma(\alpha + \beta) \tilde{F}_{(2,1)}\left(-\frac{k}{b}, \alpha; \alpha + \beta; 1 - \frac{B}{A}\right) \quad (24)$$

where $\tilde{F}_{(2,1)}$ is the regularized hypergeometric function, Virchenko et al. [20], defined by;

$$\tilde{F}_{(2,1)}(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{\Gamma(c+m)m!} z^m$$

and $(a)_m$ is as defined by (7).

Note that, in case that $A = 0$ then;

$$E(X^k) = a^k B^{\frac{k}{b}} \frac{\Gamma\left(\alpha + \frac{k}{b}\right) \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma\left(\alpha + \beta + \frac{k}{b}\right)}.$$

3.15 Mean and Variance

Using (24), the mean of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ is given by;

$$E(X) = aA^{\frac{1}{b}} \Gamma(\alpha + \beta) \tilde{F}_{(2,1)}\left(-\frac{1}{b}, \alpha; \alpha + \beta; 1 - \frac{B}{A}\right) \quad (25)$$

And the variance;

$$\text{Var}(X) = a^2 A^{\frac{2}{b}} \Gamma(\alpha + \beta) \left\{ \tilde{F}_{(2,1)}\left(-\frac{2}{b}, \alpha; \alpha + \beta; 1 - \frac{B}{A}\right) - \Gamma(\alpha + \beta) \left[\tilde{F}_{(2,1)}\left(-\frac{1}{b}, \alpha; \alpha + \beta; 1 - \frac{B}{A}\right) \right]^2 \right\}$$

Table 2 represents the mean, median, mode and variance of the selected SPBD data sets that are given in Table 1.

3.16 The Moment Generating Function

Similarly, the moment generating function of the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, $M_X(t)$, can be found to be;

$$M_X(t) = E(e^{tX}) = \Gamma(\alpha + \beta) \sum_{i=0}^{\infty} \frac{a^i A^{\frac{i}{b}}}{i!} \tilde{F}_{(2,1)}\left(-\frac{i}{b}, \alpha; \alpha + \beta; 1 - \frac{B}{A}\right) t^i.$$

Table 2 Mean, median, mode and variance of the selected SPBD data sets

Data set	Variable range		Moments			
	Minimum	Maximum	Mean	Median	Mode	Variance
1	0	1.8	0.94632	0.955627	0.984679	0.096145
2	0.38702	1.546834	0.942554	0.953689	1.01134	0.069825
3	0.673388	1.692171	0.969019	0.948343	–	0.048677
4	1.17346046	3.143355	1.80987	1.76616	1.60839	0.135081
5	1.122462	3.264052	2.36608	2.39793	2.53353	0.215075
6	0.261047	2.999082	2.14465	2.31082	–	0.501776

3.17 Harmonic Mean

The harmonic mean of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, on the same lines as that of the moment of X , is given by;

$$E\left(\frac{1}{X}\right) = \frac{\Gamma(\alpha + \beta)}{aA^{\frac{1}{b}}} \tilde{F}_{(2,1)}\left(\frac{1}{b}, \alpha, \alpha + \beta; 1 - \frac{B}{A}\right).$$

3.18 Incomplete Moments

The k -th incomplete moment of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, $I(z, k)$, is defined by;

$$I(z, k) = \int_{-\infty}^z x^k f(x) dx$$

Using the form of the pdf of X given in (15), then;

$$I(z, k) = \int_{aA^{\frac{1}{b}}}^z x^k \frac{bB^{\beta-1}}{(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C(i, j; a, b, \alpha, \beta, A, B) x^{b(\alpha+j-i)-1} dx$$

from which we have that;

$$I(z, k) = \frac{bB^{\beta-1}}{(B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{C(i, j; a, b, \alpha, \beta, A, B)}{k + b(\alpha + j - i)} \left[z^{k+b(\alpha+j-i)} - (aA^{\frac{1}{b}})^{k+b(\alpha+j-i)} \right]. \quad (26)$$

3.19 Mean Deviations

The mean deviation of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ about its mean $\mu = E(X)$, $MD(\mu)$, is given by;

$$MD(\mu) = E|X - \mu|$$

which can be found, Cordeiro et al. [21], to be;

$$MD(\mu) = 2[\mu F(\mu) - I(\mu, 1)]$$

Hence, using (13) and (26), for the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, we have that;

$$MD(\mu) = 2 \left[\mu I \left(\left(\frac{\mu}{a} \right)^b - A, \alpha, \beta \right) - \frac{bB^{\beta-1}}{(B-A)^{\alpha+\beta-1}B(\alpha, \beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{C(i, j; a, b, \alpha, \beta, A, B)}{k + b(\alpha + j - i)} \left[\mu^{k+b(\alpha+j-i)} - (aA^{\frac{1}{b}})^{k+b(\alpha+j-i)} \right] \right]$$

where μ is given from (25).

Similarly, the mean deviation of X about its median m , $MD(m)$, is given by;

$$MD(m) = aA^{\frac{1}{b}} \Gamma(\alpha + \beta) \tilde{F}_{(2,1)} \left(-\frac{1}{b}, \alpha, \alpha + \beta; 1 - \frac{B}{A} \right) - 2I \left(\left(\frac{m}{a} \right)^b - A, \alpha, \beta \right).$$

3.20 Probability Weighted Moments

The probability weighted moments of order s and r of $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, $\rho_{s,r}$, is given by;

$$\rho_{s,r} = E(X^s [f(X)]^r)$$

Using the fact that;

$$[f(x; a, b, \alpha, \beta, A, B)]^r = \frac{x^{(b-1)(r-1)} B((\alpha-1)r+1, (\beta-1)r+1)}{a^{b(r-1)} b^{(1-r)} [B(\alpha, \beta)]^r (B-A)^{r-1}} f(x; a, b, (\alpha-1)r+1, (\beta-1)r+1, A, B)$$

with the use of (24), we have that;

$$\rho_{s,r} = \frac{b^r (aA^{\frac{1}{b}})^{\frac{s+r(b-1)}{b}} \Gamma[(\alpha-1)(r+1)+1] \Gamma[(\beta-1)(r+1)+1]}{a^{r-s} [B(\alpha, \beta)]^{r+1} (B-A)^r} \tilde{F}_{(2,1)} \left(-\frac{s+r(b-1)}{b}, (\alpha-1)(r+1)+1, (r+1)(\alpha+\beta-2)+2; 1 - \frac{B}{A} \right).$$

3.21 Renyi Entropy

Let us compute the Renyi entropy as a measure of variation of the uncertainty of the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$. For $\theta > 0$ such that $\theta \neq 1$, we have for the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$ that;

$$I_X(\theta) = \frac{1}{1-\theta} \log \int_{-\infty}^{+\infty} [f(x)]^\theta dx \quad (27)$$

First, we note that;

$$\begin{aligned} [f(x; a, b, \alpha, \beta, A, B)]^\theta &= \left\{ \frac{bx^{b-1}}{a^b(B-A)^{\alpha+\beta-1}B(\alpha, \beta)} \left[\left(\frac{x}{a}\right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x}{a}\right)^b \right]^{\beta-1} \right\}^\theta \\ &= \frac{x^{(b-1)(\theta-1)} B((\alpha-1)\theta+1, (\beta-1)\theta+1)}{a^{b(\theta-1)} b^{(1-\theta)} [B(\alpha, \beta)]^\theta (B-A)^{\theta-1}} f(x; a, b, (\alpha-1)\theta+1, (\beta-1)\theta+1, A, B) \end{aligned}$$

Therefore,

$$I_X(\theta) = \frac{1}{1-\theta} \log \left\{ \left[\frac{B((\alpha-1)\theta+1, (\beta-1)\theta+1)}{a^{b(\theta-1)} b^{(1-\theta)} [B(\alpha, \beta)]^\theta (B-A)^{\theta-1}} \right] E[Y^{(b-1)(\theta-1)}] \right\}$$

where $Y \sim \text{SPBD}(a, b, (\alpha-1)\theta+1, (\beta-1)\theta+1, A, B)$. It follows, using (24) that;

$$\begin{aligned} I_X(\theta) &= \frac{1}{1-\theta} \log \left\{ \frac{B((\alpha-1)\theta+1, (\beta-1)\theta+1)}{a^{b(\theta-1)} b^{(1-\theta)} [B(\alpha, \beta)]^\theta (B-A)^{\theta-1}} \Gamma[(\alpha+\beta-1)\theta+2] \right. \\ &\quad \left. \tilde{F}_{(2,1)} \left(-\frac{(b-1)(\theta-1)}{b}, (\alpha-1)\theta+1, (\alpha-1)\theta+(\beta-1)\theta+2; 1-\frac{B}{A} \right) \right\}. \end{aligned}$$

3.22 Lorenz and Bonferroni Curves

For $0 < \pi < 1$, the Lorenz curve, $L(\pi)$, and Bonferroni curves, $B(\pi)$, for the rv $X \sim \text{SPBD}(a, b, \alpha, \beta, A, B)$, are given by, respectively;

$$L(\pi) = \frac{I(Q(\pi), 1)}{\mu}$$

and

$$B(\pi) = \frac{I(Q(\pi), 1)}{\pi \mu}$$

where $Q(\pi)$ is the quantile function of the rv X at π , and $I(z, k)$ is the incomplete moment of the rv X . Therefore, using (25) and (26), we have that;

$$L(\pi) = \frac{bB^{\beta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{C(i,j;a,b,\alpha,\beta,A,B)}{b(\alpha+j-i)+1} \left[(Q(\pi))^{b(\alpha+j-i)+1} - (aA^{\frac{1}{b}})^{b(\alpha+j-i)+1} \right]}{aA^{\frac{1}{b}} (B-A)^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta) \tilde{F}_{(2,1)} \left(-\frac{1}{b}, \alpha, \alpha + \beta; 1 - \frac{B}{A} \right)}$$

And similarly, that;

$$B(\pi) = \frac{bB^{\beta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{C(i,j;a,b,\alpha,\beta,A,B)}{b(\alpha+j-i)+1} \left[(Q(\pi))^{b(\alpha+j-i)+1} - \left(aA^{\frac{1}{b}} \right)^{b(\alpha+j-i)+1} \right]}{a\pi A^{\frac{1}{b}} (B-A)^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta) \tilde{F}_{(2,1)} \left(-\frac{1}{b}, \alpha, \alpha + \beta; 1 - \frac{B}{A} \right)}.$$

4 Parameters Estimation of the SPBD

The maximum likelihood estimation (MLE) method will be used for estimating the parameters of the SPBD. Let $x_1, x_2, \dots, x_n$ be a random sample from SPBD($a, b, \alpha, \beta, A, B$), as given by (9), then we want to estimate the parameters a, b, α, β, A , and B by maximizing the log-likelihood function, where the likelihood function $L = L(a, b, \alpha, \beta, A, B; x_1, x_2, \dots, x_n)$ can be written as;

$$L = \prod_{i=1}^n f(x_i) = \left[\frac{b}{a^b (B-A)^{\alpha+\beta-1} B(\alpha, \beta)} \right]^n \prod_{i=1}^n \left\{ x_i^{b-1} \left[\left(\frac{x_i}{a} \right)^b - A \right]^{\alpha-1} \left[B - \left(\frac{x_i}{a} \right)^b \right]^{\beta-1} \right\}$$

Let us inspect the normal equations $\frac{\partial}{\partial a} \log L = 0, \frac{\partial}{\partial b} \log L = 0, \dots, \frac{\partial}{\partial B} \log L = 0$, to see if they admit an explicit solution. We have that;

$$\frac{\partial}{\partial a} \log L = -(\alpha-1) \frac{b}{a} \sum_{i=1}^n \frac{\left(\frac{x_i}{a} \right)^b}{\left(\frac{x_i}{a} \right)^b - A} + (\beta-1) \frac{b}{a} \sum_{i=1}^n \frac{\left(\frac{x_i}{a} \right)^b}{B - \left(\frac{x_i}{a} \right)^b} - \frac{nb}{a} = 0, \quad (28)$$

$$\frac{\partial}{\partial b} \log L = \frac{n}{b} + \sum_{i=1}^n \log(x_i) + (\alpha-1) \sum_{i=1}^n \frac{\left(\frac{x_i}{a} \right)^b \log\left(\frac{x_i}{a} \right)}{\left(\frac{x_i}{a} \right)^b - A} - (\beta-1) \sum_{i=1}^n \frac{\left(\frac{x_i}{a} \right)^b \log\left(\frac{x_i}{a} \right)}{B - \left(\frac{x_i}{a} \right)^b} - n \log(a) = 0, \quad (29)$$

$$\frac{\partial}{\partial \alpha} \log L = \sum_{i=1}^n \log \left[\left(\frac{x_i}{a} \right)^b - A \right] - n \log(B-A) - n[\psi(\alpha) - \psi(\alpha + \beta)] = 0, \quad (30)$$

where ψ is the digamma function, Abramowitz and Stegun [6, p. 258],

$$\frac{\partial}{\partial \beta} \log L = \sum_{i=1}^n \log \left[B - \left(\frac{x_i}{a} \right)^b \right] - n \log(B-A) - n[\psi(\beta) - \psi(\alpha + \beta)] = 0, \quad (31)$$

and, since $aA^{\frac{1}{b}} < x < aB^{\frac{1}{b}}$, then the MLE of $aA^{\frac{1}{b}}$ and $aB^{\frac{1}{b}}$ are; respectively, $x_{1:n}$ and $x_{n:n}$; that is $\hat{a}\hat{A}^{\frac{1}{\hat{b}}} = x_{1:n}$ and $\hat{a}\hat{B}^{\frac{1}{\hat{b}}} = x_{n:n}$, and hence;

$$\hat{A} = \left(\frac{x_{1:n}}{\hat{a}} \right)^{\hat{b}}, \quad (32)$$

and

$$\hat{B} = \left(\frac{x_{n:n}}{\hat{a}} \right)^{\hat{b}} \quad (33)$$

Since Eqs. (28)–(33) are not easy to be solved explicitly, numerical technique, as Newton Rapson method or any other well-known optimization algorithm, see Shi et al. [22], may be employed to do so, or to use a well-known software package, such as maxLik, Henningsen and Toomet [23], or GAMLSS, Stasinopoulos and Rigby [24], to find the MLE of the parameters of the SPBD.

5 A Simulation Study

In order to examine the performance of the MLE method given in Sect. 4, we perform a simulation study to do so. The bias and the mean squares errors (MSE) of the estimates are the principle measures of the performance.

The statistical software R and the Absoft Pro Fortran compiler are employed for computing. The maxLik package of the statistical software R is used mainly for computing the MLEs, see Henningsen and Toomet [23] for details of this package, while the Absoft Pro Fortran is used for other needed computations.

The six miscellaneous SPBD models given in Table 1, that have different pdf's shapes and variable ranges, will be used to simulated data sets for each model, and for each data set, the bias and the MSE are computed for the MLE of the model parameters for different simulated sample sizes. The sample sizes that will be taken are 25, 50, 100, 300, 500, and 1000. In each situation, the parameters of, θ say, the first model of the six SPBD models given in Table 1, are estimated from 5000 random variates generated from the given SPBD model, and the sample mean, bias, variance, and the MSE for the parameters are computed as; $Mean(\hat{\theta}) = \frac{1}{5000} \sum_{i=1}^{5000} \hat{\theta}_i = \bar{\bar{\theta}}$ say, $Bias(\hat{\theta}) = \bar{\bar{\theta}} - \theta$, $Var(\bar{\bar{\theta}}) = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\theta}_i - \bar{\bar{\theta}})^2$, and hence $MSE(\bar{\bar{\theta}}) = Var(\bar{\bar{\theta}}) + [Bias(\hat{\theta})]^2$. This procedure is repeated for each sample size, then repeated for each SPBD model.

Table 3 shows the bias of the estimated parameters of the different simulated SPBD data sets for each sample size, while Table 4 presents the MSE of the estimated parameters of the different simulated SPBD data sets for each sample size. Both Tables 3 and 4 show, for each of the SPBD model parameters, that the bias and MSE decreases as the sample size increases. Figure 3 shows the behaviour of the MSE plots of the estimated parameters for six the SPBD simulated data sets, which shows graphically, for of the SPBD model parameters, that the

Table 3 The bias of the estimated parameters of the simulated SPBD data sets for each sample size n

n	Actual value					Bias					$\hat{A}$	$\hat{B}$
	a	b	α	β	A	B	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$		
25	1.8	2.3	1.4	3.9	0	1	-0.34878	0.329715	0.013104	0.399618	0.420087	0.601777
	1.5	3.1	0.93	2.65	0.015	1.1	0.140342	-0.0969	0.056847	-0.05489	0.464818	-0.48701
	1.5	5.75	0.43	5.65	0.01	2	-0.37808	-0.27911	0.315561	-0.30242	-0.00807	-0.19319
	2	1.3	1.6	3.8	0.5	1.8	0.039897	-0.41322	0.612345	0.366937	0.149	-0.14444
	2	1.2	2.3	1.8	0.5	1.8	-0.18231	-0.25249	0.334888	0.198619	-0.11285	0.232156
	2	0.45	2.15	0.65	0.4	1.2	0.396452	0.403079	-0.34372	0.335451	-0.1812	0.263205
50	1.8	2.3	1.4	3.9	0	1	0.512223	-0.29606	0.58745	-0.58925	0.451566	-0.34638
	1.5	3.1	0.93	2.65	0.015	1.1	0.414662	-0.15015	0.687373	-0.73152	0.294003	-0.49454
	1.5	5.75	0.43	5.65	0.01	2	0.070608	0.459072	0.278546	-0.12252	0.102261	0.156337
	2	1.3	1.6	3.8	0.5	1.8	0.153985	-0.4023	-0.08337	0.381255	-0.2772	-0.32743
	2	1.2	2.3	1.8	0.5	1.8	0.111967	0.199792	0.105013	-0.2507	0.080938	0.051812
	2	0.45	2.15	0.65	0.4	1.2	0.275	-0.28298	-0.36704	-0.26281	-0.14315	-0.26767
100	1.8	2.3	1.4	3.9	0	1	-0.10525	0.227391	-0.09405	-0.67146	0.4522	0.262165
	1.5	3.1	0.93	2.65	0.015	1.1	0.019408	-0.24609	-0.13464	0.094321	0.033752	-0.31785
	1.5	5.75	0.43	5.65	0.01	2	0.153569	-0.36599	-0.40473	-0.06738	0.191963	-0.16552
	2	1.3	1.6	3.8	0.5	1.8	-0.07852	-0.21469	-0.08842	0.090949	0.160529	-0.20546
	2	1.2	2.3	1.8	0.5	1.8	0.431635	0.23417	0.037447	0.241197	-0.10608	0.343295
	2	0.45	2.15	0.65	0.4	1.2	-0.33975	0.166262	-0.02429	-0.4521	0.334471	0.360404
300	1.8	2.3	1.4	3.9	0	1	0.274458	0.049549	0.474342	0.345484	0.368289	-0.24068
	1.5	3.1	0.93	2.65	0.015	1.1	0.424818	0.026234	-0.15182	0.237266	-0.49699	0.312069
	1.5	5.75	0.43	5.65	0.01	2	0.091822	0.012653	-0.37405	-0.31214	-0.41615	0.322652
	2	1.3	1.6	3.8	0.5	1.8	-0.34078	-0.25521	0.336573	0.489306	0.169098	0.536246
	2	1.2	2.3	1.8	0.5	1.8	-0.21183	0.072752	-0.12231	-0.05324	-0.47379	0.106958
	2	0.45	2.15	0.65	0.4	1.2	-0.15788	0.177951	0.208611	0.352764	0.153525	-0.16458

Table 3 (continued)

n	Actual value			Bias									
	a	b		α	β	A	B	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{A}$	$\hat{B}$
500	1.8	2.3		1.4	3.9	0	1	-0.37304	0.278684	-0.31915	-0.19441	0.432184	0.377181
	1.5	3.1		0.93	2.65	0.015	1.1	0.373571	-0.40604	-0.38178	-0.47487	0.248089	0.219282
	1.5	5.75		0.43	5.65	0.01	2	-0.02872	-0.32082	0.380835	-0.06513	-0.05995	0.143485
	2	1.3		1.6	3.8	0.5	1.8	0.507451	-0.11285	0.159177	0.147718	0.223398	0.356879
	2	1.2		2.3	1.8	0.5	1.8	-0.36108	0.142093	0.102362	-0.36187	-0.50241	0.268284
	2	0.45		2.15	0.65	0.4	1.2	0.094234	0.134663	-0.22689	-0.2296	0.042163	-0.00386
1000	1.8	2.3		1.4	3.9	0	1	0.204721	-0.32209	0.257236	0.110389	0.104925	0.215005
	1.5	3.1		0.93	2.65	0.015	1.1	0.098276	0.118352	0.104907	0.002979	0.410095	-0.38891
	1.5	5.75		0.43	5.65	0.01	2	-0.14047	0.252312	-0.46269	0.190593	-0.14857	0.29962
	2	1.3		1.6	3.8	0.5	1.8	0.177988	0.01111	-0.31224	-0.04958	0.227256	-0.11569
	2	1.2		2.3	1.8	0.5	1.8	0.258915	0.015733	-0.3335	-0.37711	-0.35998	-0.04232
	2	0.45		2.15	0.65	0.4	1.2	0.060013	0.171233	0.137993	0.116084	0.389542	-0.13421

Table 4 The MSE of the estimated parameters of the simulated SPBD data sets for each sample size n

n	Actual value						MSE					
	a	b	α	β	A	B	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{A}$	$\hat{B}$
25	1.8	2.3	1.4	3.9	0	1	1.342013	1.954471	0.932849	1.895358	2.084947	1.762417
	1.5	3.1	0.93	2.65	0.015	1.1	1.402194	2.396725	0.793264	1.629038	1.911926	2.325153
	1.5	5.75	0.43	5.65	0.01	2	1.477281	1.833293	1.406634	2.277118	1.563066	1.61373
	2	1.3	1.6	3.8	0.5	1.8	1.640697	2.313115	0.900516	1.306511	1.6181	1.904378
	2	1.2	2.3	1.8	0.5	1.8	0.944398	2.117512	1.005152	2.534064	1.15113	1.954144
	2	0.45	2.15	0.65	0.4	1.2	1.342938	2.332513	1.467529	1.019239	1.239475	1.407573
50	1.8	2.3	1.4	3.9	0	1	1.170272	1.657704	0.799091	1.401564	1.755979	1.654412
	1.5	3.1	0.93	2.65	0.015	1.1	1.132947	1.743292	0.598261	1.309235	1.681494	1.776936
	1.5	5.75	0.43	5.65	0.01	2	1.324648	1.781486	1.32801	2.060954	1.019974	1.361083
	2	1.3	1.6	3.8	0.5	1.8	1.31652	1.99784	0.701776	1.157632	1.311115	1.187063
	2	1.2	2.3	1.8	0.5	1.8	0.785298	1.910105	0.862128	2.282315	1.021342	1.489123
	2	0.45	2.15	0.65	0.4	1.2	0.993972	2.011772	1.162289	0.873739	1.157483	1.251442
100	1.8	2.3	1.4	3.9	0	1	0.756539	1.210055	0.607076	0.992914	1.460646	1.345788
	1.5	3.1	0.93	2.65	0.015	1.1	0.88934	1.199509	0.404822	1.201444	1.357685	1.206554
	1.5	5.75	0.43	5.65	0.01	2	1.117483	1.529878	1.037574	1.595888	0.829085	1.121212
	2	1.3	1.6	3.8	0.5	1.8	0.998634	1.752594	0.537625	0.828906	1.06102	0.939682
	2	1.2	2.3	1.8	0.5	1.8	0.565906	1.546855	0.705282	2.017984	0.717713	1.153898
	2	0.45	2.15	0.65	0.4	1.2	0.805249	1.722256	0.777865	0.696402	0.851498	1.018501
300	1.8	2.3	1.4	3.9	0	1	0.547782	0.852198	0.401467	0.644066	0.773053	1.007341
	1.5	3.1	0.93	2.65	0.015	1.1	0.711853	0.781424	0.269156	0.788936	0.916783	0.831953
	1.5	5.75	0.43	5.65	0.01	2	0.910827	1.280392	0.930746	1.260824	0.410875	0.789677
	2	1.3	1.6	3.8	0.5	1.8	0.609068	1.387463	0.38542	0.5643	0.865376	0.664865
	2	1.2	2.3	1.8	0.5	1.8	0.412669	1.172685	0.475687	1.818785	0.500591	0.891926
	2	0.45	2.15	0.65	0.4	1.2	0.483891	1.390424	0.570112	0.455555	0.530735	0.691226

Table 4 (continued)

n	Actual value				MSE							
	a	b	α	β	A	B	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{A}$	$\hat{B}$
500	1.8	2.3	1.4	3.9	0	1	0.427417	0.64704	0.332296	0.435204	0.200255	0.854437
	1.5	3.1	0.93	2.65	0.015	1.1	0.449441	0.434471	0.155422	0.575885	0.650976	0.650911
	1.5	5.75	0.43	5.65	0.01	2	0.7656	1.07478	0.704428	0.970532	0.325475	0.618804
	2	1.3	1.6	3.8	0.5	1.8	0.272669	1.19933	0.241967	0.271555	0.418878	0.422975
	2	1.2	2.3	1.8	0.5	1.8	0.278361	0.995957	0.321993	1.531532	0.367942	0.542543
1000	2	0.45	2.15	0.65	0.4	1.2	0.178667	0.82391	0.340492	0.202494	0.388572	0.487987
	1.8	2.3	1.4	3.9	0	1	0.354872	0.41543	0.213608	0.304466	0.101616	0.472349
	1.5	3.1	0.93	2.65	0.015	1.1	0.301129	0.205473	0.081764	0.27106	0.33213	0.376744
	1.5	5.75	0.43	5.65	0.01	2	0.645736	0.648055	0.658258	0.585563	0.148738	0.425584
	2	1.3	1.6	3.8	0.5	1.8	0.149529	0.628928	0.113052	0.243147	0.300697	0.311164
	2	1.2	2.3	1.8	0.5	1.8	0.178081	0.714394	0.16387	1.281251	0.198019	0.215385
	2	0.45	2.15	0.65	0.4	1.2	0.052826	0.547684	0.102232	0.057344	0.159444	0.255718

Fig. 3 Behaviour of the MSE plots of the estimated parameters for the SPBD simulated data sets

MSE decreases as the sample size increases. Hence, from the result, as the MLS plots decreases as the sample size increases, we may conclude that the MLE method seems to have high efficiency as the sample size become large.

Table 5 shows the actual values and the MLE parameter values (as the average values for the 5000 replications) of the different simulated SPBD data sets, and Fig. 4 shows visually their corresponding pdf's plots.

In conclusion, the simulation indicates that the MLE method is appropriate and can be used to estimate the parameters of the SPPBD models.

Table 5 Actual and MLE parameters values of the simulated SPBD data sets

Data set	Value	Parameters						Variable Range	
		a	b	α	β	A	B	Minimum	Maximum
1	Actual	1.8	2.3	1.4	3.9	0	1	0	1.8
	MLE	1.777778	2.339845	1.373519	3.877778	0.000111	1.01218	0.036285462	1.787000108
2	Actual	1.5	3.1	0.93	2.65	0.015	1.1	0.38702	1.546834
	MLE	1.435333	3.117889	0.923133	2.587889	0.019188	1.211111	0.403895566	1.526273077
3	Actual	1.5	5.75	0.43	5.65	0.01	2	0.673387689	1.69217121
	MLE	1.487889	5.855124	0.411889	5.444889	0.010134	2.227889	0.679167201	1.706033177
4	Actual	2	1.3	1.6	3.8	0.5	1.8	1.17346046	3.143355214
	MLE	1.993333	1.443825	1.473245	3.895556	0.46999	1.933889	1.18159022	3.147485177
5	Actual	2	1.2	2.3	1.8	0.5	1.8	1.122462048	3.264052108
	MLE	1.879889	1.213113	2.351245	1.778986	0.512333	1.933333	1.083200324	3.236995497
6	Actual	2	0.45	2.15	0.65	0.4	1.2	0.261047095	2.999081861
	MLE	2.054789	0.467478	2.035899	0.598789	0.397999	1.198758	0.286325174	3.028201867

6 Application of Fitting SPBD Model to Real-Life Data

We consider two real-life data sets in order to show the usefulness of the proposed estimation procedure to estimate and fit the SPBD model to these real-life data sets. The data sets are;

Data Set 1 Represents the waiting period of Muslim worshipers from the time of entering the mosque till the actual time of starting Alfajir pray (the early morning and first pray of the day) in Al-Mani Jamieh Mosque (Masjid no. 942), where Friday prayers are held and it accommodates more than two thousand worshipers, in Al-Waab town in Doha-Qatar. The data consists of 4539 observations recorded in this masjid for the period from 30th October 2017 till 15th January 2020. We will abbreviate this data set by main street mosque data.

Data Set 2 Represents the waiting period of Muslim worshipers from the time of entering the mosque till the actual time of starting Alfajir pray in Saeed bin Fahad Al-Dosari Mosque (Masjid no. 1031), where Friday prayers are not held and it accommodates no more than two hundred fifty worshipers, in Al-Waab town in Doha-Qatar. The data consists of 3360 observations recorded in this mosque for the period from 25th January 2015 to 20th October 2017. We will abbreviate this data set by within streets mosque data.

Table 6 presents some statistics of the observed mosque data sets.

Using both mosque data sets, the MLE method was employed to estimate the parameters of the SPBD model for each, and Table 7 shows the actual and the predicted frequencies, model parameters estimates, the Chi squares goodness of fit test for the SPB, the gamma, the exponential, the four parameters beta, and the generalized beta of the first kind distributions, as well as, the likelihood ratio test (LRT) for

Fig. 4 Plots of the actual and simulated SPBD pdf's

the nested models of the SPB distribution, namely; the four parameters beta, and the generalized beta of the first kind distributions. Figure 5, illustrating the histograms and the fitted pdfs for both main and within street mosque data sets. Now, for the main street data set case, since the p values of Chi squares goodness of fit test for the gamma, the exponential, the four parameters beta, and the generalized beta of the first kind distributions, is smaller than 0.05, and that the p value of the SPBD model equals to 0.9488, the SPBD performs better than all these distributions. Although, for the within street mosque data set, the Chi squares goodness of fit test p value of the generalized beta of the first kind distribution equals to 0.23087 indicating that this distribution can fit this data, the SPBD model perform better in this case since its p value equals to 0.96088, and since the p values of Chi squares goodness of fit test for the gamma, the exponential, and the four parameters beta, is smaller than 0.05, the SPBD performs better than all these distributions also. Next, the p values of the likelihood ratio test (LRT) for the nested models of the SPB distribution,

Table 6 Some statistics of the observed mosque data sets

Statistics	Observed	
	Main	Within
No. of observation	4539	3360
Mean	7.0986	5.2372
Standard error of mean	0.08194	0.07554
Median	5.65258	4.4779
Mode	0.685519	0.6196016
SD	5.52032	4.37859
Variance	19.172	30.474
Skewness	0.706	1.162
Standard error of skewness	0.036	0.042
Kurtosis	−0.378	1.161
Standard error of kurtosis	0.073	0.084
Minimum	0.07	0.08
Maximum	24.9	24.9
<i>Percentiles</i>		
25	2.44012	1.46321
50	5.5	4.4779
75	10.64123	7.60603

namely; the four parameters beta, and the generalized beta of the first kind distributions, are less than 0.05, indicating statistically, that SPBD preforms better, in both main and within street data sets. These finding indicates that the SPBD outperforms the gamma, exponential, the four parameters beta, and the generalized beta of the first kind distributions and provides the best fit for both main and within mosque data sets.

7 Summary

A new six parameters beta distribution is introduced, which has a more flexible shape and a wide bounded domain than the than the two (standard) and the four parameters beta distributions, and its properties consisting of, and some of its different various shapes are given to show its flexibility. Its boundaries, limits, mode, quantities, reliability and hazard functions, Renyi entropy, Lorenz and Bonferroni curves are studied. This distribution is closed under scaling and exponentiation, and has reflection symmetry property, and has some well-known distributions as special cases, such as, the two and four parameters beta, generalized modification of the Kumaraswamy, generalized beta of the first kind, the power function, Kumaraswamy power function, Minimax, exponentiated Pareto, and the generalized uniform distributions. Its order statistics, moment generating function, with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted moments, and mean deviations are derived. The maximum likelihood estimation method is used

Table 7 Observed and predicted frequencies, model parameters estimates and goodness of fit for mosque data sets

Data range	Main streets mosque						Within streets mosque					
	Observed			Predicted			Observed			Predicted		
	Proposed 6 parameters beta	Gamma	Exponential	4 parameters beta	Generalized Data of the first kind		Proposed 6 parameters beta	Gamma	Exponential	4 parameters beta	Generalized Data of the first kind	
0.0–1.0	599	589	259	596	533	523	517	380	583	611	536	
1.1–2.0	405	412	417	518	426	430	428	465	482	416	430	
2.1–3.0	373	362	460	450	375	372	374	441	398	346	367	
3.1–4.0	323	330	453	391	339	330	329	388	329	297	317	
4.1–5.0	310	306	423	339	310	285	288	329	272	257	276	
5.1–6.0	291	284	382	295	285	250	251	273	224	224	240	
6.1–7.0	260	265	337	256	263	215	217	223	185	196	208	
7.1–8.0	243	247	294	222	242	182	186	181	153	171	179	
8.1–9.0	219	229	253	193	223	160	158	145	127	148	154	
9.1–10.0	221	212	215	168	205	129	133	116	105	128	131	
10.1–11.0	188	195	182	146	188	110	111	92	86	110	111	
11.1–12.0	185	178	153	126	172	86	91	73	71	94	93	
12.1–13.0	151	162	128	110	156	75	73	58	59	80	77	
13.1–14.0	143	145	107	95	141	51	58	45	49	67	62	
14.1–15.0	135	128	89	83	126	49	45	35	40	55	50	
15.1–16.0	108	112	74	72	112	36	34	28	33	44	39	
16.1–16.0	99	96	61	62	99	26	25	22	27	35	30	
17.1–17.0	71	81	50	54	85	18	17	17	23	27	22	
18.1–18.0	68	66	41	47	72	13	11	13	19	20	16	
19.1–20.0	49	52	34	41	60	8	7	10	15	14	10	

Table 7 (continued)

Data range	Main streets mosque					Within streets mosque						
	Observed	Predicted	Gamma	Exponential	4 parameters beta	Generalized Beta of the first kind	Observed	Predicted	Gamma	Exponential	4 parameters beta	Generalized Beta of the first kind
			Proposed 6 parameters beta						Proposed 6 parameters beta			
20.1–21.0	44	38	28	36	45	48	5	4	8	13	10	6
21.1–22.0	24	26	23	31	35	36	3	2	6	11	6	3
22.1–23.0	18	16	19	27	25	25	2	1	5	9	3	2
23.1–24.0	8	7	15	23	15	14	1	0	4	7	1	1
24.1–25.0	4	1	42	158	5	4	1	0	3	40	0	0
Total	4539	4539	4539	4539	4539	4539	3360	3360	3360	3360	3360	3360
Model parameters	$\hat{a}$	25.01238	$\hat{\lambda}=0.213$	$\hat{\lambda}=0.141$	1	24.97111	a	24.20246	$\hat{\lambda}=0.273$	$\hat{\lambda}=0.191$	1	24.511778
	$\hat{b}$	2.023441	$\hat{\beta}=1.643$		1	0.999999	b	1.159141	$\hat{\beta}=1.431$		1	0.999999
	$\hat{\alpha}$	0.381451			0.7533	0.89295	α	0.799451			0.80945	0.915556
	$\hat{\beta}$	2.612456			2.00145	2.239999	β	3.901245			2.97825	3.31778
	$\hat{A}$	0.000012			0	0	A	0.000132			0	0
	$\hat{B}$	0.998912			25	1	B	1.038113			25	1
Goodness of Fit	χ^2	8.7141	736.85	416.134	53.1875	48.22419	χ^2	6.2113	101.441	112.324	52.4733	19.7677
	df	17*	21*	23	20*	18*	df	14*	21*	23	17*	16*
	p value	0.9488	0.0	0.0	0.00008	0.000141	p value	0.96088	0.0	0.0	0.000017	0.23087
Likelihood ratio test (nested)**	LRT				10.3345	8.76259	LRT				20.5047	7.61264
	df				2	2	df				2	2
	p value				0.0056	0.01251	p value				0.00004	0.02222

*The number of intervals were adjusted in order to make the expected number of observations in each interval equal to or greater than 5, which is in term effected the number of the degree of the freedom

**The 4 parameters beta distribution and the generalized beta of the first kind distribution are special cases of the SPBD, see Sect. 3.7 cases 1 and 4

Fig. 5 Histograms and the fitted pdfs for the Mosque data sets

for estimating its parameters and applied to estimate the parameters of six different simulated data sets of this distribution having different pdf shapes, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from different simulated sample sizes, which are shown to be decreasing as the sample size increases, indicating that the MLE method is appropriate and can be used to estimate the parameters of the SPPBD models. Finally, two real life data sets, represent the waiting period of Muslim worshipers from the time of entering the mosque till the actual time of starting Alfajir pray in two different mosques, are used in order to show the usefulness and the flexibility of this distribution in application to real-life data sets. The MLE method was employed using these data set to estimate the parameters of the SPBD, the gamma, the exponential, the four parameters beta, and the generalized beta of the first kind distributions, and the Chi squares goodness of fit test for these distributions, as well as, the LRT for the nested models of the SPB distribution, namely; the four parameters beta, and the generalized beta of the first kind distributions, were employed, and all the results through the p values of these tests, statistically, outperforms SPBDs over the other stated distributions.

Acknowledgements Open Access funding provided by the Qatar National Library. The publication of this article was funded by the Qatar National Library.

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