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Secondary cup and cap products in coarse geometry

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Abstract

We construct secondary cup and cap products on coarse (co-)homology theories from given cross and slant products. They are defined for coarse spaces relative to weak generalized controlled deformation retracts. On ordinary coarse cohomology, our secondary cup product agrees with a secondary product defined by Roe. For coarsifications of topological coarse (co-)homology theories, our secondary cup and cap products correspond to the primary cup and cap products on Higson dominated coronas via transgression maps. And in the case of coarse K-theory and -homology, the secondary products correspond to canonical primary products between the K-theories of the stable Higson corona and the Roe algebra under assembly and co-assembly.

Keywords: Coarse homology, Coarse cohomology, Secondary products, Coarse assembly, Coarse coassembly

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1 Introduction

Although the usefulness of multiplicative structures on (co-)homology theories is undisputed in algebraic topology, their coarse geometric counterparts have been neglected for quite a long time and Roe's secondary product on his coarse cohomology (cf. [14, Section 2.4]) remained the only one of his kind for many years.

Only recently, there has been more research on this topic, which was primarily motivated by applications to coarse index theory and thereby also to positive scalar curvature. Cross products were the main multiplicative structure of interest, in particular the cross product between the analytic structure group and K-homology (cf. [4, 9, 16, 22, 25, 26], albeit not all of these references work in a truly coarse set-up) but also the cross product between the K-theories of Roe algebras (cf. [9]). All of these cross products were complemented by slant products in [9] and, moreover, the slant product between the K-theory of the Roe algebra and the K-theory of the stable Higson corona also generalizes the pairing introduced in [7] to dualize the coarse assembly map. Further multiplicative structures are the ring and module multiplications between the K-theory of the stable Higson corona and the K-theory of the Roe algebra constructed in [18, 20], which should be understood

as cup and cap products and have applications to the index theory of twisted operators (see also [19] for a related cap product, which was based on a conjecture in [15]).

Finally, there is also the secondary (cup) product on the coarse K-theory defined in [18], which is defined for the pairs of spaces $(X, \{p\})$ if X is contractible to p in a coarse geometric sense. The purpose of the present paper is to generalize it to a vast class of coarse cohomology theories and to also introduce dual cap products between coarse homology and cohomology theories. Additionally, we define our new products relative to more general subspaces than just single points.

More precisely, the coarse (co-)homology theories under consideration are ones that satisfy the Eilenberg–Steenrod like axioms of [21] including the so-called strong homotopy axiom, but here we shall stay within the non-equivariant world, because interesting group actions are rarely compatible with the subspaces and homotopies under consideration anyway. Given three such coarse cohomology theories EX_i^* , EX_{ii}^* , EX_{iii}^* and a cross product

$$\times: EX_i^m(X, A) \otimes EX_{ii}^n(Y, B) \rightarrow EX_{iii}^{m+n}((X, A) \times (Y, B))$$

between them or two coarse homology theories EX_*^i , EX_*^{iii} , a coarse cohomology theory EX_{ii}^* and a slant product

$$/: EX_m^i((X, A) \times (Y, B)) \otimes EX_{ii}^n(Y, B) \rightarrow EX_{m-n}^{iii}(X, A)$$

between them, our secondary cup and cap products take the form

$$\smile: EX_i^m(X, A) \otimes EX_{ii}^n(X, B) \rightarrow EX_{iii}^{m+n-1}(X, A \cup B),$$

$$\frown: EX_m^i(X, A \cup B) \otimes EX_{ii}^n(X, B) \rightarrow EX_{m-n+1}^{iii}(X, A),$$

respectively, and they exist whenever the subspaces $A, B \subset X$ are weak generalized controlled deformation retracts as defined in Definition 6.1. Note that such subspaces are always non-empty, so our secondary products do not exist on the absolute groups.

In a nutshell, the construction works as follows: Recall that one standard construction of primary cup products is pulling back a cross product under the diagonal map $\Delta: (X, A \cup B) \rightarrow (X, A) \times (X, B)$, i. e., $x \cup y := \Delta^*(x \times y)$, and similarly primary cap products are obtained from slant products via $x \cap y := \Delta_*(x)/y$. If A or B is a weak generalized controlled deformation retract, then Δ is homotopic to maps $(X, A \cup B) \rightarrow (A, A) \times (X, B)$ or $(X, A \cup B) \rightarrow (X, A) \times (B, B)$, respectively, and hence, the primary cup and cap products vanish. Now, for the secondary products we combine the two homotopies to obtain a “suspended diagonal map”

$$\Gamma: (X, A \cup B) \times ([-1, 1], \{-1, 1\}) \rightarrow (X, A) \times (X, B), \quad (1)$$

whose domain equipped with a certain coarse structure is a kind of suspension of $(X, A \cup B)$. The secondary cup and cap products are then defined up to a sign by replacing Δ with Γ in the equations above and additionally applying a suspension homomorphism. This construction is compatible with the usual slogan that secondary products arise by comparing two different reasons for the vanishing of primary products.

Note also that whilst in algebraic topology one often goes the other way around and defines cross products from cup products by $x \times y := \pi_1^*x \cup \pi_2^*y$ using the two projections $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$ (and similarly slant products from cap products by $x/y := (\pi_1)_*(x \cap \pi_2^*y)$), this does not work in coarse geometry, because π_1, π_2 are not

proper and hence not coarse maps. Thus, it is indeed imperative to start our construction with the more substantial cross and slant products.

This paper does not limit itself to the construction of the secondary cup and cap products and proving their basic properties. The main results are the following comparison theorems to other primary and secondary cup and cap products.

First of all, Roe's secondary (cup) product from [14, Section 2.4], which is defined at the level of cocycles, can be directly generalized to a secondary product in ordinary coarse cohomology

$$*: HX^m(X, A; M_1) \otimes HX^n(X, B; M_2) \rightarrow HX^{m+n-1}(X, A \cup B; M_3)$$

for all coarsely connected countably generated coarse spaces X with non-empty subspaces $A, B \subset X$ which are coarsely excisive. The non-emptiness assumption on A, B is made to resolve a well-definedness issue with Roe's original definition in degrees $m = 0$ or $n = 0$, but in degrees $m, n \geq 1$ we can also recover Roe's secondary product between the absolute groups by simply taking $A = B$ to be a singleton. The coefficients are multiplied using a given homomorphism of abelian groups $M_1 \otimes M_2 \rightarrow M_3$. However, we point out that we had to slightly correct the signs in Roe's definition. Similarly, a secondary (cap) product

$$*: HX_m(X, A \cup B; M_1) \otimes HX^n(X, B; M_2) \rightarrow HX_{m-n+1}(X, A \cup B; M_3)$$

between ordinary coarse homology and ordinary coarse cohomology can be defined under the same circumstances.

Theorem A (cf. Theorems 7.6 and 7.11) *Let X be a coarsely connected countably generated coarse space, $A, B \subset X$ weak generalized controlled deformation retracts which are coarsely excisive and let $M_1 \otimes M_2 \rightarrow M_3$ be a homomorphism of abelian groups. Then, the secondary product $*$ coincides with our secondary cup product $\smile$ and the secondary product $\ast$ coincides with our secondary cap product.*

Secondly, a large class of coarse (co-)homology theories EX_*, EX^* on the category of pairs of countably generated coarse spaces of bornologically bounded geometry can be obtained by coarsifying topological (co-)homology theories E_*, E^* for σ -locally compact spaces via a Rips complex construction. If E_* or E^* even satisfies the strong excision axiom and $(\partial X, \partial A)$ is a pair of Higson dominated coronas for the pair of coarse spaces (X, A) , then there are the transgression maps

$$T_*^{X,A}: EX_{*+1}(X, A) \rightarrow E_*(\partial X, \partial A), \quad T_{*,A}^*: E^*(\partial X, \partial A) \rightarrow EX^{*+1}(X, A)$$

relating the coarse (co-)homology of the pair of coarse spaces to the topological (co-)homology of the coronas. Furthermore, a cross or slant product between topological theories induces a cross or slant product between the coarse theories. Hence, we obtain in particular a topological primary cup or cap product and a secondary coarse cup or cap product, and the next main theorem says that transgression is compatible with them.

Theorem B (cf. Theorem 8.8) *If X is a countably generated coarse spaces of bornologically bounded geometry with a Higson dominated corona ∂X and if $A, B \subset X$ are weak generalized controlled deformation retracts with corresponding coronas $\partial A, \partial B$, then the associated above-mentioned transgression maps and cup or cap products are compatible*

in the following way:

$$\begin{aligned} \forall x \in E_i^*(\partial X, \partial A), y \in E_{ii}^*(\partial X, \partial B): & \quad T_{X, A \cup B}^*(x \cup y) = T_{X, A}^*(x) \uplus T_{X, B}^*(y), \\ \forall x \in EX_*^i(X, A \cup B), y \in E_{ii}^*(\partial X, \partial B): & \quad T_*^{X, A}(x \cap T_{X, B}^*(y)) = T_*^{X, A \cup B}(x) \cap y. \end{aligned}$$

Part of the importance of this theorem stems from the fact that the transgression maps are isomorphisms in many cases, in particular for pairs of open cones $(\mathcal{O}K, \mathcal{O}L)$ over pairs of compact metric spaces (K, L) if we choose the base spaces themselves as coronas, $(\partial \mathcal{O}K, \partial \mathcal{O}L) = (K, L)$. Then, the theorem implies that the theory of our secondary cup and cap products is at least as rich as the corresponding theory of primary cup and cap products on compact metrizable spaces, see Example 8.9.

Thirdly and most importantly, for all pairs of coarsely connected proper metric spaces (X, A) and all coefficient C^* -algebras D there are the assembly and coassembly maps

$$\begin{aligned} \mu: KX_*(X, A; D) &\rightarrow K_*(C^*(X, A; D)), \\ \mu^*: K_{-*}(c(X, A; D)) &\rightarrow KX^{1+*}(X, A; D), \end{aligned}$$

respectively. On KX^* and KX_* , we have our secondary cup and cap products, whereas on the K -theory of the stable Higson coronas we have a primary cup product

$$\cup: K_{-m}(c(X, A; D)) \otimes K_{-n}(c(X, B; E)) \rightarrow K_{-m-n}(c(X, A \cup B; D \otimes E))$$

and if in addition X has coarsely bounded geometry and A is non-empty, then there is a primary cap product between the K -theory of the Roe algebra and the K -theory of the stable Higson corona,

$$\cap: K_m(C^*(X, A \cup B; D)) \otimes K_{-n}(c(X, B; E)) \rightarrow K_{m-n}(C^*(X, A; D \otimes E)).$$

Theorem C (cf. Theorem 9.1) *Let X be a coarsely connected proper metric space, $A, B \subset X$ weak generalized controlled deformation retracts and let D, E be C^* -algebras. Then,*

$$\mu^*(x \cup y) = \mu^*(x) \uplus \mu^*(y)$$

for all $x \in K_(c(X, A; D)), y \in K_*(c(X, B; E))$. If in addition X has coarsely bounded geometry, then*

$$\mu(x \cap \mu^*(y)) = \mu(x) \cap y$$

for all $x \in KX_(X, A \cup B; D), y \in K_*(c(X, B; E))$.*

The assembly map μ plays an important role in non-commutative coarse geometry. It is subject to the important coarse Baum–Connes isomorphism conjecture, and it is the index map in coarse index theory, i. e., it maps fundamental classes of elliptic operators to their coarse indices. In these fields, the additional algebraic structures provided by the theorem could turn out to be very useful.

We do not have any concrete applications at the moment, but one short-term objective of the cap products could be to develop a coarse version of Poincaré duality.

Furthermore, the author is also thankful to Alexander Engel for pointing out that the second part of Theorem C seems to explain a strange phenomenon concerning the index formula of [20, Theorem 8.7]. Let M be a complete Riemannian manifold of bounded

geometry, $K \subset M$ a non-empty compact subset and C_1, C_2 unital C^* -algebras. Furthermore, let D be the Dirac operator of a Dirac bundle $S \rightarrow M \setminus K$ of finitely generated projective Hilbert C_1 -modules and let $E \rightarrow M \setminus K$ be a bundle of finitely generated projective Hilbert- C_2 -modules of vanishing variation. Then, the index formula says that the coarse index

$$\text{ind}(D_E) \in K_*(C^*(M, K; C_1 \otimes C_2))$$

of the operator D twisted by E is equal to the cap product of the coarse index $\text{ind}(D) \in K_*(C^*(M, K; C_1))$ of D with a class $\llbracket E \rrbracket_c \in K_*(C^*(M, K; C_2))$, and a more general index formula probably holds if we replace K by arbitrary non-empty closed subsets $A, B \subset M$. Now, the strange phenomenon is as follows: The condition on the bundle E says that it can be trivialized by a smooth projection valued function $P: M \setminus K \rightarrow C_2 \otimes \mathfrak{K}$ whose variation vanishes at infinity, but this function itself is not part of the presumed data. Different choices of P might result in different classes $\llbracket E \rrbracket_c$, but the coarse index $\text{ind}(D_E)$ is independent of it according to a simple application of bordism invariance. In some cases, Theorem C provides an explanation: If K is additionally a weak generalized controlled deformation retract of M , then the coarse index $\text{ind}(D)$ is the image of a fundamental class $[D] \in KX_*(M, K; C_1)$ under the coarse assembly map μ and hence, we have

$$\text{ind}(D) \cap \llbracket E \rrbracket_c = \mu([D]) \cap \llbracket E \rrbracket_c = \mu([D] \cap \mu^*(\llbracket E \rrbracket_c)).$$

Therefore, if the class $\mu^*(\llbracket E \rrbracket_c) \in KX^{1-*}(M, K; C_2)$ is a purely topological invariant of E , i. e., independent of P , then the above equation explains why the cap product on the right hand side of the index formula $\text{ind}(D_E) = \text{ind}(D) \cap \llbracket E \rrbracket_c$ is indeed independent of P . In Sect. 10, we will give a very simple proof that this condition is true if M is a uniformly contractible manifold of bounded geometry, but it is not unreasonable to conjecture that it also holds in greater generality.

The big remaining mystery about our secondary products is the question of when we really need the subspaces $A, B \subset X$ to be weak generalized controlled deformation retracts. As we have explained above, secondary products on ordinary coarse (co-)homology can also be defined differently under the apparently unrelated condition of coarse excisiveness, and then Theorem A tells us that the secondary products agree whenever the two conditions are satisfied simultaneously. Furthermore, Theorem 9.7 will show that the secondary cap product between coarse K-homology and coarse K-theory

$$\cap: KX_m(X, A \cup B; D) \otimes KX^n(X, B; E) \rightarrow KX_{m-n+1}(X, A; D \otimes E)$$

pulls back under the coassembly map $\mu^*: K_{1-n}(C(X, B; E)) \rightarrow KX^n(X, B; E)$ to a primary cap product which, surprisingly, can be defined without any conditions on $A, B \subset X$ except that A should be non-empty. For now, the controlled deformations of X onto A and B are an essential part of the construction, but the above examples indicate that this condition should possibly not play such a big role.

The present paper is organized as follows. First of all, we should mention that *we expect the reader to be fully familiar with the coarse geometric language and axiomatic framework for coarse (co-)homology theories presented in [21]*. In fact, the contents of that paper (in the non-equivariant set-up) were originally planned to be part of the present paper, before the author realized that they are so extensive that outsourcing them into a separate publication is fully justified. In the further course of this paper, we will only recall the less intuitive definitions.

One of the new notions from [21] is generalized coarse homotopies. Section 2 generalizes them even further to generalized controlled homotopies, which are fundamental to the construction of our secondary products, and provides a toolkit for working with them. Afterward, we can formulate in Sect. 3 the framework and standing assumptions on the (co-)homology theories under consideration and the underlying categories of coarse spaces.

The further ingredients for our secondary products, namely coarse suspension and the coarse cross and slant products, will be introduced in Sects. 4 and 5. The secondary products will then be constructed in Sect. 6, where we will also prove their basic properties.

The final three main Sects. 7, 8, 9 discuss the matters leading to Theorems A, B, C, respectively.

The exposition is concluded by a brief discussion of vector bundles of vanishing variation and their K-theory classes in Sect. 10, such that one of the motivational examples given in this introduction is non-vacuous.

2 Generalized controlled and coarse homotopies

We refer the reader to [21, Section 2.1] for most of the coarse geometric terminology used in this paper. However, we need to expand more on the novel notion of *generalized coarse homotopies*, which was introduced and shown to generalize the classical coarse homotopies in [21, Section 2.2]. More precisely, this section is about introducing the even more general *generalized controlled homotopies* and developing a large toolkit for working with it, which consists mainly of non-trivial statements analogue to trivial properties of topological homotopies.

Recall that for $E \subset X \times X$ and $F \subset Y \times Y$, we write

$$E \tilde{\times} F := \{(x, z, y, w) \mid (x, y) \in E \wedge (z, w) \in F\} \subset (X \times Y) \times (X \times Y)$$

and if $E, F \subset X \times X$ and $A \subset X$, we furthermore write

$$\begin{aligned} E \circ F &:= \{(x, z) \mid \exists y: (x, y) \in E \wedge (y, z) \in F\}, \\ E \circ A &:= \{x \mid \exists y \in A: (x, y) \in E\}. \end{aligned}$$

If E is an entourage in a coarse space X and $A \subset X$, then we denote the latter set also by $\text{Pen}_E(A)$.

Definition 2.1 Let $(X, \mathcal{E}_X)$ be a coarse space and K a connected compact Hausdorff space. We denote by $\mathcal{N}(X; K)$ the set of all families $\mathcal{U} = \{U_x\}_{x \in X}$ of neighborhoods U_x of the diagonal Δ_K in $K \times K$ indexed over points x of X . For each such $\mathcal{U}$, we define the *warped Cartesian product* $X \times_{\mathcal{U}} K$ to be the set $X \times K$ equipped with the coarse structure $\mathcal{E}_{\mathcal{U}}$ generated by the entourages

$$\begin{aligned} E \tilde{\times} \Delta_K &= \{((x, s), (y, s)) \mid (x, y) \in E \wedge s \in K\} \quad \text{for } E \in \mathcal{E}_X \text{ and} \\ E_{\mathcal{U}} &:= \{((x, s), (x, t)) \mid x \in X \wedge (s, t) \in U_x\}. \end{aligned}$$

We call another such collection $\mathcal{V} = \{V_x\}_{x \in X} \in \mathcal{N}(X; K)$ a *refinement* of $\mathcal{U} = \{U_x\}_{x \in X}$ if $V_x \subset U_x$ for all $x \in X$. If $(Y, \mathcal{E}_Y)$ is another a coarse space, L another compact Hausdorff space, $f: Y \rightarrow X$ a controlled map and $g: L \rightarrow K$ continuous, then we define the *pullback*

of $\mathcal{U}$ under $f \times g$ as the collection $(f \times g)^*(\mathcal{U}) := \{(g \times g)^{-1}(U_{f(y)})\}_{y \in Y} \in \mathcal{N}(Y; L)$ of neighborhoods of the diagonal in $L \times L$.

Lemma 2.2 *With the notation as in the definition, we have:*

1. *The bounded subsets of a warped Cartesian product $X \times_{\mathcal{U}} K$ are exactly the subsets which are contained in some $A \times K$ with $A \subset X$ bounded.*
2. *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then the coarse structure $\mathcal{E}_{\mathcal{V}}$ is finer than $\mathcal{E}_{\mathcal{U}}$ and hence, the identity map $X \times_{\mathcal{V}} K \rightarrow X \times_{\mathcal{U}} K$ is a coarse map.*
3. *The warped Cartesian product with respect to the pullback $(f \times g)^*(\mathcal{U})$ has the property that $f \times g: Y \times_{(f \times g)^*(\mathcal{U})} L \rightarrow X \times_{\mathcal{U}} K$ is a controlled map. If f is not only a controlled but even a coarse map, then $f \times g$ is also a coarse map.*

Proof As K is compact and Hausdorff, 2.2 can be proven just like the special case of K being an interval in [21, Section 2.1]: Assume that $A \subset X$ is bounded and choose $x \in X$ in the same coarse component as A . Then, $A \times \{x\}$ is an entourage of X . Furthermore, fix any $s \in K$. The key observation is that there is $n \in \mathbb{N}$ such $U_x^n \circ \{s\}$ is all of K , where $U_x^n := U_x \circ \dots \circ U_x \subset K \times K$. To see this, we may assume that U_x is even an *open* neighborhood of the diagonal and consider the open subset $L := \bigcup_{n \in \mathbb{N}} U_x^n \circ \{s\} \subset K$. It is non-empty, because $s \in L$. It is also closed, because if t lies in the closure of L , then L intersects the neighborhood $U_x^{-1} \circ \{t\}$ of t and hence, $t \in U_x \circ L = L$. Connectedness of K now implies $L = K$ and from compactness of K we conclude that K is already equal to one of the open subsets $U_x^n \circ \{s\}$, as claimed above. Therefore, $A \times K = ((A \times \{x\}) \tilde{\times} \Delta_K) \circ E_{\mathcal{U}}^n \circ \{(x, s)\}$ is bounded.

Conversely, if $A \times K$ is bounded, then it is contained in

$$(E_1 \tilde{\times} \Delta_K) \circ E_{\mathcal{U}} \circ (E_2 \tilde{\times} \Delta_K) \circ E_{\mathcal{U}} \circ \dots \circ E_{\mathcal{U}} \circ (E_k \tilde{\times} \Delta_K) \circ \{(x, s)\}$$

for some entourages $E_1, \dots, E_k$ of X and some $(x, s) \in X \times I$. But then the subset $K \subset E_1 \circ \dots \circ E_k \circ \{x\}$ is bounded.

Point 2.2 follows trivially from $E_{\mathcal{V}} \subset E_{\mathcal{U}}$ and 2.2 follows from

$$(f \times g \times f \times g)(E_{(f \times g)^*(\mathcal{U})}) \subset E_{\mathcal{U}}$$

for controlledness together with Point 2.2 for properness. $\square$

Lemma 2.3 *Let X be a coarse space and K, L be compact connected Hausdorff spaces.*

1. *Given $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; K)$ and $\mathcal{V} = \{V_{x,s}\}_{(x,s) \in X \times K} \in \mathcal{N}(X \times_{\mathcal{U}} K; L)$, there is a collection $\mathcal{W} = \{W_x\}_{x \in X} \in \mathcal{N}(X; K \times L)$ such that the identity map*

$$X \times_{\mathcal{W}} (K \times L) \leftrightarrow (X \times_{\mathcal{U}} K) \times_{\mathcal{V}} L$$

is a coarse equivalence

2. *Given $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; K)$ and $\mathcal{V} = \{V_x\}_{x \in X} \in \mathcal{N}(X; L)$, then upon identifying them with the pullback collections $\{U_x\}_{(x,t) \in X \times L} \in \mathcal{N}(X \times_{\mathcal{V}} L; K)$ and $\{V_x\}_{(x,s) \in X \times K} \in \mathcal{N}(X \times_{\mathcal{U}} K; L)$, respectively, the canonical bijection*

$$(X \times_{\mathcal{U}} K) \times_{\mathcal{V}} L \leftrightarrow (X \times_{\mathcal{V}} L) \times_{\mathcal{U}} K$$

is a coarse equivalence.

3. Given $\mathcal{W} = \{W_x\}_{x \in X} \in \mathcal{N}(X; K \times L)$, there are collections $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X, K)$ and $\mathcal{V} = \{V_{x,s}\}_{(x,s) \in X \times K} \in \mathcal{N}(X \times_{\mathcal{U}} K; L)$ such that the identity map

$$(X \times_{\mathcal{U}} K) \times_{\mathcal{V}} L \rightarrow X \times_{\mathcal{W}} (K \times L)$$

is a coarse map.

Proof Note that properness of all of the maps is clear from Lemma 2.2.1 and it remains to show controlledness.

For all $x \in X$, $(s', s) \in U_x$ and $(t', t) \in V_{(x,s)}$, we have

$$((x, s', t'), (x, s, t)) = ((x, s', t'), (x, s, t')) \circ ((x, s, t'), (x, s, t)) \in (E_{\mathcal{U}} \tilde{\times} \Delta_L) \circ E_{\mathcal{V}},$$

so when we define $\mathcal{W}$ as the collection of the open neighborhoods

$$W_x := \bigcup_{(s,t) \in K \times L} (U_x \circ \{s\}) \times (V_{(x,s)} \circ \{t\}) \times (U_x \circ \{s\}) \times (V_{(x,s)} \circ \{t\})$$

and we recall that $U_x \circ \{s\} := \{s' \in K \mid (s', s) \in U_x\}$, etc., then $E_{\mathcal{W}} \subset (E_{\mathcal{U}} \tilde{\times} \Delta_L) \circ E_{\mathcal{V}} \circ E_{\mathcal{V}}^{-1} \circ (E_{\mathcal{U}} \tilde{\times} \Delta_L)^{-1}$, showing that the map from the left to the right is controlled. For the other direction, we use that $E_{\mathcal{U}} \tilde{\times} \Delta_L \subset E_{\mathcal{W}}$ and $E_{\mathcal{V}} \subset E_{\mathcal{W}}$.

The second claim is trivial.

For the third claim, we note that due to the compactness of L we can find for each $(x, s) \in X \times K$ a neighborhood $U'_{x,s} \subset K$ of s such that $U'_{x,s} \times \{t\} \times U'_{x,s} \times \{t\} \subset W_x$ for all $t \in L$ and similarly the compactness of K implies for each $(x, t) \in X \times L$ the existence of a neighborhood $V'_{x,t} \subset L$ of t such that $\{s\} \times V'_{x,t} \times \{s\} \times V'_{x,t} \subset W_x$ for all $s \in K$. Defining $U_x := \bigcup_{s \in K} U'_{x,s} \times U'_{x,s}$ and $V_x := \bigcup_{t \in L} V'_{x,t} \times V'_{x,t}$, we see that $E_{\mathcal{U}} \tilde{\times} \Delta_L \subset E_{\mathcal{W}} \supset E_{\mathcal{V}}$. $\square$

Now, we recall [21, Definition 2.14] about generalized coarse homotopies and simultaneously introduce generalized controlled homotopies.

Definition 2.4 A *generalized controlled/coarse homotopy* between two controlled/coarse maps $f, g: X \rightarrow Y$ is a controlled/coarse map $H: X \times_{\mathcal{U}} I \rightarrow Y$, where $I = [a, b]$ is a closed interval and $\mathcal{U} \in \mathcal{N}(X; I)$, which restricts to f on $X \times \{a\}$ and to g on $X \times \{b\}$.

If $f, g: (X, A) \rightarrow (Y, B)$ are controlled/coarse maps between pairs of coarse spaces, then a *generalized controlled/coarse homotopy* between them is a generalized coarse homotopy between the absolute controlled/coarse maps $f, g: X \rightarrow Y$ which takes $A \times I$ to B .

In both cases, we call f *generalized controlledly/coarsely homotopic* to g .

In [21, Section 2], we have shown that any coarse homotopy $X \times \mathbb{Z} \rightarrow X$ in the classical sense can be extended to a generalized coarse homotopy $X \times_{\mathcal{U}} [-\infty, \infty] \rightarrow X$ for a certain family $\mathcal{U} \in \mathcal{N}(X; [-\infty, \infty])$. Of course, the analogue statement for controlled homotopies holds as well. This can be utilized as the perhaps most important source of examples.

The generalized notion of coarse and controlled homotopies is much more flexible than the classical one. Another big practical advantage is that there are distinct boundaries $X \times \{a\}$, $X \times \{b\}$ on which they can be evaluated. The generalized notion behaves to the classical one approximately as synchronous combings behave to asynchronous ones in geometric group theory, as we will see later in Example 6.3.

Lemma 2.5 Let $f: Y \rightarrow X$ be a controlled/coarse map between coarse spaces, let $g_0, g_1: L \rightarrow K$ be two continuous maps between compact Hausdorff spaces and $\mathcal{U} =$

$\{U_x\}_{x \in X} \in \mathcal{N}(X; K)$. If I is a closed interval and $H: L \times I \rightarrow K$ is a homotopy between g_0, g_1 , then there is a common refinement $\mathcal{U}' \in \mathcal{N}(X; K)$ of $(f \times g_i)^*(\mathcal{U})$ ($i = 0, 1$) and $\mathcal{V}' = \{V'_y\}_{y \in Y} \in \mathcal{N}(Y; I)$ such that $f \times H: (Y \times_{\mathcal{U}'} L) \times_{\mathcal{V}'} I \rightarrow X \times_{\mathcal{U}} K$ is a generalized controlled/coarse homotopy between $f \times g_0, f \times g_1: Y \times_{\mathcal{U}'} L \rightarrow X \times_{\mathcal{U}} K$.

Proof If $\mathcal{W}$ is the pullback of $\mathcal{U}$ under the map $f \times H$, then according to Lemma 2.2.3 and Lemma 2.3.1 we have a controlled map

$$(Y \times_{\mathcal{U}'} L) \times_{\mathcal{V}'} I \xrightarrow{\text{id}} Y \times_{\mathcal{W}} (L \times I) \xrightarrow{f \times H} X \times_{\mathcal{U}} K$$

for certain collections $\mathcal{U}', \mathcal{V}'$. If necessary, $\mathcal{U}'$ can be refined to be finer than $(f \times g_i)^*(\mathcal{U})$ for $i = 0, 1$.

If in addition f is proper, then so is $f \times H$. $\square$

The lemma does not give too much information about how much finer $\mathcal{U}'$ is in comparison with the pullback collections $(f \times g_i)^*(\mathcal{U})$. Although this is almost always sufficient, we need a more sophisticated version in one special case.

Lemma 2.6 *Let X be a coarse space, I a closed interval and $\mathcal{U} \in \mathcal{N}(X; I)$. Then, there is a refinement $\mathcal{U}' \in \mathcal{N}(X; I)$ of $\mathcal{U}$ and $\mathcal{V}' \in \mathcal{N}(X; [0, 1])$ such that for all Lipschitz maps $g_0, g_1: I \rightarrow I$ the maps $\text{id}_X \times g_0, \text{id}_X \times g_1: X \times_{\mathcal{U}'} I \rightarrow X \times_{\mathcal{U}'} I$ are coarse maps and*

$$H: (X \times_{\mathcal{U}'} I) \times_{\mathcal{V}'} [0, 1] \rightarrow X \times_{\mathcal{U}'} I, \quad (x, s, t) \mapsto (x, (1-t)g_0(s) + tg_1(s))$$

is a generalized coarse homotopy between them.

In particular, if $g_0 = \text{id}_I$ and $g_1: I \rightarrow J \subset I$ is the canonical retraction onto a closed interval or a point, then $X \times_{\mathcal{U}'} J$ is a strong generalized coarse deformation retract of $X \times_{\mathcal{U}'} I$, i. e., there is a generalized coarse homotopy between the identity on $X \times_{\mathcal{U}'} I$ and the retraction $X \times_{\mathcal{U}'} I \rightarrow X \times_{\mathcal{U}'} J$ which is constant on all points of $X \times_{\mathcal{U}'} J$.

Here, we have denoted the restriction of $\mathcal{U}'$ to a collection in $\mathcal{N}(X; J)$ by the same letter and ‘strong’ is supposed to mean that the subspace $X \times_{\mathcal{U}'} J$ stays pointwise fixed throughout the homotopy.

Proof Given $\mathcal{U} = \{U_x\}_{x \in X}$, we can choose for each $x \in X$ an $\varepsilon_x > 0$ such that

$$U'_x := \{(s_1, s_2) \in I^2 \mid |s_1 - s_2| < \varepsilon_x\} \subset U_x$$

and subsequently we also define $V'_x := \{(s_1, s_2) \in [0, 1]^2 \mid |s_1 - s_2| < \varepsilon_x\}$. Then, $\mathcal{U}' := \{U'_x\}_{x \in X}$ and $\mathcal{V}' := \{V'_x\}_{x \in X}$ do the job, because $H \times H$ clearly maps $E_{\mathcal{U}'} \tilde{\times} \Delta_{[0,1]}$ into the entourage $E_{\mathcal{U}'}^m = E_{\mathcal{U}'} \circ \dots \circ E_{\mathcal{U}'}$ for $m \in \mathbb{N}$ bigger than the Lipschitz constants of g_0, g_1 , it maps $E_{\mathcal{V}'}$ into $E_{\mathcal{U}'}$ for $n \in \mathbb{N}$ bigger than the diameter of I and of course it also maps $E \tilde{\times} \Delta_{I \times [0,1]}$ into $E \tilde{\times} \Delta_I$ for all entourages E of X . $\square$

3 Framework and standing assumptions

In the present paper, we always use the axiomatic framework of [21, Section 3] for (non-equivariant) coarse (co-)homology theories on an admissible category $\mathcal{C}$ of pairs of coarse spaces. Admissible categories are full sub-categories of the category of pairs of coarse spaces and coarse maps. They are convenient, because many coarse (co-)homology theories are a priori not defined on all coarse spaces and extending them usually requires a lot of additional effort, which sometimes one simply might want to avoid.

Most of the axioms are rather canonical analogues to Eilenberg–Steenrod axioms for topological (co-)homology theories, so we do not see the necessity for recalling all of them in detail here. The mandatory axioms are homotopy, exactness and excision, and then, there are also a few optional axioms, of which only the strong homotopy axiom is of relevance here.

The exactness axiom is obvious and hence does not require any further explanation. There is also no need to review the homotopy axiom, because it is implied by the strong homotopy axiom and we will only make use of the latter. However, we should indeed review the excision and strong homotopy axiom to fix some notation.

All coarse (co-)homology theories are postulated to satisfy the *excision axiom*, which says that all excisions as defined in the following definition induce isomorphisms.

Definition 3.1 An inclusion map between objects of $\mathfrak{C}$ of the form $(X \setminus C, X \setminus A) \rightarrow (X, A)$ with $C \subset A \subset X$ is called an *excision* if for every entourage E of X , there is an entourage F of X such that $\text{Pen}_E(A) \setminus C \subset \text{Pen}_F(A \setminus C)$.

A *triad* in $\mathfrak{C}$ is a triple $(X; A, B)$ such that both (X, A) and (X, B) are objects in $\mathfrak{C}$. We call a triad $(X; A, B)$ in $\mathfrak{C}$ (*coarsely*) *excisive* in $\mathfrak{C}$ if for every entourage E of X there is an entourage F of X such that $\text{Pen}_E(A) \cap \text{Pen}_E(B) \subset \text{Pen}_F(A \cap B)$.

It is readily verified that $(X \setminus C, X \setminus A) \rightarrow (X, A)$ is an excision if and only if $(X; A, X \setminus C)$ is an excisive triad. We emphasize, however, that in this case we have $X = A \cup (X \setminus C)$, whereas general excisive triads $(X; A, B)$ do not need to satisfy $X = A \cup B$.

The other noteworthy axiom is the *strong homotopy axiom*, which holds for a coarse (co-)homology theory if generalized coarsely homotopic coarse maps (cf. Definition 2.4) induce the same map on (co-)homology. This axiom is optional, because in general we only demand the weaker *homotopy axiom*, which says that maps which are coarsely homotopic in the classical sense induce the same map.

We are going to construct our secondary products only for coarse (co-)homology theories which satisfy this strong homotopy axiom. These include the ordinary coarse (co-)homology theories and coarsifications of topological (co-)homology theories for σ -locally compact spaces, which we will analyze later on in Sects. 7, 8.

There are two main examples of coarse (co-)homology theories for which we do not know if the strong homotopy axiom holds, namely the K-theory of the Roe algebras and the K-theory of the stable Higson coronas. It would be the case if the coarse Baum–Connes conjecture and its dual, respectively, were true.

Thus, they cannot be equipped with our secondary products, but that cannot be expected anyway and does not need to bother us at all, because they have even better non-vanishing primary products instead. Also note that these primary products are defined by direct construction and not induced by cross and slant products, which is—besides the lack of generalized coarse homotopy invariance—another and more important reason why in this case the primary products do not vanish as sketched in the lines preceding Display (1) in the introduction. These primary products and their compatibility with secondary products under assembly and co-assembly will be the topic of Sect. 9.

We also have to pose some additional assumptions on the admissible category $\mathfrak{C}$. Recall that admissibility means that $\mathfrak{C}$ is a full sub-category of the category of pairs of coarse spaces which contains $\emptyset = (\emptyset, \emptyset)$ and such that if (X, A) is an object in $\mathfrak{C}$, then so are

$(X, X), (A, A), X = (X, \emptyset)$ and $A = (A, \emptyset)$. However, this is not enough to make our constructions work. On the one hand, the theory of generalized controlled homotopies will play an essential role and therefore we need $\mathfrak{C}$ to be closed under warped products with intervals. On the other hand, our secondary cup and cap products will be obtained from cross and slant products and hence, we also assume that $\mathfrak{C}$ is closed under taking products of coarse spaces. More precisely, what we need is the following.

Assumption 3.2 We assume that $\mathfrak{C}$ is an admissible category with the following properties:

- If $(X, A), (Y, B)$ are objects in $\mathfrak{C}$, then so are

$$(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y),$$

$$(X \times B \cup A \times Y, X \times B) \quad \text{and} \quad (X \times B \cup A \times Y, A \times Y).$$

- If (X, A) is an object in $\mathfrak{C}$, $I = [a, b]$ is a closed interval with subsets $I_2 \subset I_1 \subset I$ which are closed and have finitely many components (i. e., I_1, I_2 are finite disjoint unions of closed intervals and points) and $\mathcal{U} \in \mathcal{N}(X; I)$, then

$$(X, A) \times_{\mathcal{U}} (I, I_i) := (X \times_{\mathcal{U}} I, X \times I_i \cup A \times I)$$

for $i = 1, 2$ and

$$(X \times I_1 \cup A \times I, X \times I_2 \cup A \times I)$$

equipped with the subspace coarse structure of $X \times_{\mathcal{U}} I$ are also objects of $\mathfrak{C}$.

4 Coarse suspension

As a first step towards the secondary products, we construct the coarse suspension homomorphisms, which relate the (co-)homology of a pair of coarse spaces (X, A) with the (co-)homology of a warped product $(X, A) \times_{\mathcal{U}} (I, \partial I)$. But this works only if the collection $\mathcal{U} = \{U_x\}_{x \in X}$ makes the interval I become arbitrary wide as we go far away from A , that is, the U_x become arbitrarily small, because otherwise the inclusion of the subspace $X \times \partial I \cup A \times I$ into $X \times_{\mathcal{U}} I$ would be a coarse equivalence and hence the (co-)homology of $(X, A) \times_{\mathcal{U}} (I, \partial I)$ would vanish by the exactness axiom.

Definition 4.1 Let (X, A) be a pair of coarse spaces and $I \subset \mathbb{R}$ a closed interval. (In particular, the endpoints are finite and not $\pm\infty$.) We say that $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; I)$ satisfies the *suspension condition* if for each $\varepsilon > 0$ there is a penumbra $\text{Pen}_\varepsilon(A)$ of A such that $\sup_{(s,t) \in U_x} |s - t| \leq \varepsilon$ for all $x \in X \setminus \text{Pen}_\varepsilon(A)$. The set of all such $\mathcal{U}$ is denoted by $\mathcal{N}^\sigma(X, A; I)$.

Remark 4.2 Several things should be pointed out about this definition.

1. If $\mathcal{U}$ satisfies the suspension condition and $\mathcal{U}'$ is a refinement of $\mathcal{U}$, then $\mathcal{U}'$ also satisfies the suspension condition.
2. The set $\mathcal{N}^\sigma(X, A; I)$ is non-empty if and only if there is a sequence of entourages $E_1, E_2, \dots$ of X such that $X = \bigcup_{i=1}^\infty \text{Pen}_{E_i}(A)$. In this case, each family in $\mathcal{N}(X; I)$ has a refinement in $\mathcal{N}^\sigma(X, A; I)$.

In practice, this is not a big issue, as many interesting admissible categories comprise only countably generated coarse spaces.

3. The suspension condition can only be satisfied if A is non-empty, unless X itself is empty. More generally, it can only be satisfied if each coarse component of X intersects A nontrivially. Coincidentally, this is the only case which we will be interested in later on.
4. If one of the coarse components of X is disjoint from A , then our coarse suspensions will not exist. As a fallback option there is always the classical notion of coarse suspension, i. e., taking the product with $\mathbb{R}$, but it has a different scope of application. We briefly discuss the relation between the two notions of coarse suspension at the end of this section.

Lemma 4.3 *Let (X, A) be a pair of coarse spaces, I a closed interval with boundary $\partial I = \{e_1, e_2\}$ (we do not specify which of e_1, e_2 is which end) and $\mathcal{U} \in \mathcal{N}^\sigma(X, A; I)$. Then, the inclusion*

$$(X, A) \rightarrow (X \times \partial I \cup A \times I, X \times \{e_2\} \cup A \times I), \quad x \mapsto (x, e_1),$$

where the spaces on the right are equipped with the subspace coarse structure of $X \times_{\mathcal{U}} I$, is an excision and hence induces isomorphisms on coarse (co-)homology.

Proof Any entourage of $X \times_{\mathcal{U}} I$ is contained in one of the form

$$E = (E_1 \bar{\times} \Delta_I) \circ E_{\mathcal{U}} \circ (E_2 \bar{\times} \Delta_I) \circ E_{\mathcal{U}} \circ \cdots \circ E_{\mathcal{U}} \circ (E_n \bar{\times} \Delta_I) \circ E_{\mathcal{U}}$$

with $n \in \mathbb{N}$ and entourages $E_1, \dots, E_n$ of X . Let E' be an entourage of X such that $\sup_{(s,t) \in U_x} |s - t| \leq \frac{|e_1 - e_2|}{n+1}$ for all $x \in X \setminus \text{Pen}_{E'}(A)$ and define the entourage

$$F' := E' \cup (E_1 \circ E') \cup (E_1 \circ E_2 \circ E') \cup \cdots \cup (E_1 \circ \cdots \circ E_n \circ E')$$

of X . Then, for each $(x, e_1) \in \text{Pen}_E(X \times \{e_2\})$ we can find points $x_0 = x, x_1, \dots, x_n \in X$ and $s_0 = e_1, s_1, \dots, s_n = e_2 \in I$ such that $(x_{i-1}, x_i) \in E_i$ and $(s_{i-1}, s_i) \in U_{x_i}$ for all $i = 1, \dots, n$. But this is only possible if $|s_{i-1} - s_i| \geq \frac{|e_1 - e_2|}{n}$ for at least one i , which implies $x_i \in \text{Pen}_{E'}(A)$ and therefore $x \in \text{Pen}_{F'}(A)$. This implication shows

$$\text{Pen}_E(X \times \{e_2\} \cup A \times I) \cap (X \times \{e_1\}) \subset \text{Pen}_{F'}(A \times \{e_1\})$$

for the entourage $F := (F' \cup (E_1 \circ \cdots \circ E_n)) \bar{\times} \Delta_I$, which is exactly the excisiveness condition of [21, Definition 3.2] for the spaces under consideration. $\square$

Lemma 4.4 *Let (X, A) be a pair of coarse spaces in $\mathfrak{C}$, I a closed interval with boundary $\partial I = \{e_1, e_2\}$ and $\mathcal{U} \in \mathcal{N}^\sigma(X, A; I)$. Then, for any coarse homology theory EX_* or coarse cohomology theory EX^* on $\mathfrak{C}$ satisfying the strong homotopy axiom there are homomorphisms*

$$\begin{aligned} \sigma_* : \text{EX}_*(X, A) &\rightarrow \text{EX}_{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)), \\ \sigma^* : \text{EX}^{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) &\rightarrow \text{EX}^*(X, A) \end{aligned}$$

such that σ_ is a right inverse to*

$$\text{EX}_{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) \xrightarrow{\partial} \text{EX}_*(X \times \partial I \cup A \times I, X \times \{e_2\} \cup A \times I) \cong \text{EX}_*(X, A)$$

and σ^ is a left inverse to*

$$\text{EX}^*(X, A) \cong \text{EX}^*(X \times \partial I \cup A \times I, X \times \{e_2\} \cup A \times I) \xrightarrow{\delta} \text{EX}^{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)),$$

where the connecting homomorphisms are those associated with the triple

$$(X \times_{\mathcal{U}} I, \quad X \times \partial I \cup A \times I, \quad X \times \{e_2\} \cup A \times I)$$

and the isomorphisms are those induced by the excision from Lemma 4.3. Furthermore, they are natural under coarse maps $f: (Y, B) \rightarrow (X, A)$, homeomorphisms $g: J \rightarrow I$ of intervals which preserve the chosen distinction of the end points and the associated coarse map on the warped products

$$f \times g: (Y, B) \times_{\mathcal{V}} (J, \partial J) \rightarrow (X, A) \times_{\mathcal{U}} (I, \partial I)$$

from Lemma 2.2.3, with $\mathcal{V}$ being a refinement of $(f \times g)^*(\mathcal{U})$ which automatically is an element of $\mathcal{N}^\sigma(Y, B; J)$. Also, each $\mathcal{U}$ has a refinement $\mathcal{U}'$ for which σ_*, σ^* are isomorphisms.

Proof According to Lemma 2.6, there is a refinement $\mathcal{U}'$ of $\mathcal{U}$ such that

$$(X, A) \times_{\mathcal{U}'} (I, \{e_2\}) = (X \times_{\mathcal{U}'} I, X \times \{e_2\} \cup A \times I) \quad (2)$$

is generalized coarsely homotopy equivalent to (X, X) . Indeed, the generalized coarse homotopy from that lemma between the identity and the retraction $X \times_{\mathcal{U}'} I \rightarrow X \rightarrow X \times_{\mathcal{U}'} I, (x, s) \mapsto (x, e_2)$ preserves the subspace $X \times \{e_2\} \cup A \times I$. Therefore, EX_* and EX^* vanish on the pair (2).

Naturality of the connecting homomorphisms associated with the triples

$$(X \times_{\mathcal{U}'} I, \quad X \times \partial I \cup A \times I, \quad X \times \{e_2\} \cup A \times I)$$

under the coarse map $X \times_{\mathcal{U}'} I \rightarrow X \times_{\mathcal{U}} I$ together with the isomorphisms induced by the excisions from Lemma 4.3 yields the two commutative triangles

$$\begin{array}{ccccc} EX_{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) & \longrightarrow & EX_*(X, A) & & EX^{*+1}((X, A) \times_{\mathcal{U}'} (I, \partial I)) \\ \uparrow & \nearrow \cong & & \nearrow \cong & \uparrow \\ EX_{*+1}((X, A) \times_{\mathcal{U}'} (I, \partial I)) & & EX^*(X, A) & \longrightarrow & EX^{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) \end{array}$$

where the diagonal maps are isomorphisms due to the property of $\mathcal{U}'$ mentioned above and the long exact sequences for triples. This shows the existence of one-sided inverses for $\mathcal{U}$ and two-sided inverses for $\mathcal{U}'$.

To prove naturality of this construction, we pick a common refinement $\mathcal{V}'$ of $\mathcal{V}$ and $(f \times g)^*(\mathcal{U}')$ with the property of Lemma 2.6 and then the claim follows directly from the naturality of connecting homomorphisms under the maps of the diagram

$$\begin{array}{ccc} Y \times_{\mathcal{V}} J & \longrightarrow & X \times_{\mathcal{U}} I \\ \uparrow & & \uparrow \\ Y \times_{\mathcal{V}'} J & \longrightarrow & X \times_{\mathcal{U}'} I \end{array}$$

relative to certain subspaces. In particular, the case $f = \text{id}_X, g = \text{id}_I$ shows that the maps σ_* and σ^* are independent of the choice of the refinement $\mathcal{U}'$ in the construction. $\square$

Definition 4.5 We call $(X, A) \times_{\mathcal{U}} (I, \partial I)$ as in Lemma 4.4 a *suspension* of (X, A) . If e_1 is the lower endpoint of the interval I , then we call the homomorphisms σ_* and σ^* from Lemma 4.4 the associated *suspension homomorphisms/isomorphisms*.

Note that under the condition of Remark 4.2.2, the definition of the suspension homomorphisms can be extended to all $\mathcal{U} \in \mathcal{N}(X; I)$ by choosing a refinement $\mathcal{U}' \in \mathcal{N}^\sigma(X, A; I)$ of $\mathcal{U}$ and composing the suspension homomorphisms for $\mathcal{U}'$ with the homomorphisms induced by the refinement. However, they are harder to work with, because the connecting homomorphisms, to which they are supposed to be one-sided inverses, do not exist for $\mathcal{U}$.

The following lemma tells us how the choice of the distinguished endpoints and the order of consecutive suspensions affects the signs.

Lemma 4.6 *Let EX_* be a coarse homology or EX^* a coarse cohomology theory on $\mathfrak{C}$ satisfying the strong homotopy axiom. Furthermore, let (X, A) be a pair of coarse spaces in $\mathfrak{C}$, $I = [a, b]$ a closed interval and $\mathcal{U} \in \mathcal{N}^\sigma(X, A; I)$.*

1. *If J is another closed interval, $f: J \rightarrow I$ is an orientation preserving and $g: J \rightarrow I$ an orientation reversing homeomorphism, then there is a common refinement $\mathcal{V}$ of $(\text{id} \times f)^*(\mathcal{U})$ and $(\text{id} \times g)^*(\mathcal{U})$ such that the homomorphisms on EX_* or EX^* induced by the two coarse maps*

$$\text{id}_X \times f, \text{id}_X \times g: (X, A) \times_{\mathcal{V}} (J, \partial J) \rightarrow (X, A) \times_{\mathcal{U}} (I, \partial I)$$

are negatives of one another.

2. *If we interchange the roles of e_1 and e_2 in Lemma 4.4, then the homomorphisms σ_* and σ^* will change signs. In particular, if e_1 is the upper endpoint of the interval, then the resulting maps are exactly the negative of the suspension homomorphisms.*
3. *If J is another closed interval and $\mathcal{V} \in \mathcal{N}^\sigma(X, A; J)$, then the suspensions with respect to I commute with the suspensions with respect to J up to the sign -1 , that is, the diagrams*

$$\begin{array}{ccc} EX_*(X, A) & \xrightarrow{\sigma_*} & EX_{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) \\ \downarrow \sigma_* & & \downarrow \sigma_* \\ EX_{*+1}((X, A) \times_{\mathcal{V}} (J, \partial J)) & \xrightarrow{\sigma_*} & EX_{*+2}((X, A) \times_{\mathcal{U}} (I, \partial I) \times_{\mathcal{V}} (J, \partial J)) \\ EX^{*+2}((X, A) \times_{\mathcal{U}} (I, \partial I) \times_{\mathcal{V}} (J, \partial J)) & \xrightarrow{\sigma^*} & EX^{*+1}((X, A) \times_{\mathcal{V}} (J, \partial J)) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ EX^{*+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) & \xrightarrow{\sigma^*} & EX^*(X, A) \end{array}$$

commute up to the sign -1 .

Proof Consider the coarse space $Y := X \times_{\mathcal{U}} I \cup_{X \times \{a\}} X \times_{\mathcal{U}} I$ obtained by gluing together two copies of $X \times_{\mathcal{U}} I$ along the common subspace $X \times \{a\}$ and equipping it with the canonical coarse structure for which the two canonical inclusions $\iota_1, \iota_2: X \times_{\mathcal{U}} I \rightarrow Y$ are inclusions as coarse subspaces, that is, it is generated by entourages of the form $(\iota_i \times \iota_i)(E)$ for entourages E of $X \times_{\mathcal{U}} I$ and $i = 1, 2$. Then, there are three canonical coarse maps $p, p_1, p_2: Y \rightarrow X \times_{\mathcal{U}} I$: The map p is simply defined to be the identity on both halves $\text{im}(\iota_i)$ of Y , whereas p_i maps only $\text{im}(\iota_i)$ identically to $X \times_{\mathcal{U}} I$ and projects the other half $\text{im}(\iota_{3-i})$ onto the subspace $X \times \{a\}$.

Note that the coarse space Y can be identified with some warped Cartesian product $X \times_{\mathcal{W}} K$ for the interval $K = I \cup_{\{a\}} I$ and it contains the two subspaces

$$\begin{aligned} Z &:= \iota_1(X \times \{a, b\} \cup A \times I) \cup \iota_2(X \times_{\mathcal{W}} I), \\ W &:= \iota_1(X \times \{a, b\} \cup A \times I) \cup \iota_2(X \times \{a, b\} \cup A \times I). \end{aligned}$$

Then, Assumption 3.2 tells us that $(Y, Z), (Y, W), (Z, W)$ are also objects in $\mathfrak{C}$ up to this identification. Furthermore, we note that the two inclusions $\iota_1: (X, A) \times_{\mathcal{W}} (I, \partial I) \rightarrow (Y, Z)$ and $\iota_2: (X, A) \times_{\mathcal{W}} (I, \partial I) \rightarrow (Z, W)$ are obviously excisions. With these identifications, it is now clear that the long exact sequence in EX_* associated with the triple (Y, Z, W) decomposes into the split short exact sequences

$$\begin{array}{ccccc} & & \overset{p_2}{\curvearrowright} & & \overset{\iota_1}{\curvearrowright} \\ EX_*((X, A) \times_{\mathcal{W}} (I, \partial I)) & \xrightarrow{\iota_2} & EX_*(Y, W) & \xrightarrow{p_1} & EX_*((X, A) \times_{\mathcal{W}} (I, \partial I)) \end{array}$$

and hence, we get the vertical isomorphisms in the diagram

$$\begin{array}{ccccc} EX_*((X, A) \times_{\mathcal{V}} (J, \partial J)) & \xrightarrow{(\text{id} \times h)_*} & EX_*(Y, W) & \xrightarrow{p} & EX_*((X, A) \times_{\mathcal{W}} (I, \partial I)) \\ & \searrow & \uparrow & & \nearrow \\ & ((\text{id} \times f)_*, (\text{id} \times g)_*) & \downarrow & & + \\ & & EX_*((X, A) \times_{\mathcal{W}} (I, \partial I))^2 & & \end{array}$$

which are inverse to each other and make the right part commute. The map h and commutativity of the left part is obtained as follows. We can choose some homeomorphism $h: J \rightarrow K$ such that $p_1 \circ (\text{id} \times h) = \text{id} \times f'$ and $p_2 \circ (\text{id} \times h) = \text{id} \times g'$ for some continuous maps $f', g': (J, \partial J) \rightarrow (I, \partial I)$ which are homotopic to f and g , respectively, and $p \circ (\text{id} \times h) = \text{id} \times h'$ with a continuous map $h': (J, \partial J) \rightarrow (I, \{b\}) \subset (I, \partial I)$ which is homotopic to the constant map with image b . Then, using Lemma 2.5 shows that there is a common refinement $\mathcal{V}$ of $(\text{id} \times f)^*(\mathcal{W})$ and $(\text{id} \times g)^*(\mathcal{W})$ (among others) such that $p_1 \circ (\text{id} \times h)$, $p_2 \circ (\text{id} \times h)$ and $p \circ (\text{id} \times h)$ are generalized coarsely homotopic to $\text{id} \times f$, $\text{id} \times g$ and $\text{id} \times \text{const}_b$, respectively, as maps $(X, A) \times_{\mathcal{V}} (J, \partial J) \rightarrow (X, A) \times_{\mathcal{W}} (I, \partial I)$. Therefore, the left part of the diagram commutes, too, and the composition of the horizontal arrows factor through $EX_*(X \times \{b\}, X \times \{b\}) = 0$. The homological version of the first claim follows, and the cohomological version is completely dual.

To prove the second claim, we choose an orientation preserving homeomorphism $f: J \rightarrow I$ and an orientation reversing homeomorphism $g: J \rightarrow I$ and let $\mathcal{V}$ be the associated collection from the first statement, which automatically also satisfies the suspension condition. Then, naturality of the homomorphisms from Lemma 4.4 shows that we have a commutative diagram

$$\begin{array}{ccc} EX_*(X, A) & \xrightarrow{\sigma} & EX_{*+1}((X, A) \times_{\mathcal{W}} (I, \partial I)) \\ \sigma \downarrow & \searrow \sigma & \uparrow (\text{id} \times f)_* \\ EX_{*+1}((X, A) \times_{\mathcal{W}} (I, \partial I)) & \xleftarrow{(\text{id} \times g)_*} & EX_{*+1}((X, A) \times_{\mathcal{V}} (J, \partial J)) \end{array}$$

where the horizontal and the diagonal σ are associated with choosing the lower endpoints of the interval and the vertical one is associated with choosing the upper endpoint. The

homological version of the second property now follows from the first, and dually we obtain the cohomological version.

For the third claim, we choose a homeomorphism $g: (I, \partial I) \times (J, \partial J) \rightarrow (I, \partial I) \times (J, \partial J)$ which rotates the rectangle $I \times J$ by ninety degrees up to the obvious distortion. It is homotopic to the identity, and hence, Lemma 2.5 in combination with Lemma 2.3.3 gives us refinements $\mathcal{U}', \mathcal{V}'$ of $\mathcal{U}, \mathcal{V}$ such that

$$\text{id}_X \times g, \text{id}_{X \times I \times J}: (X, A) \times_{\mathcal{U}'} (I, \partial I) \times_{\mathcal{V}'} (J, \partial J) \rightarrow (X, A) \times_{\mathcal{U}} (I, \partial I) \times_{\mathcal{V}} (J, \partial J)$$

are two coarse maps which are generalized coarsely homotopic and hence induce the same map on EX_*, EX^* . Composing the double suspension

$$\text{EX}_*(X, A) \xrightarrow{\sigma} \text{EX}_{*+1}((X, A) \times_{\mathcal{U}'} (I, \partial I)) \xrightarrow{\sigma} \text{EX}_{*+2}((X, A) \times_{\mathcal{U}'} (I, \partial I) \times_{\mathcal{V}'} (J, \partial J))$$

with $(\text{id}_{X \times I \times J})_*$ yields the top right composition of the first diagram by naturality of the suspension map and similarly composing it with $(\text{id}_X \times g)_*$ yields the negative of the bottom left composition, because here the orientation of one of the intervals is reversed and hence part 2 gives us a sign -1 . For EX^* , we just have to dualize the proof. $\square$

To conclude the section, let us briefly discuss the relation of Definition 4.5 to the classical notion of coarse suspension, that is, the space $X \times \mathbb{R}$ (or, equivalently, $X \times \mathbb{Z}$) equipped with the product coarse structure. Recall that the purpose of the latter is that the coarse Mayer–Vietoris boundary sequences associated with the decomposition into the two flasque spaces $X \times (-\infty, 0]$ and $X \times [0, \infty)$ yield suspension isomorphisms

$$\text{EX}_{*+1}(X \times \mathbb{R}) \cong \text{EX}_*(X), \quad \text{EX}^*(X) \cong \text{EX}^{*+1}(X \times \mathbb{R})$$

for all coarse (co-)homology theories EX_* or EX^* that satisfy the flasqueness axiom, i. e., vanish on flasque spaces. This axiom is only optional in our axiomatic framework from [21, Section 3], and we do not make use of it in the present publication. In contrast, the suspensions from Definition 4.5 do not need the flasqueness axiom, but they make use of the strong homotopy axiom instead. Hence, the two notions of suspension are good for different purposes.

Let us consider a special case in which we can directly compare the two suspension spaces. Assume that there is a controlled function $\rho: X \rightarrow [1, \infty)$ and define $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; [-1, 1])$ by $U_x := \{(s, t) \in [-1, 1]^2 \mid |s - t| \cdot \rho(x) \leq 1\}$. Then, the bijection

$$\begin{aligned} Z := \{(x, s) \in X \times \mathbb{R} \mid -\rho(x) \leq s \leq \rho(x)\} &\rightarrow X \times_{\mathcal{U}} [-1, 1] \\ (x, s) &\mapsto \left(x, \frac{s}{\rho(x)}\right) \end{aligned}$$

is a coarse equivalence. Now, we assume furthermore that $A \subset X$ is a subspace and that ρ has the property that for each $R > 0$ there is a penumbra $\text{Pen}_E(A)$ such that $\rho(x) \geq R$ for all $x \in X \setminus \text{Pen}_E(A)$, then we even have $\mathcal{U} \in \mathcal{N}^\sigma(X, A; I)$.

Let us furthermore define

$$\begin{aligned} Y^+ &:= \{(x, s) \in X \times \mathbb{R} \mid s \geq \rho(x)\}, \\ Y^- &:= \{(x, s) \in X \times \mathbb{R} \mid s \leq -\rho(x)\}, \\ Y &:= Y^+ \cup Y^- \quad \partial Z := Z \cap Y \end{aligned}$$

and let $B^\pm, C$ denote the intersections of $A \times \mathbb{R}$ with $Y^\pm, Z$, respectively. Then, the inclusion

$$(Z, \partial Z \cup C) \subset (X \times \mathbb{R}, Y \cup (A \times \mathbb{R}))$$

is clearly a coarse excision and we have a bijective coarse equivalence

$$(Z, \partial Z \cup C) \rightarrow (X, A) \times_{\mathcal{U}} ([-1, 1], \{-1, 1\}).$$

Both of them induce isomorphisms on all coarse (co-)homology theories EX_*, EX^* . Together with the inclusion $(X, A) \times \mathbb{R} \subset (X \times \mathbb{R}, Y \cup (A \times \mathbb{R}))$, they induce homomorphisms

$$EX_*((X, A) \times \mathbb{R}) \rightarrow EX_*((X, A) \times_{\mathcal{U}} ([-1, 1], \{-1, 1\})),$$

$$EX^*((X, A) \times_{\mathcal{U}} ([-1, 1], \{-1, 1\})) \rightarrow EX^*((X, A) \times \mathbb{R})$$

and we claim that they are isomorphisms if the EX_*, EX^* satisfy the flasqueness axiom.

By the long exact sequences for triples, this claim is equivalent to vanishing of the coarse (co-)homology groups of the pair $(Y \cup (A \times \mathbb{R}), A \times \mathbb{R})$. To prove the latter, note that the two inclusions

$$(Y^-, A^-) \subset (Y \cup (A \times \mathbb{R}), Y^+ \cup (A \times \mathbb{R})),$$

$$(Y^+, A^+) \subset (Y^+ \cup (A \times \mathbb{R}), A \times \mathbb{R})$$

are excisions due to the assumption on ρ . Hence, the (co-)homology groups of the right hand sides are isomorphic to the (co-)homology of the left hand sides, which in turn vanish, because $Y^\pm, A^\pm$ are flasque. Evoking the long exact sequence for triples again, the claim follows.

In a nutshell, one could say that the classical notion of suspension does not see the “boundary” $Y \subset X \times \mathbb{R}$, because its contribution to (co-)homology vanishes due to flasqueness.

Proving that the two suspension maps agree under the isomorphisms constructed above is now a rather standard argument which compares Mayer–Vietoris boundary maps with connecting homomorphisms in long exact sequences of triples. The entire geometric idea behind this argument will appear in the proof of Theorem 7.6, and hence, we leave the details here to the reader.

5 Coarse cross and slant products

We will now introduce our notions of cross and slant products between coarse homology and coarse cohomology theories. This is essentially done by turning diagrams similar to the ones found in [5, Chapter VII], which describe the compatibility of cross and slant products with connecting homomorphisms, into axioms. An instant consequence of these axioms will be the compatibility of cross and slant products with suspension.

In topology, cross and slant products in the relative case are usually subject to a certain excisiveness condition. Let us note right away that the corresponding coarse geometric analogue is always satisfied.

Lemma 5.1 *If $(X, A), (Y, B)$ are objects in $\mathfrak{C}$, then $(X \times Y; A \times Y, X \times B)$ is an excisive triad in $\mathfrak{C}$. In particular, the inclusions $i: A \times (Y, B) \rightarrow (A \times Y \cup X \times B, X \times B)$ and $j: (X, A) \times B \rightarrow (A \times Y \cup X \times B, A \times Y)$ are excisions.*

Proof If E and F are entourages in X and Y , respectively, then

$$\begin{aligned} \text{Pen}_{E \tilde{\times} F}(A \times Y) \cap \text{Pen}_{E \tilde{\times} F}(X \times B) &= \text{Pen}_E(A) \times Y \cap X \times \text{Pen}_F(B) \\ &= \text{Pen}_E(A) \times \text{Pen}_F(B) \\ &= \text{Pen}_{E \tilde{\times} F}(A \times B). \end{aligned}$$

□

Definition 5.2 A *cross product* between two coarse cohomology theories $\text{EX}_i^*, \text{EX}_{ii}^*$ on $\mathfrak{C}$ with values in a third one EX_{iii}^* is a collection of transformations

$$\times = \times^{m,n}: \text{EX}_i^m(X, A) \otimes \text{EX}_{ii}^n(Y, B) \rightarrow \text{EX}_{iii}^{m+n}((X, A) \times (Y, B))$$

for all $m, n \in \mathbb{Z}$ which are natural under morphisms of $\mathfrak{C} \times \mathfrak{C}$ and which are compatible with the connecting homomorphisms $\delta_i, \delta_{ii}, \delta_{iii}$ of the three cohomology theories in the sense that the diagram

$$\begin{array}{ccc} \text{EX}_i^m(A) \otimes \text{EX}_{ii}^n(Y, B) & \xrightarrow{\times} & \text{EX}_{iii}^{m+n}(A \times (Y, B)) \\ \downarrow \delta_i \otimes \text{id} & & \downarrow (i^*)^{-1} \cong \\ & & \text{EX}_{iii}^{m+n}(A \times Y \cup X \times B, X \times B) \\ & & \downarrow \delta_{iii} \\ \text{EX}_i^{m+1}(X, A) \otimes \text{EX}_{ii}^n(Y, B) & \xrightarrow{\times} & \text{EX}_{iii}^{m+n+1}((X, A) \times (Y, B)) \end{array}$$

commutes and the diagram

$$\begin{array}{ccc} \text{EX}_i^m(X, A) \otimes \text{EX}_{ii}^n(B) & \xrightarrow{\times} & \text{EX}_{iii}^{m+n}((X, A) \times B) \\ \downarrow \text{id} \otimes \delta_{ii} & & \downarrow (j^*)^{-1} \cong \\ & & \text{EX}_{iii}^{m+n}(A \times Y \cup X \times B, A \times Y) \\ & & \downarrow \delta_{iii} \\ \text{EX}_i^{m+1}(X, A) \otimes \text{EX}_{ii}^n(Y, B) & \xrightarrow{\times} & \text{EX}_{iii}^{m+n+1}((X, A) \times (Y, B)) \end{array}$$

commutes up to a sign $(-1)^m$. The isomorphisms in the diagram are induced by the excisions from Lemma 5.1. We shall denote such cross products simply by $\times: \text{EX}_i^* \otimes \text{EX}_{ii}^* \rightarrow \text{EX}_{iii}^*$.

Definition 5.3 A *slant product* between a coarse homology theory EX_*^i and a coarse cohomology theory EX_{ii}^* with values in another coarse homology theory EX_*^{iii} is a collection of transformations of the form

$$/ := /_m^n: \text{EX}_m^i((X, A) \times (Y, B)) \otimes \text{EX}_{ii}^n(Y, B) \rightarrow \text{EX}_{m-n}^{iii}(X, A)$$

for all $m, n \in \mathbb{Z}$ which are natural with respect to both pairs of spaces in the sense that the corresponding transformations

$$\text{EX}_m^i((X, A) \times (Y, B)) \rightarrow \text{Hom}(\text{EX}_{ii}^n(Y, B), \text{EX}_{m-n}^{iii}(X, A))$$

are natural under morphisms of $\mathfrak{C} \times \mathfrak{C}$ and which are compatible with the connecting homomorphisms $\partial^i, \delta_{ii}, \partial^{iii}$ of the three theories in the sense that the diagram

$$\begin{array}{ccc}
EX_m^i((X, A) \times (Y, B)) \otimes EX_{ii}^n(Y, B) & \xrightarrow{/} & EX_{m-n}^{iii}(X, A) \\
\downarrow \partial^i \otimes \text{id} & & \downarrow \partial^{iii} \\
EX_{m-1}^i(A \times Y \cup X \times B, X \times B) \otimes EX_{ii}^n(Y, B) & & \\
\downarrow (i_*)^{-1} \cong & & \\
EX_{m-1}^i(A \times (Y, B)) \otimes EX_{ii}^n(Y, B) & \xrightarrow{/} & EX_{m-n-1}^{iii}(A)
\end{array}$$

commutes and the diagram

$$\begin{array}{ccc}
EX_m^i((X, A) \times (Y, B)) \otimes EX_{ii}^n(B) & \xrightarrow{\text{id} \otimes \delta_{ii}} & EX_m^i((X, A) \times (Y, B)) \otimes EX_{ii}^{n+1}(Y, B) \\
\downarrow \partial^i \otimes \text{id} & & \downarrow / \\
EX_{m-1}^i(A \times Y \cup X \times B, A \times Y) \otimes EX_{ii}^n(B) & & \\
\downarrow (j_*)^{-1} \cong & & \downarrow / \\
EX_{m-1}^i((X, A) \times B) \otimes EX_{ii}^n(B) & \xrightarrow{/} & EX_{m-n-1}^{iii}(X, A)
\end{array}$$

commutes up to a sign $(-1)^m$. Again, the isomorphisms in the diagram are induced by the excisions from Lemma 5.1. We shall denote such slant products simply by $/: EX_*^i \otimes EX_{ii}^* \rightarrow EX_*^{iii}$.

Note that the signs in both definitions are chosen in accordance with the usual sign heuristics, in which connecting homomorphisms are symbols of order 1:

$$\begin{aligned}
\delta_{iii}(x \times y) &= (\delta_i x) \times y, & \delta_{iii}(x \times y) &= (-1)^{\deg(x)} \cdot x \times (\delta_{ii} y), \\
\partial^{iii}(x/y) &= (\partial^i x)/y, & (\partial^i x)/y &= (-1)^{\deg(x)} \cdot x/(\delta_{ii} y).
\end{aligned}$$

Lemma 5.4 Let (X, A) and (Y, B) be pairs of coarse spaces in $\mathfrak{C}$ and let $p_X: X \times Y \rightarrow X$, $p_Y: X \times Y \rightarrow Y$ be the canonical controlled projections. If I is a closed interval and $\mathcal{U} \in \mathcal{N}^\sigma(X, A; I)$, we define

$$\widetilde{\mathcal{U}} := (p_X \times \text{id}_I)^* \mathcal{U} \in \mathcal{N}^\sigma((X, A) \times Y; I) \subset \mathcal{N}^\sigma((X, A) \times (Y, B); I)$$

and then $((X, A) \times_{\mathcal{U}} (I, \partial I)) \times (Y, B)$ is canonically coarsely equivalent to $((X, A) \times (Y, B)) \times_{\widetilde{\mathcal{U}}} (I, \partial I)$. Furthermore, the associated suspension homomorphisms defined in Definition 4.5 are compatible with cross and slant products in the sense that the diagrams

$$\begin{array}{ccc}
EX_i^{m+1}((X, A) \times_{\mathcal{U}} (I, \partial I)) \otimes EX_{ii}^n(Y, B) & \xrightarrow{\times} & EX_{iii}^{m+n+1}(((X, A) \times (Y, B)) \times_{\widetilde{\mathcal{U}}} (I, \partial I)) \\
\downarrow \sigma_i^* \otimes \text{id} & & \downarrow \sigma_{iii}^* \\
EX_i^m(X, A) \otimes EX_{ii}^n(Y, B) & \xrightarrow{\times} & EX_{iii}^{m+n}((X, A) \times (Y, B))
\end{array}$$

and

$$\begin{array}{ccc} \mathrm{EX}_{m-1}^i((X, A) \times (Y, B)) \otimes \mathrm{EX}_{\mathrm{ii}}^n(Y, B) & \xrightarrow{\quad / \quad} & \mathrm{EX}_{m-n-1}^{\mathrm{iii}}(X, A) \\ \downarrow \sigma_*^i \otimes \mathrm{id} & & \downarrow \sigma_*^{\mathrm{iii}} \\ \mathrm{EX}_m^i(((X, A) \times (Y, B)) \times_{\mathcal{U}} (I, \partial I)) \otimes \mathrm{EX}_{\mathrm{ii}}^n(Y, B) & \xrightarrow{\quad / \quad} & \mathrm{EX}_{m-n}^{\mathrm{iii}}((X, A) \times_{\mathcal{U}} (I, \partial I)) \end{array}$$

commute. Similarly, if J is a closed interval, $\mathcal{V} \in \mathcal{N}^\sigma(Y, B; J)$ and $\tilde{\mathcal{V}} := (p_Y \times \mathrm{id}_J)^* \mathcal{V}$, then we have the diagrams

$$\begin{array}{ccc} \mathrm{EX}_{\mathrm{ii}}^{n+1}((Y, B) \times_{\mathcal{V}} (J, \partial J)) \otimes \mathrm{EX}_i^m(X, A) & \xrightarrow{\quad \times \quad} & \mathrm{EX}_{\mathrm{iii}}^{m+n+1}(((X, A) \times (Y, B)) \times_{\tilde{\mathcal{V}}} (J, \partial J)) \\ \downarrow \mathrm{id} \otimes \sigma_{\mathrm{ii}}^* & & \downarrow \sigma_{\mathrm{iii}}^* \\ \mathrm{EX}_i^m(X, A) \otimes \mathrm{EX}_{\mathrm{ii}}^n(Y, B) & \xrightarrow{\quad \times \quad} & \mathrm{EX}_{\mathrm{iii}}^{m+n}((X, A) \times (Y, B)) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{EX}_{m-1}^i((X, A) \times (Y, B)) \otimes \mathrm{EX}_{\mathrm{ii}}^{n+1}((Y, B) \times_{\mathcal{V}} (J, \partial J)) & \xrightarrow{\quad \mathrm{id} \otimes \sigma_{\mathrm{ii}}^* \quad} & \mathrm{EX}_{m-1}^i((X, A) \times (Y, B)) \otimes \mathrm{EX}_{\mathrm{ii}}^n(Y, B) \\ \downarrow \sigma_*^i \otimes \mathrm{id} & & \downarrow / \\ \mathrm{EX}_m^i(((X, A) \times (Y, B)) \times_{\tilde{\mathcal{V}}} (J, \partial J)) \otimes \mathrm{EX}_{\mathrm{ii}}^{n+1}((Y, B) \times_{\mathcal{V}} (J, \partial J)) & \xrightarrow{\quad / \quad} & \mathrm{EX}_{m-n-1}^{\mathrm{iii}}(X, A) \end{array}$$

commuting up to the sign $(-1)^m$.

Proof The first statement about $\tilde{\mathcal{U}}$ and the warped products is obvious.

Let $I = [e_1, e_2]$. If $\mathcal{U}'$ is the refinement of $\mathcal{U}$ with the properties of Lemma 2.6 (such that in particular $X \times \{e_2\}$ is canonically a strong generalized coarse deformation retract of $X \times_{\mathcal{U}'} I$), then $\tilde{\mathcal{U}}' := (p_X \times \mathrm{id}_I)^*(\mathcal{U}')$ is a refinement of $\tilde{\mathcal{U}}$ with the analogous properties. By naturality of the cross products under the coarse maps induced by these refinements, it is sufficient to prove the first diagram for $\mathcal{U}'$ and $\tilde{\mathcal{U}}'$ instead of $\mathcal{U}$ and $\tilde{\mathcal{U}}$. Recall that for these refinements the suspension homomorphisms are even isomorphisms.

For this task, we use the abbreviations

$$\begin{aligned} Z &:= X \times \partial I \cup A \times I, & \tilde{Z} &:= X \times Y \times \partial I \cup (A \times Y \cup X \times B) \times I, \\ C &:= X \times \{e_2\} \cup A \times I, & \tilde{C} &:= X \times Y \times \{e_2\} \cup (A \times Y \cup X \times B) \times I, \end{aligned}$$

where the left two are coarse subspaces of $X \times_{\mathcal{U}'} I$ and the right two of $(X \times Y) \times_{\tilde{\mathcal{U}}'} I$. Recall that by passing to the refinements $\mathcal{U}'$ and $\tilde{\mathcal{U}}'$ of $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively, the suspension homomorphisms become isomorphisms inverse to the connecting homomorphisms associated with the triples $(X \times_{\mathcal{U}'} I, Z, C)$ and $((X \times Y) \times_{\tilde{\mathcal{U}}'} I, \tilde{Z}, \tilde{C})$ under the isomorphisms induced by the excisions $(X, A) \cong (X, A) \times \{e_1\} \subset (Z, C)$ and $(X, A) \times (Y, B) \cong (X, A) \times (Y, B) \times \{e_1\} \subset (\tilde{Z}, \tilde{C})$. Note furthermore that the inclusion

$$(Z, C) \times (Y, B) \subset (Z \times Y \cup (X \times_{\mathcal{U}'} I) \times Y, C \times Y \cup (X \times_{\mathcal{U}'} I) \times B) \cong (\tilde{Z}, \tilde{C})$$

is an excision, too, because for all entourages E of $X \times_{\mathcal{U}'} I$ and F of Y we have

$$\begin{aligned} & \text{Pen}_{E \times F}(Z \times Y) \cap \text{Pen}_{E \times F}(C \times Y \cup (X \times_{\mathcal{U}'} I) \times B) \\ &= \text{Pen}_E(Z) \times Y \cap (\text{Pen}_E(C) \times Y \cup (X \times_{\mathcal{U}'} I) \times \text{Pen}_F(B)) \\ &= \text{Pen}_E(C) \times Y \cup \text{Pen}_E(Z) \times \text{Pen}_F(B) \\ &= \text{Pen}_{E \times F}(C \times Y \cup Z \times B). \end{aligned}$$

Now, consider the following diagram, in which all arrows not labeled as cross products or connecting homomorphisms are induced by inclusions.

$$\begin{array}{ccccc} \begin{array}{c} EX_i^m(X, A) \\ \otimes EX_{ii}^n(Y, B) \end{array} & \xrightarrow{\quad \times \quad} & \begin{array}{c} EX_{iii}^{m+n}((X, A) \\ \times (Y, B)) \end{array} & & \\ \uparrow \cong & & \nearrow & & \uparrow \cong \\ \begin{array}{c} EX_i^m(Z, C) \\ \otimes EX_{ii}^n(Y, B) \end{array} & \xrightarrow{\quad \times \quad} & \begin{array}{c} EX_{iii}^{m+n}((Z, C) \\ \times (Y, B)) \end{array} & \xleftarrow{\quad \cong \quad} & EX_{iii}^{m+n}(\tilde{Z}, \tilde{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \begin{array}{c} EX_i^m(Z) \otimes \\ EX_{ii}^n(Y, B) \end{array} & \xrightarrow{\quad \times \quad} & EX_{iii}^{m+n}(Z \times (Y, B)) & \xleftarrow{\quad \cong \quad} & EX_{iii}^{m+n}(\tilde{Z}, X \times B \times I) \\ \downarrow \delta_i \otimes \text{id} & & & & \downarrow \delta_{iii} \\ \begin{array}{c} EX_i^{m+1}((X, A) \\ \times_{\mathcal{U}'}(I, \partial I)) \\ \otimes EX_{ii}^n(Y, B) \end{array} & \xrightarrow{\quad \times \quad} & \begin{array}{c} EX_{iii}^{m+n+1}((X, A) \\ \times (Y, B)) \\ \times_{\mathcal{U}'}(I, \partial I) \end{array} & & \end{array}$$

The upper two third of the diagram commutes trivially or by naturality of the cross products. Some of the arrows are marked as isomorphisms, because we know that they are induced by excisions. If we invert all of them, then the upper two third of the diagram still commutes and the lower third becomes commutative by the postulated compatibility of the cross products with connecting homomorphisms. Thus, the cross products are compatible with the left and right vertical compositions, which are exactly the inverses to the suspension isomorphisms.

The claimed commutativity of the second and third diagram (up to the signs $+1$, $(-1)^m$, respectively) is proven completely analogously. The fourth diagram should better first be rewritten as

$$\begin{array}{ccc} EX_{m-1}^i((X, A) \times (Y, B)) & \xrightarrow{\quad / \quad} & \text{Hom}(EX_{ii}^n(Y, B), EX_{m-n-1}^{iii}(X, A)) \\ \downarrow \sigma_*^i & & \downarrow (\sigma_{ii}^*)^* \\ EX_m^i(((X, A) \times (Y, B)) \times_{\mathcal{V}'}(J, \partial J)) & \xrightarrow{\quad / \quad} & \text{Hom}(EX_{ii}^{n+1}((Y, B) \times_{\mathcal{V}'}(J, \partial J)), \\ & & EX_{m-n-1}^{iii}(X, A)) \end{array}$$

and then commutativity up to the sign $(-1)^m$ can also be proven in exactly the same manner. $\square$

6 Secondary cup and cap products from weak generalized controlled deformation retracts

In this section, we finally define our secondary cup and cap products and prove their basic properties. All constructions and proofs are closely modeled after [18, Section 8]. The main difference is that here we work solely in the coarse geometric world, whereas in that reference the secondary product was constructed first in the topological set-up and then coarsified in a later section, which is similar to what we will do in Sect. 8.

Definition 6.1 We call a subset A of a coarse space X a *weak generalized controlled deformation retract* if there is $\mathcal{U} \in \mathcal{N}^\sigma(X; [0, 1])$ and a generalized controlled homotopy

$$H: (X, A) \times_{\mathcal{U}} ([0, 1], \{1\}) \rightarrow (X, A) \quad (3)$$

between the identity map $H_0 := H|_{(X,A) \times \{0\}} = \text{id}_{(X,A)}$ and, as (3) already encodes, a controlled map $H_1 := H|_{(X,A) \times \{1\}}: (X, A) \rightarrow (A, A) \subset (X, A)$.

The word ‘generalized’ refers to the usage of a generalized controlled homotopy and the word ‘weak’ indicates that the restriction of $H_1: X \rightarrow A$ to A is not necessarily the identity.

Example 6.2 The big advantage of allowing this weakness is its flexibility, which can be seen, for example, in the case of open cones over compact metric spaces (cf. [8, Section 3.1]). If L is a closed subspace of a compact metric space K , then the open cone $A := \mathcal{O}L$ is a weak generalized controlled deformation retract of the open cone $X := \mathcal{O}K$, where H can simply be chosen to be a contraction along straight lines to the apex of the cone and we do not have to worry about reaching the identity on A at time 1 or any other time.

Example 6.3 Let G be a finitely generated group with word metric d and neutral element e . Assume that a generalized controlled contraction

$$H: (G, \{e\}) \times_{\mathcal{U}} ([0, 1], \{1\}) \rightarrow (G, \{e\})$$

with $\mathcal{U} = \{U_g\}_{g \in G} \in \mathcal{N}(G; [0, 1])$ is given. Then, there are $R, S \in \mathbb{N}$ such that $H \times H$ maps $E_1 \hat{\times} \Delta_{[0,1]}$ into E_R and $E_{\mathcal{U}}$ into E_S . For each $g \in G$, we pick $0 = t_{g,0} < t_{g,1} < \dots < t_{g,N_g} = 1$ such that $(t_{g,n-1}, t_{g,n}) \in U_g$ and hence $d(H(g, t_{g,n-1}), H(g, t_{g,n})) \leq S$ for all $n = 1, \dots, N_g$. Thus, we find a eventually constant function $\gamma_g: \mathbb{N} \rightarrow G$ such that $\gamma_g(S \cdot n) = H(g, t_{g,n})$ for all $n = 1, \dots, N_g$, $\gamma_g(m) = g$ for all $m \geq S \cdot N_g$ and $d(\gamma_g(m+1), \gamma_g(m)) \leq 1$ for all $m \in \mathbb{N}$. It is now easy to see that γ_g, γ_h are asynchronous $(R + 3S)$ -fellow-travellers for all $g, h \in G$ with $d(g, h) \leq 1$ and therefore G is asynchronously combable as defined in [2, Definition 2.6].

It would be interesting to know if or under which conditions the converse also holds true, but it is not clear, how a construction of a generalized controlled contraction should work. The problem is that each two combing pathes in an asynchronously combable group can be reparametrized such that they stay close to each other, but all of these reparametrizations together are a priori not assumed to be compatible in any way.

Note that we did not include the collection $\mathcal{U}$ and the homotopy H itself into the data of weak generalized controlled deformation retracts. This is, because up to refining $\mathcal{U}$

sufficiently the homotopy is unique up to another generalized controlled homotopy, as the special case $f = \text{id}$ of the following lemma shows.

Lemma 6.4 *Let*

$$\begin{aligned} H &: (X, A) \times_{\mathcal{U}} ([0, 1], \{1\}) \rightarrow (X, A), \\ H' &: (X', A') \times_{\mathcal{U}'} ([0, 1], \{1\}) \rightarrow (X', A') \end{aligned}$$

be two generalized controlled homotopies associated with weak generalized controlled deformation retracts $A \subset X$ and $A' \subset X'$, respectively, and let $f: (X, A) \rightarrow (X', A')$ be a coarse map. Then, there is a common refinement $\mathcal{V} \in \mathcal{N}^\sigma(X; [0, 1])$ of $\mathcal{U}$ and $(f \times \text{id})^ \mathcal{U}'$ such that*

$$f \circ H, H' \circ (f \times \text{id}): (X, A) \times_{\mathcal{V}} ([0, 1], \{1\}) \rightarrow (X', A')$$

are generalizedly controlledly homotopic via a homotopy of coarse maps $\tilde{H}_t$ which agree with f on $(X, A) \times \{0\}$.

Proof Consider the topological homotopy

$$h: [0, 1] \times [-1, 1] \rightarrow [0, 1]^2, \quad (s, t) \mapsto \begin{cases} (s(1+t), s) & t \leq 0, \\ (s, s(1-t)) & t \geq 0. \end{cases}$$

According to Lemma 2.2.3 together with parts 2.3 and 2.3 of Lemma 2.3, there are $\mathcal{V}, \mathcal{V}'$ such that

$$\tilde{H} := H' \circ (f \circ H \times \text{id}_{[0,1]}) \circ (\text{id}_X \times h): (X \times_{\mathcal{V}} [0, 1]) \times_{\mathcal{V}'} [-1, 1] \rightarrow X$$

is a controlled map. Of course, $\mathcal{V}$ can be chosen to be a common refinement of $\mathcal{U}$ and $(f \times \text{id})^* \mathcal{U}'$.

We have $\tilde{H}(x, s, -1) = H'(f \circ H(x, 0), s) = H'(f(x), s)$ and $\tilde{H}(x, s, +1) = H'(f \circ H(x, s), 0) = f \circ H(x, s)$. Furthermore, for each $t \in [-1, 1]$ we have $\tilde{H}(x, 0, t) = H'(f \circ H(x, 0), 0) = f(x)$ and

$$\tilde{H}(x, 1, t) = \begin{cases} H'(f \circ H(x, 1+t), 1) \in A' & t \leq 0 \\ H'(\underbrace{f \circ H(x, 1)}_{\in A}, 1-t) \in A' & t \geq 0. \end{cases}$$

As $\tilde{H}$ clearly maps $A \times [0, 1] \times [-1, 1]$ into A' , too, the claim follows. $\square$

A priori, the collection $\mathcal{U}$ in Definition 6.1 is not required to fulfill the suspension condition from Definition 4.1, although this property will be needed in the following. To this end, we prove the next lemma, which shows that it can always be assumed after refining $\mathcal{U}$ sufficiently.

Lemma 6.5 *Let $A \subset X$ be a weak generalized controlled deformation retract. Then, $\mathcal{N}^\sigma(X, A; I)$ is nonempty for any closed interval I . Consequently, any collection in $\mathcal{N}(X; I)$ has a refinement in $\mathcal{N}^\sigma(X, A; I)$.*

Proof By affine linear reparametrization, we may assume without loss of generality that $I = [0, 1]$.

To construct $\mathcal{V} \in \mathcal{N}^\sigma(X, A; [0, 1])$, we start with $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; [0, 1])$ and a generalized controlled homotopy H as in Definition 6.1. For each $x \in X$, we choose $\delta_x > 0$ such that

$$V_x := \{(s_1, s_2) \in [0, 1]^2 \mid |s_1 - s_2| < \delta_x\} \subset U_x.$$

Clearly, $\mathcal{V} := \{V_x\}_{x \in X}$ is a refinement of $\mathcal{U}$ and it remains to show that we have $\mathcal{V} \in \mathcal{N}^\sigma(X, A; [0, 1])$.

Assume the contrary, i. e., that there is an $\varepsilon > 0$ such that for each penumbra $\text{Pen}_E(A)$ of A there exists $x_E \in X \setminus \text{Pen}_E(A)$ with $\delta_{x_E} \geq \varepsilon$. Then,

$$\{(x_E, 0), (x_E, 1)\} \mid E \subset X \times X \text{ entourage}\}$$

is an entourage of $X \times_{\mathcal{U}} [0, 1]$, because it is contained in $E_{\mathcal{V}}^k \subset E_{\mathcal{U}}^k$ for $k \in \mathbb{N}$ with $k\varepsilon > 1$. Applying the generalized controlled homotopy H , we see that

$$F := \{(x_E, H(x_E, 1)) \mid E \subset X \times X \text{ entourage}\}$$

is an entourage of X such that all x_E are contained in $\text{Pen}_F(A)$. For $E = F$, this yields a contradiction, because x_F was chosen such that $x_F \notin \text{Pen}_F(A)$.

The second part is clear: if $\{V_x\}_{x \in X} \in \mathcal{N}^\sigma(X, A; I)$ and $\mathcal{U} = \{U_x\}_{x \in X} \in \mathcal{N}(X; I)$, then $\{U_x \cap V_x\}_{x \in X} \in \mathcal{N}(X, A; I)$ is a refinement of $\mathcal{U}$. $\square$

Definition 6.6 We call a triad $(X; A, B)$ in $\mathfrak{C}$ a *deformation triad* if $A, B \subset X$ are weak generalized controlled deformation retracts.

Lemma 6.7 If $(X; A, B)$ is a deformation triad and H^A and H^B are associated generalized controlled homotopies, then

$$\begin{aligned} \Gamma^{X, A, B}: (X, A \cup B) \times_{\mathcal{U}} ([-1, 1], \{ -1, 1 \}) &\rightarrow (X, A) \times (X, B) \\ (x, t) &\mapsto \begin{cases} (H^A(x, -t), x) & t \leq 0, \\ (x, H^B(x, t)) & t \geq 0 \end{cases} \end{aligned}$$

is a coarse map for sufficiently fine $\mathcal{U} \in \mathcal{N}^\sigma(X, A \cup B; [-1, 1])$. In this case, we say that $\mathcal{U}$ supports $\Gamma^{X, A, B}$.

Proof The collection $\mathcal{U}$ is obtained in the obvious way from the collections

$$\begin{aligned} \mathcal{U}^A &\in \mathcal{N}^\sigma(X, A; [0, 1]) \subset \mathcal{N}^\sigma(X, A \cup B; [0, 1]), \\ \mathcal{U}^B &\in \mathcal{N}^\sigma(X, B; [0, 1]) \subset \mathcal{N}^\sigma(X, A \cup B; [0, 1]) \end{aligned}$$

defining the coarse structures on the domains of H^A and H^B , respectively. Here, we have chosen $\mathcal{U}^A, \mathcal{U}^B$ according to Lemma 6.5 and then it is clear that $\mathcal{U}$ also satisfies the suspension condition. The map Γ is controlled, because H^A, H^B are, and it is proper, because there is always one component of $\Gamma^{X, A, B}(x, t)$ equal to x . $\square$

The map $\Gamma^{X, A, B}$ defined in this lemma will from now on be omnipresent. Therefore, it is convenient to introduce some abbreviations to simplify the formulas.

Notation 6.8 For the remainder of this paper, we shall use the letter I exclusively for the interval $[-1, 1]$. Furthermore, we define $I_+ := [0, 1]$ and $I_- := [-1, 0]$.

Given any real number $t \in \mathbb{R}$, we let $t^+ := \max(t, 0)$ and $t^- := -\min(t, 0)$. Then, for any map of the form $H: X \times I_+ \rightarrow X$ we define the maps

$$H_{\pm}: X \times I \rightarrow X, \quad (x, t) \mapsto H(x, t^{\pm})$$

and for each $t \in [0, 1]$ we write H_t for $H(-, t): X \rightarrow X$.

Using this notation, we can rewrite the map $\Gamma^{X,A,B}$ very concisely as

$$\Gamma^{X,A,B} = (H_-^A, H_+^B): (x, t) \mapsto (H_{t^-}^A(x), H_{t^+}^B(x)).$$

Definition 6.9 Let $(X; A, B)$ be a deformation triad in $\mathfrak{C}$ and define $\Gamma^{X,A,B}$ as in Lemma 6.7 with $\mathcal{U} \in \mathcal{N}^{\sigma}(X, A \cup B; I)$ supporting it.

- Given a cross product $\times: \text{EX}_i^* \otimes \text{EX}_{ii}^* \rightarrow \text{EX}_{iii}^*$ between coarse cohomology theories on $\mathfrak{C}$ satisfying the strong homotopy axiom, the associated secondary cup product $\mathbb{U}$ is defined as $(-1)^m$ times the composition

$$\begin{aligned} \text{EX}_i^m(X, A) \otimes \text{EX}_{ii}^n(X, B) &\xrightarrow{\times} \text{EX}_{iii}^{m+n}((X, A) \times (X, B)) \\ &\xrightarrow{(\Gamma^{X,A,B})^*} \text{EX}_{iii}^{m+n}((X, A \cup B) \times_{\mathcal{U}} (I, \partial I)) \\ &\xrightarrow{\sigma^*} \text{EX}_{iii}^{m+n-1}(X, A \cup B). \end{aligned}$$

- Given a slant product $/: \text{EX}_*^i \otimes \text{EX}_{ii}^* \rightarrow \text{EX}_{iii}^*$ between coarse homology and cohomology theories on $\mathfrak{C}$ satisfying the strong homotopy axiom, the associated secondary cap product $\mathbb{M}$ is defined as $(-1)^{m+1}$ times the composition

$$\begin{aligned} \text{EX}_m^i(X, A \cup B) \otimes \text{EX}_{ii}^n(X, B) &\rightarrow \\ &\xrightarrow{\sigma_* \otimes \text{id}} \text{EX}_{m+1}^i((X, A \cup B) \times_{\mathcal{U}} (I, \partial I)) \otimes \text{EX}_{ii}^n(X, B) \\ &\xrightarrow{(\Gamma^{X,A,B})_* \otimes \text{id}} \text{EX}_{m+1}^i((X, A) \times (X, B)) \otimes \text{EX}_{ii}^n(X, B) \\ &\xrightarrow{/} \text{EX}_{m-n+1}^{iii}(X, A). \end{aligned}$$

Lemma 6.10 (Naturality and well-definedness) Let $(X; A, B)$, $(X'; A', B')$ be deformation triads in $\mathfrak{C}$ and let $f: X \rightarrow X'$ be a coarse map with $f(A) \subset A'$ and $f(B) \subset B'$. Then, the secondary cross and slant products are natural under f in the sense that $f^*(x \mathbb{U} y) = (f^*x) \mathbb{U} (f^*y)$ and $f_*(x \mathbb{M} f^*y) = (f_*x) \mathbb{M} y$.

In particular, the special case $f = \text{id}$ shows that the definition of the secondary cup and cap products is independent of the choice of H^A, H^B and $\mathcal{U}$.

Proof First, we note that the secondary products are independent of the choice of $\mathcal{U}$, because the suspension maps are natural under refinement of $\mathcal{U}$. Then, if we construct $\Gamma^{X,A,B}$ and $\Gamma^{X',A',B'}$ from arbitrary generalized coarse homotopies associated with the four deformation pairs, Lemma 6.4 tells us that the coarse maps

$$\begin{aligned} (f \times f) \circ \Gamma^{X,A,B}, \quad \Gamma^{X',A',B'} \circ (f \times \text{id}_I): \\ (X, A \cup B) \times_{\mathcal{U}} (I, \partial I) \rightarrow (X', A') \times (X', B') \end{aligned}$$

products $\times: EX_i^* \otimes EX_{ii}^* \rightarrow EX_v^*$, $\times: EX_{ii}^* \otimes EX_{iii}^* \rightarrow EX_{iv}^*$, $\times: EX_i^* \otimes EX_{iv}^* \rightarrow EX_{vi}^*$,
 $\times: EX_v^* \otimes EX_{iii}^* \rightarrow EX_{vi}^*$ which are associative in the sense that the diagram

$$\begin{array}{ccc} EX_i^* \otimes EX_{ii}^* \otimes EX_{iii}^* & \xrightarrow{\text{id} \otimes \times} & EX_i^* \otimes EX_{iv}^* \\ \downarrow \times \otimes \text{id} & & \downarrow \times \\ EX_v^* \otimes EX_{iii}^* & \xrightarrow{\times} & EX_{vi}^* \end{array}$$

commutes in the obvious sense, that is,

$$\begin{array}{ccc} EX_i^l(X, A) \otimes EX_{ii}^m(Y, B) \otimes EX_{iii}^n(Z, C) & \xrightarrow{\text{id} \otimes \times} & EX_i^l(X, A) \otimes EX_{iv}^{m+n}((Y, B) \times (Z, C)) \\ \downarrow \times \otimes \text{id} & & \downarrow \times \\ EX_v^{l+m}((X, A) \times (Y, B)) \otimes EX_{iii}^n(Z, C) & \xrightarrow{\times} & EX_{vi}^{l+m+n}((X, A) \times (Y, B) \times (Z, C)) \end{array}$$

commutes for all pairs of coarse spaces (X, A) , (Y, B) , (Z, C) in $\mathfrak{C}$ and all $l, m, n \in \mathbb{Z}$. Then, the corresponding secondary cup products are also associative in the sense that

$$\begin{array}{ccc} EX_i^l(X, A) \otimes EX_{ii}^m(X, B) \otimes EX_{iii}^n(X, C) & \xrightarrow{\text{id} \otimes \cup} & EX_i^l(X, A) \otimes EX_{iv}^{m+n-1}(X, B \cup C) \\ \downarrow \cup \otimes \text{id} & & \downarrow \cup \\ EX_v^{l+m-1}(X, A \cup B) \otimes EX_{iii}^n(X, C) & \xrightarrow{\cup} & EX_{vi}^{l+m+n-2}(X, A \cup B \cup C) \end{array}$$

commutes for every coarse space X with subspaces $A, B, C \subset X$ such that $(X; A, B)$, $(X; B, C)$, $(X; A \cup B, C)$, $(X; A, B \cup C)$ are deformation triads in $\mathfrak{C}$ and all $l, m, n \in \mathbb{Z}$.

We omit the proof, because it is just a generalization of a coarse geometric analogue of [18, Theorem 8.4] and we prefer to work out the details in the proof of the following theorem about the associativity of secondary cap with secondary cup products, which works completely analogously.

Theorem 6.13 (Associativity of $\frown$ with $\cup$) Let $EX_*^i, EX_v^*, EX_{ii}^{vi}$ be coarse homology theories and $EX_{ii}^*, EX_{iii}^*, EX_{iv}^*$ be coarse cohomology theories, all satisfying the strong homotopy axiom, and assume that there are slant products $/: EX_*^i \otimes EX_{iii}^* \rightarrow EX_v^*$, $/: EX_*^i \otimes EX_{iv}^* \rightarrow EX_{ii}^{vi}$, $/: EX_v^* \otimes EX_{ii}^* \rightarrow EX_{ii}^{vi}$ and a cross products $\times: EX_{ii}^* \otimes EX_{iii}^* \rightarrow EX_{iv}^*$ which are associative in the sense that the diagrams

$$\begin{array}{ccc} EX_k^i((X, A) \times (Y, B) \times (Z, C)) \otimes EX_{ii}^m(Y, B) \otimes EX_{iii}^n(Z, C) & \xrightarrow{\text{id} \otimes \times} & EX_k^i((X, A) \times (Y, B) \times (Z, C)) \otimes EX_{iv}^{m+n}((Y, B) \times (Z, C)) \\ \downarrow \text{id} \otimes \text{flip} & & \downarrow / \\ EX_k^i((X, A) \times (Y, B) \times (Z, C)) \otimes EX_{iii}^n(Z, C) \otimes EX_{ii}^m(Y, B) & & EX_k^i((X, A) \times (Y, B) \times (Z, C)) \otimes EX_{iv}^{m+n}((Y, B) \times (Z, C)) \\ \downarrow / \otimes \text{id} & & \downarrow / \\ EX_{k-n}^v((X, A) \times (Y, B)) \otimes EX_{ii}^m(Y, B) & \xrightarrow{/} & EX_{k-m-n}^{vi}(X, A) \end{array}$$

commute up to a sign $(-1)^{mn}$ for all pairs of coarse spaces (X, A) , (Y, B) , (Z, C) in $\mathfrak{C}$ and all $l, m, n \in \mathbb{Z}$. Then, the corresponding secondary cap and cup products are also associative

in the sense that

$$\begin{array}{ccc}
 \begin{array}{c} \mathrm{EX}_k^i(X, A \cup B \cup C) \\ \otimes \mathrm{EX}_{\mathrm{ii}}^m(X, B) \otimes \mathrm{EX}_{\mathrm{iii}}^n(X, C) \end{array} & \xrightarrow{\mathrm{id} \otimes \mathbb{U}} & \begin{array}{c} \mathrm{EX}_k^i(X, A \cup B \cup C) \\ \otimes \mathrm{EX}_{\mathrm{iv}}^{m+n-1}(X, B \cup C) \end{array} \\
 \downarrow \mathrm{id} \otimes \mathrm{flip} & & \downarrow \mathbb{M} \\
 \begin{array}{c} \mathrm{EX}_k^i(X, A \cup B \cup C) \\ \otimes \mathrm{EX}_{\mathrm{iii}}^n(X, C) \otimes \mathrm{EX}_{\mathrm{ii}}^m(X, B) \end{array} & & \\
 \downarrow \mathbb{M} \otimes \mathrm{id} & & \downarrow \\
 \begin{array}{c} \mathrm{EX}_{k-n+1}^v(X, A \cup B) \otimes \mathrm{EX}_{\mathrm{ii}}^m(X, B) \end{array} & \xrightarrow{\mathbb{M}} & \begin{array}{c} \mathrm{EX}_{k-m-n+2}^{\mathrm{vi}}(X, A) \end{array}
 \end{array}$$

commutes up to a sign $(-1)^{(m-1)(n-1)}$ for every coarse space X with subspaces $A, B, C \subset X$ such that $(X; A, B)$, $(X; B, C)$, $(X; A \cup B, C)$, $(X; A, B \cup C)$ are deformation triads in $\mathfrak{C}$ and all $l, m, n \in \mathbb{Z}$.

Again, the signs are in accordance with the sign heuristics, because in the formulas $x/(y \times z) = (-1)^{\deg(y)\deg(z)}(x/z)/y$ and $x \mathbb{M} (y \mathbb{U} z) = (-1)^{(\deg(y)-1)(\deg(z)-1)}(x \mathbb{M} z) \mathbb{M} y$, the symbols y, z are being interchanged.

The important step in the proofs of both theorems is the following lemma.

Lemma 6.14 *Let X be a coarse space with subspaces $A, B, C \subset X$ such that $(X; A, B)$, $(X; B, C)$, $(X; A \cup B, C)$, $(X; A, B \cup C)$ are deformation triads in $\mathfrak{C}$. Furthermore, let J also denote the interval $[-1, 1]$, just like I does. Here, we use the two different letters to distinguish the two copies of the same interval. Then, there are $\mathcal{U} \in \mathcal{N}^\sigma(X, A \cup B; I)$, $\mathcal{V} \in \mathcal{N}^\sigma(X, B \cup C; J)$ and $\mathcal{W} \in \mathcal{N}^\sigma(X, A \cup B \cup C; I \times J)$ such that the diagram*

$$\begin{array}{ccc}
 \begin{array}{c} (X, A \cup B \cup C) \times_{\mathcal{W}} \\ ((I, \partial I) \times (J, \partial J)) \end{array} & \xrightarrow{\Gamma^{X, A, B \cup C} \times \mathrm{id}_{(J, \partial J)}} & \begin{array}{c} (X, A) \times \\ ((X, B \cup C) \times_{\mathcal{V}} (J, \partial J)) \end{array} \\
 \downarrow \tilde{\Gamma}^{X, A \cup B, C} & & \downarrow \mathrm{id}_{(X, A)} \times \Gamma^{X, B, C} \\
 \begin{array}{c} ((X, A \cup B) \times_{\mathcal{U}} (I, \partial I)) \\ \times (X, C) \end{array} & \xrightarrow{\Gamma^{X, A, B} \times \mathrm{id}_{(X, C)}} & (X, A) \times (X, B) \times (X, C)
 \end{array}$$

commutes up to generalized coarse homotopy, where $\tilde{\Gamma}^{X, A \cup B, C}$ denotes $\Gamma^{X, A \cup B, C} \times \mathrm{id}_{(I, \partial I)}$ up to the obvious identifications. That is, the horizontal Γ -maps take the I -interval as input and the vertical ones use J .

Proof Consider the map $H: X \times I \times J \times [0, 4] \rightarrow X^3$ which is given by

$$\begin{aligned}
 H(x, s, t, 0) &:= (H_s^A(x), H_{t-}^B \circ H_{s+}^{B \cup C}(x), H_{t+}^C \circ H_{s+}^{B \cup C}(x)) \\
 H(x, s, t, 1) &:= (H_{s-}^A(x), H_{s+}^B \circ H_{t-}^B \circ H_{s+}^{B \cup C}(x), H_{t+}^C \circ H_{s+}^{B \cup C}(x)) \\
 H(x, s, t, 2) &:= (H_{s-}^A(x), H_{s+}^B \circ H_{t-}^B(x), H_{t+}^C(x)) \\
 H(x, s, t, 3) &:= (H_{s-}^A \circ H_{t-}^{A \cup B}(x), H_{s+}^B \circ H_{t-}^B \circ H_{t-}^{A \cup B}(x), H_{t+}^{\mathcal{C}}(x)) \\
 H(x, s, t, 4) &:= (H_{s-}^A \circ H_{t-}^{A \cup B}(x), H_{s+}^B \circ H_{t-}^{A \cup B}(x), H_{t+}^{\mathcal{C}}(x))
 \end{aligned}$$

if the homotopy parameter is one of the integers between 0 and 4 and interpolate between these values in the obvious way on $[0, 4] \setminus \{0, 1, 2, 3, 4\}$. So for example, for $u \in [0, 1]$ we

have

$$H(x, s, t, u) = (H_{s-}^A(x), H_{u,s+}^B \circ H_{t-}^B \circ H_{s+}^{B \cup C}(x), H_{t+}^C \circ H_{s+}^{B \cup C}(x)).$$

It is straightforward to verify that it maps the set

$$((A \cup B \cup C) \times I \times J \cup X \times (\partial I \times J \cup I \times \partial J)) \times [0, 4]$$

into $(A \times X^2) \cup (X \times B \times X) \cup (X^2 \times C)$. For $s = -1$ and $t = +1$, this follows easily from H_1^A having image in A and H_1^C having image in C , respectively. We check the other cases exemplarily for $u \in [0, 1]$, the rest being similar.

- If $t = -1$, then the middle component lies in $\text{im}(H_{u,s+}^B \circ H_1^B) \subset H_{u,s+}^B(B) \subset B$.
- If $s = +1$ or $x \in B \cup C$, then $H_{s+}^{B \cup C}(x)$ lies in B or in C . In the first case, the middle component lies in B , and in the second case, the third component lies in C .
- If $x \in A$, then the first component is $H_{s-}^A(x) \in A$.

Thus, the construction gives rise to a map of pairs

$$H: (X, A \cup B \cup C) \times (I, \partial I) \times (J, \partial J) \times [0, 4] \rightarrow (X, A) \times (X, B) \times (X, C)$$

and we have

$$\begin{aligned} & (\text{id}_{(X,A)} \times \Gamma^{X,B,C}) \circ (\Gamma^{X,A,B \cup C} \times \text{id}_{(J,\partial J)})(x, s, t) \\ &= (\text{id}_{X,A} \times \Gamma^{X,B,C})(H_{s-}^A(x), H_{s+}^{B \cup C}(x), t) \\ &= (H_{s-}^A(x), H_{t-}^B \circ H_{s+}^{B \cup C}(x), H_{t+}^C \circ H_{s+}^{B \cup C}(x)) = H(x, s, t, 0) \end{aligned}$$

as well as

$$\begin{aligned} & (\Gamma^{X,A,B} \times \text{id}_{X,C}) \circ \tilde{\Gamma}^{X,A \cup B,C}(x, s, t) \\ &= (\Gamma^{X,A,B} \times \text{id}_{X,C})(H_{t-}^{A \cup B}(x), s, H_{t+}^C(x)) \\ &= (H_{s-}^A \circ H_{t-}^{A \cup B}(x), H_{s+}^B \circ H_{t-}^{A \cup B}(x), H_{t+}^C(x)) = H(x, s, t, 4) \end{aligned}$$

for all $x \in X, s, t \in [-1, 1]$. The collections of neighborhoods $\mathcal{U}, \mathcal{V}$ are chosen as in Lemma 6.7 and by using Lemmas 2.2, 2.3 in the same way as before we also obtain as sufficiently fine $\mathcal{W}$ such that the maps in the diagram are coarse maps and H is a generalized controlled homotopy between the two compositions. Note furthermore that for all s, t, u at least one component of $H(x, s, t, u)$ is equal to x and therefore H is also proper. $\square$

Proof (Theorem 6.13) Consider the diagram in Fig. 1, in which the single squares commute up to the signs which are displayed within them. We explain it in a bit more detail, starting at the bottom right corner:

Commutativity of the bottom right square up to the sign $(-1)^{mn}$ is the postulated associativity between the given slant and cross products.

Next, $\mathcal{U} \in \mathcal{N}^\sigma(X, A \cup B; I)$ and $\mathcal{V} \in \mathcal{N}^\sigma(X, B \cup C; J)$ are chosen fine enough such that

$$\begin{aligned} \Gamma^{X,A,B}: (X, A \cup B) \times_{\mathcal{U}} (I, \partial I) &\rightarrow (X, A) \times (Y, B), \\ \Gamma^{X,B,C}: (X, B \cup C) \times_{\mathcal{V}} (J, \partial J) &\rightarrow (X, B) \times (Y, C) \end{aligned}$$

become coarse maps, and then the middle bottom and middle right square commute by naturality of the cross and slant products, respectively.

The top right and bottom left square commute up to the signs $(-1)^{k+2}$, $+1$, respectively, by Lemma 5.4. The commutativity of the middle square follows from Lemma 6.14 for a certain $\mathcal{W} \in \mathcal{N}^\sigma(X, A \cup B \cup C; I \times J)$ which depends on $\mathcal{U}$ and $\mathcal{V}$.

Lemma 2.3 tells us that by refining $\mathcal{W}$ we can assume that $X \times_{\mathcal{W}} (I \times J)$ is canonically coarsely equivalent to $(X \times_{\mathcal{U}'} I) \times_{\mathcal{V}'} J$ and $(X \times_{\mathcal{V}'} J) \times_{\mathcal{U}'} I$ for some $\mathcal{U}' \in \mathcal{N}^\sigma(X, A \cup B \cup C; I)$ and $\mathcal{V}' \in \mathcal{N}^\sigma(X, A \cup B \cup C; J)$ and by additionally choosing them finer than $\mathcal{U}, \mathcal{V}$, respectively, we can ensure that

$$\begin{aligned} \Gamma^{X,A,B \cup C}: (X, A \cup B \cup C) \times_{\mathcal{U}'} (I, \partial I) &\rightarrow (X, A) \times (X, B \cup C) \quad \text{and} \\ \Gamma^{X,A \cup B,C}: (X, A \cup B \cup C) \times_{\mathcal{V}'} (J, \partial J) &\rightarrow (X, A \cup B) \times (X, C) \end{aligned}$$

are coarse maps. Then, the middle top and middle left square commute by naturality of the suspension maps and the top left square commutes up to the sign -1 by Lemma 4.6.3.

The maps on the left hand side compose to $(-1)^{k+1} \cdot \mathfrak{M}$, the maps at the bottom compose to $(-1)^{k-n+2} \cdot \mathfrak{M}$, the maps at the top compose to $(-1)^{k+1} \cdot \mathfrak{M}$ and the maps on the right compose to $((-1)^m \cdot \mathfrak{U})^*$. Collecting all the signs mentioned above, we end up with $(-1)^{(m-1)(n-1)}$, which was to be shown. $\square$

7 Secondary cup and cap products on ordinary coarse (co-)homology

The first secondary product in coarse geometry ever was the one Roe constructed on his coarse cohomology HX^* in [14, Section 2.4]. The multiplication maps $HX^m(X) \otimes HX^n(X) \rightarrow HX^{m+n-1}(X)$ are explicitly constructed on the level of cocycles, but they are only well-defined for $m, n \geq 1$. The issues with well-definedness disappear if one considers the secondary cup product not on the absolute cohomology groups but on the groups $HX^*(X, \{o\})$ relative to a base-point $o \in X$ instead.

Roe actually only considered the absolute groups for proper metric spaces and with coefficients in $\mathbb{R}$, but it is indeed straightforward to generalize them to a coarse cohomology theory $HX^*(-, -; M)$ on the admissible category of pairs of coarsely connected countably generated coarse spaces, which happens to satisfy Assumption 3.2, for any choice of abelian group M as coefficients. It is called the *ordinary coarse cohomology* and satisfies the strong homotopy axiom, see [21, Section 4.2].

In Definition 7.5 below, we also give a straightforward generalization of Roe's secondary product to maps

$$*: \mathrm{HX}^m(X, A; M_1) \otimes \mathrm{HX}^n(X, B; M_2) \rightarrow \mathrm{HX}^{m+n-1}(X, A \cup B; M_3)$$

for all excisive triads $(X; A, B)$ and abelian groups M_1, M_2, M_3 together with a bilinear map $M_1 \times M_2 \rightarrow M_3: (r, s) \mapsto r \cdot s$, e. g., $M_1 = M_2 = M_3 = R$ could be a ring and $\cdot$ the multiplication map on R . However, it should be pointed out that we use different signs in our definition than Roe did in his, because Roe's claim in [14, Example 5.28(v)] is only true for our corrected choice of signs.

The purpose of this section is to show that the secondary product $*$ agrees with our secondary cup product $\mathbb{U}$ obtained from a canonical cross product if $(X; A, B)$ is an excisive deformation triad. Note that it is somewhat of a mystery why the two products $*$ and $\mathbb{U}$ can only be constructed under different, seemingly unrelated assumptions, namely excisive triads versus deformation triads, although the products are equal.

Dual to the ordinary coarse cohomology, there is also an ordinary coarse homology $\mathrm{HX}_*(-, -; M)$. In a special case, its definition has already been given in [23, page 453], and the general version was developed in [3, Section 6.3]. We refer to [21, Section 4.1] for proofs of basic properties, including that it is a coarse homology theory satisfying the strong excision axiom. A secondary product between ordinary coarse homology and ordinary coarse cohomology can be defined in a similar manner to Roe's product at the level of cycles. We will show that it agrees with the secondary cap product obtained from a canonical slant product.

We start by recalling the definition of ordinary coarse cohomology, which can be written down for arbitrary coarse spaces, but the proofs of the excision axiom in [21, Lemma 4.10] and a similar statement in Lemma 7.2 below work only for coarsely connected countably generated ones. Hence, we restrict ourselves to the admissible category of coarsely connected countably generated coarse spaces, on which ordinary coarse cohomology is indeed a coarse cohomology theory satisfying the strong excision axiom, see [21, Section 4.2].

Definition 7.1 Let X be a coarse space and M an abelian group. For $m \in \mathbb{N}$, we define $\mathrm{CX}^m(X; M)$ as the group of all functions $\varphi: X^{m+1} \rightarrow M$ whose support $\mathrm{supp}(\varphi)$ intersects each penumbra of the multidagonal in a precompact subset, i. e., a finite union of bounded subsets. For $m \in \mathbb{Z} \setminus \mathbb{N}$, we define $\mathrm{CX}^m(X; M) := 0$. These groups together with the Alexander–Spanier coboundary maps $\delta: \mathrm{CX}^m(X; M) \rightarrow \mathrm{CX}^{m+1}(X; M)$, that is

$$\delta\varphi(x_0, \dots, x_{m+1}) := \sum_{i=0}^{m+1} (-1)^i \varphi(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1}), \quad (4)$$

constitute a cochain complex. If $A \subset X$ is a subspace, then we define the cochain complex $\mathrm{CX}^*(X, A; M)$ as the kernel of the surjective cochain map $\mathrm{CX}^*(X; M) \rightarrow \mathrm{CX}^*(A; M)$ and its cohomology is the *ordinary coarse cohomology* $\mathrm{HX}^*(X, A; M)$ of the pair (X, A) with coefficients in M .

In the upcoming constructions, excisiveness enters the game through the following subcomplexes: Given a countably generated coarse space X with subspaces $A, B \subset X$, we define

$$\mathrm{CX}^*(X, A + B; M) := \ker(\mathrm{CX}^*(X; M) \rightarrow \mathrm{CX}^*(A, M) \oplus \mathrm{CX}^*(B; M)).$$

It contains $CX^*(X, A \cup B; M)$ as a subcomplex.

Lemma 7.2 *Assume that $(X; A, B)$ is an excisive triad of coarsely connected countably generated coarse spaces. Then, the inclusion*

$$\iota: CX^*(X, A \cup B; M) \xrightarrow{\subseteq} CX^*(X, A + B; M) \quad (5)$$

is a cochain homotopy equivalence.

Proof As the coarse structure on X is assumed to be countably generated, there is a sequence $E_0 \subset E_1 \subset E_2 \subset \dots$ of entourages such that each entourage is contained in one of them. Let $F_0 \subset F_1 \subset F_2 \subset \dots$ be the corresponding sequence of entourages given to us by the excisiveness condition of A, B , that is, $\text{Pen}_{E_n}(A) \cap \text{Pen}_{E_n}(B) \subset \text{Pen}_{F_n}(A \cap B)$ for all $n \in \mathbb{N}$. We may assume that all E_n, F_n are symmetric.

To each nonempty finite subset $Q \subset X$, we assign a “barycenter” $b(Q) \in X$ as follows: If $Q \subset A \cup B$ and $Q \cap A \neq \emptyset \neq Q \cap B$, then we let $n \in \mathbb{N}$ be the smallest number such that $Q \times Q \subset E_n$, which exists because X is coarsely connected. It follows that $Q \subset \text{Pen}_{E_n}(A) \cap \text{Pen}_{E_n}(B) \subset \text{Pen}_{F_n}(A \cap B)$ and we pick $b(Q) \in A \cap B$ such that $Q \cap \text{Pen}_{F_n}(\{b(Q)\}) \neq \emptyset$ and hence $Q \subset \text{Pen}_{E_n \circ F_n}(\{b(Q)\})$. In all other cases, we just pick any $b(Q) \in Q$. We may assume $b(\{q\}) = q$ for all $q \in X$, e. g., simply by demanding $E_0 = F_0 = \Delta_X$.

Now, the “barycentric subdivision” of $\mathbf{x} = (x_0, \dots, x_m) \in X^{m+1}$ is the collection of $(m+1)$ -tuples

$$\mathbf{x}_\sigma := (x_{\sigma(0)}, b(\{x_{\sigma(0)}, x_{\sigma(1)}\}), b(\{x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}\}), \dots, b(\{x_{\sigma(0)}, \dots, x_{\sigma(m)}\}))$$

for all permutations $\sigma \in S_{m+1}$. Note that the barycenters have been chosen in such a way that if every two points of $\mathbf{x}$ are at most E_n apart, then every two points of each $\mathbf{x}_\sigma$ are at most $F_n \circ E_n \circ E_n \circ F_n$ apart. In other words: for each penumbra P of the multidagonal in X^{p+1} there is another penumbra P' such that $\mathbf{x} \in P \implies \mathbf{x}_\sigma \in P'$. This implies that for each coarse cochain $\varphi \in CX^m(X; M)$ the map

$$S\varphi: X^{m+1} \rightarrow M, \quad \mathbf{x} \mapsto \sum_{\sigma \in S_{m+1}} \pm \varphi(\mathbf{x}_\sigma)$$

is also a coarse cochain. Note furthermore that if $\mathbf{x} \in (A \cup B)^{m+1}$, then $\mathbf{x}_\sigma \in A^{m+1} \cup B^{m+1}$ for each σ , so $S\varphi \in CX^m(X, A \cup B; M)$ for all $\varphi \in CX^m(X, A + B; M)$.

Now, we all know from our first course on algebraic topology that there is a certain set of signs which nobody dares to write down for which the standard calculations show that $S: CX^*(X, A + B; M) \rightarrow CX^*(X, A \cup B; M)$ is in fact a cochain map. Furthermore, the same properties of our barycenters ensure that the standard construction known from algebraic topology also gives us cochain homotopies $S \circ \iota \simeq \text{id}$ and $\iota \circ S \simeq \text{id}$. $\square$

Remark 7.3 If X is only countably generated but not coarsely connected, then the barycentric subdivision can still be performed for $(m+1)$ -tuples $\mathbf{x} = (x_0, \dots, x_m)$ whose vertices $x_0, \dots, x_m$ all lie in the same connected component of X . This will be the case when we reuse the construction later on in Lemmas 7.8 and 8.3.

We can apply the lemma right away to the excisive triad from Lemma 5.1. It is then straightforward to check that we really have a cochain map in the following definition and obtain a well-defined cross product which satisfies the axioms of Definition 5.2.

Definition 7.4 Let (X, A) and (Y, B) be coarsely connected and countably generated coarse spaces. Then, the cross product

$$\begin{aligned} \mathrm{HX}^m(X, A; M_1) \otimes \mathrm{HX}^n(Y, B; M_2) &\rightarrow \mathrm{HX}^{m+n}(X \times Y, A \times Y + X \times B; M_3) \\ &\cong \mathrm{HX}^{m+n}((X, A) \times (Y, B); M_3) \end{aligned}$$

is induced by the cochain map

$$\mathrm{CX}^*(X, A; M_1) \otimes \mathrm{CX}^*(Y, B; M_2) \rightarrow \mathrm{CX}^*(X \times Y, A \times Y + X \times B; M_3),$$

which takes the tensor product of $\varphi \in \mathrm{CX}^m(X, A; M_1)$ and $\psi \in \mathrm{CX}^n(Y, B; M_2)$ to

$$\varphi \times \psi : ((x_0, y_0), \dots, (x_{m+n}, y_{m+n})) \mapsto \varphi(x_0, \dots, x_m) \cdot \psi(y_m, \dots, y_{m+n}).$$

Thus, Definition 6.9 gives rise to a secondary cup product

$$\cup : \mathrm{HX}^m(X, A; M_1) \otimes \mathrm{HX}^n(X, B; M_2) \rightarrow \mathrm{HX}^{m+n-1}(X, A \cup B; M_3)$$

on ordinary coarse cohomology for all deformation triads $(X; A, B)$ of coarsely connected countably generated coarse spaces.

Next, we introduce the generalization of Roe's secondary product. To this end, let $\varphi \cup \psi$ denote the usual cup product of functions $\varphi : X^{m+1} \rightarrow M_1$ and $\psi : X^{m+1} \rightarrow M_2$, that is,

$$\varphi \cup \psi : X^{m+n+1} \rightarrow M_3, \quad (x_0, \dots, x_{m+n}) \mapsto \varphi(x_0, \dots, x_m) \cdot \psi(x_m, \dots, x_{m+n}).$$

Note that $\varphi \cup \psi$ is already a coarse cochain if one of φ, ψ is a coarse cochain. Furthermore, for any function φ on X^{m+1} and any point $o \in X$ we define the function $s_o \varphi : (x_1, \dots, x_m) \mapsto \varphi(o, x_0, \dots, x_m)$ on X^m .

Definition 7.5 Let $(X; A, B)$ be an excisive triad of coarsely connected countably generated coarse spaces with A, B nonempty and choose base-points $a \in A, b \in B$. Then, the *Roe secondary (cup) product* of two coarse cohomology classes $[\varphi] \in \mathrm{HX}^m(X, A; M_1)$ and $[\psi] \in \mathrm{CX}^n(X, B; M_2)$ is the class $[\varphi] * [\psi] \in \mathrm{HX}^{m+n-1}(X, A \cup B; M_3)$ represented by the coarse cocycle

$$\varphi * \psi := (-1)^{m+1}(s_a \varphi) \cup \psi + \varphi \cup (s_b \psi) \in \mathrm{CX}^{m+n-1}(X, A + B; M_3).$$

Although $s_a \varphi, s_b \psi$ do not satisfy the support condition of coarse cochains, $\varphi * \psi$ clearly does and lies in the claimed cochain group. It is readily verified that $\varphi * \psi$ is even a cocycle and that the resulting cohomology class is independent of the choice of the base-points a, b . The details of the calculations work exactly as in the proof of [14, Proposition 2.33], just that our definition of the product differs from Roe's, as mentioned above, by the sign $(-1)^{m+1}$. What is needed in the calculations is that the Alexander–Spanier coboundary map (4) satisfies $\delta(\varphi \cup \psi) = (\delta \varphi) \cup \psi + (-1)^m \varphi \cup (\delta \psi)$ and we have $\delta s_a \varphi + s_a \delta \varphi = \varphi$ for $m \geq 1$ as well as $\delta s_a \varphi + s_a \delta \varphi = \varphi - \varphi(a)$ for $m = 0$ and similarly for s_b, ψ . The summands $\varphi(a)$ if $m = 0$ and $\psi(b)$ if $n = 0$ appearing in the latter formulas are actually the reason why Roe's secondary product is not well defined on the absolute cohomology groups in these degrees, but in our case they vanish because φ, ψ are cocycles relative to A, B , respectively.

Theorem 7.6 *Let $(X; A, B)$ be an excisive deformation triad of coarsely connected countably generated coarse spaces. Then, the associated secondary products $*$ and $\mathbb{U}$ agree.*

Proof The first step in the proof is to reformulate the suspension homomorphism in a Mayer–Vietoris like fashion. As before, we write $I := [-1, 1]$, $I_+ := [0, 1]$, $I_- := [-1, 0]$ and let $\mathcal{U} \in \mathcal{N}^\sigma(X, A \cup B; I)$ be a collection which supports $\Gamma^{X, A, B}$.

Then, Lemma 2.6 provides us with a refinement $\mathcal{U}' \in \mathcal{N}^\sigma(X, A \cup B; I)$ of $\mathcal{U}$ such that in particular all of the inclusions

$$X \times \{-1\} \subset X \times_{\mathcal{U}'} I_- \subset X \times_{\mathcal{U}'} I \supset X \times_{\mathcal{U}'} I_+ \supset X \times \{1\}$$

are strong generalized coarse deformation retracts in the canonical way and, furthermore,

$$f: X \times_{\mathcal{U}'} I_+ \rightarrow X \times_{\mathcal{U}'} I, \quad (x, s) \mapsto (x, 2s - 1)$$

is a coarse equivalence which is canonically generalized coarsely homotopic to the inclusion. We then obtain the diagram

$$\begin{array}{ccc} \mathrm{HX}^*((X, A \cup B) \times_{\mathcal{U}'} (I, I_- \cup \{1\}); M_3) & \xrightarrow{(\mathrm{incl})^*} & \mathrm{HX}^*((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_3) \\ \downarrow (\mathrm{incl})^* \cong & \nwarrow f^* & \downarrow \sigma^* \cong \\ \mathrm{HX}^*((X, A \cup B) \times_{\mathcal{U}'} (I_+, \partial I_+); M_3) & \xrightarrow[\cong]{\sigma^*} & \mathrm{HX}^{*-1}(X, A \cup B; M_3) \end{array}$$

where the upper left triangle commutes by homotopy invariance and the lower right triangle commutes by naturality of the suspension homomorphisms (Lemma 4.4). The two suspension homomorphisms and the left vertical arrow are in fact isomorphisms by the above-mentioned properties of $\mathcal{U}'$. Thus, all other arrows in the diagram are also isomorphisms.

Now, as $(X; A, B)$ was assumed to be an excisive triad, so are

$$\begin{aligned} (X \times_{\mathcal{U}'} I_+; \quad X \times \{+1\} \cup B \times I_+, \quad A \times I_+) \quad \text{and} \\ (X \times_{\mathcal{U}'} I_-; \quad X \times \{-1\} \cup A \times I_-, \quad B \times I_-). \end{aligned}$$

Thus, Lemma 7.2 and the choice of $\mathcal{U}'$ imply that the cohomologies of the complexes

$$\begin{aligned} \mathrm{CX}_+^* &:= \mathrm{CX}^*(X \times_{\mathcal{U}'} I_+, X \times \{1\} \cup B \times I_+ + A \times I_+; M_3), \\ \mathrm{CX}_-^* &:= \mathrm{CX}^*(X \times_{\mathcal{U}'} I_-, X \times \{-1\} \cup A \times I_- + B \times I_-; M_3) \end{aligned}$$

vanish and hence, the induced upper horizontal arrow κ' in the commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
CX^*((X, A \cup B) \times_{\mathcal{U}'} (I_+, \partial I_+); M_3) & \xrightarrow{\kappa'} & CX_{\cup}^* \\
\downarrow & & \downarrow \\
CX^*((X, A \cup B) \times_{\mathcal{U}'} (I_+, \{1\}); M_3) & \xrightarrow{(\text{incl}, 0)} & CX_+^* \oplus CX_-^* \\
\downarrow & & \downarrow \\
CX^*(X \times \partial I_+ \cup (A \cup B) \times I_+, & \xrightarrow{(\iota_+)^*} & CX^*(X, A + B) \\
X \times \{1\} \cup (A \cup B) \times I_+; M_3) & & \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$(\iota_+)^* - (\iota_-)^*$

induces an isomorphism on cohomology, because the cohomology of the middle cochain complexes vanish and the bottom horizontal map is induced by an excision. Here, $CX_{\cup}^*$ is simply the kernel of the map below it and $\iota_{\pm}: X \rightarrow X \times_{\mathcal{U}_{\pm}} I_{\pm}$ denote the inclusions as the subspace $X \times \{0\}$.

Furthermore, the cochain complexes $CX^*((X, A \cup B) \times_{\mathcal{U}'} (I, I_- \cup \{1\}); M_3)$ and $CX^*((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_3)$ map canonically to $CX_{\cup}^*$ and we obtain a diagram

$$\begin{array}{ccc}
CX^*((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_3) & & \\
\downarrow f^* \cong & \swarrow \cong & \searrow \cong \\
& CX^*((X, A \cup B) \times_{\mathcal{U}'} (I, I_- \cup \{1\}); M_3) & \xrightarrow{\kappa} CX_{\cup}^* \\
& \swarrow \cong & \searrow \cong \\
CX^*((X, A \cup B) \times_{\mathcal{U}'} (I_+, \partial I_+); M_3) & &
\end{array}$$

κ'

where the left square triangle commutes up to cochain homotopy (see [21, Lemma 4.9]) and the right two triangles commute trivially. In particular, the outer triangle commutes in cohomology and combining this with the naturality of connecting homomorphisms under f^* and the diagram of exact sequences above, we see that the diagram

$$\begin{array}{ccc}
HX^{*-1}(X \times \partial I \cup (A \cup B) \times I, & \xrightarrow{\cong} & HX^{*-1}(X, A + B) \\
X \times \{1\} \cup (A \cup B) \times I) & & \\
\downarrow \delta \cong & & \downarrow \delta \cong \\
HX^*((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_3) & \xrightarrow[\cong]{\kappa_*} & HX_{\cup}^*
\end{array}$$

commutes. At the left, we have the connecting homomorphism in the long exact sequence of a triple and at the right we have the Mayer–Vietoris like connecting homomorphism coming from the right exact column of the diagram above. Both of them are isomorphism by what we have seen before. Moreover, we also know that the canonical map κ_* at the bottom and the excision at the top are also isomorphisms.

With this diagram at hand, proving the claim is a simple calculation. Let $\varphi \in CX^m(X, A; M_1)$ and $\psi \in CX^n(X, B; M_2)$ be coarse cocycles. Then, $\varphi * \psi$ is the image of

$$(\varphi \cup (H_+^B)^*(s_b \psi), (-1)^m (H_-^A)^*(s_a \varphi) \cup \psi) \in CX_+^{m+n-1} \oplus CX_-^{m+n-1}$$

under $(\iota_+)^* - (\iota_-)^*$ and it is mapped to

$$(-1)^m \cdot (\varphi \cup (H_+^B)^* \psi, (H_-^A)^* \varphi \cup \psi) \in CX_{\cup}^{m+n} \subset CX_+^{m+n} \oplus CX_-^{m+n}$$

under the coboundary map. But the latter is exactly the image of

$$(-1)^m \cdot (\Gamma^{X,A,B})^*(\varphi \times \psi) \in CX^{m+n}(X \times_{\mathcal{U}'} I, X \times \{-1\} \cup A \\ \times I + X \times \{1\} \cup B \times I; M_3)$$

under the canonical map κ_* , showing that $(-1)^m \cdot (\Gamma^{X,A,B})^*([\varphi] \times [\psi]) = \delta([\varphi] * [\psi])$. By applying the left inverse σ^* of δ to this equation, we obtain $[\varphi] \cup [\psi] = [\varphi] * [\psi]$. $\square$

Now, we turn our attention to ordinary coarse homology and first recall its definition.

Definition 7.7 Let X be a coarse space and M an abelian group. For $m \in \mathbb{N}$, we define $CX_m(X; M)$ as the group of all infinite formal sums $c = \sum_{\mathbf{x} \in X^{p+1}} m_{\mathbf{x}} \mathbf{x}$ such that

- the set $\text{supp}(c) := \{\mathbf{x} \in X^{m+1} \mid m_{\mathbf{x}} \neq 0\}$ is a penumbra of the multidagonal;
- the set $\text{supp}(c) \cap K$ is finite for all bounded $K \subset X^{m+1}$, or equivalently, for all precompact K .

For $m \in \mathbb{Z} \setminus \mathbb{N}$, we define $CX_m(X; M) := 0$. These groups together with the boundary maps $\partial: CX_m(X; M) \rightarrow CX_{m-1}(X; M)$ which are defined on the single summands by

$$\partial(x_0, \dots, x_m) := \sum_{i=0}^m (-1)^i (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$$

constitute a chain complex. If $A \subset X$ is a coarse subspace, then $CX_*(A; M)$ is a subcomplex of $CX_*(X; M)$ and we define the *coarse chain complex* $CX_*(X, A; M) := CX_*(X; M)/CX_*(A; M)$. Its homology is the *ordinary coarse homology* $HX_*(X, A; M)$ of the pair of coarse spaces (X, A) with coefficients in M .

For the construction of slant products, we need to make use of the excisiveness of Lemma 5.1 again, just as it was the case for cross products. For subspaces $A, B \subset X$, we define

$$CX_*(X, A + B; M) := CX_*(X; M)/(CX_*(A; M) + CX_*(B; M)).$$

The proof of the following lemma is completely dual to the proof of Lemma 7.2, but the support condition on coarse cycles implies that we only need to barycentrically subdivide $(m+1)$ -tuples as in Remark 7.3, so there is no need to assume coarse connectedness.

Lemma 7.8 Assume that $(X; A, B)$ is an excisive triad of countably generated coarse spaces. Then, the canonical quotient map

$$\pi: CX_*(X, A + B; M) \rightarrow CX_*(X, A \cup B; M)$$

is a chain homotopy equivalence. $\square$

Definition 7.9 Let (X, A) and (Y, B) be pairs of countably generated coarse spaces. Then, the slant product

$$HX_m((X, A) \times (X, B); M_1) \otimes HX^n(Y, B; M_2) \\ \cong HX_m(X \times Y, A \times Y + X \times B; M_1) \otimes HX^n(Y, B; M_2) \\ \rightarrow HX_{m-n}(X, A; M_3)$$

is induced by the chain map

$$CX_*(X \times Y, A \times Y + X \times B; M_1) \rightarrow CX_*(X, A; M_3) \otimes (CX^*(Y, B; M_2))^*$$

which is defined on the single summands $m_1 \cdot (x_0, y_0, \dots, x_m, y_m)$ of a coarse chain in $CX_m(X \times Y, A \times Y + X \times B; M_1)$ and coarse cochains $\varphi \in CX^n(Y, B; M_2)$ by

$$(m_1 \cdot (x_0, y_0, \dots, x_m, y_m))/\varphi := (m_1 \cdot \varphi(y_m, \dots, y_{m-n}))(x_0, \dots, x_{m-n}),$$

yielding a coarse chain in $CX_{m-n}(X, A; M_3)$.

Again it is straightforward to check that we really have a chain map in the definition and obtain a well-defined slant product satisfying the axioms of Definition 5.3. The computations rely on the observation that $(\partial c)/\varphi = \partial(c/\varphi) + (-1)^m c/(\delta\varphi)$ if m is the degree of the chain c . Strictly speaking, for the slant product to truly fit into Definition 5.3 we actually want HX^* to be a coarse cohomology theory, so we should restrict ourselves to category of coarsely connected countably generated coarse spaces as in [21, Section 4.2].

This definition differs from the one found in most textbooks in so far as our cocycle swallows not the first $n + 1$ entries but the last $n + 1$ ones and this in reversed order. The change is necessary to fulfill the sign conventions of Definition 5.3. To be precise, our definition is the right one for slanting away cohomology classes from the right, whereas many textbooks prefer to slant them away from the left.

Dually to the Roe secondary cup product, we can now define a secondary cap product as follows. For any infinite formal linear combination $c = \sum_{\mathbf{x} \in X^{m+1}} m_{\mathbf{x}} \mathbf{x}$ of $(m + 1)$ -tuples and any point $o \in X$, we define the infinite formal linear combination of $(m + 2)$ -tuples $r_o c := \sum_{\mathbf{x} \in X^{m+1}} m_{\mathbf{x}}(o, \mathbf{x})$. Also, we define the cap product of any linear combination c as above with all $m_{\mathbf{x}} \in M_1$ and any $\varphi: X^{n+1} \rightarrow M_2$ with $n \leq m$ as the linear combination

$$c \cap \varphi := \sum_{(x_0, \dots, x_m) \in X^{m+1}} (m_{(x_0, \dots, x_m)} \cdot \varphi(x_m, \dots, x_{m-n}))(x_0, \dots, x_{m-n})$$

with coefficients in M_3 . Again, this cap product differs from the one usually found in textbooks. Note that the equality $(\partial c) \cap \varphi = \partial(c \cap \varphi) + (-1)^m c \cap (\delta\varphi)$ holds for all m and we have $\partial r_o c + r_o \partial c = c$ for $m \geq 1$ as well as $\partial r_o c + r_o \partial c = c - (o)$ for $m = 0$. In the following, $o = a$ will be a point in A and hence, the summand (o) will be divided out when passing to the coarse chain complex relative to A .

Definition 7.10 Let $(X; A, B)$ be an excisive triad of coarsely connected countably generated coarse spaces with A, B nonempty and choose base-points $a \in A, b \in B$. Then, the *Roe secondary cap product* of a coarse homology class $[c] \in HX_m(X, A \cup B; M_1)$ represented by a coarse cycle $c \in CX_m(X, A + B; M_1)$ and a coarse cohomology class $[\varphi] \in HX^n(X, B; M_2)$ is the class $[c] * [\varphi] \in HX_{m-n+1}(X, A \cup B; M_3)$ represented by the coarse cycle

$$c * \varphi := -r_a(c \cap \varphi) - (-1)^m c \cap (s_b \varphi) \in CX_{m-n+1}(X, A; M_3).$$

It is again straightforward to check that $c * \varphi$ is indeed a well-defined coarse cycle in the claimed chain group and that the resulting homology class is independent of the choice of the base-points a, b . Coarse connectedness is needed, because otherwise the first summand might not be supported in a penumbra of the multidagonal.

Theorem 7.11 *Let $(X; A, B)$ be an excisive deformation triad of coarsely connected countably generated coarse spaces. Then, the associated secondary products $\ast$ and $\cap$ agree.*

Proof Unfortunately, only part of the proof is truly dual to the proof of Theorem 7.6 and some work remains to be done. The complexes and diagrams can be dualized without problems, and we obtain a short exact sequence of chain complexes

$$0 \rightarrow CX_*(X, A + B; M_1) \xrightarrow{((\iota_+)_*, -(\iota_-)_*)} CX_*^+ \oplus CX_*^- \rightarrow CX_*^\cup \rightarrow 0$$

and a canonical map $CX_*^\cup \rightarrow CX_*((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_1)$ such that the diagram in homology

$$\begin{array}{ccc} HX_{*+1}^\cup & \longrightarrow & HX_{*+1}((X, A \cup B) \times_{\mathcal{U}'} (I, \partial I); M_1) \\ \partial \cong \downarrow & & \downarrow \cong \partial \\ HX_*(X, A + B; M_1) & \xrightarrow{\cong} & HX_*(X \times \partial I \cup (A \cup B) \times I, \\ & & X \times \{1\} \cup (A \cup B) \times I) \end{array}$$

commutes.

The diagram allows us to compute the suspension homomorphism, but at this time the calculations become more complicated than in the case of the secondary cup products. To this end, we start with a coarse cycle $c \in CX_m(X, A + B; M_1)$. Now, there are chain contractions $P^\pm: CX_*(X, A + B; M_1) \rightarrow CX_{*+1}^\pm$ for $(\iota_\pm)^*$, i.e., $\partial P^\pm + P^\pm \partial = (\iota_\pm)^*$. For example, we can take the prism operators constructed in the proof of [21, Lemma 4.4] with $H = \text{id}$. Then, using the diagram above we see that $\sigma_*[c]$ is represented by the coarse cycle

$$P^+c - P^-c \in CX_{m+1}(X \times_{\mathcal{U}'} I, X \times \{-1\} \cup A \times I + X \times \{1\} \cup B \times I; M_1).$$

It follows that

$$\begin{aligned} [c] \cap [\varphi] &= ((\Gamma^{X,A,B})_*[P^+c - P^-c]) / [\varphi] \\ &= [((\text{pr}_X \times H_B)_*(P^+c)) / \varphi - ((H_A \times \text{pr}_X)_*(P^-c)) / \varphi]. \end{aligned}$$

Just as in Definition 7.10, it is straightforward to verify that

$$d := (-1)^m((\text{pr}_X \times H_B)_*(P^+c)) / (s_b\varphi) - r_a(((H_A \times \text{pr}_X)_*(P^-c)) / \varphi)$$

is a coarse cochain in $CX_{m-n}(X, A; M_3)$ and then the calculation

$$\begin{aligned} \partial d &= (-1)^m((\text{pr}_X \times H_B)_*(\partial P^+c)) / (s_b\varphi) + ((\text{pr}_X \times H_B)_*(P^+c)) / (\delta s_b\varphi) \\ &\quad + r_a\partial(((H_A \times \text{pr}_X)_*(P^-c)) / \varphi) + ((H_A \times \text{pr}_X)_*(P^-c)) / \varphi \\ &= (-1)^m((\text{pr}_X \times H_B)_*(\iota_+)_*c) / (s_b\varphi) + ((\text{pr}_X \times H_B)_*(P^+c)) / \varphi \\ &\quad + r_a(((H_A \times \text{pr}_X)_*(\partial P^-c)) / \varphi) + ((H_A \times \text{pr}_X)_*(P^-c)) / \varphi \\ &= (-1)^m c \cap (s_b\varphi) + ((\text{pr}_X \times H_B)_*(P^+c)) / \varphi \\ &\quad + r_a(((H_A \times \text{pr}_X)_*(\iota_-)_*c) / \varphi) + ((H_A \times \text{pr}_X)_*(P^-c)) / \varphi \\ &= (-1)^m c \cap (s_b\varphi) + ((\text{pr}_X \times H_B)_*(P^+c)) / \varphi \\ &\quad + r_a(c \cap \varphi) + ((H_A \times \text{pr}_X)_*(P^-c)) / \varphi \end{aligned}$$

finishes the proof. $\square$

8 Transgression maps

A big class of coarse (co-)homology theories is obtained by coarsifying topological (co-)homology theories for σ -locally compact spaces. We refer to [21, Section 5] for a very detailed introduction into this topic, which even treats the more general equivariant case. It thereby widely generalizes the construction for the absolute groups in [8, Section 4]. Here, we shall only recall the rough outlines of the construction, based on the general perception that σ -locally compact spaces are countable direct limits of locally compact Hausdorff spaces in some sense. The category of pairs of σ -locally compact spaces, on which we will consider (co-)homology theories in the sense of Eilenberg and Steenrod [6], has the pairs $(\mathcal{X}, \mathcal{A})$ of σ -locally compact spaces with $\mathcal{A} \subset \mathcal{X}$ closed as objects, and its morphisms are the proper and continuous σ -maps. The purpose of this section is to show that the associated transgression maps relate the secondary products on the coarse space to the primary products on its corona.

In a nutshell, the coarsification procedure is as follows: Let (X, A) be a pair of countably generated coarse spaces of bornologically bounded geometry. The latter condition means that there exist discretizations $(X', A') \subset (X, A)$, that is, X', A' are locally finite (i. e., bounded subsets of X', A' are finite) and the inclusions $X' \subset X$ and $A' \subset A$ are coarse equivalences (cf. [21, Definition 2.4]). The Rips complex construction applied to such a discretization yields a countable directed system of pairs of locally compact Hausdorff spaces $(P_n(X'), P_n(A'))$ which constitutes a pair of σ -locally compact spaces $(\mathcal{P}(X'), \mathcal{P}(A'))$. Now, assume that E_* is a homology theory or E^* is a cohomology theory on the category of pairs of σ -locally compact spaces and proper continuous maps which satisfies the homotopy, exactness and excision axioms. Recall that the homotopy axiom says that this theory is invariant under homotopies which are themselves proper continuous maps and the excision axiom says that topological excisions induce isomorphisms. Here, an inclusion $(\mathcal{X} \setminus \mathcal{C}, \mathcal{A} \setminus \mathcal{C}) \subset (\mathcal{X}, \mathcal{A})$ of pairs of σ -locally compact spaces (where $\mathcal{C} \subset \mathcal{A} \subset \mathcal{X}$ with $\mathcal{A}$ closed and $\mathcal{C}$ open) is called a topological excision, if the closure of $\mathcal{C}$ is contained in the interior of $\mathcal{A}$. Given such a topological homology or cohomology theory, the associated coarse homology and cohomology groups are defined by

$$EX_*(X, A) := E_*(\mathcal{P}(X'), \mathcal{P}(A')) \quad \text{and} \quad EX^*(X, A) := E^*(\mathcal{P}(X'), \mathcal{P}(A')),$$

respectively. They are independent of the choice of discretizations up to canonical isomorphism and together with certain induced maps they constitute a coarse (co-)homology theory satisfying the strong homotopy axiom (see [21, Theorem 5.15]).

Now, in order to introduce the products on the topological side we first have to generalize the notion of topological excisiveness. We call $(\mathcal{X}; \mathcal{A}, \mathcal{B})$ with $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ a (topologically) excisive triad of σ -locally compact spaces if the interiors of $\mathcal{A}, \mathcal{B}$ in $\mathcal{A} \cup \mathcal{B}$ cover all of $\mathcal{A} \cup \mathcal{B}$. In this case, the two inclusions $(\mathcal{A}, \mathcal{A} \cap \mathcal{B}) \subset (\mathcal{A} \cup \mathcal{B}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \subset (\mathcal{A} \cup \mathcal{B}, \mathcal{A})$ are topological excisions and hence induce isomorphisms under E_* and E^* . Generalizing this notion, we call $(\mathcal{X}; \mathcal{A}, \mathcal{B})$ an E_* -excisive triad or an E^* -excisive triad if the two inclusions of pairs induce isomorphisms under E_* or E^* , respectively (cf. [5, Proposition and Definition 8.1]). Important examples for this notion are triads of Rips complexes of coarsely excisive triads, as the following reformulation of the excisiveness part of [21, Theorem 5.15] illustrates.

Lemma 8.1 *If $(X'; A', B')$ is a coarsely excisive triad of countably generated locally finite coarse spaces, then $(\mathcal{P}(X'); \mathcal{P}(A'), \mathcal{P}(B'))$ is an E_* - and E^* -excisive triad of σ -locally compact spaces for all homology and cohomology theories E_*, E^* . $\square$*

Furthermore, we say that two pairs of σ -locally compact spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are E_* -productable or E^* -productable if $(\mathcal{A} \times \mathcal{Y} \cup \mathcal{X} \times \mathcal{B}; \mathcal{A} \times \mathcal{Y}, \mathcal{X} \times \mathcal{B})$ is an E_* - or an E^* -excisive triad, respectively, and hence, the two inclusions

$$\begin{aligned} i: (\mathcal{A} \times \mathcal{Y}, \mathcal{A} \times \mathcal{B}) &\rightarrow (\mathcal{A} \times \mathcal{Y} \cup \mathcal{X} \times \mathcal{B}, \mathcal{X} \times \mathcal{B}), \\ j: (\mathcal{X} \times \mathcal{B}, \mathcal{A} \times \mathcal{B}) &\rightarrow (\mathcal{A} \times \mathcal{Y} \cup \mathcal{X} \times \mathcal{B}, \mathcal{A} \times \mathcal{Y}) \end{aligned}$$

induce isomorphisms. In contrast to the coarse excisiveness from Lemma 5.1, it is well-known that this analogous condition is not true for arbitrary topological spaces. Luckily, we do not have to worry about this issue, because we will show that it is true in the two cases relevant to us, namely for:

- pairs of Rips complexes $(\mathcal{P}(X'), \mathcal{P}(A'))$, $(\mathcal{P}(Y'), \mathcal{P}(B'))$ and arbitrary E_*, E^* , cf. Corollary 8.4;
- whenever E_*, E^* are single space (co-)homology theories, i. e., they satisfy the so-called strong excision axiom, see our discussion before Theorem 8.8.

Definition 8.2 A cross product $\times: E_i^* \otimes E_{ii}^* \rightarrow E_{iii}^*$ between cohomology theories for σ -locally compact spaces consists of a family of natural homomorphisms

$$\times: E_i^m(\mathcal{X}, \mathcal{A}) \otimes E_{ii}^n(\mathcal{Y}, \mathcal{B}) \rightarrow E_{iii}^{m+n}((\mathcal{X}, \mathcal{A}) \times (\mathcal{Y}, \mathcal{B}))$$

for all E_{iii}^* -productable pairs of σ -locally compact spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ which satisfy the obvious topological analogues of the diagrams in Definition 5.2.

Similarly, a slant product $/: E_i^* \otimes E_{ii}^* \rightarrow E_{*}^{iii}$ between homology and cohomology theories for σ -locally compact spaces consists of a family of natural transformations

$$/: E_m^i((\mathcal{X}, \mathcal{A}) \times (\mathcal{Y}, \mathcal{B})) \otimes E_{ii}^n(\mathcal{Y}, \mathcal{B}) \rightarrow E_{m-n}^{iii}(\mathcal{X}, \mathcal{A})$$

for all E_i^* -productable pairs of σ -locally compact spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ which satisfy the obvious topological analogues of the diagrams in Definition 5.3.

We now use the ideas of [9, Section 4.6] to turn them into coarse cross and slant products. To this end, we need to pass from the product of Rips complexes to the Rips complex of products.

Lemma 8.3 *Let (X', A') and (Y', B') be pairs of countably generated locally finite coarse spaces. Then, the canonical inclusion maps*

$$\begin{aligned} \mathcal{P}(X') \times \mathcal{P}(Y') &\subset \mathcal{P}(X' \times Y'), & \mathcal{P}(A') \times \mathcal{P}(Y') &\subset \mathcal{P}(A' \times Y'), \\ \mathcal{P}(X') \times \mathcal{P}(B') &\subset \mathcal{P}(X' \times B'), & \mathcal{P}(A') \times \mathcal{P}(B') &\subset \mathcal{P}(A' \times B'), \\ \mathcal{P}(A') \times \mathcal{P}(Y') \cup \mathcal{P}(X') \times \mathcal{P}(B') &\subset \mathcal{P}(A' \times Y') \cup \mathcal{P}(X' \times B'), \\ \mathcal{P}(A' \times Y') \cup \mathcal{P}(X' \times B') &\subset \mathcal{P}(A' \times Y' \cup X' \times B') \end{aligned}$$

are all homotopy equivalences in the category of σ -locally compact spaces, i. e., the witnessing homotopies are also proper continuous σ -maps.

Proof All but the last one are a direct consequence of the simple construction in [9, Lemma 4.43].

For the last one, we use the barycentric subdivision from the proof of Lemma 7.2, taking Remark 7.3 into account: The simplex of $\mathcal{P}(A' \times Y' \cup X' \times B')$ spanned by an $(m+1)$ -tuple $\mathbf{x} \in (A' \times Y' \cup X' \times B')^{m+1}$ can be mapped piecewise linearly to the union of simplices in $\mathcal{P}(A' \times Y') \cup \mathcal{P}(X' \times B')$ spanned by the $(m+1)$ -tuples $\mathbf{x}_\sigma \in (A' \times Y')^{m+1} \cup (X' \times B')^{m+1}$. This yields a proper continuous σ -map which is a homotopy inverse to the inclusion, where the homotopies are simply linear interpolation. $\square$

Corollary 8.4 *Let (X', A') and (Y', B') be pairs of countably generated locally finite coarse spaces. Then, $(\mathcal{P}(X'), \mathcal{P}(A'))$ and $(\mathcal{P}(Y'), \mathcal{P}(B'))$ are E_* - and E^* -productable for all homology and cohomology theories E_*, E^* .*

Proof Lemmas 5.1 and 8.1 together show that

$$(\mathcal{P}(A' \times Y' \cup X' \times B'); \quad \mathcal{P}(A' \times Y'), \quad \mathcal{P}(X' \times B'))$$

is an E_* - and an E^* -excisive triad. The isomorphisms which are necessary to prove that

$$(\mathcal{P}(A') \times \mathcal{P}(Y') \cup \mathcal{P}(X') \times \mathcal{P}(B'); \quad \mathcal{P}(A') \times \mathcal{P}(Y'), \quad \mathcal{P}(X') \times \mathcal{P}(B'))$$

is an E_* - and an E^* -excisive triad can now be tracked down using the isomorphisms induced by the homotopy equivalences in the lemma. $\square$

Definition 8.5 For all pairs (X, A) and (Y, B) of countably generated coarse spaces of bornologically bounded geometry, we choose discretizations $(X', A') \subset (X, A)$ and $(Y', B') \subset (Y, B)$. Corollary 8.4 then ensures that the topological cross and slant products from Definition 8.2 exist for the pairs $(\mathcal{P}(X'), \mathcal{P}(A'))$ and $(\mathcal{P}(Y'), \mathcal{P}(B'))$. Lemma 8.3 allows us to identify them with the homomorphisms appearing in Definitions 5.2 and 5.3. It is clear from naturality of the topological cross and slant products that these resulting homomorphisms are independent of the chosen discretization. The family of all of them constitute a coarse cross product $\times: EX_i^* \otimes EX_{ii}^* \rightarrow EX_{iii}^*$ or a coarse slant product $/: EX_i^* \otimes EX_{ii}^* \rightarrow EX_{ii}^*$, respectively, and it is called the *coarsification* of the topological cross or slant product.

The purpose of this section is to analyze the resulting secondary cup and cap products of these coarsified cross and slant products. The first step is to reformulate them in a more topological way. As before, we write $I := [-1, 1]$. Note that we can define the following topological analogue to Definition 6.9.

Definition 8.6 Let $\mathcal{X}$ be a σ -locally compact space, $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ closed subspaces and

$$\begin{aligned} \tilde{H}^{\mathcal{A}} &: (\mathcal{X}, \mathcal{A}) \times ([0, 1], \{1\}) \rightarrow (\mathcal{X}, \mathcal{A}) \\ \tilde{H}^{\mathcal{B}} &: (\mathcal{X}, \mathcal{B}) \times ([0, 1], \{1\}) \rightarrow (\mathcal{X}, \mathcal{B}) \end{aligned}$$

continuous homotopies with $\tilde{H}^{\mathcal{A}}|_{\mathcal{X} \times \{0\}} = \text{id} = \tilde{H}^{\mathcal{B}}|_{\mathcal{X} \times \{0\}}$. Then,

$$\tilde{F}^{\mathcal{X}, \mathcal{A}, \mathcal{B}} := (\tilde{H}_-^{\mathcal{A}}, \tilde{H}_+^{\mathcal{B}}): (\mathcal{X}, \mathcal{A} \cup \mathcal{B}) \times (I, \partial I) \rightarrow (\mathcal{X}, \mathcal{A}) \times (\mathcal{X}, \mathcal{B})$$

is a proper continuous σ -map. Given a topological cross product $\times: E_i^* \otimes E_{ii}^* \rightarrow E_{iii}^*$, we define the associated topological secondary cup product $\mathbb{U}$ as $(-1)^m$ times the composition

$$\begin{aligned} E_i^m(\mathcal{X}, \mathcal{A}) \otimes E_{ii}^n(\mathcal{X}, \mathcal{B}) &\xrightarrow{\times} E_{iii}^{m+n}((\mathcal{X}, \mathcal{A}) \times (\mathcal{X}, \mathcal{B})) \\ &\xrightarrow{(\tilde{\Gamma}^{\mathcal{X}, \mathcal{A}, \mathcal{B}})^*} EX_{iii}^{m+n}((\mathcal{X}, \mathcal{A} \cup \mathcal{B}) \times (I, \partial I)) \\ &\xrightarrow{\sigma_{iii}^*} E_{iii}^{m+n-1}(\mathcal{X}, \mathcal{A} \cup \mathcal{B}). \end{aligned}$$

and given a topological slant product $/: E_i^i \otimes E_{ii}^* \rightarrow E_{iii}^*$ we define the associated secondary cap product $\mathbb{M}$ as $(-1)^{m+1}$ times the composition

$$\begin{aligned} E_m^i(\mathcal{X}, \mathcal{A} \cup \mathcal{B}) \otimes E_{ii}^n(\mathcal{X}, \mathcal{B}) &\rightarrow \\ &\xrightarrow{\sigma_*^i \otimes \text{id}} E_{m+1}^i((\mathcal{X}, \mathcal{A} \cup \mathcal{B}) \times (I, \partial I)) \otimes E_{ii}^n(\mathcal{X}, \mathcal{B}) \\ &\xrightarrow{(\tilde{\Gamma}^{\mathcal{X}, \mathcal{A}, \mathcal{B}})_* \otimes \text{id}} E_{m+1}^i((\mathcal{X}, \mathcal{A}) \times (\mathcal{X}, \mathcal{B})) \otimes E_{ii}^n(\mathcal{X}, \mathcal{B}) \\ &\xrightarrow{\downarrow} E_{m-n+1}^{iii}(\mathcal{X}, \mathcal{A}). \end{aligned}$$

The topological suspension isomorphisms in this definition are defined in the obvious way and they exist, because the inclusion

$$(\mathcal{X}, \mathcal{A} \cup \mathcal{B}) \times \{0\} \subset (\mathcal{X} \times \partial I \cup (\mathcal{A} \cup \mathcal{B}) \times I, \mathcal{X} \times \{1\} \cup (\mathcal{A} \cup \mathcal{B}) \times I)$$

is homotopy equivalent to a topological excision. The topological products fulfil the analogue properties to those of the coarse products proven in Sect. 6, but their proofs are a lot simpler, because we do not need to cope with the warped products.

Lemma 8.7 *Let $(X; A, B)$ be a deformation triad of countably generated coarse spaces of bornologically bounded geometry and $(X'; A', B') \subset (X; A, B)$ a discretization. Then, for the triad $(\mathcal{P}(X'); \mathcal{P}(A'), \mathcal{P}(B'))$ of σ -locally compact spaces there are induced continuous homotopies as in Definition 8.6. Furthermore, the associated topological secondary products canonically identify with the coarse secondary products for $(X; A, B)$ associated with the coarsified cross and slant products from Definition 8.5.*

Proof Let $H^A: (X, A) \times_{\mathcal{U}_A} ([0, 1], \{1\}) \rightarrow (X, A)$ be an associated generalized controlled homotopy with $H_0^A = \text{id}_X$, and we may assume that it restricts to a controlled homotopy $(X', A') \times_{\mathcal{U}} ([0, 1], \{1\}) \rightarrow (X', A')$. Then, the construction of [21, Lemma 5.13] provides us with a discretization (Z_A, D_A) of $(X, A) \times_{\mathcal{U}_A} ([0, 1], \{1\})$ which contains $(X', A') \cup \{0, 1\}$ and is contained in $(X', A') \cup [0, 1]$ and with a proper continuous map

$$(\mathcal{P}(X'), \mathcal{P}(A')) \times ([0, 1], \{1\}) \rightarrow (\mathcal{P}(Z_A), \mathcal{P}(D_A)),$$

which extends the canonical inclusion $(\mathcal{P}(X'), \mathcal{P}(A')) \times \{0, 1\} \subset (\mathcal{P}(Z_A), \mathcal{P}(D_A))$. Note that its composition with $\mathcal{P}(H^A): (\mathcal{P}(Z_A), \mathcal{P}(D_A)) \rightarrow (\mathcal{P}(X'), \mathcal{P}(A'))$ is a continuous homotopy

$$\tilde{H}^A: (\mathcal{P}(X'), \mathcal{P}(A')) \times ([0, 1], \{1\}) \rightarrow (\mathcal{P}(X'), \mathcal{P}(A'))$$

with $\tilde{H}|_{\mathcal{P}(X') \times \{0\}} = \text{id}$.

Similarly, we pick a controlled homotopy H^B associated with (X, B) and turn it into a continuous homotopy $\tilde{H}^B$ for $(\mathcal{P}(X'), \mathcal{P}(B'))$. Combining these two constructions,

we obtain a discretization (Z, D) of $(X, A \cup B) \times_{\mathcal{U}} (I, \partial I)$ and a proper continuous map $i: (\mathcal{P}(X'), \mathcal{P}(A') \cup \mathcal{P}(B')) \times (I, \partial I) \rightarrow (\mathcal{P}(Z), \mathcal{P}(D))$ extending the canonical inclusion of $(\mathcal{P}(X'), \mathcal{P}(A')) \times \partial I$ into $(\mathcal{P}(Z), \mathcal{P}(D))$ such that we have

$$\begin{aligned} \tilde{\Gamma}^{X,A,B} &:= (\tilde{H}_+^A, \tilde{H}_+^B) = \mathcal{P}(\Gamma^{X,A,B}) \circ i: \\ &(\mathcal{P}(X'), \mathcal{P}(A') \cup \mathcal{P}(B')) \times (I, \partial I) \rightarrow (\mathcal{P}(X'), \mathcal{P}(A')) \times (\mathcal{P}(X'), \mathcal{P}(B')). \end{aligned}$$

The identification of the coarse secondary cup product with the topological one is now given by the diagram

$$\begin{array}{ccc} \mathrm{EX}_I^m(X, A) \otimes \mathrm{EX}_{\mathrm{ii}}^n(X, B) & \xlongequal{\quad} & \mathrm{E}_I^m(\mathcal{P}(X'), \mathcal{P}(A')) \otimes \mathrm{E}_{\mathrm{ii}}^n(\mathcal{P}(X'), \mathcal{P}(B')) \\ \downarrow & & \downarrow \\ \mathrm{EX}_{\mathrm{iii}}^{m+n}((X, A) \times (X, B)) & \xrightarrow{\cong} & \mathrm{E}_{\mathrm{iii}}^{m+n}((\mathcal{P}(X'), \mathcal{P}(A')) \times (\mathcal{P}(X'), \mathcal{P}(B'))) \\ \downarrow (\Gamma^{X,A,B})^* & & \downarrow (\tilde{\Gamma}^{X,A,B})^* \\ \mathrm{EX}_{\mathrm{iii}}^{m+n}((X, A \cup B) \times_{\mathcal{U}} (I, \partial I)) & \xrightarrow{i^*} & \mathrm{E}_{\mathrm{iii}}^{m+n}((\mathcal{P}(X'), \mathcal{P}(A') \cup \mathcal{P}(B')) \times (I, \partial I)) \\ \downarrow \sigma_{\mathrm{iii}}^* & & \downarrow \sigma_{\mathrm{iii}}^* \\ \mathrm{EX}_{\mathrm{iii}}^{m+n-1}(X, A \cup B) & \xrightarrow{\cong} & \mathrm{E}_{\mathrm{iii}}^{m+n-1}(\mathcal{P}(X'), \mathcal{P}(A') \cup \mathcal{P}(B')) \end{array}$$

where the second and fourth horizontal arrows are isomorphisms induced by homotopy equivalences from Lemma 8.3. Commutativity of the upper square is the definition of the coarsification of the cross product. The middle square commutes by definition of $\tilde{\Gamma}^{X,A,B}$, and the lower square commutes by naturality of the connecting homomorphism under i .

The proof for the identification of the secondary cap products is completely dual. $\square$

We now turn our attention to transgression maps (cf. [21, Section 5.3]). They only exist for the so-called *single-space (co-)homology theories* (cf. [21, Definition 5.16], that is, (co-)homology theories E_*, E^* for σ -locally compact spaces which satisfy the *strong excision axiom*: For all pairs of σ -locally compact spaces $(\mathcal{X}, \mathcal{A})$, there are natural isomorphisms $E_*(\mathcal{X}, \mathcal{A}) \cong E_*(\mathcal{X} \setminus \mathcal{A})$ and $E^*(\mathcal{X}, \mathcal{A}) \cong E^*(\mathcal{X} \setminus \mathcal{A})$ and hence, these theories can be expressed completely by their absolute groups. The most important examples are K-theory, K-homology and Alexander–Spanier cohomology.

Note also that we do not have to worry about excisiveness conditions for single space (co-)homology theories E_*, E^* , because for every two closed subspaces $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ the triad $(\mathcal{X}; \mathcal{A}, \mathcal{B})$ is E_* - and E^* -excisive.

Let us briefly recall the definition of the transgression maps. Let (X, A) be a pair of countably generated coarse spaces of bornologically bounded geometry and $(X', A') \subset (X, A)$ a discretization. Canonically associated with X , there is a compact Hausdorff space $\partial_h X$, the so-called *Higson corona*. By a *Higson dominated corona*, we then mean a compact Hausdorff space ∂X together with a natural surjection $\partial_h X \rightarrow \partial X$. It has the property that we can glue it to the Rips complex $\mathcal{P}(X')$ to obtain a σ -compactification $\overline{\mathcal{P}(X')}$ of $\mathcal{P}(X')$ with corona $\partial X = \overline{\mathcal{P}(X')} \setminus \mathcal{P}(X')$. The subspace A then also has an associated Higson-dominated corona $\partial A \subset \partial X$, namely $\partial A = \partial X \cap \overline{\mathcal{P}(A')}$. With all this data, the transgression maps are the connecting homomorphisms for the pair $(\overline{\mathcal{P}(X')} \setminus \overline{\mathcal{P}(A')}, \partial X \setminus \partial A)$, that is,

$$T_*^{X,A}: \mathrm{EX}_{*+1}(X, A) \cong E_{*+1}(\mathcal{P}(X') \setminus \mathcal{P}(A')) \rightarrow E_*(\partial X \setminus \partial A) \cong E_*(\partial X, \partial A)$$

and dually $T_{X,A}^*: E^*(\partial X, \partial A) \rightarrow EX^{*+1}(X, A)$. The main result of this section is the following.

Theorem 8.8 *Given a topological cross product $\times: E_i^* \otimes E_{ii}^* \rightarrow E_{iii}^*$ or a topological slant product $/: E_i^* \otimes E_{ii}^* \rightarrow E_{ii}^{iii}$ between single space (co-)homology theories for σ -locally compact spaces, the induced secondary cup and cap products on coarse spaces (if existent) and the induced primary cup and cap products on the coronas are related by the transgression maps:*

$$\begin{aligned} \forall x \in E_i^*(\partial X, \partial A), y \in E_{ii}^*(\partial X, \partial B): \quad & T_{X,A \cup B}^*(x \cup y) = T_{X,A}^*(x) \cup T_{X,B}^*(y), \\ \forall x \in EX_*^i(X, A \cup B), y \in E_{ii}^*(\partial X, \partial B): \quad & T_*^{X,A}(x \cap T_{X,B}^*(y)) = T_*^{X,A \cup B}(x) \cap y. \end{aligned}$$

Example 8.9 Every compact metric space K is a Higson dominated corona of the open cone $\mathcal{O}K$ over itself and if moreover $L \subset K$ is a closed subset, then L is the corresponding Higson dominated corona of the subspace $\mathcal{O}L \subset \mathcal{O}K$. Combining [8, Section 3.1, Theorem 5.7, Lemma 4.17] and the five-lemma to pass from the reduced to the relative case shows that the transgression maps $T_*^{\mathcal{O}K, \mathcal{O}L}, T_{\mathcal{O}K, \mathcal{O}L}^*$ are isomorphisms.

Now, if two closed subsets $L, M \subset K$ are given then $(\mathcal{O}K, \mathcal{O}L, \mathcal{O}M)$ is a deformation triad by Example 6.2. Therefore, Theorem 8.8 shows that the secondary cup and cap products of $(\mathcal{O}K, \mathcal{O}L, \mathcal{O}M)$ can be identified canonically with the primary cup and cap products of (K, L, M) . This shows that our theory of coarse secondary products is at least as rich as the theory of the primary cup and cap products on compact metrizable spaces.

Instead of proving Theorem 8.8 directly, we note that by Lemma 8.7 it is nothing but the special case $\mathcal{X} = \mathcal{P}(X'), \mathcal{A} = \mathcal{P}(A'), \mathcal{B} = \mathcal{P}(B'), \partial \mathcal{X} = \partial X, \partial \mathcal{A} = \partial A, \partial \mathcal{B} = \partial B$ of the following theorem.

Theorem 8.10 *Let $(\mathcal{X}; \mathcal{A}, \mathcal{B})$ be a topological deformation triad of σ -locally compact spaces, that is, the assumptions of Definition 8.6 hold. Furthermore, let $\overline{\mathcal{X}}$ be a σ -locally compact space which contains $\mathcal{X}$ as an open subset and let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be closed subspaces of $\overline{\mathcal{X}}$ such that $\mathcal{A} = \overline{\mathcal{A}} \cap \mathcal{X}$ and $\mathcal{B} = \overline{\mathcal{B}} \cap \mathcal{X}$. We define $\partial \mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{X}, \partial \mathcal{A} = \overline{\mathcal{A}} \setminus \mathcal{A}$ and $\partial \mathcal{B} = \overline{\mathcal{B}} \setminus \mathcal{B}$.*

1. *If $\times: E_i^* \otimes E_{ii}^* \rightarrow E_{iii}^*$ is a cross product between single-space cohomology theories for σ -locally compact spaces, then the connecting homomorphisms*

$$\begin{aligned} \delta_i: E_i^*(\partial \mathcal{X} \setminus \partial \mathcal{A}) &\rightarrow E_i^{*+1}(\mathcal{X} \setminus \mathcal{A}), \\ \delta_{ii}: E_{ii}^*(\partial \mathcal{X} \setminus \partial \mathcal{B}) &\rightarrow E_{ii}^{*+1}(\mathcal{X} \setminus \mathcal{B}), \\ \delta_{iii}: E_{iii}^*(\partial \mathcal{X} \setminus (\partial \mathcal{A} \cup \partial \mathcal{B})) &\rightarrow E_{iii}^{*+1}(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \end{aligned}$$

are compatible with the primary and secondary cup product in the sense that $\delta_{iii}(x \cup y) = \delta_i(x) \cup \delta_{ii}(y)$ for all $x \in E_i^(\partial \mathcal{X} \setminus \partial \mathcal{A})$ and $y \in E_{ii}^*(\partial \mathcal{X} \setminus \partial \mathcal{B})$.*

2. *If $/: E_i^* \otimes E_{ii}^* \rightarrow E_{ii}^{iii}$ is a slant product between single-space cohomology theories for σ -locally compact spaces, then the connecting homomorphisms*

$$\begin{aligned} \partial^i: E_{*+1}^i(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) &\rightarrow E_*^i(\partial \mathcal{X} \setminus (\partial \mathcal{A} \cup \partial \mathcal{B})) \\ , \delta_{ii}: E_{ii}^*(\partial \mathcal{X} \setminus \partial \mathcal{B}) &\rightarrow E_{ii}^{*+1}(\mathcal{X} \setminus \mathcal{B}), \\ \partial^{iii}: E_{*+1}^{iii}(\mathcal{X} \setminus \mathcal{A}) &\rightarrow E_*^{iii}(\partial \mathcal{X} \setminus \partial \mathcal{A}) \end{aligned}$$

are compatible with the primary and secondary cap product in the sense that $\partial^{\text{iii}}(x \cap \delta_{\text{ii}}(y)) = \partial^{\text{i}}(x) \cap y$ for all $x \in E_{*+1}^{\text{i}}(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B}))$ and $y \in E_{\text{ii}}^*(\partial \mathcal{X} \setminus \partial \mathcal{B})$.

The notation already suggests that we have the special case in mind in which $\overline{\mathcal{X}}$ is a σ -compactification of $\mathcal{X}$ and $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ are the closures of $\mathcal{A}, \mathcal{B}$, respectively, in $\overline{\mathcal{X}}$, but we are not restricted to this case.

Proof In the following, we let $\overset{\circ}{J}$ denote the interior of J for any interval J . Consider the diagrams of function algebras with exact rows in Fig. 2.

All solid vertical maps are defined by pulling back functions along proper continuous maps between pairs of σ -locally compact spaces, even the inclusion of function algebras which is obtained by pulling back along the identity. We already know the notations $\tilde{H}^A, \tilde{H}^B$ for the associated continuous homotopies and the resulting proper continuous map $\tilde{\Gamma}^{\mathcal{X}, \mathcal{A}, \mathcal{B}}$ and continuous map $\tilde{H}^{\mathcal{A}}$. Furthermore, Δ denotes the diagonal map. The other continuous σ -maps are defined as follows:

$$\begin{aligned} \gamma: \mathcal{X} \times [-1, 1] &\rightarrow \mathcal{X} \times [-1, 1], & (x, s) &\mapsto (H^{\mathcal{A}}(x, s^-), 2s^+ - 1), \\ \beta: \mathcal{X} \times [0, 1] &\rightarrow \mathcal{X} \times \mathcal{X}, & (x, s) &\mapsto (H^{\mathcal{A}}(x, s^-), x). \end{aligned}$$

It is straightforward to check that they map the function algebras as stated. Furthermore, the γ^* in the bottom left corner of the first upper diagram is simply the canonical $*$ -homomorphism

$$C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \overset{\circ}{I}) \cong C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \overset{\circ}{I}_+) \subset C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \overset{\circ}{I}),$$

which is, of course, homotopic to the identity.

The two dashed arrows are both the one which is induced by γ , that is, the one which makes the lower half of the upper diagram commute. It maps a function $f \in C_0(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B}))$ to the class of the function

$$((x, s) \mapsto f(H^{\mathcal{A}}(x, s^-)) \cdot (1 - s^+)) \in C_b(\mathcal{X} \times I, \mathcal{X} \times \partial I \cup \mathcal{A} \times I),$$

because for all $f \in C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times [-1, 1))$ the function which maps (x, s) to

$$\begin{aligned} & f(\gamma(x, s)) - f(H^{\mathcal{A}}(x, s^-), -1) \cdot (1 - s^+) \\ &= \begin{cases} 0 & s \leq 0, \\ f(x, 2s^+ - 1) - f(x, -1) \cdot (1 - s^+) & s \geq 0 \end{cases} \end{aligned}$$

is an element of the ideal $C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \dot{I})$.

As the upper half of the upper diagram also commutes by definition, we just have to check commutativity of the middle diagram. For the right half, this follows trivially from $\beta \circ \text{ev}_1 = \Delta$ and $H_-^{\mathcal{A}} \circ \text{ev}_1 = \text{id}$. The upper left square commutes, because for $f \in C_0(\mathcal{X} \setminus \mathcal{A})$ and $g \in C_0(\overline{\mathcal{X} \setminus \mathcal{B}})$ we have

$$(\tilde{\Gamma}^{\mathcal{X}, \mathcal{A}, \mathcal{B}})^*(f \otimes g) - \beta^*(f \otimes g): (x, s) \mapsto \begin{cases} 0 & x \leq 0 \\ f(x) \cdot (g(H^{\mathcal{B}}(x, s^+)) - g(x)) & x \geq 0 \end{cases}$$

and this function lies in the ideal $C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times (-1, 1])$. For the bottom left square of the middle diagram, we use that for $f \in C_0(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B}))$ the function

$$(x, s) \mapsto f(H^{\mathcal{A}}(x, s^-)) - f(H^{\mathcal{A}}(x, s^-)) \cdot (1 - s^+) = s^+ \cdot f(x)$$

lies in $C_0((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times (-1, 1])$.

Dualizing these diagrams, we obtain the diagram of spaces at the bottom of Fig. 2, where the inclusions going right (i. e., the inclusions of the first into the third and the fourth into the fifth column) are inclusions of open subsets and the inclusions going left are inclusions of closed subsets. Furthermore, the spaces in the second and fourth column are the disjoint union of their left and right neighbors. This is exactly the data which yields the four upper commuting squares in the diagram

$$\begin{array}{ccccc} E_{m+1}^i((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \dot{I}) & \xrightarrow[\cong]{\partial^i} & E_m^i(\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) & \xrightarrow{\partial^i} & E_{m-1}^i(\partial \mathcal{X} \setminus (\partial \mathcal{A} \cup \partial \mathcal{B})) \\ \parallel & & \uparrow & & \uparrow \\ E_{m+1}^i((\mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})) \times \dot{I}) & \xrightarrow{\partial^i} & E_m^i(\mathcal{Y}_2) & \xrightarrow{\partial^i} & E_{m-1}^i(\mathcal{Y}_4) \\ \downarrow \tilde{\Gamma}^{\mathcal{X}, \mathcal{A}, \mathcal{B}} & & \downarrow & & \downarrow \Delta_* \\ E_{m+1}^i((\mathcal{X} \setminus \mathcal{A}) \times (\mathcal{X} \setminus \mathcal{B})) & \xrightarrow{\partial^i} & E_m^i((\mathcal{X} \setminus \mathcal{A}) \times (\partial \mathcal{X} \setminus \partial \mathcal{B})) & \xrightarrow{\partial^i} & E_{m-1}^i((\partial \mathcal{X} \setminus \partial \mathcal{A}) \times (\partial \mathcal{X} \setminus \partial \mathcal{B})) \\ \downarrow / \delta_{\text{ii}}(y) & & \downarrow / y & & \downarrow / y \\ E_{m-n}^{\text{iii}}(\mathcal{X} \setminus \mathcal{A}) & \xrightarrow{\quad} & E_{m-n}^{\text{iii}}(\mathcal{X} \setminus \mathcal{A}) & \xrightarrow{\partial^{\text{iii}}} & E_{m-n-1}^{\text{iii}}(\partial \mathcal{X} \setminus \partial \mathcal{A}) \end{array}$$

and the bottom right square commutes, whereas the bottom left square commutes up to a sign $(-1)^{m+1}$ for all $y \in E_{\text{ii}}^n(\partial \mathcal{X} \setminus \partial \mathcal{B})$ by the axioms of slant products. The secondary cap product with $\delta_{\text{ii}}(y)$ is by definition $(-1)^{m+1}$ times the composition from the top middle entry along the left side to the bottom left/middle entry. The composition along the right hand side on the other hand is the primary cap product with y itself. The second claim follows.

We omit the proof of the first claim, because it is completely analogous to the proof of the second claim and also very similar to the proof of [18, Theorem 9.1] $\square$

9 Assembly and coassembly

The relative versions of assembly and coassembly with coefficients in a C^* -algebra D ,

$$\begin{aligned}\mu: KX_*(X, A; D) &\rightarrow K_*(C^*(X, A; D)) \quad \text{and} \\ \mu^*: K_{-*}(c(X, A; D)) &\rightarrow KX^{1+*}(X, A; D),\end{aligned}$$

respectively, are natural transformations of coarse homology and cohomology theories on the category of proper metric spaces. Here, in contrast to [21], we use the standard convention that metrics only take finite values, which is equivalent to metric spaces being coarsely connected. We refer to [21, Sections 6.3, 6.4] for their definition and note that the technical Assumption 6.8 made there is always true in the non-equivariant case, which is the only case we are interested in here. We assume that the reader is already familiar with this theory and will only recall the relevant definitions in a very superficial manner.

The goal of this section is to show that canonical primary cup and cap products

$$\cup: K_{-m}(c(X, A; D)) \otimes K_{-n}(c(X, B; E)) \rightarrow K_{-m-n}(c(X, A \cup B; D \otimes E)), \quad (6)$$

$$\cap: K_m(C^*(X, A \cup B; D)) \otimes K_{-n}(c(X, B; E)) \rightarrow K_{m-n}(C^*(X, A; D \otimes E)) \quad (7)$$

correspond to our secondary cup and cap products

$$\cup: KX^{1+m}(X, A; D) \otimes KX^{1+n}(X, B; D) \rightarrow KX^{1+m+n}(X, A \cup B; D \otimes E), \quad (8)$$

$$\cap: KX_m(X, A \cup B; D) \otimes KX^{1+n}(X, B; D) \rightarrow KX_{m-n}(X, A; D \otimes E) \quad (9)$$

under assembly and co-assembly. Regarding the tensor products on the right hand sides, we have to mention that throughout this section we deviate from the usual conventions and use the symbol $\otimes$ for the *maximal* tensor product of C^* -algebras instead of the minimal one.

Theorem 9.1 *Let $(X; A, B)$ be a deformation triad of proper metric spaces and let D, E be C^* -algebras. Then,*

$$\mu^*(x \cup y) = \mu^*(x) \cup \mu^*(y)$$

for all $x \in K_{-m}(c(X, A; D)), y \in K_{-n}(c(X, B; E))$. If in addition X has coarsely bounded geometry, then

$$\mu(x \cap \mu^*(y)) = \mu(x) \cap y$$

for all $x \in KX_m(X, A \cup B; D), y \in K_{-n}(c(X, B; E))$.

Here, coarsely bounded geometry denotes the usual notion of bounded geometry for metric spaces: A space X has coarsely bounded geometry if it contains a coarsely dense uniformly locally finite subset $X' \subset X$, which we call a uniform discretization. Uniform finiteness means that for each $R > 0$ there is an upper bound on the number of points in $B_R(x) \cap X'$ for all $x \in X'$.

Before we can prove this theorem, we have to briefly recall the definitions of the objects appearing here and introduce the primary cup and cap products.

We start with the stable Higson coronas, which were first introduced in [7]. A continuous function $f: X \rightarrow D$ from a proper metric space X into a C^* -algebra D is said to have *vanishing variation* if the function

$$\text{Var}_E(f): X \rightarrow [0, \infty], \quad x \mapsto \sup\{\|f(x) - f(y)\| \mid (x, y) \in E\}$$

vanishes at infinity for every entourage E . Let $\ell^2 := \ell^2(\mathbb{N})$ denote our favorite infinite dimensional separable Hilbert space and $\mathfrak{K} := \mathfrak{K}(\ell^2)$ the C^* -algebra of compact operators on it. The *stable Higson compactification* of a proper metric space X with coefficients in a C^* -algebra D is the C^* -algebra $\bar{c}(X; D)$ of all bounded continuous functions $f: X \rightarrow D \otimes \mathfrak{K}$ of vanishing variation. It contains $C_0(X) \otimes D \otimes \mathfrak{K}$ as an ideal and the quotient C^* -algebra $c(X; D) := \bar{c}(X; D)/C_0(X) \otimes D \otimes \mathfrak{K}$ is called the *stable Higson corona*. Given furthermore a closed subspace $A \subset X$, we can define the relative versions of these C^* -algebras as

$$\begin{aligned} \bar{c}(X, A; D) &:= \ker(\bar{c}(X; D) \rightarrow \bar{c}(A; D)), \\ c(X, A; D) &:= \ker(c(X; D) \rightarrow c(A; D)). \end{aligned}$$

Definition 9.2 Given any proper metric space X with closed subspaces $A, B \subset X$ and C^* -algebras D, E , the primary cap product (6) is defined as the composition of the exterior product with the homomorphism induced by the multiplication $*$ -homomorphism

$$\begin{aligned} \nabla: c(X, A; D) \otimes c(X, B; E) &\rightarrow c(X, A \cup B; D \otimes E) \\ [f] \otimes [g] &\mapsto [x \mapsto f(x) \otimes g(x) \in D \otimes \mathfrak{K} \otimes E \otimes \mathfrak{K} \cong D \otimes E \otimes \mathfrak{K}]. \end{aligned}$$

The definition involved the choice of an identification $\mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$, but this is unique up to homotopy. Hence, the cup product is unambiguous.

The coarse K-theory $KX^*(-, -; D)$ with coefficients in D is defined by coarsification as in the previous section of the cohomology theory for σ -locally compact spaces defined by

$$K^*(\mathcal{X}, \mathcal{A}; D) := K_{-*}(C_0(\mathcal{X} \setminus \mathcal{A}) \otimes D).$$

Here, $C_0(\mathcal{X} \setminus \mathcal{A})$ is a σ - C^* -algebra and hence use Phillips' K-theory for σ - C^* -algebras on the right hand side (see [11], but also [12]). The latter can be equipped with an exterior tensor product (see [18, Section 5]) and if E is another C^* -algebra, then the resulting cross products

$$\times: K^m(\mathcal{X}, \mathcal{A}; D) \otimes K^n(\mathcal{Y}, \mathcal{B}; E) \rightarrow K^{m+n}((\mathcal{X}, \mathcal{A}) \times (\mathcal{Y}, \mathcal{B}); D \otimes E)$$

indeed constitute a cross product in the sense of Definition 8.2. Therefore, we also obtain cross products

$$\times: KX^*(-, -; D) \otimes KX^*(-, -; E) \rightarrow KX^*(-, -; D \otimes E)$$

in the sense of Definition 5.2, which induce the secondary cup products (8) as in the previous section.

Now, the *uncoarsified co-assembly map* is nothing but the connecting homomorphism $K_{-*}(c(X, A; D)) \rightarrow K^{1+*}(X, A; D)$ associated with the short exact sequence

$$0 \rightarrow C_0(X \setminus A) \otimes D \otimes \mathfrak{K} \rightarrow \bar{c}(X, A; D) \rightarrow c(X, A; D) \rightarrow 0.$$

In order to obtain the coarsified co-assembly map μ^* , we need to perform the same construction on the pair of the Rips complexes $(\mathcal{P}(X'), \mathcal{P}(A'))$ of a discretization $(X', A') \subset (X, A)$ instead of on (X, A) itself. To this end, we equip $\mathcal{P}(X')$ with the metric constructed in [21, Lemma 6.10] whose restriction to each finite scale Rips complex $P_n(X')$ is a proper metric and for which all the inclusions $X \supset X' \subset X_m \subset X_n$ are isometric coarse equivalences for all $0 \leq m \leq n$. Then, we have $c(X, A; D) \cong \varprojlim_{n \in \mathbb{N}} c(P_{E_n}(X'), P_{E_n}(A'); D)$ and we define the σ - C^* -algebra $\bar{c}(\mathcal{P}(X'), \mathcal{P}(A'); D) := \varprojlim_{n \in \mathbb{N}} \bar{c}(P_{E_n}(X'), P_{E_n}(A'); D)$. The *co-assembly map* μ^* is defined as the connecting homomorphism in K-theory for σ - C^* -algebras associated with the short exact sequence

$$0 \rightarrow C_0(\mathcal{P}(X') \setminus \mathcal{P}(A')) \otimes D \otimes \mathfrak{K} \rightarrow \bar{c}(\mathcal{P}(X'), \mathcal{P}(A'); D) \rightarrow c(X, A; D) \rightarrow 0.$$

How to prove the first part of Theorem 9.1 The special case of the theorem with X being coarsely contractible to $A = B = \{p\}$ was the main result of [18]. In order to prove Theorem 9.1, one only needs to generalize [18, Theorem 9.1], which is a straightforward task. The essential ideas which are necessary to do so can be found in our proof of Theorem 8.10 above. $\square$

Next up, we briefly recall the definition of the relative Roe algebras $C^*(X, A; D)$ with coefficients in D . We fix an ample representation $\rho_X: C_0(X) \rightarrow \mathfrak{B}(H_X)$ on a separable Hilbert space H_X and consider the Hilbert- D -module $H_X \otimes D$ equipped with the representation $\rho_X \otimes \text{id}_D$ of $C_0(X)$. In the following, we will consider adjointable operators $T \in \mathfrak{B}(H_X \otimes D)$ on the Hilbert- D -module $H_X \otimes D$. We say that:

- T has *finite propagation* if $\text{supp}(T)$ is an entourage in $X \times X$. In this case, the *propagation* $\text{prop}(T)$ of T is the smallest $R \geq 0$ such that $d(x, y) \leq R$ for all $(x, y) \in \text{supp}(T)$. (Recall that the support of an operator $T: \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ between two Hilbert modules $\mathfrak{H}_{1,2}$ equipped with representations $\rho_{1,2}: C_0(X_{1,2}) \rightarrow \mathfrak{B}(\mathfrak{H}_{1,2})$ is the largest closed subset $\text{supp}(T) \subset X_1 \times X_2$ such that $\rho_2(g) \circ T \circ \rho_1(f) = 0$ for all $f \in C_0(X_1)$ and $g \in C_0(X_2)$ with $\text{supp}(f) \times \text{supp}(g) \cap \text{supp}(T) = \emptyset$.)
- T is *locally compact* if $(\rho_X(f) \otimes \text{id}_D) \circ T, T \circ (\rho_X(f) \otimes \text{id}_D) \in \mathfrak{K}(H_X \otimes D) = \mathfrak{K}(H_X) \otimes D$ for all $f \in C_0(X)$.
- T is *supported near* A if there is $R \geq 0$ such that $\text{supp}(T) \subset \text{Pen}_R(A) \times \text{Pen}_R(A)$, where $\text{Pen}_R(A) := \text{Pen}_{E_R}(A)$ denotes the closed R -neighborhood of A or equivalently $(\rho_X(f) \otimes \text{id}_D) \circ T = 0 = T \circ (\rho_X(f) \otimes \text{id}_D)$ for all functions $f \in C_0(X)$ with $\text{dist}(\text{supp}(f), A) \geq R$.

Now, we can define the following *Roe algebras*: The C^* -algebra $C^*(X; D) \subset \mathfrak{B}(H_X \otimes D)$ is the norm closure of all locally compact operators of finite propagation. It contains the ideal $C^*(A \subset X; D) \subset C^*(X; D)$ which is the norm closure of all locally compact operators of finite propagation which are supported near A . Furthermore, the quotient $C^*(X, A; D) := C^*(X; D)/C^*(A \subset X; D)$ is the relative Roe algebra. Note that $C^*(X, \emptyset; D) = C^*(X; D)$.

If Y is another proper metric spaces and $\rho_Y: C_0(Y) \rightarrow \mathfrak{B}(H_Y)$ an ample representation on another separable Hilbert space H_Y , then an isometry $V: H_X \rightarrow H_Y$ is said to *cover* a coarse map $\alpha: X \rightarrow Y$ if $(\text{id} \times \alpha)(\text{supp}(V)) \subset Y \times Y$ is an entourage. Adjoining with $V \otimes \text{id}_D$ yields a $*$ -homomorphism $\text{Ad}_{V \otimes \text{id}_D}: C^*(X; D) \rightarrow C^*(Y; D)$. If α maps $A \subset X$ to $B \subset Y$, then $\text{Ad}_{V \otimes \text{id}_D}(C^*(A \subset X; D)) \subset C^*(B \subset Y; D)$ and we also obtain a $*$ -homomorphism between the relative Roe algebras.

Covering isometries always exist and the induced maps on K-theory

$$\alpha_* := (\text{Ad}_{V \otimes \text{id}_D})_* : K_*(C^*(X, A; D)) \rightarrow K_*(C^*(Y, B; D))$$

are independent of the choice of V . Applied to $\alpha = \text{id}$, one can deduce that the K-theory of the Roe algebras is independent of the choice of ρ_X, H_X up to canonical isomorphism. Furthermore, for an inclusion map $\alpha: A \subset X$ we have $\text{im}(\text{Ad}_{V \otimes \text{id}_D}) \subset C^*(A \subset X; D)$ and the induced map on K-theory is a canonical natural isomorphism $K_*(C^*(A; D)) \cong K_*(C^*(A \subset X; D))$.

We can now introduce the cap product (7) as a generalization of the module multiplication defined in [20, Section 8].

Lemma 9.3 *Let X be a proper metric space of coarsely bounded geometry, $A, B \subset X$ closed subspaces with A non-empty and D, E C^* -algebras. Choose an ample representation ρ_X of $C_0(X)$ on a separable Hilbert space H_X and assume that the Roe algebras $C^*(X, A \cup B; D)$ and $C^*(X, A; D \otimes E)$ have been constructed using the representations $\rho_X \otimes \text{id}_D$ on $H_X \otimes D$ and $\rho_X \otimes \text{id}_{D \otimes E \otimes \ell^2}$ on $H_X \otimes D \otimes E \otimes \ell^2$, respectively. Furthermore, ρ_X gives rise to a representation $C_0(X) \otimes E \otimes \mathfrak{K}$ on $H_X \otimes E \otimes \ell^2$ which extends by strong continuity to a representation $\bar{\rho}_X$ of $\mathcal{M}(C_0(X) \otimes E \otimes \mathfrak{K})$. Then,*

$$\begin{aligned} \mathfrak{m}: C^*(X, A \cup B; D) \otimes \bar{\mathfrak{c}}(X, B; E) &\rightarrow C^*(X, A; D \otimes E) \\ [T] \otimes [f] &\mapsto [(T \otimes \text{id}_{E \otimes \ell^2}) \circ (\bar{\rho}_X(f) \otimes \text{id}_D)] \end{aligned}$$

defines a $*$ -homomorphism.

Proof The lemma is a generalization of the first part of [20, Lemma 8.5], just without the gradings. Thanks to the fact that ρ_X always extends to a representation of all bounded Borel functions, the same arguments can be used in our case where X is not necessarily a manifold and H_X not necessarily an L^2 -space. The proof shows that the commutators $[T \otimes \text{id}_{E \otimes \ell^2}, \bar{\rho}_X(f) \otimes \text{id}_D]$ are compact operators for all $T \in C^*(X; D)$ and all $f \in \bar{\mathfrak{c}}(X; E)$. This step requires the assumption of coarsely bounded geometry on X . Now, as A is non-empty the compact operators are contained in $C^*(A \subset X; D \otimes E)$ and hence, we obtain a $*$ -homomorphism

$$C^*(X; D) \otimes \bar{\mathfrak{c}}(X; E) \rightarrow C^*(X, A; D \otimes E) \quad (10)$$

by the universal property of the maximal tensor product, which we denoted by $\otimes$. It clearly maps $C^*(A \subset X; D) \otimes \bar{\mathfrak{c}}(X; E)$ to zero and the same is true for $C^*(B \subset X; D) \otimes \bar{\mathfrak{c}}(X, B; E)$. To see the latter, note that for all $R \geq 0$ each element of $\bar{\mathfrak{c}}(X, B; E)$ and can be represented by a function f which vanishes on an R -neighborhood of B . Therefore, if T is supported near B , we just have to choose R sufficiently large and obtain $(T \otimes \text{id}_{E \otimes \ell^2}) \circ \bar{\rho}_X(f) = 0$. We now use the fact that

$$C^*(A \subset X; D) + C^*(B \subset X; D) = C^*(A \cup B \subset X; D) \quad (11)$$

and see that $\mathfrak{m}$ is well-defined as the quotient of a restriction of (10). $\square$

To avoid possible confusion about an alleged necessity for excisiveness, we note that (11) holds for *all* triads: If $T \in C^*(X; D)$ is supported in $\text{Pen}_R(A \cup B) \times \text{Pen}_R(A \cup B)$ and

$P \in \mathfrak{B}(H_X \otimes D)$ is the projection corresponding to the characteristic function of $\text{Pen}_R(A)$, then $PT \in C^*(A \subset X; D)$ and $(1 - P)T \in C^*(B \subset X; D)$. Excisiveness would only be necessary for the other condition for coarse Mayer–Vietoris, i. e., $C^*(A \subset X; D) \cap C^*(B \subset X; D) = C^*(A \cap B \subset X; D)$, which we do not need here.

Definition 9.4 Given a triad $(X; A, B)$ of proper metric spaces of coarsely bounded geometry with A non-empty and C^* -algebras D, E , the cap product (7) is defined as the composition of the exterior product with the homomorphism induced by m .

Lemma 9.5 The cap product is natural under coarse maps $\alpha: X \rightarrow X'$ taking the subspaces A, B into subspaces A', B' in the sense that

$$\alpha_*(x \cap \alpha^*y) = (\alpha_*x) \cap y$$

for all $x \in K_m(C^*(X, A \cup B; D))$ and $y \in K_{-n}(c(X', B'; E))$.

Proof It is easy to see that it suffices to show that $(V \otimes \text{id}_{E \otimes \ell^2}) \circ \bar{\rho}_X(g) - \bar{\rho}_{X'}(f) \circ (V \otimes \text{id}_{E \otimes \ell^2})$ is compact for all $g \in \bar{c}(X; D)$ and $f \in \bar{c}(X'; E)$ such that the (possibly non-continuous) function $g - \alpha^*f$ converges to zero at infinity. This can be done exactly as in the proof of [9, Theorem 4.28]. $\square$

The second part of [20, Lemma 8.5] also directly generalizes to our case, yielding the following

Lemma 9.6 Let X be a proper metric space of coarsely bounded geometry with non-empty closed subspaces $A, B, C \subset X$ and let D, E, F be C^* -algebras. Then, the diagram

$$\begin{array}{ccc} C^*(X, A \cup B \cup C; D) \otimes c(X, B; E) \otimes c(X, C; F) & \xrightarrow{m \otimes \text{id}} & C^*(X, A \cup C; D \otimes E) \otimes c(X, C; F) \\ \text{id} \otimes \nabla \downarrow & & \downarrow m \\ C^*(X, A \cup B \cup C; D) \otimes c(X, B \cup C; E \otimes F) & \xrightarrow{m} & C^*(X, A; D \otimes E \otimes F) \end{array}$$

commutes after adequately identifying the underlying Hilbert modules. Consequently, the cup and cap products (6), (7) are associative: $x \cap (y \cup z) = (x \cap y) \cap z$. $\square$

The coarse K-homology $KX_*(-, -; D)$ with coefficients in D is defined by coarsification of the K-homology with coefficients in D . We define the latter for pairs of second countable locally compact Hausdorff spaces via E-theory for C^* -algebras by

$$K_*(X, A; D) := E_*(C_0(X \setminus A), D)$$

and, subsequently, if a pair $(\mathcal{X}, \mathcal{A})$ of σ -locally compact spaces is given by a sequence of pairs (X_r, A_r) (with $A_r = \mathcal{A} \cap X_r$) of locally compact spaces which is second countable, then we define it by

$$K_*(\mathcal{X}, \mathcal{A}; D) := \varinjlim_{r \in \mathbb{N}} K_*(X_r, A_r; D).$$

As the finite scale Rips complexes of countable discrete coarse spaces are always second countable, the coarsification process of Sect. 8 still works out fine for K-homology. Using

the fact that on the cohomological side, there are canonical natural epimorphisms

$$K^*(\mathcal{X}, \mathcal{A}; D) \rightarrow \varprojlim_{r \in \mathbb{N}} E_{-*}(\mathbb{C}, C_0(X_r \setminus A_r) \otimes D),$$

which are compatible with cross products, the composition product in E-theory immediately gives rise to slant products

$$/: K_m((\mathcal{X}, \mathcal{A}) \times (\mathcal{Y}, \mathcal{B}); D \otimes E) \otimes K^n(\mathcal{Y}, \mathcal{B}; E) \rightarrow K_{m-n}(\mathcal{X}, \mathcal{A}; D) \quad (12)$$

in the sense of Definition 8.2, and the cross and slant products are compatible in a topological analogue sense to the condition in Theorem 6.13. This provides us with all the data needed to obtain the secondary cap products and show that they are even associative with the secondary cup products.

The coarse assembly map μ can only be defined by using other pictures of K-homology. One such picture is provided by the localization algebras introduced in [24]. See also [17] for a comprehensive exposition on this topic, but be aware that the “localized Roe algebras” introduced there are slightly bigger than Yu’s original algebras. We briefly recall their definition, but we additionally implement the coefficients.

With the notation introduced above, we denote by $C_L^*(X; D)$ the sub-C*-algebra of $C_b([1, \infty), C^*(X; D))$ generated by all uniformly continuous families $(T_t)_{t \geq 1}$ of locally compact adjointable operators of finite propagation with $\text{prop}(T_t) \xrightarrow{t \rightarrow \infty} 0$. It contains the ideal $C_L^*(A \subset X; D)$ generated by those families for which there is a function $\varepsilon: [1, \infty) \rightarrow (0, \infty)$ with $\varepsilon(t) \xrightarrow{t \rightarrow \infty} 0$ such that T_t is supported in the $\varepsilon(t)$ -neighborhood of $A \times A$ for all $t \geq 1$ and we define the relative localization algebra $C_L^*(X, A; D) := C_L^*(X; D)/C_L^*(A \subset X; D)$.

Now, as discussed in the context of [21, Assumption 6.8], which was only formulated as an assumption because of the unclear situation in the equivariant set-up, there are canonical natural isomorphisms

$$\Delta: K_*(C_L^*(X, A; D)) \cong K_*(X, A; D).$$

Here, naturality is with respect to uniformly continuous coarse maps $\alpha: (X, A) \rightarrow (Y, B)$, which induce homomorphisms between the domains of Δ as follows: We say that a uniformly continuous family of isometries $V: [1, \infty) \rightarrow \mathfrak{B}(H_X, H_Y)$, $t \mapsto V_t$ covers α if

$$\sup\{d_Y(y, \alpha(x)) \mid (y, x) \in \text{supp}(V_t)\} \begin{cases} < \infty & \text{for all } t \geq 1 \text{ and} \\ \rightarrow 0 & \text{for } t \rightarrow \infty. \end{cases} \quad (13)$$

Such families of covering isometries always exist, because we have assumed that ρ_Y is ample (compare [13, Proposition 3.2], [17, Theorem 6.6.3]), and adjoining with $V \otimes \text{id}_D$ gives rise to the induced maps

$$\alpha_* := (\text{Ad}_{V \otimes \text{id}_D})_*: K_*(C_L^*(X, A; D)) \rightarrow K_*(C_L^*(Y, B; D)).$$

As a special case of functoriality, the inclusions $A \subset X$ induce canonical natural isomorphisms $K_*(C_L^*(A; D)) \cong K_*(C_L^*(A \subset X; D))$, which is in complete analogy to what we wrote about of Roe algebras.

Now, the *uncoarsified assembly map* is simply the map

$$K_*(X, A; D) \cong K_*(C_L^*(X, A; D)) \xrightarrow{(\text{ev}_1)_*} K_*(C^*(X, A; D))$$

induced by evaluation at 1, and this construction is readily coarsified: Again, we let $(X', A') \subset (X, A)$ be a discretization and equip the Rips complexes $\mathcal{P}_n(X')$ with the metric from [21, Lemma 6.10] such that all the inclusions $X \supset X' \subset P_n(X) \subset P_m(X)$ (for $0 \leq m \leq n$) are isometric coarse equivalences. By exploiting the functoriality of Roe and localization algebras under coarse and uniformly continuous coarse maps, respectively, the *assembly map* μ is simply defined as

$$\begin{aligned} \mu: KX_*(X, A; D) &\cong \varinjlim_{r \in \mathbb{N}} K_*(C_L^*(P_r(X'), P_r(A'); D)) \\ &\xrightarrow{(\text{ev}_1)_*} \varinjlim_{r \in \mathbb{N}} K_*(C^*(P_r(X'), P_r(A'); D)) \cong K_*(C^*(X, A; D)), \end{aligned}$$

and we see that this definition is independent of the choice of refinement. Furthermore, by using a sufficiently thinned out discretization (X', A') , the fourth part of [21, Lemma 6.10] then directly implies that the assembly map μ is natural under coarse maps.

We have now introduced all objects appearing in the second part of Theorem 9.1, but instead of proving it directly, we prove a stronger localized version. It involves the following localized cap product. To this end, we note that similarly to (11) we also have $C_L^*(A \subset X; D) + C_L^*(B \subset X; D) = C_L^*(A \cup B \subset X; D)$ for arbitrary triads $(X; A, B)$ of proper metric spaces and hence, the construction of the multiplication map m in Lemma 9.3 can be performed pointwise over $[1, \infty)$ to obtain a $*$ -homomorphism

$$m_L: C_L^*(X, A \cup B; D) \otimes c(X, B; E) \rightarrow C_L^*(X, A; D \otimes E).$$

It induces a natural primary cap product

$$\cap: K_m(C_L^*(X, A \cup B; D)) \otimes K_{-n}(c(X, B; E)) \rightarrow K_m(C_L^*(X, A; D \otimes E)). \quad (14)$$

Theorem 9.7 *Let $(X; A, B)$ be a deformation triad of proper metric space of coarsely bounded geometry and let D, E be C^* -algebras. Then, the coarsification*

$$\cap: KX_m(X, A \cup B; D) \otimes K_{-n}(c(X, B; E)) \rightarrow KX_{m-n}(X, A; D \otimes E)$$

of (14) corresponds to the secondary cap product under coassembly in the sense that $x \frown \mu^(y) = x \cap y$ for all $x \in KX_m(X, A \cup B; D)$, $y \in K_{-n}(c(X, B; E))$.*

Proof (Second part of Theorem 9.1) The claim follows immediately from Theorem 9.7, because the cap product (14) clearly corresponds to the cap product on the K-theory of the Roe-algebras under evaluation at 1. $\square$

In order to prove Theorem 9.7, we first need to pass from the E-theoretic slant product to a slant product on the K-theory of the localization algebras. In the absolute case without coefficients, this was done in [9, Section 4.3]. We shall briefly recall the constructions, simultaneously adapting them to the relative case with coefficients. The proofs work exactly the same way in this more general set-up.

Let (X, A) and (Y, B) be pairs of proper metric spaces and $\rho_X: C_0(X) \rightarrow \mathfrak{B}(H_X)$, $\rho_Y: C_0(Y) \rightarrow \mathfrak{B}(H_Y)$ be ample representations as before. The tensor product of ρ_Y ,

and the canonical representations of E on itself and of $\mathfrak{K}$ on ℓ^2 is a non-degenerate representation of $C_0(Y, E \otimes \mathfrak{K})$ on the Hilbert module $H_Y \otimes E \otimes \ell^2$. By [1, Theorem II.7.3.9], it extends uniquely to a strictly continuous representation

$$\tilde{\rho}_Y: \mathcal{M}(C_0(Y, E \otimes \mathfrak{K})) \rightarrow \mathfrak{B}(H_Y \otimes E \otimes \ell^2)$$

of the multiplier algebra. Furthermore, we shall assume that the localization algebra $C_L^*(X \times Y; D)$ has been constructed using the representation $\rho_X \otimes \rho_Y \otimes \text{id}_D$ on the Hilbert module $H_X \otimes H_Y \otimes D$ and $C_L^*(X; D \otimes E)$ has been constructed using the representation $\tilde{\rho} := \rho_X \otimes \text{id}_{H_Y \otimes D \otimes E \otimes \ell^2}$ on $H_X \otimes H_Y \otimes D \otimes E \otimes \ell^2$. The latter is an ideal in the sub- C^* -algebra

$$E_L^*(X; D \otimes E) \subset C_b([1, \infty), \mathfrak{B}(H_X \otimes H_Y \otimes D \otimes E \otimes \ell^2))$$

generated by all bounded and uniformly continuous functions $T: t \mapsto T_t$ such that the propagation $\text{prop}(T_t)$ with respect to the representation $\tilde{\rho}_X$ is finite for all $t \geq 1$ and tends to zero as $t \rightarrow \infty$.

Now, if Y has bounded geometry, then implementing coefficients into [9, Lemma 4.8(iii)] shows that there is a canonical $*$ -homomorphism

$$\begin{aligned} \psi_L: C_L^*(X \times Y; D) \otimes c(Y; E) &\rightarrow E_L^*(X; D \otimes E)/C_L^*(X; D \otimes E) \\ T \otimes [f] &\mapsto [t \mapsto (T_t \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_X \otimes D} \otimes \tilde{\rho}_Y(f))]. \end{aligned}$$

It maps $C_L^*(X \times B \subset X \times Y; D) \otimes c(Y; B; E)$ to zero, because each element of $c(Y; B; E)$ can be represented by a function f which vanishes on the 1-neighborhood $\text{Pen}_1(B)$ of B and for each $T \in C_L^*(X \times B \subset X \times Y; D)$ the T_t are supported within $\text{Pen}_1(X \times B) \times \text{Pen}_1(X \times B)$ for t large enough. Furthermore, it maps $C_L^*(A \times Y \subset X \times Y; D) \otimes c(Y; E)$ to the subset of equivalence classes represented by elements of the ideal $E_L^*(A \subset X; D \otimes E) \subset E_L^*(X; D \otimes E)$ which we define as the closure of all T for which there is in addition a function $\varepsilon: [1, \infty) \rightarrow (0, \infty)$ with $\varepsilon(t) \xrightarrow{t \rightarrow \infty} 0$ such that T_t is supported in the $\varepsilon(t)$ -neighborhood of $A \times A$ for all $t \geq 1$. Let $E_L^*(X, A; D \otimes E) := E_L^*(X; D \otimes E)/E_L^*(A \subset X; D \otimes E)$ and note that it contains $C_L^*(X, A; D \otimes E)$ canonically as an ideal, because $C_L^*(A \subset X; D \otimes E) = E_L^*(A \subset X; D \otimes E) \cap C_L^*(X; D \otimes E)$. Then, the above considerations show that ψ_L induces a $*$ -homomorphism

$$\psi_L: C_L^*((X, A) \times (Y, B); D) \otimes c(Y, B; E) \rightarrow E_L^*(X, A; D \otimes E)/C_L^*(X, A; D \otimes E),$$

which we denote by the same letter.

Definition 9.8 (Relative version with coefficients of [9, Definition 4.9]) We define a slant product between the K-theory of the localization algebra and the K-theory of the stable Higson corona as $(-1)^m$ times the composition

$$\begin{aligned} K_m(C_L^*((X, A) \times (Y, B); D)) \otimes K_{1-n}(c^{\text{red}}(Y, B; E)) \\ \rightarrow K_{m+1-n}(C_L^*((X, A) \times (Y, B); D) \otimes c^{\text{red}}(Y, B; E)) \\ \xrightarrow{(\psi_L)_*} K_{m+1-n}(E_L^*(X, A; D \otimes E)/C_L^*(X, A; D \otimes E)) \\ \xrightarrow{\partial} K_{m+1-n}(C_L^*(X, A; D \otimes E)), \end{aligned}$$

where the first arrow is the external tensor product and the last arrow is the boundary map of K-theory.

In the next lemma, we have to assume that Y has continuously bounded geometry (cf. [9, Definition 4.1 (b)]), which is a notion of bounded geometry for metric spaces which is modeled after bounded geometry for complete Riemannian manifolds and implies coarsely bounded geometry. It says the following: For every $r > 0$ and $R > 0$ there exists a constant $K_{r,R} > 0$ such that

- for every $r > 0$ there is a subset $\hat{Y}_r \subset Y$ such that $Y = \bigcup_{\hat{y} \in \hat{Y}_r} B_r(\hat{y})$ and such that for all $r, R > 0$ and $y \in Y$ the number $\#(\hat{Y}_r \cap \bar{B}_R(y))$ is bounded by $K_{r,R}$ and
- for all $\alpha > 0$ we have $\limsup_{r \rightarrow 0} K_{r,\alpha r} < \infty$.

Important to us is that this property holds for all Rips complexes $P_n(Y')$ equipped with the metric from [21, Lemma 6.10] of uniformly locally finite proper metric spaces Y' : In this case, the Rips complexes are locally finite and finite dimensional and hence it is easy to write down subsets $\hat{Y}_r \subset P_n(Y')$ witnessing the continuously bounded geometry, e. g., the subset of all points whose barycentric coordinates are multiples of $\frac{1}{N}$ for $N \in \mathbb{N}$ large enough.

Lemma 9.9 (Relative version with coefficients of [9, Theorem 4.13]) *Let Y have continuously bounded geometry. Then, the slant product from 9.8 and the slant product coming from E-theory are related via the uncoarsified coassembly map μ^* and the isomorphism Δ by the commutative diagram*

$$\begin{array}{ccc} K_m(C_L^*((X, A) \times (Y, B); D)) \otimes K_{1-n}(\mathfrak{c}^{\text{red}}(Y, B; E)) & \xrightarrow{\quad / \quad} & K_{m-n}(C_L^*(X, A; D \otimes E)) \\ \Delta \otimes \mu^* \downarrow & & \Delta \downarrow \cong \\ K_m((X, A) \times (Y, B); D) \otimes K^n(Y, B; E) & \xrightarrow{\quad / \quad} & K_{m-n}(X, A; D \otimes E). \end{array}$$

Proof Continuously bounded geometry of Y implies that there is a $*$ -homomorphism

$$\begin{aligned} \gamma_L: C_L^*(X \times Y; D) \otimes C_0(Y; E \otimes \mathfrak{K}) &\rightarrow C_L^*(X; D \otimes E)/C_0([1, \infty), C^*(X; D \otimes E)) \\ T \otimes [f] &\mapsto [t \mapsto (T_t \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_X \otimes D} \otimes \bar{\rho}_Y(f))] \end{aligned}$$

as one easily sees by implementing coefficients into the arguments leading to [9, Display (4.10)]. It clearly vanishes for $T \in C_L^*(X \times B \subset X \times Y; B)$ and $f \in C_0(Y \setminus B; E \otimes \mathfrak{K})$, because for every $\varepsilon > 0$ the support of T_t will be contained in $f^{-1}(-\varepsilon, \varepsilon) \times f^{-1}(-\varepsilon, \varepsilon)$ for t large enough and hence, $\|(T_t \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_X \otimes D} \otimes \bar{\rho}_Y(f))\| \leq \|T\| \cdot \varepsilon$. It clearly also maps $C_L^*(A \times Y \subset X \times Y; B) \otimes C_0(Y; E \otimes \mathfrak{K})$ to the ideal $C_L^*(A \subset X; D \otimes E)/C_0([1, \infty), C^*(A \subset X; D \otimes E))$ and hence we obtain an induced $*$ -homomorphism

$$\gamma_L: C_L^*((X, A) \times (Y, B); D) \otimes C_0(Y, B; E \otimes \mathfrak{K}) \rightarrow \frac{C_L^*(X, A; D \otimes E)}{C_0([1, \infty), C^*(X, A; D \otimes E))}.$$

The proof can then be finished just like in [9, Section 4.3] by showing that the two slant products agree with the composition

$$\begin{aligned} K_m(C_L^*((X, A) \times (Y, B); D)) \otimes K^n(Y, B; E) \\ \rightarrow K_{m-n}(C_L^*((X, A) \times (Y, B); D) \otimes C_0(Y, B; E \otimes \mathfrak{K})) \end{aligned}$$

$$\begin{aligned} & \xrightarrow{(\tau_L)_*} K_{m-n} \left(\frac{C_L^*(X, A; D \otimes E)}{C_0([1, \infty), C^*(X, A; D \otimes E))} \right) \\ & \cong K_{m-n}(C_L^*(X, A; D \otimes E)). \end{aligned}$$

□

Proof of Theorem 9.7 Let $X' \subset X$ be a uniform discretization such that $A' := A \cap X'$ and $B' := B \cap X'$ are also uniform discretizations. We may also assume that X' was chosen in a δ -separated manner for some $\delta > 0$, that is, the distance between distinct points of X' is at least δ . For each $r \geq 0$, we introduce the short notations $X_r := P_r(X')$, $A_r := P_r(A')$, $B_r := P_r(B')$ for the associated Rips complexes at scale r . As in the last few sections, we also write $I := [-1, 1]$, $I_+ := [0, 1]$, $I_- := [-1, 0]$.

In Lemma 8.7, we had reformulated the coarse secondary products as special cases of the topological secondary products. Recall that its proof involved the proper continuous σ -map $\tilde{\Gamma}^{X,A,B} = (\tilde{H}_-^A, \tilde{H}_+^B): \mathcal{P}(X') \times I \rightarrow \mathcal{P}(X') \times \mathcal{P}(X')$. In order to express the topological secondary product in terms of localization algebras and covering isometries, we thus have to metrize every $X_r \times I$ in such a way that all the restrictions of $\tilde{\Gamma}^{X,A,B}$ of the form $X_r \times I \rightarrow X_R \times X_R$ are uniformly continuous coarse maps. This can be done as follows. First, we equip $X \times I$ with the largest metric such that all the slices $X \times \{t\} \subset X \times I$ are isometric to X and the points $(x, s), (x, t)$ have distance at most 1 if $(s, t) \in U_x$. We do not care that this metric is not proper (it induces the topology which is the disjoint union topology of the slices $X \times \{t\}$), but it obviously induces the coarse structure of $X \times_{\mathcal{U}} I$ and its restriction to the discretization $Z \subset X \times_{\mathcal{U}} I$ appearing in the proof of Lemma 8.7 is a proper metric. Now, we may furthermore assume that the discretization $Z \subset X \times_{\mathcal{U}} I$ containing $X' \cup \{-1, 0, 1\}$ is also δ -separated, because otherwise we can simply thin it out. Thus, if we equip all the Rips complexes with the metrics from [21, Lemma 6.10], then all the restrictions of $\mathcal{P}(\Gamma^{X,A,B}): \mathcal{P}(Z) \rightarrow \mathcal{P}(X') \times \mathcal{P}(X')$ to finite scale Rips complexes are uniformly continuous coarse maps by the fourth part of that lemma. As $\tilde{\Gamma}^{X,A,B}$ is the composition of $\mathcal{P}(\Gamma^{X,A,B})$ with the embedding $i: \mathcal{P}(X') \times I \rightarrow \mathcal{P}(Z)$, the pull-back metric on $\mathcal{P}(X') \times I$ does the job. We denote the subspaces $X_r \times I$ equipped with the restrictions of this metric by $X_r \times_{\mathcal{U}} I$. Recall in particular that Z contained $X' \times \{0\}$, so the subspace $\mathcal{P}(X') \times \{0\} \subset \mathcal{P}(X') \times I$ is isometric to $\mathcal{P}(X')$ by construction.

We do have some degree of freedom in the choice of the representations in the construction of the various localization algebras. We chose ρ_r to be the canonical ample representation of $C_0(X_r)$ on $H_{X_r} := \ell^2(X_r^{\mathbb{Q}}) \otimes \ell^2$ by multiplication operators, where $X_r^{\mathbb{Q}} \subset X_r$ denotes the subset of all points whose barycentric coordinates with respect to the simplex they lie in are rational. The big advantage of this particular representation is that H_X has an obvious canonical basis of vectors which are supported at single points, which makes working with covering isometries much easier later on. For the same reason, we choose for each closed interval J the canonical representation $\rho_J: C(J) \rightarrow \mathfrak{B}(\ell^2(J \cap \mathbb{Q}))$ by multiplication. In the following, J is always one of $I, I_{\pm}$.

We now consider a fixed $r \geq 0$ and choose $R \geq 0$ large enough such that $\tilde{\Gamma}^{X,A,B}$ maps $X_r \times I$ into $X_R \times X_R$. For these particular fixed r, R , we use the representations indicated in the following table to construct the various localization algebras.

C*-algebra	Representation	Hilbert module
$C_L^*(X_r; D)$	$\rho_r \otimes \text{id}_D$	$H_{X_r} \otimes D$
$C_L^*(X_r; D \otimes E)$	$\rho_r \otimes \text{id}_{D \otimes E \otimes \ell^2}$	$H_{X_r} \otimes D \otimes E \otimes \ell^2$
$C_L^*(X_r \times_{\mathcal{U}} J; D)$	$\rho_r \otimes \rho_J \otimes \text{id}_D$	$H_{X_r} \otimes \ell^2(J \cap \mathbb{Q}) \otimes D$
$C_L^*(X_R \times X_R; D)$	$\rho_R \otimes \rho_R \otimes \text{id}_D$	$H_{X_R} \otimes H_{X_R} \otimes D$
$E_L^*(X_R; D \otimes E)$	$\rho_R \otimes \text{id}_{H_{X_r} \otimes D \otimes E \otimes \ell^2}$	$H_{X_R} \otimes H_{X_R} \otimes D \otimes E \otimes \ell^2$
$C_L^*(X_R; D \otimes E)$		

Of course, all of the derive ideals and quotients as well as the corresponding Roe algebras are assumed to be constructed using the same representations. Note also that we use different representation for $C_L^*(X_r; D \otimes E)$ and $C_L^*(X_R; D \otimes E)$ even if $r = R$.

With these choices of representations, we have *canonical* inclusions of C*-algebras

$$C_L^*(X_r \times_{\mathcal{U}} I_{\pm}; D) \subset C_L^*(X_r \times I_{\pm} \subset X_r \times_{\mathcal{U}} I; D)$$

with $C_L^*(X_r \times_{\mathcal{U}} I_+; D) \cap C_L^*(X_r \times_{\mathcal{U}} I_-; D) \cong C_L^*(X_r; D)$. The quotients of these C*-algebras associated with closed subspaces clearly satisfy the analogue statements. More precisely, we are interested in the C*-algebras

$$\begin{aligned} L &:= C_L^*(X_r, A_r \cup B_r) \times_{\mathcal{U}} (I, \partial I; D), \\ L_{\pm} &:= C_L^*((X_r, A_r \cup B_r) \times_{\mathcal{U}} (I_{\pm}, \{\pm 1\}); D), \\ L_0 &:= C_L^*(X_r, A_r \cup B_r; D) \end{aligned}$$

and for these the above inclusions induce *-monomorphisms $L_{\pm} \hookrightarrow L$ and $L_0 \hookrightarrow L_{\pm}$, which we write as inclusions, and then we have $L_0 = L_+ \cap L_-$, too. Furthermore, the three C*-algebras

$$\begin{aligned} L_{\pm}^{\subseteq} &:= C_L^*((X_r, A_r \cup B_r) \times (I_{\pm}, \{\pm 1\})) \subset X_r \times_{\mathcal{U}} I; D), \\ L_0^{\subseteq} &:= C_L^*((X_r, A_r \cup B_r) \times \{0\}) \subset X_r \times_{\mathcal{U}} I; D) \end{aligned}$$

can be identified canonically with ideals in L which satisfy $L = L_+^{\subseteq} + L_-^{\subseteq}$ and $L_0 = L_+^{\subseteq} \cap L_-^{\subseteq}$.

Given any C*-algebra C with subalgebras C_0, C_1 , we define the two-sided mapping cylinder

$$\text{Zyl}(C, C_0, C_1) := \{f \in C([0, 1], C) \mid f(0) \in C_0, f(1) \in C_1\}.$$

Furthermore, we recall that the mapping cone of a *-epimorphism $\varphi: C_1 \rightarrow C_2$ is

$$\text{Cone}(\varphi) := \{(T, f) \in C_1 \oplus C_0([0, 1], C_2) \mid \varphi(T) = f(0)\}.$$

We can then consider the following diagram, in which all arrows marked with $\cong$ induce isomorphisms on K-theory:

$$\begin{array}{ccccc} C_0((0, 1), L_-^{\subseteq}/L_0^{\subseteq}) & \longrightarrow & \text{Cone}(L_-^{\subseteq} \rightarrow L_-^{\subseteq}/L_0^{\subseteq}) & \xleftarrow{\cong} & L_0^{\subseteq} \\ \uparrow \cong & & \uparrow \cong & & \uparrow \text{ev}_0 \cong \\ C_0((0, 1), L) & \longrightarrow & \text{Zyl}(L, L_-^{\subseteq}, L_+^{\subseteq}) & \xleftarrow{\cong} & C([0, 1], L_0^{\subseteq}) \\ \parallel & & \uparrow \cong & & \uparrow \text{incl. as const. fu.} \cong \\ C_0((0, 1), L) & \longrightarrow & \text{Zyl}(L, L_-, L_+) & \xleftarrow{\cong} & L_0 \end{array}$$

The left and middle vertical arrows from the second to the first row are induced by the canonical $*$ -epimorphism $L \rightarrow L/L_+^\subseteq \cong L_-^\subseteq/L_0^\subseteq$. They induce isomorphisms on K-theory, because homotopy invariance of K-homology implies $K_*(L_+^\subseteq; D) \cong K_*((X, A \cup B) \times (I_+, \{1\}; D) = 0$ and hence, the respective kernels $C_0((0, 1), L_+^\subseteq)$ and $C_0((0, 1), L_+^\subseteq)$ of the two arrows have zero K-theory, too.

The other arrows are defined in the obvious way. Recall that the first two rows are actually well known, as well as the fact that the arrows from the third to the second column in these rows induce isomorphisms on K-theory: The first row is the one inducing the connecting homomorphism $K_{m+1}(L_-^\subseteq/L_0^\subseteq) \rightarrow K_m(L_0^\subseteq)$ and the second row induces the boundary map in the Mayer–Vietoris sequence associated with the decomposition $L = L_+^\subseteq + L_-^\subseteq$ with $L_0 = L_+^\subseteq \cap L_-^\subseteq$ (cf. [10, Exercise 4.10.21]).

Using the fact that the inclusions $L_\pm \subset L_\pm^\subseteq$ and $L_0 \subset L_0^\subseteq$ induce isomorphisms in K-theory, it is easy to deduce that the arrows from the third to the second row (and hence also the arrow going left in the third row) induce isomorphisms as well.

Now, note that the connecting homomorphism $K_{m+1}(L_-^\subseteq/L_0^\subseteq) \rightarrow K_m(L_0^\subseteq)$ induced by the first row identifies with the connecting homomorphism in the K-homology $K_*(-, -; D)$ associated with the triple

$$(X_r \times [-1, 0], X_r \times \{-1, 0\} \cup (A_r \cup B_r) \times [-1, 0], X_r \times \{-1\} \cup (A_r \cup B_r) \times [-1, 0]).$$

Therefore, combining it with orientation preserving homeomorphism $[-1, 0] \rightarrow I$ we see that the top row and left column together induce exactly the *negative* of the inverse of suspension

$$\sigma_*: K_m(X_r, A_r \cup B_r; D) \rightarrow K_{m+1}((X_r, A_r \cup B_r) \times (I, \partial I); D).$$

The sign appeared because we have flipped the roles of the endpoints of the interval. Hence, the bottom row of the diagram is (up to sign) a reformulation of suspension and it will enable us to compare the cap products.

To finish the proof, we construct the following diagram.

$$\begin{array}{ccccc} C_0(0, 1) \otimes L & \xrightarrow{\quad} & \text{Zyl}(L, L_-, L_+) & \xleftarrow{\cong} & L_0 \otimes c(X_R, B_R; E) \\ \otimes c(X_R, B_R; E) & & \otimes c(X_R, B_R; E) & & \\ \downarrow \text{id}_{C_0(0,1)} \otimes & & \downarrow \mathfrak{M} & & \downarrow \mathfrak{M}_0 \\ \psi_L \circ ((\tilde{f}^{X,A,B})_* \otimes \text{id}) & & & & \\ C_0(0, 1) \otimes & \xrightarrow{\quad} & \text{Cone} \left(\begin{array}{c} E_L^*(X_R, A_R; D \otimes E) \\ \downarrow \\ E_L^*(X_R, A_R; D \otimes E) \\ C_L^*(X_R, A_R; D \otimes E) \end{array} \right) & \xleftarrow{\cong} & C_L^*(X_R, A_R; D \otimes E) \\ \frac{E_L^*(X_R, A_R; D \otimes E)}{C_L^*(X_R, A_R; D \otimes E)} & & & & \end{array}$$

The left vertical arrow is obtained by applying first

$$(\tilde{f}^{X,A,B})_*: L = C_L^*(X_r, A_r \cup B_r) \times_{\mathcal{U}} (I, \partial I; D) \rightarrow C_L^*((X_R, A_R) \times (X_R, B_R); D)$$

and then

$$\psi_L: C_L^*((X_R, A_R) \times (X_R, B_R); D) \otimes c(X_R, B_R; E) \rightarrow \frac{E_L^*(X_R, A_R; D \otimes E)}{C_L^*(X_R, A_R; D \otimes E)}$$

pointwise. The right vertical arrow $\mathfrak{M}_0$ is the composition of

$$\mathfrak{m}: C_L^*(X_r, A_r \cup B_r; D) \otimes c(X_r, B_r; E) \rightarrow C_L^*(X_r, A_r; D \otimes E) \subset C_L^*(X_R, A_R; D \otimes E)$$

with the canonical identification $\iota_{r,R}^*: \mathfrak{c}(X_R, B_R; E) \cong \mathfrak{c}(X_r, B_r; E)$ and the $*$ -homomorphism

$$(\iota_{r,R})_*: C_L^*(X_r, A_r; D \otimes E) \rightarrow C_L^*(X_R, A_R; D \otimes E)$$

induced by the inclusion $\iota_{r,R}: X_r \rightarrow X_R$.

It remains to combine these two constructions to obtain the middle vertical arrow $\mathfrak{M}$, which is still a lot of work. At this point, it is very helpful to choose a very specific covering isometry yielding $(\tilde{F}^{X,A,B})_*$. We may assume that the discretization $Z \subset X \times_{\mathcal{Q}} I$ is even contained in $X \times (I \cap \mathbb{Q})$ and then $\tilde{F}^{X,A,B} = (\tilde{H}_-^A, \tilde{H}_+^B)$ maps $X_r^{\mathbb{Q}} \times (I \cap \mathbb{Q})$ into $X_R^{\mathbb{Q}} \times X_R^{\mathbb{Q}}$. This allows us to directly write down the isometry

$$\begin{aligned} V: \underbrace{\ell^2(X_r^{\mathbb{Q}}) \otimes \ell^2 \otimes \ell^2(I \cap \mathbb{Q})}_{=H_{X_r}} &\rightarrow \underbrace{\ell^2(X_R^{\mathbb{Q}}) \otimes \ell^2}_{=H_{X_R}} \otimes \underbrace{\ell^2(X_R^{\mathbb{Q}}) \otimes \ell^2}_{=H_{X_R}} \\ \delta_x \otimes \delta_n \otimes \delta_t &\mapsto \delta_{\tilde{H}_-^A(x,t)} \otimes \delta_n \otimes \delta_{\tilde{H}_+^B(x,t)} \otimes \delta_{\beta(t)}. \end{aligned}$$

Here, $\{\delta_x\}$, $\{\delta_n\}$ and $\{\delta_t\}$ denote the obvious bases of the ℓ^2 space and $\beta: I \cap \mathbb{Q} \rightarrow \mathbb{N}$ denotes some fixed bijection. In fact, it is rather irrelevant that we have defined V using the isometry $\delta_n \otimes \delta_t \mapsto \delta_n \otimes \delta_{\beta(t)}$ from $\ell^2 \otimes \ell^2(I \cap \mathbb{Q})$ to $\ell^2 \otimes \ell^2$, as any other one also has the same properties that we use in the following.

Note first of all that the constant family of isometries $t \mapsto V$ clearly covers $\tilde{F}^{X,A,B}$, because the associated function (13) is the zero function. Hence, we can define the induced map $(\tilde{F}^{X,A,B})_*$ as adjoining with the constant families $V_D := V \otimes \text{id}_D$ or $V_{D \otimes E} := V \otimes \text{id}_{D \otimes E \otimes \ell^2}$, depending on the coefficient algebras.

The second important property is the following. Let

$$\begin{aligned} \bar{\rho}_R: C_b(X_R, E \otimes \mathfrak{K}) &\rightarrow \mathfrak{B}(H_{X_R} \otimes E \otimes \ell^2), \\ \bar{\rho}_{r,I}: C_b(X_r \times I, E \otimes \mathfrak{K}) &\rightarrow \mathfrak{B}(H_{X_r} \otimes \ell^2(I \cap \mathbb{Q}) \otimes E \otimes \ell^2) \end{aligned}$$

be the canonical representations. Then, we have

$$(\text{id}_{H_{X_R} \otimes D} \otimes \bar{\rho}_R(f)) \circ V_{D \otimes E} = V_{D \otimes E} \circ (\bar{\rho}_{r,I}((\tilde{H}_+^B|_{X_r})^* f) \otimes \text{id}_D)$$

for all $f \in \bar{\mathfrak{c}}(X_R; E)$ and therefore

$$\begin{aligned} (V_D T V_D^* \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_R \otimes D} \otimes \bar{\rho}_R(f)) \\ = V_{D \otimes E} \circ (T \otimes \text{id}_{E \otimes \ell^2}) \circ (\bar{\rho}_{r,I}((\tilde{H}_+^B|_{X_r})^* f) \otimes \text{id}_D) \circ V_{D \otimes E}^* \end{aligned} \quad (15)$$

holds for all f as above and $T \in \mathfrak{B}(H_{X_r} \otimes \ell^2(I \cap \mathbb{Q}) \otimes D)$. Similarly, we have

$$\begin{aligned} (V_D T V_D^* \otimes \text{id}_{E \otimes \ell^2}) \circ (\rho_R(g) \otimes \text{id}_{H_R \otimes D \otimes E \otimes \ell^2}) \\ = V_{D \otimes E} \circ (T \otimes \text{id}_{E \otimes \ell^2}) \circ ((\rho_r \otimes \rho_I)((\tilde{H}_-^A|_{X_r})^* g) \otimes \text{id}_{D \otimes E \otimes \ell^2}) \circ V_{D \otimes E}^* \end{aligned}$$

and

$$\begin{aligned} (\rho_R(g) \otimes \text{id}_{H_R \otimes D \otimes E \otimes \ell^2}) \circ (V_D T V_D^* \otimes \text{id}_{E \otimes \ell^2}) \\ = V_{D \otimes E} \circ ((\rho_r \otimes \rho_I)((\tilde{H}_-^A|_{X_r})^* g) \otimes \text{id}_{D \otimes E \otimes \ell^2}) \circ (T \otimes \text{id}_{E \otimes \ell^2}) \circ V_{D \otimes E}^* \end{aligned}$$

for all $g \in C_0(X_R)$ and $T \in \mathfrak{B}(H_{X_r} \otimes \ell^2(I \cap \mathbb{Q}) \otimes D)$.

On $X_r \times_{\mathcal{U}} I_+$ the map $\tilde{H}_-^A$ agrees with the projection $\pi(x, t) := x$. Therefore, if $T \in C^*(X_r \times_{\mathcal{U}} I_+; D)$ then

$$\begin{aligned} & (V_D TV_D^* \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_R \otimes D} \otimes \tilde{\rho}_R(f)) \circ (\rho_R(g) \otimes \text{id}_{H_R \otimes D \otimes E \otimes \ell^2}) \\ &= V_{D \otimes E} \circ (T \otimes \text{id}_{E \otimes \ell^2}) \circ (\tilde{\rho}_{r,I}(\underbrace{(\tilde{H}_+^B|_{X_r})^* f}_{\in C_0(X_r \times_{\mathcal{U}} I_+) \otimes E \otimes \mathfrak{K}}) \cdot \pi^* g) \otimes \text{id}_D) \circ V_{D \otimes E}^* \end{aligned}$$

is compact for all $g \in C_0(X_R)$ due to the locally compactness of T . If additionally g is compactly supported and T has finite propagation, then we can choose a function $g' \in C_0(X_r \times_{\mathcal{U}} I_+)$ which is constantly one on the $\text{prop}(T)$ -neighborhood of the compact subset $\text{supp}(\pi^* g)$ and thus $((\rho_r \otimes \rho_I)(\pi^* g) \otimes \text{id}_D) \circ T = ((\rho_r \otimes \rho_I)(\pi^* g) \otimes \text{id}_D) \circ T \circ ((\rho_r \otimes \rho_I)(g') \otimes \text{id}_D)$. Then, we also obtain that

$$\begin{aligned} & (\rho_R(g) \otimes \text{id}_{H_R \otimes D \otimes E \otimes \ell^2}) \circ (V_D TV_D^* \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_R \otimes D} \otimes \tilde{\rho}_R(f)) \\ &= V_{D \otimes E} \circ (\rho_r \otimes \rho_I)(\pi^* g) \otimes \text{id}_{D \otimes E \otimes \ell^2} \\ & \quad \circ (T \otimes \text{id}_{E \otimes \ell^2}) \circ (\tilde{\rho}_{r,I}(\underbrace{(\tilde{H}_+^B|_{X_r})^* f}_{\in C_0(X_r \times_{\mathcal{U}} I_+) \otimes E \otimes \mathfrak{K}}) \cdot g') \otimes \text{id}_D) \circ V_{D \otimes E}^* \end{aligned}$$

is compact. What we have just shown is that $(V_D TV_D^* \otimes \text{id}_{E \otimes \ell^2}) \circ (\text{id}_{H_R \otimes D} \otimes \tilde{\rho}_R(f))$ is locally compact for all $T \in C^*(X_r \times_{\mathcal{U}} I_+; D)$ and $f \in \mathfrak{c}(X_R; E)$ and therefore

$$\psi_L((\tilde{F}^{X,A,B})_*[T] \otimes [f]) = 0 \text{ for all } [T] \in L_+ \text{ and } [f] \in \mathfrak{c}(X_R; E). \quad (16)$$

Note that the above computations went through so easily, because T was an element of $C^*(X_r \times_{\mathcal{U}} I_+; D)$ and not just of $C^*(X_r \times_{\mathcal{U}} I_+ \subset X_r \times_{\mathcal{U}} I; D)$. This is the reason why we have passed from the ideal $L_+^{\subseteq}$ to the subalgebra L_+ in the first big diagram of this proof. Similar reasoning in the next paragraph also explains why we have passed from $L_-^{\subseteq}$ to L_- .

In order to see what happens with L_- , we first make the following observation. Recall that if a uniformly continuous family of isometries V covers a uniformly continuous coarse map $\alpha: (X, A) \rightarrow (Y, B)$ in the sense of (13), then adjoining with $V \otimes D$ maps $C_L^*(X, A; D)$ into $C_L^*(Y, B; D)$ and $E_L^*(X, A; D)$ into $E_L^*(Y, B; D)$. If, however, α is only a uniformly continuous controlled map (i. e., it lacks properness), then we can perform the same definitions and the same proofs will still show that adjoining with $V \otimes D$ still maps $E_L^*(X, A; D)$ into $E_L^*(Y, B; D)$, but it does not map $C_L^*(X, A; D)$ into $C_L^*(Y, B; D)$ anymore.

We exploit this fact in our case as follows. Note that our constant family of isometries $V: H_{X_r} \otimes \ell^2(I \cap \mathbb{Q}) \rightarrow H_{X_R} \otimes H_{X_R}$ not only covers the uniformly continuous coarse map $\tilde{F}^{X,A,B}$ with respect to the representations $\rho_r \otimes \rho_I$ and $\rho_R \otimes \rho_R$, but its restriction to $H_{X_r} \otimes \ell^2(I_- \cap \mathbb{Q})$ also covers the uniformly continuous controlled map $\tilde{H}_-^A|_{X_r \times_{\mathcal{U}} I_-}$ with respect to the representations $\rho_r \otimes \rho_{I_-}$ and $\rho_R \otimes \text{id}_{H_{X_R}}$. Thus, adjoining with $V \otimes \text{id}_{D \otimes E \otimes \ell^2}$ yields a $*$ -homomorphism

$$(\tilde{H}_-^A|_{X_r \times_{\mathcal{U}} I_-})_*: C_L^*((X_r, A_r) \times_{\mathcal{U}} (I_-, \{-1\}); D \otimes E) \rightarrow E_L^*(X_R, A_R; D \otimes E).$$

Composing it with the $*$ -homomorphism

$$m_L: L_- \otimes \mathfrak{c}(X_r, B_r) \times_{\mathcal{U}} I_-; E \rightarrow C_L^*((X_r, A_r) \times_{\mathcal{U}} (I_-, \{-1\}); D \otimes E),$$

which is associated with the space $X_r \times_{\mathcal{U}} I_-$ and its subspaces $X_r \times \{-1\} \cup A_r \times I_-$ and $B_r \times I_-$, as well as with the $*$ -homomorphism $\pi^*: \mathfrak{c}(X_R, B_R; E) \rightarrow \mathfrak{c}((X_r, B_r) \times_{\mathcal{U}} I_-; E)$ induced by the projection $\pi(x, t) = x$, which happens to agree with $\tilde{H}_+^B$ on $X_r \times_{\mathcal{U}} I_-$, we obtain a $*$ -homomorphism

$$\mathfrak{M}_-: (\tilde{H}_-^A|_{X_r \times_{\mathcal{U}} I_-})_* \circ (\mathfrak{m}_L \otimes \pi^*): L_- \otimes \mathfrak{c}(X_R, B_R; E) \rightarrow E_L^*(X_R, A_R; D \otimes E).$$

Note that the restriction of V to $H_{X_r} \cong H_{X_r} \otimes \ell^2(\{0\})$ also covers the inclusion $\iota_{r,R}: X_r \rightarrow X_R$ with respect to the representations ρ_r and $\rho_R \otimes \text{id}_{H_{X_r}}$, and if define $(\iota_{r,R})_*$ in the definition of $\mathfrak{M}_0$ as adjoining with $V \otimes \text{id}_{D \otimes E \otimes \ell^2}$, then $\mathfrak{M}_-$ clearly extends $\mathfrak{M}_0$. Furthermore, (15) immediately implies that $\mathfrak{M}_-$ lifts the restriction of

$$\psi_L \circ ((\tilde{\Gamma}^{X,A,B})_* \otimes \text{id}): L \otimes \mathfrak{c}(X_R, B_R; E) \rightarrow \frac{E_L^*(X_R, A_R; D \otimes E)}{C_L^*(X_R, A_R; D \otimes E)}$$

to $L_- \otimes \mathfrak{c}(X_R, B_R; E)$. Combining these two properties and taking (16) into account, we see that a $*$ -homomorphism $\mathfrak{M}$ making the second diagram commute is obtained by applying $\psi_L \circ ((\tilde{\Gamma}^{X,A,B})_* \otimes \text{id})$ pointwise over $(0, 1]$ and $\mathfrak{M}_-$ over $\{0\}$.

Now, what the two diagrams together give us in K-theory is the commutative rectangle

$$\begin{array}{ccc} K_{m+1}((X_r, A_r \cup B_r) \times (I, \partial I); D) & \xrightarrow[-(\sigma_*)^{-1}]{\cong} & K_m(X_r, A_r \cup B_r; D) \\ \otimes K_{-n}(\mathfrak{c}(X_R, B_R; E)) & & \otimes K_{-n}(\mathfrak{c}(X_R, B_R; E)) \\ \downarrow (\tilde{\Gamma}^{X,A,B})_* \otimes \text{id} & & \downarrow (\iota_{r,R})_* \circ \cap \circ (\text{id} \otimes \iota_{r,R}^*) = \cap \circ ((\iota_{r,R})_* \otimes \text{id}) \\ K_{m+1}(X_R, A_R) \times (X_R, B_R); D) & & \\ \otimes K_{-n}(\mathfrak{c}(X_R, B_R; E)) & & \\ \downarrow (\psi_L)_* \circ \times & & \downarrow \\ K_{m+1-n} \left(\frac{E_L^*(X_R, A_R; D \otimes E)}{C_L^*(X_R, A_R; D \otimes E)} \right) & \xrightarrow{\partial} & K_{m-n}(X_R, A_R; D \otimes E) \end{array}$$

which is obviously natural under enlarging R and (if R is large enough) also natural under enlarging r . Recall that the two arrows in the bottom left corner together are nothing but $(-1)^{m+1}$ times the slant product.

Now, if we use the canonical identification $\mathfrak{c}(X, B; E) \cong \mathfrak{c}(X_R, B_R; E)$, then we can first let R go to infinity and subsequently also let r go to infinity and the diagram becomes the following.

$$\begin{array}{ccc} KX_{m+1}((X, A \cup B) \times_{\mathcal{U}} (I, \partial I); D) & \xrightarrow[-(\sigma_*)^{-1}]{\cong} & K_m(X_r, A_r \cup B_r; D) \\ \otimes K_{-n}(\mathfrak{c}(X, B; E)) & & \otimes K_{-n}(\mathfrak{c}(X, B; E)) \\ \downarrow (\Gamma^{X,A,B})_* \otimes \text{id} & & \downarrow \cap \\ KX_{m+1}(X_R, A_R) \times (X_R, B_R); D) & \xrightarrow{(-1)^{m+1} \cdot /} & KX_{m-n}(X_R, A_R; D \otimes E) \\ \otimes K_{-n}(\mathfrak{c}(X_R, B_R; E)) & & \end{array}$$

The composition of the left vertical and the two horizontal arrows is exactly the secondary cap product, because even the sign $(-1)^m = -(-1)^{m+1}$ is the one we need. $\square$

10 Bundles of vanishing variation and coassembly

Now that all the main results are proven, it remains to show a statement that we made in the introduction about K-theory classes of bundles of vanishing variation. Let us start directly by recalling this notion from [20, Section 8] and explaining the problem.

Let M be a complete Riemannian manifold of bounded geometry and C a unital C^* -algebra. We define the subalgebra

$$\bar{c}^\infty(M; C) := \{f \in C_b^\infty(M; C \otimes \mathfrak{K}) \mid \text{grad}(f) \text{ vanishes at } \infty\}$$

of $\bar{c}(M; C)$.

Definition 10.1 ([20, Definition 8.4]) Let $P \in \bar{c}^\infty(M; C)$ be self-adjoint projection valued outside of a compact subset $K \subset M$. The bundle $E \rightarrow M \setminus K$ whose fiber at x is the finitely generated projective Hilbert- C -module $\text{im}(P(x)) \subset C \otimes \ell^2$ is called a *C-bundle of vanishing variation defined outside of the compact subset K* . Its K-theory class $\llbracket E \rrbracket_c \in K_0(\bar{c}(M; C))$ is the class of the projection $\bar{P} \in \bar{c}(M; C) = \bar{c}(M; K)$ determined by P .

Now, recall the following fact: If B is a compact Hausdorff space, $P_1, P_2 \in \mathcal{C}(B; C \otimes \mathfrak{K})$ are self-adjoint projections and $E_1, E_2 \rightarrow B$ the bundles of finitely generated projective C -modules defined by the images of the projections P_1, P_2 , then any bundle isomorphism $E_1 \cong E_2$ gives rise to a Murray–von Neumann equivalence between P_1, P_2 , i. e., an element $v \in \mathcal{C}(B; C \otimes \mathfrak{K})$ with $v^*v = P_1, vv^* = P_2$. Therefore, the K-theory class of a projection $P \in \mathcal{C}(B; C \otimes \mathfrak{K})$ depends only on the isomorphism class of the associated bundle of finitely generated projective C -modules.

For C -bundles $E_1, E_2 \rightarrow M \setminus K$ of vanishing variation defined by projections P_1, P_2 as in Definition 10.1, however, an isomorphism of Hilbert- C -module bundles $E_1 \cong E_2$ only gives rise to a Murray–von Neumann equivalence in $C_b(M; C \otimes \mathfrak{K})/C_0(M) \otimes C \otimes \mathfrak{K}$ between $\bar{P}_1, \bar{P}_2$, but not necessarily in $\bar{c}(M; C)$. Hence, we cannot conclude $\llbracket E_1 \rrbracket_c = \llbracket E_2 \rrbracket_c$. Basically, there is no good reason to assume that the class $\llbracket E \rrbracket_c$ is a topological invariant after all, i. e., depends only on the isomorphism class of E as a bundle of finitely generated projective Hilbert- C -modules. Indeed, in light of Proposition 10.2 below one would only expect this property to hold if the coarse co-assembly map is injective, but that is a whole other story. In fact, in duality to the assembly map it is usually surjectivity of the co-assembly map which is more likely to hold than injectivity (cf., e. g., [8, Section 5.2]).

Nevertheless, this appears to be in contrast to the index formula for twisted operators $\text{ind}(D_E) = \text{ind}(D) \cap \llbracket E \rrbracket_c$ from [20, Theorem 8.7], whose left hand side only depends on the isomorphism class of E (and the operator D , of course) but not on P itself. As we have explained in the introduction, an independent explanation of why the cap product on the right hand side, too, depends only on the isomorphism class of E is given by Theorem C, if one can show that $\mu^*(\llbracket E \rrbracket_c) \in KX^1(M, K; C)$ is a topological invariant. We prove the following proposition, which states that the latter property holds for uniformly contractible manifolds of bounded geometry. It is not unreasonable to conjecture that it also holds in greater generality, but unfortunately our simple proof cannot be directly transferred to the Rips complexes.

Proposition 10.2 *Let M be a uniformly contractible manifold of bounded geometry and $K \subset M$ compact. Then, the image of the K-theory class $\llbracket E \rrbracket_c$ of a C -bundle $E \rightarrow M \setminus K$ of vanishing variation under the coarse co-assembly map*

$$\mu^*: K_0(\bar{c}(M; C)) = K_0(\bar{c}(M, K; C)) \rightarrow KX^1(M, K; C)$$

depends only on the isomorphism class of E .

Proof Bounded geometry of M allows us to choose a discretization $M' \subset M$ and an $R > 0$ such that $\{B_R(y)\}_{y \in M'}$ is a locally finite open cover of M . Then, any subordinate partition of unity $\{\chi_y\}_{y \in M'}$ gives rise to a proper continuous map

$$f: M \rightarrow P_{2R}(M'), x \mapsto \sum_{y \in M'} \chi_y(x) \cdot x.$$

Moreover, we may assume that the discretization and the partition of unity have been chosen in such a way that $K' := M' \cap K$ is R -dense in K and the partial sum $\sum_{y \in K'} \chi_y$ is 1 on K . Hence, f maps K into $P_{2R}(K')$.

Now, uniform contractibility of M implies that f gives us a proper homotopy equivalence $M \simeq \mathcal{P}(M')$ of σ -locally compact spaces (cf. proof of [7, Theorem 4.8]). Thus, $KX^{1-*}(M, p; C) \cong K^{1-*}(M, p; C)$, but not necessarily $KX^{1-*}(M, K; C) \cong K^{1-*}(M, K; C)$, because K might have non-trivial topology. We obtain the following commutative diagram, in which the left vertical arrow is induced by a canonical inclusion α of C^* -algebras, the arrows from the first to the second column are connecting homomorphisms in K -theory and the other arrows are induced by f or the inclusion $\iota: (M, p) \subset (M, K)$.

$$\begin{array}{ccccc} K_*(c(M, K; C)) & \xrightarrow{\mu^*} & KX^{1-*}(M, K; C) & \xrightarrow[\cong]{\iota^*} & KX^{1-*}(M, p; C) \\ \downarrow \alpha_* & \searrow & \downarrow f^* & & \downarrow f^* \cong \\ K_*\left(\frac{C_b(M, K; C \otimes \mathfrak{K})}{C_0(M \setminus K) \otimes C \otimes \mathfrak{K}}\right) & \longrightarrow & K^{1-*}(M, K; C) & \xrightarrow{\iota^*} & K^{1-*}(M, p; C) \end{array}$$

The Murray–von Neumann equivalence mentioned earlier implies that $\alpha_*[[E]]_c$ depends only on the isomorphism class of E . Therefore, the diagram shows that the same is true for $\mu^*[[E]]_c$. $\square$

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