J. L. Cisneros-Molina , A. Menegon *Res Math Sci(2024)11:19*
https://doi.org/10.1007/s40687-024-00431-4

Research in
the Mathematical Sciences

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Milnor fibration theorem for differentiable maps

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The first author was supported by UNAM-DGAPA-PAPIIT-IN105121. The second author had partial support from CNPq (Brazil). Data sharing is not applicable to this article as no datasets were generated or analysed during the current study. The authors declare that there are no conflicts of interest

Abstract

In Cisneros-Molina et al. (São Paulo J Math Sci, 2023. https://doi.org/10.1007/s40863-023-00370-y) it was proved the existence of fibrations à la Milnor (in the tube and in the sphere) for real analytic maps $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$, where $n\ge k\ge 2$, with non-isolated critical values. In the present article we extend the existence of the fibrations given in Cisneros-Molina et al. (São Paulo J Math Sci, 2023. https://doi.org/10.1007/s40863-023-00370-y) to differentiable maps of class C^ℓ, $\ell\ge 2$, with possibly non-isolated critical value. This is done using a version of Ehresmann fibration theorem for differentiable maps of class C^ℓ between smooth manifolds, which is a generalization of the proof given by Wolf (Michigan Math J 11:65–70, 1964) of Ehresmann fibration theorem. We also present a detailed example of a non-analytic map which has the aforementioned fibrations.

Keywords: Milnor fibration, d-regularity, Differentiable singularities

Mathematics Subject Classification: Primary 14D06, 14B05, 14J17, 32S55, 58K05, 58K15

1 Introduction

One of the most important tools to study singularities of analytic maps and spaces is given by the fibration theorems à la Milnor [10,12,13,15,16].

In the case of a real analytic map $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$ with $n\ge k\ge 2$ and with an isolated critical value, it was proved in [18, Theorem 1.3] that if f has Thom's a_f property, one has fibration on the tube:

$$f\colon\mathbb{B}^n_\epsilon\cap f^{-1}(\mathbb{S}^{k-1}_\delta)\longrightarrow\mathbb{S}^{k-1}_\delta, \tag{1}$$

where $\mathbb{B}^n_\epsilon$ is the closed ball in $\mathbb{R}^n$ of radius ϵ around the origin. This is known as Milnor-Lê fibration. Moreover, using Milnor's vector field [16, Lemma 11.3] one also has an equivalent fibration on the sphere:

$$\phi\colon\mathbb{S}^{n-1}_\epsilon\setminus f^{-1}(0)\to\mathbb{S}^{k-1}, \tag{2}$$

where $\mathbb{S}^{k-1}$ is the sphere of radius 1 around $0\in\mathbb{R}^k$. However, there is no control on the projection map ϕ.

The question of whether we can take the projection ϕ of (2) to be the natural one, $\phi=f/\|f\|$ was answered in [7] for analytic functions with an isolated critical value,

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by introducing the concept of d-regularity (see also [3,4,8]). The d-regularity condition actually springs from [6] and is defined by means of a canonical pencil as follows: For every line $0 \in \ell \subset \mathbb{R}^k$ consider the set

$$X_\ell = \{x \in \mathbb{R}^n \,|\, f(x) \in \ell\}.$$

This is a pencil of real analytic varieties intersecting at $f^{-1}(0)$ and smooth away from it. The map f is said to be *d-regular at* 0 if there exists $\epsilon_0 > 0$ such that every $X_\ell \setminus V$ is transverse to every sphere centred at 0 and contained in $\mathbb{B}_{\epsilon_0}$, whenever the intersection is not empty.

The case of a real analytic map $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ with $n \geq k \geq 2$ and with non-isolated critical value was studied in [5,9,19]. If f has the transversality property [5, Definition 2.1] (compare with the definition of ρ-regularity in [9,19]), then for each $\epsilon > 0$ sufficiently small there is a smooth locally trivial fibration on the tube:

$$f\colon \mathbb{B}_\epsilon \cap f^{-1}(\mathbb{S}^{k-1}_\delta \setminus \Delta_\epsilon) \to \mathbb{S}^{k-1}_\delta \setminus \Delta_\epsilon \tag{3}$$

where Δ_ϵ is the image by f of the critical points of f in the interior of $\mathbb{B}^n_\epsilon$. It was proved in [5, Theorem 3.12] that if f admits a linearization $h : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ (as in [5, Definition 3.11]) such that $h^{-1} \circ f$ is d-regular. Then the map

$$\phi_{f,h}\colon \mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\Delta_\epsilon) \to \mathbb{S}^{k-1} \setminus \mathcal{A}_h \tag{4}$$

defined by

$$\phi_{f,h}(x) = \frac{h^{-1} \circ f(x)}{\|h^{-1} \circ f(x)\|},$$

is the projection of a smooth locally trivial fibration, and this is equivalent to fibration (3) above.

In this paper we envisage the case of (possibly non-analytic) functions $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ of class C^ℓ, $\ell \geq 2$, with possibly non-isolated critical value. We extend the fibration theorems of [5] for these maps: if f has the transversality property, then there is a fibration on tube (3); if in addition f has linear discriminant and is d-regular, then there is a fibration on sphere (6). This is done using a version of Ehresmann Fibration Theorem for differentiable maps of class C^ℓ between smooth manifolds given in Sect. 6. It is an open question whether these two fibrations are equivalent, as in the analytic case. If f has arbitrary discriminant with the transversality property and admits a linearization $h : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $h^{-1} \circ f$ is d-regular, then there is fibration (4) on the sphere.

Notice that the discriminant Δ_ϵ can have real codimension 1 and in that case its complement splits into finitely many connected components, say $S_1, \ldots, S_r$. As it was pointed out in [5], the topology of the fibres $f^{-1}(t) \cap \mathbb{B}_\epsilon$ can change for values in different S_i. It would be very interesting to determine how the topology changes as we move from one sector to another. This can clearly be seen in the example given in Sect. 5.

Remark 1.1 As in [5], throughout this paper, we will assume that f is *locally surjective*, that is, the image by f of every neighbourhood of the origin in $\mathbb{R}^n$ contains an open neighbourhood of the origin in $\mathbb{R}^k$, and we shall not mention it all the time. Nevertheless, it is easy to see that in the general case the same results hold if one intersects the bases of the locally trivial fibrations with their image. This choice is to avoid a heavy notation.

2 Fibration on the tube

Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $n > k \geq 2$, be a map of class C^ℓ with $\ell \geq 2$ and a critical point at 0.

Assume that f is locally surjective (see Remark 1.1). In what follows, for $0 < \epsilon$ we shall consider the restriction f_ϵ of f to the closed ball $\mathbb{B}^n_\epsilon$ of radius ϵ around 0 in $\mathbb{R}^n$.

Denote by Σ_ϵ the intersection of the critical set of f with the ball $\mathbb{B}_\epsilon$ and set $\Delta_\epsilon := f_\epsilon(\Sigma_\epsilon)$, which we call the *discriminant* of f_ϵ. It may depend on the choice of the radius ϵ, as shown in [9].

Also denote by $\Sigma_\epsilon(\mathbb{S}^{n-1}_\epsilon)$ the set of critical points in $\mathbb{S}^{n-1}_\epsilon$ of the restriction $f_\epsilon|_{\mathbb{S}^{n-1}_\epsilon}$. Set $\hat{\Sigma}_\epsilon := \Sigma_\epsilon \cup \Sigma_\epsilon(\mathbb{S}^{n-1}_\epsilon)$ and denote by $\hat{\Delta}_\epsilon := f(\hat{\Sigma}_\epsilon)$ which we call the *extended discriminant*[1] of f.

Following [17, §IV.4.4] and [2, Corollary 2.2], one can prove that the restriction of f to the tube

$$f: \mathbb{B}_\epsilon \cap f^{-1}(\mathbb{S}^{k-1}_\delta \setminus \hat{\Delta}_\epsilon) \to \mathbb{S}^{k-1}_\delta \setminus \hat{\Delta}_\epsilon, \tag{5}$$

is a locally trivial fibration, where $\mathbb{S}^{k-1}_\delta = \partial\mathbb{B}^k_\delta$. Hence, in this general setting there is always a fibration on the tube.

Definition 2.1 We say that f has the *transversality property in the ball* $\mathbb{B}^n_\epsilon$ if there exist $0 < \delta \ll \epsilon$ such that for every $y \in \mathbb{B}^k_\delta \setminus \Delta_\epsilon$ the fibre $f^{-1}(y)$ is transverse to the sphere $\mathbb{S}^{n-1}_\epsilon$.

So, if f satisfies the *transversality property*, there is no contribution to the extended discriminant $\hat{\Delta}_\epsilon$ by points on the sphere $\mathbb{S}^{n-1}_\epsilon$; the extended discriminant is just the discriminant Δ_ϵ of f in $\mathbb{B}^n_\epsilon$. Then we get:

Theorem 2.2 *Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $n > k \geq 2$ be a map of class C^ℓ, $\ell \geq 2$ with a critical point at* 0 *and* $\dim(f^{-1}(0)) > 0$. *The map f has the transversality property in the ball $\mathbb{B}^n_\epsilon$ if and only if it admits local Milnor-Lê fibrations in tubes over the complement of the discriminant Δ_ϵ.*

3 Differentiable maps with linear discriminant

In this section, we extend the concept of d-regularity to real differentiable maps with linear discriminant. Then we show that in this context, d-regularity guarantees a fibration on the sphere.

First, let us recall some definitions.

We say that a map $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ of class C^ℓ, $\ell \geq 2$ has *linear discriminant in the ball* $\mathbb{B}^n_\epsilon$ if Δ_ϵ is a union of line segments with one endpoint at $0 \in \mathbb{R}^k$ and there exists $\eta > 0$, called a *linearity radius* for Δ_ϵ, such that each of these line segments intersects $\mathbb{S}^{k-1}_\eta$, that is, if

$$\Delta_\epsilon \cap \mathbb{B}^k_\eta = \operatorname{Cone}\left(\Delta_\epsilon \cap \mathbb{S}^{k-1}_\eta\right).$$

In this case, we set

$$\mathcal{A}_\eta := \Delta_\epsilon \cap \mathbb{S}^{k-1}_\eta.$$

[1] In [17, §IV.4.4] $\hat{\Sigma}_\epsilon$ is called the *apparent contour at the source* and Δ_ϵ is called the *apparent contour at the target* or just *apparent contour*.

Also, let $\pi : \mathbb{S}^{k-1}_\eta \to \mathbb{S}^{k-1}$ be the radial projection onto the unit sphere $\mathbb{S}^{k-1}$ and set

$$\mathcal{A} = \pi(\mathcal{A}_\eta).$$

For each point $\theta \in \mathbb{S}^{k-1}_\eta$, let $\mathcal{L}_\theta \subset \mathbb{R}^k$ be the open ray in $\mathbb{R}^k$ from the origin that contains the point θ. Set:

$$E_\theta := f^{-1}(\mathcal{L}_\theta).$$

We say that f is *d-regular in the ball* $\mathbb{B}^n_\epsilon$ if E_θ intersects the sphere $\mathbb{S}^{n-1}_{\epsilon'}$ transversely in $\mathbb{R}^n$, for every ϵ' with $0 < \epsilon' \le \epsilon$ and for every $\theta \in \mathbb{S}^{k-1}_\eta \setminus \mathcal{A}_\eta$.

The following proposition is a straightforward generalization of [7, Proposition 3.2].

Proposition 3.1 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a map of class C^ℓ with $\ell \geq 1$, with linear discriminant. Then f is d-regular in the ball $\mathbb{B}^n_\epsilon$ if and only if the C^ℓ-map*

$$\phi_{\epsilon'} = \frac{f}{\|f\|} : \mathbb{S}^{n-1}_{\epsilon'} \setminus f^{-1}(\Delta_\epsilon) \longrightarrow \mathbb{S}^{k-1} \setminus \mathcal{A}$$

is a submersion for every sphere $\mathbb{S}^{n-1}_{\epsilon'}$ with $0 < \epsilon' < \epsilon$.

Now we can state the main result of this paper:

Theorem 3.2 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ with $n \geq k \geq 2$ be a map of class C^ℓ with $\ell \geq 2$. Suppose f has linear discriminant and the transversality property in the ball $\mathbb{B}^n_\epsilon$. If f is d-regular in the ball $\mathbb{B}^n_\epsilon$, then the map*

$$\phi_\epsilon = \frac{f}{\|f\|} : \mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\Delta_\epsilon) \to \mathbb{S}^{k-1} \setminus \mathcal{A} \tag{6}$$

is a locally trivial fibration of class $C^{\ell-1}$.

In order to prove Theorem 3.2 we first need the following:

Lemma 3.3 *Let $f : X \to Y$ and $g : Y \to Z$ be $C^{\ell-1}$-locally trivial fibrations with $2 \leq l \leq \infty$, between smooth manifolds possibly with boundary. Then $g \circ f : X \to Z$ is a $C^{\ell-1}$-locally trivial fibration.*

We will prove Lemma 3.3 in Sect. 6. Now we will prove Theorem 3.2.

Proof of Theorem 3.2 Set

$$\mathcal{M} := \mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\Delta_\epsilon).$$

Notice that $\mathcal{M}$ is an open submanifold of $\mathbb{S}^{n-1}_\epsilon$ since Δ_ϵ is closed in $\mathbb{R}^k$. Consider the following decomposition

$$\mathcal{M} = \left(\mathcal{M} \cap f^{-1}(\mathbb{B}^k_\delta)\right) \cup \left(\mathcal{M} \setminus f^{-1}(\mathring{\mathbb{B}}^k_\delta)\right),$$

where $\mathring{\mathbb{B}}^k_\delta$ is the interior of the closed ball $\mathbb{B}^k_\delta$. Both pieces are submanifolds with boundary of $\mathcal{M}$ of dimension $n-1$, and their intersection is the common boundary submanifold of dimension $n-2$

$$\mathbb{S}^{n-1}_\epsilon \cap f^{-1}(\mathbb{S}^{k-1}_\delta \setminus \Delta_\epsilon) = \left(\mathcal{M} \cap f^{-1}(\mathbb{B}^k_\delta)\right) \cap \left(\mathcal{M} \setminus f^{-1}(\mathring{\mathbb{B}}^k_\delta)\right).$$

We are going to show that the restriction of ϕ to each of these components is a $C^{\ell-1}$-fibre bundle and that these two fibre bundles coincide on the common boundary submanifold $\mathbb{S}^{n-1}_\epsilon \cap f^{-1}(\mathbb{S}^{k-1}_\delta \setminus \Delta_\epsilon)$, so they can be glued into a global fibre bundle.

The restriction of f given by $f_1 : \mathcal{M} \cap f^{-1}(\mathbb{B}^k_\delta) \to \mathbb{B}^k_\delta \setminus \Delta_\epsilon$ is proper since $\mathbb{S}^{n-1}_\epsilon \cap f^{-1}(\mathbb{B}^k_\delta)$ is compact, and since f has the transversality property in the ball $\mathbb{B}^n_\epsilon$ it is a submersion, and by Ehresmann fibration theorem (Theorem 6.3) it is a $C^{\ell-1}$-fibre bundle. Now consider the radial projection $\tilde{\pi} : \mathbb{B}^k_\delta \setminus \Delta_{f,\eta} \to \mathbb{S}^{k-1} \setminus \mathcal{A}$ which is a (trivial and smooth) fibre bundle. The restriction

$$\phi_1 : \left(\mathcal{M} \cap f^{-1}(\mathbb{B}^k_\delta)\right) \to \mathbb{S}^{k-1} \setminus \mathcal{A}$$

of ϕ is given by the composition $\tilde{\pi} \circ f_1$. By Lemma 3.3 the composition $\phi_1 = \tilde{\pi} \circ f_1$ is a $C^{\ell-1}$-locally trivial fibration.

So now we just have to show that the restriction:

$$\phi_2 : \mathcal{M} \setminus f^{-1}(\mathring{\mathbb{B}}^k_\delta) \to \mathbb{S}^{k-1} \setminus \mathcal{A}$$

is a $C^{\ell-1}$-fibration. We have that ϕ_2 is proper since $\mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\mathring{\mathbb{B}}^k_\delta)$ is compact.

Since f is d-regular, by Proposition 3.1 the map $\phi : \mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\Delta_\epsilon) \to \mathbb{S}^{k-1} \setminus \mathcal{A}$ has no critical points. So ϕ_2 is a submersion restricted to the interior $\mathcal{M} \setminus f^{-1}(\mathbb{B}^k_\delta)$ of $\mathcal{M} \setminus f^{-1}(\mathring{\mathbb{B}}^k_\delta)$. Since ϕ_1 and ϕ_2 coincide on the boundary $\mathcal{M} \cap f^{-1}(\mathbb{S}^{k-1}_\delta)$, we already saw that ϕ_1 restricted to this boundary is a submersion. The result follows from the Ehresmann fibration theorem for manifolds with boundary (Theorem 6.3). □

Remark 3.4 In the articles [3,4] we proved that when $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ is analytic with a critical point at $0 \in \mathbb{R}^n$ and $0 \in \mathbb{R}^k$ is an isolated critical value, fibrations (3) and (6) are equivalent. The proof uses [3, Proposition 3.5] (which is the analytic version of a corollary by Milnor [16, Corollary 3.4]) which says that there exists a neighbourhood of the origin of $\mathbb{R}^n$ such that the gradients of two non-negative analytic functions cannot point in opposite directions. This result is proved using the Analytic Curve Selection Lemma; thus, the proof does not extend to the case when f is non-analytic.

Question 3.5 *Are fibrations (3) and (6) equivalent, as in the analytic case? We do not know the answer.*

4 Differentiable maps with arbitrary discriminant

As in the analytic case, we want to extend the concept of d-regularity, allowing some maps to become d-regular after a homeomorphism on the target space. We start recalling some definitions from [5].

Given $\eta > 0$ and $\theta \in \mathbb{S}^{k-1}_\eta$, recall the set $\mathcal{L}_\theta \subset \mathbb{R}^k$, which is the open segment of line that starts in the origin and ends at the point θ.

We say that a restriction $h_\eta : \mathbb{B}^k_\eta \to h(\mathbb{B}^k_\eta)$ of a homeomorphism $h : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ is a *conic homeomorphism* if:

(i) For each $\theta \in \mathbb{S}^{k-1}_\eta$ the image $h_\eta(\mathcal{L}_\theta)$ is a path in $\mathbb{R}^k$ of class C^ℓ with $\ell \geq 1$;
(ii) The inverse map h^{-1} of h is of class C^ℓ with $\ell \geq 1$ outside the origin;
(iii) The map h^{-1} is a submersion outside the origin.

To simplify the notation, set $\mathcal{B}^k_\eta := h(\mathbb{B}^k_\eta)$.

We say that a conic homeomorphism $h : \mathbb{B}^k_\eta \to \mathcal{B}^k_\eta$ is a *linearization* for f_ϵ if

$$h^{-1}(\Delta_\epsilon \cap \mathcal{B}^k_\eta) = \mathrm{Cone}(h^{-1}(\Delta_\epsilon \cap \partial\mathcal{B}^k_\eta)).$$

Given a linearization h for f_ϵ, we say that f is *d_h-regular in $\mathbb{B}^n_\epsilon$* if the composition $h^{-1}\circ f$ is d-regular. Set

$$\mathcal{A}_{h,\eta} := h^{-1}(\Delta_\epsilon \cap \partial\mathcal{B}^k_\eta) = h^{-1}(\Delta_\epsilon)\cap \mathbb{S}^{k-1}_\eta .$$

As before, let $\pi : \mathbb{S}^{k-1}_\eta \to \mathbb{S}^{k-1}$ be the radial projection onto the unit sphere $\mathbb{S}^{k-1}$ and set $\mathcal{A}_h = \pi(\mathcal{A}_{h,\eta})$. Then the following theorem is a straightforward generalization of [5, Theorem 3.12].

Theorem 4.1 *Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a map of class C^ℓ with the transversality property in $\mathbb{B}^n_\epsilon$, and suppose it admits a linearization $h: \mathbb{B}^k_\eta \to \mathcal{B}^k_\eta$ making f d_h-regular in $\mathbb{B}^n_\epsilon$. Then the map*

$$\phi_{h,\epsilon} = \frac{h^{-1}\circ f}{\|h^{-1}\circ f\|} : \mathbb{S}^{n-1}_\epsilon \setminus f^{-1}(\Delta_f) \to \mathbb{S}^{k-1}\setminus \mathcal{A}_h$$

is a $C^{\ell-1}$-locally trivial fibration.

5 An example of a non-analytic d_h-regular map

Consider the real function $\varsigma: \mathbb{R}\to\mathbb{R}_+$ given by:

$$\varsigma(t) := \begin{cases} e^{-1/t} & \text{if} t > 0; \\ 0 & \text{if} t \le 0. \end{cases}$$

It is a classic example of a function that is smooth and non-analytic.

Now define $\alpha: \mathbb{R}^n \to \mathbb{R}$ by $\alpha(x) = 1 - \|x-\bar{1}\|^2$ where $\bar{1} := (1, 0, \ldots, 0)$.

So the function $f : \mathbb{R}^n \to \mathbb{R}_+$ given by $f(x) := \varsigma(\alpha(x))$, is smooth and non-analytic at the origin.

Notice that:

- $f(x) = 0$ if and only if $\|x-\bar{1}\| \ge 1$;
- $f(x) = t$ for some $t > 0$ if and only if $t \le e^{-1}$ and

$$\|x-\bar{1}\|^2 = \frac{1}{\ln t} + 1.$$

So we have that:

(i) $\mathrm{Im}(f) = [0, e^{-1}]$;
(ii) $V(f) = \mathbb{R}^n \setminus \mathring{\mathbb{B}}^n(\bar{1}; 1)$, where $\mathring{\mathbb{B}}^n(\bar{1}; 1)$ is the open ball of radius 1 around the point $\bar{1}$;
(iii) $f^{-1}(t) = \mathbb{S}^{n-1}\left(\bar{1}; \sqrt{\frac{1}{\ln t}+1}\right)$, where $\mathbb{S}^{n-1}\left(\bar{1}; \sqrt{\frac{1}{\ln t}+1}\right)$ is the $(n-1)$-sphere around $\bar{1}$ of radius $\sqrt{\frac{1}{\ln t}+1}$, for any $0 < t < e^{-1}$;
(iv) $f^{-1}(e^{-1}) = \{\bar{1}\}$.

The gradient of α is given by

$$\nabla\alpha(x) = -2(x-\bar{1}).$$

The derivative of ς is given by

$$\varsigma'(t) = \begin{cases} \frac{1}{t^2}e^{-\frac{1}{t}} & t > 0 \\ 0 & t \le 0. \end{cases}$$

So by the chain rule, the gradient vector of f at a point $x \in \mathbb{R}^n$ is given by:

$$\nabla f(x) = \varsigma'(\alpha(x)\nabla\alpha(x) = \begin{cases} -\frac{2}{\alpha(x)^2} f(x)(x-\bar{1}), & \text{if} \quad \|x-\bar{1}\| < 1, \\ \bar{0}, & \text{if} \quad \|x-\bar{1}\| \geq 1, \end{cases}$$

where $\bar{0} := (0, 0, \ldots, 0)$.

Hence the critical set and the discriminant of f are given by:

$$\Sigma_f = V(f) \cup \{\bar{1}\}, \qquad \Delta_f = \{0, e^{-1}\}.$$

Now we consider a function g analogous to the function f. Define $\beta\colon \mathbb{R}^n \to \mathbb{R}$ by $\beta(x) = 4 - \|x - \bar{2}\|^2$ where $\bar{2} := (2, 0, \ldots, 0)$. Define $g : \mathbb{R}^n \to \mathbb{R}_+$ by $g(x) := \varsigma(\beta(x))$.

In this case we have:

(1) $\mathrm{Im}(g) = [0, e^{-\frac{1}{4}}]$;
(2) $V(g) = \mathbb{R}^2 \setminus \mathring{\mathbb{B}}^n(\bar{2}; 2)$;
(3) $g^{-1}(t) = \mathbb{S}^{n-1}\left(\bar{2}; \sqrt{\frac{1}{\ln t} + 4}\right)$, for any $0 < t < e^{-\frac{1}{4}}$;
(4) $g^{-1}(e^{-\frac{1}{4}}) = \{\bar{2}\}$.

Doing a computation analogous to that for f, we get that the critical set and the discriminant of g are given by:

$$\Sigma_g = V(g) \cup \{\bar{2}\}, \qquad \Delta_g = \{0, e^{-\frac{1}{4}}\}.$$

Finally, set the map $\Psi\colon \mathbb{R}^n \to \mathbb{R}^2$ given by $\Psi := (f, g)$. It is a smooth map that is not analytic at the origin.

We have that:

(a) $\mathrm{Im}(\Psi) \subset [0, e^{-1}] \times [0, e^{-\frac{1}{4}}]$;
(b) $V(\Psi) = V(g) = \mathbb{R}^n \setminus \mathring{\mathbb{B}}^n(\bar{2}; 2)$, since $V(g) \subset V(f)$;
(c) $\Psi^{-1}(t_1, t_2) = \mathbb{S}^{n-1}\left(\bar{1}; \sqrt{\frac{1}{\ln t_1} + 1}\right) \cap \mathbb{S}^{n-1}\left(\bar{2}; \sqrt{\frac{1}{\ln t_2} + 4}\right)$, for any $t_1 \neq 0$ and $t_2 \neq 0$. Notice that if the two spheres $f^{-1}(t_1)$ and $g^{-1}(t_2)$ are transverse, the intersection is either homeomorphic to a sphere $\mathbb{S}^{n-2}$ or empty. If they are tangent, the intersection is a point;
(d) $\Psi^{-1}(0, t_2) = \left(\mathbb{R}^n \setminus \mathring{\mathbb{B}}^n(\bar{1}; 1)\right) \cap \mathbb{S}^{n-1}\left(\bar{2}; \sqrt{\frac{1}{\ln t_2} + 4}\right)$, for any $t_2 \neq 0$. Notice that this is homeomorphic to a ball $\mathbb{B}^{n-1}$, except when $t_2 = e^{-\frac{1}{4}}$ we get the point $\{\bar{2}\}$.
(e) $\Psi^{-1}(t_1, 0) = \mathbb{S}^{n-1}\left(\bar{1}; \sqrt{\frac{1}{\ln t_1} + 1}\right) \cap \left(\mathbb{R}^n \setminus \mathring{\mathbb{B}}^n(\bar{2}; 2)\right)$, for any $t_1 \neq 0$. Notice that this is the empty set.

The Jacobian matrix of Ψ at a point $x = (x_1, \ldots, x_n)$ is given by the following matrix

$$\begin{bmatrix} -\frac{2}{\alpha(x)^2} f(x)(x_1 - 1) & -\frac{2}{\alpha(x)^2} f(x) x_2 & \ldots & -\frac{2}{\alpha(x)^2} f(x) x_n \\ -\frac{2}{\beta(x)^2} g(x)(x_1 - 2) & -\frac{2}{\beta(x)^2} g(x) x_2 & \ldots & -\frac{2}{\beta(x)^2} g(x) x_n \end{bmatrix}.$$

So we have that the critical set and the discriminant are given by

$$\Sigma_\Psi = V(f) \cup \{x_2 = \cdots = x_n = 0\}, \quad \Delta_\Psi = \{(0, t_2) \mid 0 \leq t_2 \leq e^{-1/4}\} \cup \mathcal{C},$$

where $\mathcal{C}$ is the curve in $\mathbb{R}^2$ given by:

$$\mathcal{C}(s) := \begin{cases} \left(e^{-\frac{1}{s(2-s)}}, e^{-\frac{1}{s(4-s)}}\right) & \text{if} s > 0; \\ (0, 0) & \text{if} s \leq 0. \end{cases}$$

In particular, Ψ **does not have** linear discriminant (see Fig. 1).

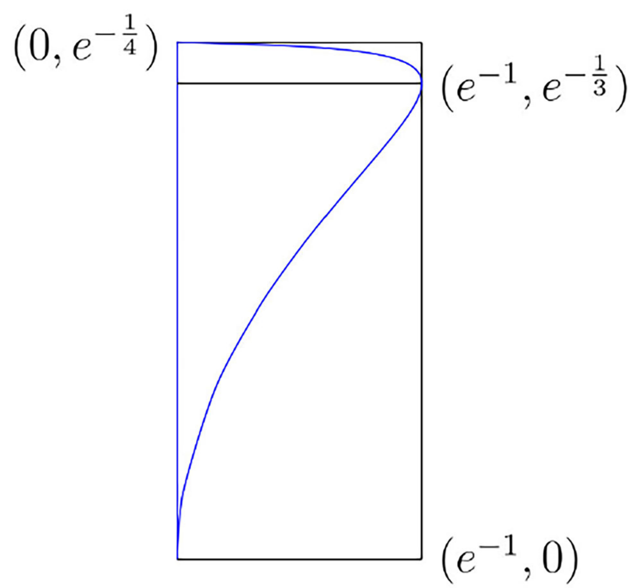

Fig. 1 The blue closed curve is the discriminant

Remark 5.1 The discriminant Δ_Ψ divides the plane in two connected components. Here we can see the phenomenon described in Introduction: by (c), over points inside the discriminant, the fibres are spheres $\mathbb{S}^{n-2}$, while over points outside the discriminant, the fibres are empty.

We claim that Ψ has the transversality property, so there is no further contribution to the discriminant by critical points of Ψ restricted to the sphere $\mathbb{S}^{n-1}_\epsilon$. Let $\mathbb{B}_\epsilon$ be the closed n-ball of small radius $\epsilon > 0$ centred at the origin $\bar{0} \in \mathbb{R}^n$. Consider the $(n-1)$-sphere $\mathbb{S}^{n-1}(\bar{1}; 1)$ of radius 1 centred at $\bar{1}$. Firstly, we want to find the equation of the $(n-2)$-sphere which is the intersection of the $(n-1)$-spheres $\mathbb{S}^{n-1}_\epsilon$ and $\mathbb{S}^{n-1}(\bar{1}; 1)$ which, respectively, have the equations

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \epsilon^2 \tag{7}$$

$$(x_1 - 1)^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{8}$$

Getting x_2^2 from (7) and substituting in (8) we get that $x_1 = \frac{\epsilon^2}{2}$, so the intersection is the $(n-2)$-sphere with equation

$$x_2^2 + \cdots + x_n^2 = \epsilon^2 - \frac{\epsilon^4}{4}. \tag{9}$$

Now we want to compute the radius r of the $(n-1)$-sphere $\mathbb{S}^{n-1}(\bar{2}; r)$ with equation

$$(x_1 - 2)^2 + x_2^2 + \cdots + x_n^2 = r_2^2 \tag{10}$$

which intersects the hyperplane $x_1 = \frac{\epsilon^2}{2}$ on the $(n-2)$-sphere given by (9). Substituting $x_1 = \frac{\epsilon^2}{2}$ and (9) in (10) we obtain that $r^2 = 4 - \epsilon^2$. The image of the $(n-1)$-sphere $\mathbb{S}^{n-1}(\bar{2}; r)$ under g is $e^{-\frac{1}{4-r^2}}$. Any $(n-1)$-sphere $\mathbb{S}^{n-1}(\bar{2}; r')$ of radius $r' > r > 0$ intersects any $(n-1)$-sphere $\mathbb{S}^{n-1}(\bar{1}; r'')$ with $0 < r'' \leq 1$ in either, an $(n-2)$-sphere **contained** in the interior of the n-ball $\mathbb{B}^n_\epsilon$ or the empty set. Taking $\delta \leq e^{-\frac{1}{r(4-r)}}$ we get that the fibre

$\Psi^{-1}(t_1, t_2)$ with $(t_1, t_2) \in \mathbb{B}^k_\delta \setminus \Delta_\Psi$ is either contained in the interior of the n-ball $\mathbb{B}^n_\epsilon$, or it is the empty set; hence, Ψ has the transversality property.

Consider the homeomorphism $h\colon (0,1) \times (0,1) \to (0, e^{-1}) \times (0, e^{-\frac{1}{3}})$ given by

$$h(u, v) := \left(e^{\frac{1}{u(u-2)}}, e^{\frac{1}{v(v-4)}}\right),$$

with inverse

$$h^{-1}(u, v) := \left(1 - \sqrt{1 + \frac{1}{\ln u}},\ 2 - \sqrt{4 + \frac{1}{\ln v}}\right).$$

For $\eta < e^{-1}$ the restriction of h to $\mathbb{B}^2_\eta \cap \big((0,1) \times (0,1)\big)$ is a conic homeomorphism that gives a linearization for Ψ, since h takes the segment $\mathcal{L}_{\frac{\pi}{2}}$ to itself and the segment $\mathcal{L}_{\frac{\pi}{4}}$ onto the curve $\mathcal{C}$ (see the small rectangle in Fig. 1).

Set $E_\theta := (h^{-1} \circ \Psi)^{-1}(\mathcal{L}_\theta)$ for $\theta \in (0, \frac{\pi}{2}]$. For any $\theta \in (\pi/4, \pi/2)$, one can check that E_θ is a manifold homeomorphic to the cylinder $\mathbb{S}^{n-2} \times (0,1)$ that intersects the sphere $\mathbb{S}_{\epsilon'}$ transversally, for any $\epsilon' \leq \epsilon$, with ϵ small enough as above, and for $\theta \in [0, \pi/4)$ we have that E_θ is empty. Moreover:

$$E_{\frac{\pi}{4}} = \{x_2 = \cdots = x_n = 0\}$$

and $E_{\frac{\pi}{2}}$ is a manifold homeomorphic to a disk $\mathbb{D}^n$ that intersects the sphere $\mathbb{S}_{\epsilon'}$ transversally, for any $\epsilon' \leq \epsilon$. Hence Ψ is d_h-regular. Alternatively, one can check this by using Proposition 3.8 of [3] for the composition $h^{-1} \circ \Psi$.

6 An extension of Ehresmann fibration theorem

In this section we give an extension of Ehresmann fibration theorem proved by Wolf in [20] to prove Lemma 3.3. Analogous results are given by Ekedahl [11] and McKay [14, Corollary 7].

We follow Section 2 of [20] to give the necessary definitions to enunciate Wolf's theorem. In [20] the results are stated for smooth manifolds and smooth maps between them. Here we also deal with smooth (C^∞) manifolds, but the maps may be only of class C^ℓ for $2 \leq l \leq \infty$.

Let E and B be smooth manifolds and $\varphi\colon E \to B$ a submersion of class C^ℓ with $2 \leq l \leq \infty$. Since φ is a submersion, for any $b \in B$ the fibre $\varphi^{-1}(b)$ is a submanifold of E of dimension $\dim E - \dim B$.

Given $x \in E$, the *vertical space* V_x at x is the subspace of T_xE defined by

$$V_x = \{\, v \in T_xE \mid D_x\varphi(v) = 0 \,\},$$

that is, the space tangent to the fibre $\varphi^{-1}(\varphi(x))$. One has that $\dim V_x = \dim E - \dim B$. The *vertical distribution* is $\mathcal{V} = \{V_x\}_{x \in E}$. An *Ehresmann connection* for φ is a distribution $\mathcal{H} = \{H_x\}_{x \in E}$ on E that is complementary to $\mathcal{V}$, *i.e.*, $T_xE = V_x \oplus H_x$ for every $x \in E$. So $D_x\varphi$ restricts to a linear isomorphism from H_x onto $T_{\varphi(x)}B$. The space H_x is the *horizontal space* at x. Notice that using a Riemannian metric on E it is always possible to construct an Ehresmann connection taking the orthogonal complement of the vertical distribution.

Fix an Ehresmann connection $\mathcal{H}$ of $\varphi\colon E \to B$. A tangent vector $v \in T_xE$ is *horizontal* (respectively, *vertical*) if $v \in H_x$ (respectively, $v \in V_x$); a sectionally smooth curve in E is *horizontal* (respectively, *vertical*) if each of its tangent vectors is *horizontal* (respectively, *vertical*). We make the convention that all sectionally smooth curves are parametrised so as to be regular (nowhere vanishing tangent vector) on each smooth arc.

Let $\alpha(t)$, $t \in [0,1]$, be a sectionally smooth curve in $\varphi(E) \subset B$. Given $x \in \varphi^{-1}(\alpha(0))$, there is at most one sectionally smooth *horizontal curve* $\alpha_x(t)$, $t \in [0,1]$, in E such that:

(a) $\alpha_x(0) = x$, and
(b) $\varphi \circ \alpha_x = \alpha$.

If it exists, α_x is called the *horizontal lift* of α to x. If α_x exists for every $x \in \varphi^{-1}(\alpha(0))$, then we say that α *has horizontal lifts*. In such case, the *translation of the fibres along* α is the map

$$\begin{aligned} \rho_\alpha \colon \varphi^{-1}(\alpha(0)) &\to \varphi^{-1}(\alpha(1)), \\ x &\mapsto \alpha_x(1). \end{aligned}$$

Following [20, Lemma 2.2] we get the following lemma.

Lemma 6.1 *Let E and B be smooth manifolds and $\varphi \colon E \to B$ a submersion of class C^ℓ with $2 \le l \le \infty$. If $u \in \varphi(E)$, then $\varphi^{-1}(u)$ is a closed C^ℓ-submanifold of E. If $\rho \colon \varphi^{-1}(u) \to \varphi^{-1}(v)$ is a translation relative to an Ehresmann connection for φ, then ρ is of class $C^{\ell-1}$.*

Proof Since φ is a C^ℓ-submersion, by the Rank Theorem [1, 2.5.15 Rank Theorem] $F_u := \varphi^{-1}(u)$ is a closed C^ℓ-submanifold of E for any $u \in \varphi(E)$. The tangent bundle TF_u of F_u is a $C^{\ell-1}$-manifold [1, 3.3.10 Theorem]; hence, the tangent spaces T_xF_u depend differentiably of class $C^{\ell-1}$ on $x \in F_u$. Thus, given an Ehresmann connection $\mathcal{H}$ the horizontal subspaces H_x depend differentiably of class $C^{\ell-1}$ on x.

Since E and B are smooth manifolds, we can take the curve α to be sectionally smooth. Thus, its derivative α', which is a curve on the tangent bundle TB, is also sectionally smooth. Given $x \in \varphi^{-1}(\alpha(0))$, lifting the vector field α' to a *horizontal* vector field α'_x on E using the Ehresmann connection $\mathcal{H}$ and the differential $D\varphi$, which is of class $C^{\ell-1}$, we loose one degree of differentiability, but since α' is of class C^∞, its lifting α'_x is also of class C^∞. View the Ehresmann connection as a system of ordinary differential equations. In local coordinates the coefficients are of class $C^{\ell-1}$ because H_x depends differentiably of class $C^{\ell-1}$ on x; thus, the solution curve at time t depends differentiably of class $C^{\ell-1}$ on the initial data. □

Taking Lemma 6.1 into account, one can follow the proofs of [20, Proposition 2.3 and Corollary 2.5] to obtain the following theorem, where $\varphi \colon E \to B$ instead of being a smooth fibre bundle, is a differentiable locally trivial fibre bundle of class $C^{\ell-1}$: the projection is a map of class C^ℓ, but the local trivializations are $C^{\ell-1}$-diffeomorphisms.

Theorem 6.2 *([20, Corollary 2.5]) Let $\varphi \colon E \to B$ be a submersion of class C^ℓ with $2 \le l \le \infty$, where E and B are paracompact and B connected.*[2] *Then the following statements are equivalent:*

(i) *$\varphi \colon E \to \varphi(E)$ is a $C^{\ell-1}$-fibre bundle.*
(ii) *There exists an Ehresmann connection for φ, relative to which every sectionally smooth curve in $\varphi(E)$ has horizontal lifts.*
(iii) *If $\mathcal{H}$ is an Ehresmann connection for φ, then every sectionally smooth curve in $\varphi(E)$ has horizontal lifts relative to $\mathcal{H}$.*

[2] The hypothesis in [20] of E being connected is not used in the proof, and it works without it.

As a corollary, we get the following version of Ehresmann fibration theorem, which corresponds to [20, Corollary 2.4].

Theorem 6.3 *(Ehresmann Fibration Theorem) Let $\varphi\colon E \to B$ be a proper submersion of class C^ℓ with $2 \leq l \leq \infty$, where E and B are paracompact. Then $\varphi\colon E \to \varphi(E)$ is a $C^{\ell-1}$-fibre bundle.*

It is easy to extend Theorem 6.3 when E is a manifold with boundary ∂E asking that the restriction $\phi|_{\partial E}\colon \partial E \to B$ is also a submersion and applying Theorem 6.3 to the restriction of φ to the interior of E and to the restriction of φ to the boundary ∂E.

Proof of Lemma 3.3 Since g is a $C^{\ell-1}$-locally trivial fibration, by Theorem 6.2 there exists an Ehresmann connection $\mathcal{H}^g$ for g, relative to which every sectionally smooth curve in $g(Y) \subset Z$ has horizontal lifts. For any $y \in Y$ we have $T_yY = V_y^g \oplus H_y^g$ where V_y^g and H_y^g are, respectively, the vertical and horizontal subspaces of T_yY. Recall that V_y^g is the tangent space of the fibre $g^{-1}(g(y))$ at y and that H_y^g projects isomorphically onto $T_{g(y)}Z$ under D_yg.

Analogously, there exists an Ehresmann connection $\mathcal{H}^f$ for f, relative to which every sectionally smooth curve in $f(X) \subset Y$ has horizontal lifts. For any $x \in X$ we have $T_xX = V_x^f \oplus H_x^f$ where V_x^f and H_x^f are, respectively, the vertical and horizontal subspaces of T_xX. Recall that V_x^f is the tangent space of the fibre $f^{-1}(f(x))$ at x and that H_x^f projects isomorphically onto $T_{f(x)}Y = V_{f(x)}^g \oplus H_{f(x)}^g$ under D_xf. This isomorphism induces a direct sum decomposition $H_x^f = \tilde{H}_x^f \oplus H_x^{g\circ f}$, where $\tilde{H}_x^f$ and $H_x^{g\circ f}$ correspond, respectively, to $V_{f(x)}^g$ and $H_{f(x)}^g$. Hence we have $T_xX = V_x^f \oplus \tilde{H}_x^f \oplus H_x^{g\circ f}$. Set $V_x^{g\circ f} = V_x^f \oplus \tilde{H}_x^f$, then we have $T_xX = V_x^{g\circ f} \oplus H_x^{g\circ f}$ and we claim that $V_x^{g\circ f}$ is the vertical space of $g \circ f$ at x and that the distribution $\mathcal{H}^{g\circ f} = \{H_x^{g\circ f}\}_{x\in X}$ is an Ehresmann connection for $g \circ f$. Firstly, it is easy to see that $H_x^{g\circ f}$ is mapped isomorphically onto $T_{g(f(x))}Z$ under $D_x(g \circ f)$

$$D_{f(x)}g\big(D_xf(H_x^{g\circ f})\big) = D_{f(x)}g(H_{f(x)}^g) = T_{g(f(x))}Z.$$

To see that $V_x^{g\circ f}$ is the vertical space of $g \circ f$ at x we need to check two cases: 1) if $v \in V_x^f$ we have that $D_xf(v) = 0$, then $D_{f(x)}g\big(D_xf(v)\big) = D_{f(x)}g(0) = 0$, 2) if $v \in \tilde{H}_x^f$ then $D_xf(v) \in V_y^g$ and $D_{f(x)}g\big(D_xf(v)\big) = 0$.

Let $z \in (g \circ f)(X) \subset Z$, $x \in (g \circ f)^{-1}(z)$ and $y = f(x) \in g^{-1}(z) \subset Y$. Let $\alpha\colon I \to Z$ be a sectionally smooth curve in $(g \circ f)(X) \subset Z$ with $\alpha(0) = z$, and let $\alpha_y\colon I \to f(X) \subset Y$ be its horizontal lift relative to $\mathcal{H}^g$, so we have that $\alpha_y(0) = y$ and $g \circ \alpha_y = \alpha$. Now let $\alpha_x\colon I \to X$ be the horizontal lift of α_y relative to $\mathcal{H}^f$, so we have that $\alpha_x(0) = x$ and $f \circ \alpha_x = \alpha_y$. Thus we have $g \circ f \circ \alpha_x = g \circ \alpha_y = \alpha$, so α_x is a lift of α by $g \circ f$. To conclude the proof we need to check that α_x is a horizontal lift relative to the Ehresmann connection $\mathcal{H}^{g\circ f}$. Since $\alpha_x\colon I \to X$ is the horizontal lift of α_y relative to $\mathcal{H}^f$ we have that $\alpha_x'(t) \in H_{\alpha_x(t)}^f = \tilde{H}_{\alpha_x(t)}^f \oplus H_{\alpha_x(t)}^{g\circ f}$ for every $t \in I$. We claim that $\alpha_x'(t) \in H_{\alpha_x(t)}^{g\circ f}$, suppose this is not true that $\alpha_x'(t) \in \tilde{H}_{\alpha_x(t)}^f$, then $D_{\alpha_x(t)}f(\alpha_x'(t)) = \alpha_y'(t) \in V_{f(\alpha_x(t))}^g$, but this contradicts the fact that α_y is a horizontal lift of α relative to the Ehresmann connection $\mathcal{H}^g$. □

Acknowledgements
We thank Prof. José Seade for many enlightening conversations, his interest and remarks on this work. We also thank the two anonymous referees for their careful reading of the article and their suggestions which greatly improve it.

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Received: 25 August 2023 Accepted: 29 January 2024
Published online: 05 March 2024

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