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On the cohomology of certain subspaces of $\mathrm{Sym}^n(\mathbb{P}^1)$ and Occam's razor for Hodge structures

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Abstract

Vakil and Matchett-Wood (Discriminants in the Grothendieck ring of varieties, 2013. arXiv:1208.3166) made several conjectures on the topology of symmetric powers of geometrically irreducible varieties based on their computations on motivic zeta functions. Two of those conjectures are about subspaces of $\mathrm{Sym}^n(\mathbb{P}^1)$. In this note, we disprove one of them and prove a stronger form of the other, thereby obtaining (counter)examples to the principle of Occam's razor for Hodge structures.

1 Introduction

For a smooth and proper variety X over $\mathbb{C}$, the Hodge–Deligne polynomial determines the Hodge numbers, but that is no longer the case when X is not smooth or proper. To elaborate, for any variety X over $\mathbb{C}$, the compactly supported cohomology groups $H^i_c(X,\mathbb{Q})$ carry Deligne's mixed Hodge structures. One defines the *Hodge–Deligne polynomial* as:

$$HD(x,y) := \sum_{p,q} e_{p,q}x^p y^q.$$

Here, $e_{p,q}$ are *virtual Hodge–Deligne numbers*, defined in terms of pure Hodge structures that the associated grades for the weight filtration on $H^*_c(X,\mathbb{Q})$ are equipped with

$$e_{p,q} = \sum_i (-1)^i h^{p,q}\left(gr_W^{p+q} H^i_c(X,\mathbb{Q})\right).$$

When X is smooth and proper, one has $e_{p,q} = (-1)^i h_{p,q}(H^i(X,\mathbb{Q}))$. There are many examples where the simplest possibility holds, i.e. there is a simplest Hodge structure on $H^i_c(X,\mathbb{Q})$ for all i in agreement with the virtual Hodge structure. In [9], Vakil and Wood dub this well-known principle as 'Occam's razor for Hodge structures'. This principle led them to conjecture about the stable rational cohomology of certain subspaces of $\mathrm{Sym}^m(\mathbb{P}^1)$, the m-fold symmetric product of $\mathbb{P}^1_{\mathbb{C}}$. [1,2] This note provides examples (from Conjectures G' and H' of [9]).

[1]Note that the conjectures are in the arXiv version of the paper, and not the published version [10].
[2]In [6] Kupers and Miller proved Conjectures G and H of [9].

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We now fix some notations and state the conjectures of Vakil and Wood. For a complex quasiprojective variety X, let $\mathrm{Sym}^m(X)$ denote the n-fold symmetric product, i.e.

$$\mathrm{Sym}^m(X) := X^m/\mathfrak{S}_m,$$

where the symmetric group over m letters, $\mathfrak{S}_m$ acts on X^m by permuting its factors. For a partition λ of m, let $w_\lambda(X)$ denote the locally closed subset of $\mathrm{Sym}^m(X)$ with multiplicities precisely λ. For example, $w_{1^m}(X)$ is the space of unordered configuration of m points on X corresponding to the partition $m = \underbrace{1 + 1 \cdots + 1}_{m \text{ times}}$.

Conjecture H' of [9] states that the values of $i > 0$ for which

$$\lim_{n\to\infty} \dim H^i(w_{1^n 2^2}(\mathbb{P}^1_{\mathbb{C}}); \mathbb{Q}) \neq 0$$

is periodic in i, and the nonzero limits equal 1. Conjecture G' of [9] states that

$$\lim_{n\to\infty} \dim H^i(w_{1^n 2^3}(\mathbb{P}^1_{\mathbb{C}}); \mathbb{Q}) = \begin{cases} 1 & i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Our main theorem disproves Conjecture G' of [9].

Theorem A *We have*

$$\lim_{n\to\infty} H^i(w_{1^n 2^3}(\mathbb{P}^1_{\mathbb{C}}); \mathbb{Q}) = \begin{cases} 1 & \text{for } i = 0, 1, 2, \\ 2 & \text{for } i > 2. \end{cases}$$

Furthermore, $H^i(w_{1^n 2^2}(\mathbb{P}^1); \mathbb{Q})$ *is pure of weight* $-2i$ *and Hodge type* $(-i, -i)$ *for all* i. □

The following corollary to Theorem A is a refinement of the statement of Conjecture G' of [9].

Corollary 1 *We have*

$$\lim_{n\to\infty} \dim H^i(w_{1^n 2^2}(\mathbb{P}^1_{\mathbb{C}}); \mathbb{Q}) \cong \begin{cases} 1 & \text{for } i = 0, 2k+1, k \geq 0, \\ 2 & \text{for } i = 4k, k \geq 1, \\ 0 & \text{for } i = 4k+2, k \geq 0. \end{cases}$$

Furthermore, $H^i(w_{1^n 2^2}(\mathbb{P}^1); \mathbb{Q})$ *is pure of weight* $-2i$ *and Hodge type* $(-i, -i)$ *for all* $0 \leq i \leq n+2$. □

A question along the lines of the conjectures based on the Occam's razor of Hodge structures would be: can one determine the rational cohomology of a variety over $\mathbb{C}$ by counting the number of $\mathbb{F}_q$ points of that variety? The answer, in general, is in negative. In fact, the conjectures in [9] were made on the very basis of such point-counts. The Grothendieck–Lefschetz trace formula (see [5]) allows one to count the number of $\mathbb{F}_q$ points of a variety X from its topology when X is a reasonably nice variety. On a larger scale, the Weil conjectures form a bridge between the topology of $X(\mathbb{C})$ and the number theoretic properties of $X(\mathbb{F}_q)$. However, there is no sufficient criterion to cross that bridge and go from $|X(\mathbb{F}_q)|$ to the rational Betti numbers of $X(\mathbb{C})$.

The existing literature on the (co)homology of configuration spaces is already extremely rich. Instead of attempting to add that, the purpose of this note is to give rather simple examples of varieties X for which points counts, or the principle of Occam's razor of Hodge structure, do not carry us across the said bridge from $|X(\mathbb{F}_q)|$ (that Vakil and Wood computed) to the rational Betti numbers of $X(\mathbb{C})$ (which we compute in this note).

2 Cohomological stability of some locally closed strata of $\mathrm{Sym}^m(\mathbb{P}^1)$

In this section, we prove Theorem A and Corollary 1. We will prove the theorem and the corollary together. Our proof can be outlined through the following steps:

1. Describe the spaces $w_{1^n23}(\mathbb{P}^1)$ and $w_{1^n22}(\mathbb{P}^1)$ as fibre-bundles over $PConf_n\mathbb{P}^1$ and $UConf_2(\mathbb{P}^1)$, respectively, with fibres isomorphic to $UConf_n\mathbb{C}^\times$. Here, for a space X we define $UConf_nX := w_{1^n}(X)$ and $PConf_nX := (X^n - \text{union of all diagonals})$.
2. Invoke [1, Corollary 2] to compute $H_c^*(UConf_n\mathbb{C}^\times;\mathbb{Q})$, the compactly supported rational cohomology of $UConf_n\mathbb{C}^\times$.
3. Analyse the Serre spectral sequence for a fibration to compute $H^*(w_{1^n22}(\mathbb{P}^1);\mathbb{Q})$ and $H^*(w_{1^n23}(\mathbb{P}^1);\mathbb{Q})$.

Now a few words about the proof. Steps 2 and 3 constitute the core of the proof. The cohomology $H^*(UConf_n\mathbb{C}^\times;\mathbb{Q})$ is very well known; for example, Cohen [2] does it by computing the homology of free E_n algebras; Totaro, by considering the Leray spectral sequence for the inclusion $PConf_nX \hookrightarrow X^n$, and describing ring structure and the weight filtration on its E_2 page, when X is a smooth algebraic variety over $\mathbb{C}$ (see [8] and the references therein), etc. With the aim of computing (a) $H^*(UConf_n\mathbb{C}^\times;\mathbb{Q})$ with the weights, and (b) bypassing the significant combinatorial complexities that arise if we approach $H^*(UConf_n\mathbb{C}^\times;\mathbb{Q})$ via $H^*(PConf_n\mathbb{C}^\times;\mathbb{Q})$; we use the spectral sequence constructed in [1], which in turn has been developed from Deligne's theory of cohomological descent (see [4]).

Since we only consider cohomology with $\mathbb{Q}$-coefficients, the field of coefficients will be almost always suppressed. Moreover, we implicitly use the fact that on a nice topological space X, if A_X denotes the constant sheaf of R-modules with stalks isomorphic to the R-module A, then the singular cohomology $H^i(X;A)$ and the sheaf cohomology $H^i(X, A_X)$ are isomorphic. As such, we use $H^*(X)$ to denote the sheaf cohomology $H^*(X,\mathbb{Q}_X)$ and by extension, the singular cohomology $H^*(X;\mathbb{Q})$. And all varieties are defined over $\mathbb{C}$.

Proof of Theorem A and Corollary 1 **Step 1** For any positive integer n define the maps

$$\begin{aligned}\pi : w_{1^n23}(\mathbb{P}^1) &\to PConf_2(\mathbb{P}^1)\\ \{x_1,\dots,x_n\}, a, a, b, b, b &\mapsto (a,b),\end{aligned} \tag{2.1}$$

where the fibres are

$$\pi^{-1}(a,b) = UConf_n(\mathbb{P}^1 - \{a,b\}) \cong UConf_n\mathbb{C}^\times,$$

and

$$\begin{aligned}\upsilon : w_{1^n22}(\mathbb{P}^1) &\to UConf_2(\mathbb{P}^1)\\ \Big(\{x_1,\dots,x_n\}, a, a, b, b\Big) &\mapsto \{a,b\},\end{aligned} \tag{2.2}$$

where the fibres are

$$\upsilon^{-1}\{a,b\} = UConf_n(\mathbb{P}^1 - \{a,b\}) \cong UConf_n\mathbb{C}^\times.$$

Note that we have the following commutative diagram:

$$\begin{array}{ccc} w_{1^n23}(\mathbb{P}^1) & \xrightarrow{\hat{\tau}} & w_{1^n22}(\mathbb{P}^1)\\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\upsilon}\\ PConf_2(\mathbb{P}^1) & \xrightarrow{\tau} & UConf_2(\mathbb{P}^1)\end{array} \tag{2.3}$$

where $\hat{\tau} : w_{1^n 23}(\mathbb{P}^1) \to w_{1^n 22}(\mathbb{P}^1)$ is induced by

$$\tau : PConf_2(\mathbb{P}^1) \to UConf_2(\mathbb{P}^1),$$

the $\mathbb{Z}/2$-quotient map defined by swapping the coordinates of the points in $PConf_2(\mathbb{P}^1)$; and

$$\hat{\tau}\Big|_{\pi^{-1}(a,b)} : \pi^{-1}(a,b) \to \upsilon^{-1}(\{a,b\})$$

is identity.

Step 2 We compute $H_c^*(UConf_n\mathbb{C}^\times)$ by directly quoting Corollary 2 from [1]:
Let X be a connected locally compact Hausdorff topological space. Then, there exists a spectral sequence:

$$\begin{aligned} E_1^{p,q} &= \bigoplus_{l+m=q} \left(H_c^l(X^p) \otimes sgn_p\right)^{\mathfrak{S}_p} \otimes H_c^m(\mathrm{Sym}^{n-2p}X) \\ &= \bigoplus_{l+m=q} \bigoplus_{i+j=p} \left(\mathrm{Sym}^i H_c^{\mathrm{odd}}(X) \otimes \Lambda^j H_c^{\mathrm{even}}(X)\right)^{(l)} \otimes H_c^m(\mathrm{Sym}^{n-2p}X) \\ &\implies H_c^{p+q}(UConf_n(X)), \end{aligned} \tag{2.4}$$

where sgn_p denotes the sign representation of $\mathfrak{S}_p$ on $H_c^l(X^p)$, induced by $\mathfrak{S}_p$ acting on X^p by permuting the coordinates; and it follows from, say, [7] that

$$\left(H_c^l(X^p) \otimes sgn_p\right)^{\mathfrak{S}_p} \cong \bigoplus_{i+j=p} \left(\mathrm{Sym}^i H_c^{\mathrm{odd}}(X) \otimes \Lambda^j H_c^{\mathrm{even}}(X)\right)^{(l)},$$

where

$$H_c^{\mathrm{odd}}(X) := \bigoplus_k H_c^{2k+1}(X), \quad H_c^{\mathrm{even}}(X) := \bigoplus_k H_c^{2k}(X),$$

and $\left(\mathrm{Sym}^i H_c^{\mathrm{odd}}(X) \otimes \Lambda^j H_c^{\mathrm{even}}(X)\right)^{(l)}$ denotes the $l^{(\mathrm{th})}$-graded summand of the cohomology $\mathrm{Sym}^i H_c^{\mathrm{odd}}(X) \otimes \Lambda^j H_c^{\mathrm{even}}(X)$.

The strategy behind (2.4) can be summarised briefly as follows:

1. Observe that if $T_p := X^p \times \mathrm{Sym}^{n-2p}X$, then for all $p \geq 0$ there are natural *face maps*

$$\begin{aligned} &f_p : T_p \to \mathrm{Sym}^n X(x_1, x_2, \ldots, x_p), \\ &\quad \{x_1', x_2', \ldots, x_{n-2p}'\} \mapsto \{x_1, x_1, x_2, x_2, \ldots, x_p, x_p, x_1', x_2', \ldots, x_{n-2p}'\} \end{aligned}$$

   i.e. by raising the coordinates of the points of X^p to multiplicity 2, and then forgetting the ordering of the resulting n-tuple.
2. This results in an augmented cosemisimplicial object in the category of locally constant sheaves on $UConf_nX$:

$$j_!\mathbb{Q}_{UConf_nX} \to f_{\bullet_*}\mathbb{Q}_{T_\bullet},$$

   where $j : UConf_nX \hookrightarrow \mathrm{Sym}^n X$ is the inclusion. See [4] for further details, and [3] for a modern treatment of it.
3. In turn, one shows that there's a quasi-isomorphism of locally constant sheaves

$$j_!\mathbb{Q}_{UConf_nX} \xrightarrow{\cong} \left(f_{p_*}\mathbb{Q}_{T_p} \otimes sgn_p\right)^{\mathfrak{S}_p},$$

   where $\mathfrak{S}_p$ acts on $T_p = X^p \times \mathrm{Sym}^{n-2p}X$ by permuting the first p factors, and sgn_p denotes the sign representation of $\mathfrak{S}_p$ on $f_{p_*}\mathbb{Q}_{T_p}$ (see, for example, [1]).

A simple but important takeaway from the above discussion is that those X for which all the differentials vanish and the spectral sequence in (2.4) degenerates on its E_1 page, every cohomology class in $H_c^*(UConf_nX)$ is a linear combination of classes of the form $\alpha \otimes \beta$ where $\alpha \in \left(H_c^l(X^p) \otimes sgn_p\right)^{\mathfrak{S}_p}$, which we dub as the 'alternating part' of a cohomology class, and $\beta \in H_c^m(\mathrm{Sym}^{n-2p}X)$, which we dub as the 'symmetric part' of a cohomology class. Also observe that when X is a quasiprojective algebraic variety over $\mathbb{C}$, all the face maps are algebraic morphisms, and the spectral sequence in (2.4) is a spectral sequence of mixed Hodge structures.

Now put $X = \mathbb{C}^\times$ in (2.4). Then, for $p \geq 1$, the spectral sequence (2.4) reads as:

$$E_1^{p,q} = \begin{cases} \left(\mathrm{Sym}^{p-1}H_c^1(\mathbb{C}^\times) \otimes H_c^2(\mathbb{C}^\times)\right) \otimes \left(\mathrm{Sym}^{n-2p}H_c^2(\mathbb{C}^\times)\right), & q = 2n-3p+1, \\ \left(\mathrm{Sym}^{p}H_c^1(\mathbb{C}^\times)\right) \otimes \left(\mathrm{Sym}^{n-2p}H_c^2(\mathbb{C}^\times)\right) & \\ \bigoplus \left(\mathrm{Sym}^{p-1}H_c^1(\mathbb{C}^\times) \otimes H_c^2(\mathbb{C}^\times)\right) & \\ \otimes \left(\mathrm{Sym}^{n-2p-1}H_c^2(\mathbb{C}^\times) \otimes H_c^1(\mathbb{C}^\times)\right), & q = 2n-3p, \\ \left(\mathrm{Sym}^{p}H_c^1(\mathbb{C}^\times)\right) \otimes \left(\mathrm{Sym}^{n-2p-1}H_c^2(\mathbb{C}^\times) \otimes H_c^1(\mathbb{C}^\times)\right), & q = 2n-3p-1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

and for $p = 0$ one has

$$E_1^{0,q} = \begin{cases} \mathrm{Sym}^{n}H_c^2(\mathbb{C}^\times), & q = 2n, \\ \mathrm{Sym}^{n-1}H_c^2(\mathbb{C}^\times) \otimes H_c^1(\mathbb{C}^\times), & q = 2n-1, \end{cases}$$

with the differentials going horizontally $E_1^{p,q} \to E_1^{p+1,q}$ (see Fig. 1). The differentials clearly vanish, and the spectral sequence degenerates on the E_1 page. Furthermore, one can read off the weights from the explicit description of the terms $E_1^{p,q}$ of the spectral sequence in (2.5) by noting that $H^1(\mathbb{C}^\times)$ is pure of weight -2 and Hodge type $(-1,-1)$. Letting $\mathbb{Q}(1)$ denote the Tate Hodge structure of weight -2 and Hodge type $(-1,-1)$, and using Poincaré duality and the universal coefficient theorem, in that order, we obtain for all i:

$$\begin{aligned} &H^i(UConf_n\mathbb{C}^\times;\mathbb{Q}) \\ &\cong \begin{cases} \left(\mathrm{Sym}^{q}H^1(\mathbb{C}^\times) \otimes H^0(\mathbb{C}^\times)\right) \bigotimes \mathrm{Sym}^{n-2(q+1)}H^0(\mathbb{C}^\times) & \\ \bigoplus \mathrm{Sym}^{q}H^1(\mathbb{C}^\times) \bigotimes \left(\mathrm{Sym}^{n-2q-1}H^0(\mathbb{C}^\times) \otimes H^1(\mathbb{C}^\times)\right), & i = 2q+1, \\ \mathrm{Sym}^{q}H^1(\mathbb{C}^\times) \bigotimes \mathrm{Sym}^{n-2q}H^0(\mathbb{C}^\times) & \\ \bigoplus \left(\mathrm{Sym}^{q-1}H^1(\mathbb{C}^\times) \otimes H^0(\mathbb{C}^\times)\right) \bigotimes \left(\mathrm{Sym}^{n-2q-1}H^0(\mathbb{C}^\times) \otimes H^1(\mathbb{C}^\times)\right), & i = 2q, \end{cases} \end{aligned} \tag{2.6}$$

and in particular, we have:

$$H^i(UConf_n\mathbb{C}^\times;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-i), & i = 0, n, \\ \mathbb{Q}(-i)^2, & 0 < i < n. \end{cases} \tag{2.7}$$

□

$$
q \uparrow \quad
\begin{array}{r|cccc}
2n & \mathbb{Q} & 0 & & \\
2n-1 & \mathbb{Q} & 0 & & \\
2n-2 & 0 & \mathbb{Q} & 0 & \\
2n-3 & 0 & \mathbb{Q}^2 & 0 & \\
2n-4 & & \mathbb{Q} & 0 & \\
2n-5 & & 0 & \mathbb{Q} & 0 \\
2n-6 & & 0 & \mathbb{Q}^2 & 0 \\
 & & 0 & \mathbb{Q} & 0 \\
 & & & 0 & \mathbb{Q} \\
 & & & & \mathbb{Q} \\
 & & & & \mathbb{Q} \\
\hline
 & 0 & 1 & 2 & 3 \quad \cdots\cdots
\end{array}
$$
$$p \rightarrow$$

Fig. 1 E_1 page spectral sequence converging to $H_c^*(UConf_n(\mathbb{C}^\times);\mathbb{Q})$

Finally, we record a list of the bases of $H^i(UConf_n\mathbb{C}^\times;\mathbb{Q})$ for each i, in terms of the cohomology classes in $H^*(\mathbb{C}^\times;\mathbb{Q})$. Let $\omega \in H^1(\mathbb{C}^\times;\mathbb{Z})$ correspond to, in terms of de Rham cohomology, the integral holomorphic one-form $\frac{dz}{z}$, and we continue to denote its image in $H^1(\mathbb{C}^\times;\mathbb{Z}) \otimes \mathbb{Q}$ by ω; and let $\nVdash$ denote a generator of $H^0(\mathbb{C}^\times)$. Then, plugging these in (2.6) we get

$$
\begin{aligned}
H^{2k}(UConf_n\mathbb{C}^\times) &= \mathbb{Q}\{\omega^k \otimes \nVdash^{n-2k},\ \omega^{k-1}\nVdash \otimes \nVdash^{n-2k-1}\omega\},\\
H^{2k+1}(UConf_n\mathbb{C}^\times) &= \mathbb{Q}\{\omega^k \otimes \nVdash^{n-2k-1}\omega,\ \omega^k\nVdash \otimes \nVdash^{n-2k-2}\},
\end{aligned}
\tag{2.8}
$$

where the terms preceding $\otimes$ are the 'alternating parts' of these cohomology classes and the terms succeeding $\otimes$ are the 'symmetric parts'. We should also keep in mind that any cohomology class of the form $\omega^i\nVdash^j \otimes \beta$, for some β in the 'symmetric part', vanishes whenever $j \geq 2$, because as noted earlier, the 'alternating part' comes from $\left(H_c^l(X^p)\otimes sgn_p\right)^{\mathfrak{S}_p}$ for some l and p, and in its Künneth decomposition, $H^0(\mathbb{C}^\times)$ can occur only at most once. Decomposing the generators of $H^*(UConf_n\mathbb{C}^\times)$ into their 'alternating' and 'symmetric' parts will play a crucial role in the endgame.

Step 3 Now we use the Serre spectral sequence for the fibre bundles

$$\pi : w_{1^n 23}(\mathbb{P}^1) \to PConf_2(\mathbb{P}^1)$$

and

$$\upsilon : w_{1^n 22}(\mathbb{P}^1) \to UConf_2(\mathbb{P}^1),$$

and for simplicity we introduce the following notations that will be used for the rest of this paper:

$$\begin{aligned} \hat{E}_n &:= w_{1^n 23}(\mathbb{P}^1), \quad E_n := w_{1^n 22}(\mathbb{P}^1), \\ \hat{B} &:= PConf_2(\mathbb{P}^1), \quad B := UConf_2(\mathbb{P}^1), \\ &\qquad\qquad F_n := UConf_n \mathbb{C}^{\times}. \end{aligned}$$

Case 1 (Proving Theorem A) The space $\hat{B} = \left(\mathbb{P}^1 \times \mathbb{P}^1 - \text{diagonal}\right)$ is isomorphic to a $\mathbb{C}$-bundle over $\mathbb{P}^1$, and is therefore simply connected. And we have

$$H^i(\hat{B}) \cong \begin{cases} \mathbb{Q}(-\frac{i}{2}), & i = 0, 2, \\ 0, & \text{otherwise.} \end{cases}$$

The Serre spectral sequence for $\pi : \hat{E}_n \to \hat{B}$ is given by

$$E_2^{p,q} = H^p(\hat{B}, \mathrm{R}^q \pi_* \mathbb{Q}_{F_n}) = H^p(\hat{B}) \otimes H^q(F_n) \implies H^{p+q}(\hat{E}_n) \tag{2.9}$$

where the second equality follows from the fact that the locally constant sheaf $\mathrm{R}^q \pi_* \mathbb{Q}_{F_n}$ is actually a constant sheaf, $\hat{B}$ being simply connected. Paired with (2.7), the spectral sequence (2.9) reads as follows:

$$\begin{array}{c|ccc} 1 & \mathbb{Q} & & \mathbb{Q} \\ 0 & \mathbb{Q} & & \mathbb{Q} \\ \hline & 0 & 1 & 2 \end{array} \tag{2.10}$$

The only differentials which can be nonzero are

$$d_2^{0,q} : E_2^{0,q} = H^0(\hat{B}) \otimes H^q(F_n) \to E_2^{2,q-1} = H^2(\hat{B}) \otimes H^{q-1}(F_n). \tag{2.11}$$

To understand the differentials, we first consider the well-known case of $n = 1$. In that case, we are dealing with $w_{123}(\mathbb{P}^1)$, which is a $\mathbb{C}^{\times}$-bundle on $PConf_2\mathbb{P}^1$, and observe that

$$w_{123}(\mathbb{P}^1) \cong PConf_3\mathbb{P}^1.$$

The cohomology of the latter can be deduced easily using several well-known methods. A quick way is to observe that the action of $PGL_2(\mathbb{C})$ on $\mathbb{P}^1$ by Möbius transformations is sharply 3-transitive. Therefore,

$$H^i(PConf_3\mathbb{P}^1) \cong H^i(PGL_2(\mathbb{C})) \cong \begin{cases} \mathbb{Q}, & i = 0, 3, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, thinking of $PConf_3\mathbb{P}^1$ as a $\mathbb{C}^\times$-bundle on $\hat{B} = PConf_2\mathbb{P}^1$, the resulting Serre spectral sequence is given by

$$E_2^{p,q} = H^p(PConf_2\mathbb{P}^1) \otimes H^q(\mathbb{C}^\times) \cong \begin{cases} \mathbb{Q}, & p = 0, 2 \text{ and } q = 0, 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the only differential that might be nonzero is $\delta : H^0(\hat{B}) \otimes H^1(\mathbb{C}^\times) \to H^2(\hat{B}) \otimes H^0(\mathbb{C}^\times)$:

$$\begin{tikzcd}
4 & \vdots & & \vdots \\
3 & \mathbb{Q}^2 \arrow[drr] & & \mathbb{Q}^2 \\
2 & \mathbb{Q}^2 \arrow[drr] & & \mathbb{Q}^2 \\
1 & \mathbb{Q}^2 \arrow[drr] & & \mathbb{Q}^2 \\
0 & \mathbb{Q} & & \mathbb{Q} \\
 & 0 & 1 & 2
\end{tikzcd}$$

Armed with the knowledge of $H^*(PConf_3\mathbb{P}^1)$, we see that δ is indeed an isomorphism of $\mathbb{Q}$-vector spaces. Therefore,

$$\delta(\omega) = ce \otimes \mathbb{1} \tag{2.12}$$

for some $c \in \mathbb{Q}^\times$, and where $\mathbb{1} \in H^0(\mathbb{C}^\times)$ as before, and e denotes a generator of the $\mathbb{Q}$-vector space $H^2(\hat{B})$.

The differentials $d_2^{p,q}$ in (2.11) are induced by δ. Indeed, given any fibre bundle $F \to E \to B$, its Serre spectral sequence is related to that of the fibre bundles

$$F^n \to E^{\times_B n} \to B$$

and

$$\mathrm{Sym}^n F \to \mathrm{Sym}^n_B E \to B,$$

by the naturality properties that their respective spectral sequences satisfy. To ease our path towards computing $d_2^{p,q}$ in (2.11), let us write down these relations explicitly in the case when $n = 2$ (for general n it follows likewise), and when the base is simply connected (which is our case here).

We have the following diagram of fibre bundles:

$$\begin{tikzcd}
F^2 \arrow[r] \arrow[d, "p_i"] & E \times_B E \arrow[r] \arrow[d] & B \arrow[d, equal] \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

where p_i denotes projection to the i^{th} factor, $i = 1, 2$. By naturality properties of their respective Serre spectral sequences, the following diagram commutes:

$$\begin{tikzcd}
H^p(B) \otimes H^q(F) \arrow[r, "p_i^*"] \arrow[d, "\delta_2^{p,q}"'] & H^p(B) \otimes H^q(F^2) \arrow[d, "\tilde{d}_2^{p,q}"] \\
H^{p+2}(B) \otimes H^{q-1}(F) \arrow[r, "p_i^*"] & H^{p+2}(B) \otimes H^{q-1}(F^2)
\end{tikzcd}$$

where $\delta_2^{p,q}$ and $\widetilde{d}_2^{p,q}$ are the differentials in the spectral sequences for the respective bundles, and p_i^* is an abuse of notation that really denotes the map $id_B \otimes p_i^*$. Then,

$$\begin{aligned}
&\widetilde{d}_2^{p,q}(\beta \otimes (\alpha_1 \otimes \alpha_2)) \\
&\quad = \widetilde{d}_2^{p,q}\Big(\beta \otimes (p_1^*(\alpha_1) \smile p_2^*(\alpha_2))\Big) = \widetilde{d}_2^{p,q}\Big(\beta \otimes p_1^*(\alpha_1)\Big) \smile p_2^*(\alpha_2) \\
&\qquad + (-1)^{|\beta||\alpha_1|}\Big(\beta \otimes \alpha_1\Big) \smile \widetilde{d}_2^{p,q} p_2^*(\alpha_2) \qquad \text{(by Leibniz rule)} \\
&\quad = p_1^*\delta_2^{p,q}(\beta \otimes \alpha_1) \smile p_2^*(\alpha_2) \\
&\qquad + (-1)^{|\beta||\alpha_1|}\Big(\beta \otimes \alpha_1\Big) \smile p_2^*\delta_2^{p,q}(\alpha_2) \qquad \text{(by the commutative diagram above)}
\end{aligned} \tag{2.13}$$

and this is enough for our purposes because we have achieved expressing $\widetilde{d}_2^{p,q}$ entirely in terms of $\delta_2^{p,q}$ and p_i^*. We make a similar argument for the following pair of fibre bundles:

$$\begin{array}{ccccc}
F^2 & \longrightarrow & E \times_B E & \longrightarrow & B \\
\downarrow{\scriptstyle \rho} & & \downarrow & & \| \\
\mathrm{Sym}^2 F & \longrightarrow & \mathrm{Sym}_B^2 E & \longrightarrow & B
\end{array}$$

where the map ρ forgets the order of a tuple. Then, as before, naturality implies that the following diagram commutes:

$$\begin{array}{ccc}
H^p(B) \otimes H^q(F^2) & \xrightarrow{\rho_*} & H^p(B) \otimes H^q(\mathrm{Sym}^2 F) \\
\downarrow{\scriptstyle \widetilde{d}_2^{p,q}} & & {\scriptstyle d_2^{p,q}}\downarrow \\
H^{p+2}(B) \otimes H^{q-1}(F) & \xrightarrow{\rho_*} & H^{p+2}(B) \otimes H^{q-1}(F^2)
\end{array}$$

Recall that elements of $H^*(\mathrm{Sym}^2 F)$ are linear combinations of elements of the form $\alpha_1\alpha_2$, where the product is alternating if and only if both α_1 and α_2 have odd cohomology degrees, and symmetric otherwise (see [7] for further details). Therefore,

$$\begin{aligned}
&d_2^{p,q}(\beta \otimes \alpha_1\alpha_2) \\
&\quad = d_2^{p,q}(\beta \otimes \rho_*(\alpha_1 \otimes \alpha_2)) \\
&\quad = \widetilde{d}_2^{p,q}\rho_*(\beta \otimes \alpha_1 \otimes \alpha_2) \\
&\quad = \rho_*\widetilde{d}_2^{p,q}(\beta \otimes \alpha_1 \otimes \alpha_2) \qquad \text{(by the commutative diagram above.)}
\end{aligned} \tag{2.14}$$

Now the terms in the spectral sequence (2.5) (as (2.4), or its proof in [1] shows in details) come from considering cohomology of the spaces $(\mathbb{C}^\times)^p \times \mathrm{Sym}^{n-2p}\mathbb{C}^\times$ for various values of p. Therefore, plugging $E = w_{123}$, $B = PConf_2\mathbb{P}^1$ and $F = \mathbb{C}^\times$, we have explicit formula for the Serre spectral sequence of the bundles

$$(\mathbb{C}^\times)^p \times \mathrm{Sym}^{n-2p}\mathbb{C}^\times \to E^{\times_B p} \times \mathrm{Sym}_B^{n-2p} E \to B,$$

which in turn gives us (again, by naturality) the formula for the differentials in (2.11). In particular, what is sufficient for our purposes is to know where ω is mapped to under the differentials; and we have

$$\begin{aligned}
&\widetilde{d}_2(\omega \otimes \omega) = \delta(\omega) \otimes \omega - \omega \otimes \delta(\omega), && \text{by (2.13)}, \\
&d_2(\omega^2) = 0, && \text{by (2.14)}.
\end{aligned}$$

Recalling the generators listed in (2.8), a straightforward computation gives us:

$$
\begin{aligned}
d_2^{0,2k}\left(\omega^k \otimes \nVdash^{n-2k}\right) &= \begin{cases} \lambda_1 e \otimes (\nVdash \otimes \nVdash^{n-2}), & k=1, \\ 0, & k \geq 2 \end{cases} \\
d_2^{0,2k}\left(\omega^{k-1}\nVdash \otimes \nVdash^{n-2k}\omega\right) &= \lambda_2 e \otimes \left(\omega^{k-1}\nVdash \otimes \nVdash^{n-2k+1}\right) \\
&\qquad \text{(where the other term.} \omega^{k-2}\nVdash^2 \otimes \nVdash^{n-2k}\omega = 0\text{)}, \\
d_2^{0,2k+1}\left(\omega^k \otimes \nVdash^{n-2k-1}\omega\right) &= \lambda_3 e \otimes \left(\omega^k \otimes \nVdash^{n-2k}\right), \\
d_2^{0,2k+1}\left(\omega^k\nVdash \otimes \nVdash^{n-2k-2}\right) &= 0 \quad \text{(because } \omega^{k-1}\nVdash^2 \otimes \nVdash^{n-2k-2} = 0\text{)},
\end{aligned} \tag{2.15}
$$

where $\lambda_i \in \mathbb{Q}^\times$ for all $1 \leq i \leq 3$. In particular, all the differentials in (2.11) are $\mathbb{Q}$-linear maps of rank 1. Therefore, the E_2 page of the spectral sequence in (2.9) results in an E_3 page that looks like:

$$
\begin{array}{c|ccc}
4 & \vdots & & \vdots \\
3 & \mathbb{Q} & & \mathbb{Q} \\
2 & \mathbb{Q} & & \mathbb{Q} \\
1 & \mathbb{Q} & & \mathbb{Q} \\
0 & \mathbb{Q} & & 0 \\
\hline
 & 0 & 1 & 2
\end{array} \tag{2.16}
$$

and all the differentials $E_3^{p,q} \to E_3^{p+3,q-2}$ vanish. So the spectral sequence (2.16) degenerates on the E_3 page, and this completes our proof of Theorem A.

Case 2 (Proving Corollary 1) Now we turn our focus to the fibre bundle

$$UConf_n\mathbb{C}^\times \to w_{1^n22}(\mathbb{P}^1) \xrightarrow{\upsilon} UConf_2(\mathbb{P}^1),$$

and recall that we set up the notations $E_n = w_{1^n22}(\mathbb{P}^1)$, $B = UConf_2\mathbb{P}^1$ and $F_n = UConf_n\mathbb{C}^\times$. The fundamental group of B is $\mathbb{Z}/2\mathbb{Z}$. Also, noting that B is a Zariski open dense subvariety of $\mathrm{Sym}^2\mathbb{P}^1 \cong \mathbb{P}^2$ (and thus, connected), with its complement $\mathrm{Sym}^2\mathbb{P}^1 - B$ a smooth conic (the discriminant locus of a quadratic form), it follows from, say, the long exact sequence of cohomology that $H^*(B) \cong \mathbb{Q}$.

Now observe that since $\mathbb{Z}/2\mathbb{Z}$ is a finite group and we are concerned with cohomology with $\mathbb{Q}$-coefficients, and thanks to the commutative diagram (2.3), the Serre spectral sequence for this fibre bundle is simply the term-wise $\mathbb{Z}/2\mathbb{Z}$ invariants of (2.9). More precisely, if $\Gamma(X, \bullet)$ denotes the global section functor on a space X, as well as the invariants under X when X is a group, then in the derived category of locally constant sheaves on E_n we have the following:

$$
\begin{aligned}
R\Gamma(E_n, \mathbb{Q}_{E_n}) &= R\Gamma(\mathbb{Z}/2\mathbb{Z}, R\Gamma(\hat{E}_n, \hat{\tau}^*\mathbb{Q}_{E_n}) && (2.17) \\
&= \Gamma(\mathbb{Z}/2\mathbb{Z}, R\Gamma(\hat{E}_n, \hat{\tau}^*\mathbb{Q}_{E_n})) && (2.18) \\
&= \Gamma(\mathbb{Z}/2\mathbb{Z}, R\Gamma(\hat{E}_n, \mathbb{Q}_{\hat{E}_n})) && (2.19) \\
&= \Gamma(\mathbb{Z}/2\mathbb{Z}, R\Gamma(\hat{B}, R\pi_*\mathbb{Q}_{F_n})), && (2.20)
\end{aligned}
$$

where (2.17) follows from the definition of $\hat{\tau}$ in the commutative diagram (2.3); (2.17) to (2.18) follow from the fact that there are no Ext groups because $\mathbb{Z}/2\mathbb{Z}$ is a finite group and $R\Gamma(\hat{E}_n, \hat{\tau}^*\mathbb{Q}_{E_n})$ is a complex of locally constant sheaves of vector spaces over $\mathbb{Q}$; (2.19) follows from the fact that $\hat{\tau}^*\mathbb{Q}_{E_n} \cong \mathbb{Q}_{\hat{E}_n}$; and $R\Gamma(\hat{B}, R\pi_*\mathbb{Q}_{F_n})$ is the complex that gives rise to the Serre spectral sequence in (2.9), which explains (2.20). On the other hand,

$$R\Gamma(E_n; \mathbb{Q}_{E_n}) = R\Gamma(B, R\upsilon_*\mathbb{Q}_{F_n}) \tag{2.21}$$

gives the Serre spectral sequence for the fibre bundle $F_n \to E_n \to B$. Comparing (2.20) and (2.21), we get that

$$E_2^{p,q}(F_n \to E_n \to B) \cong \left(E_2^{p,q}(F_n \to \hat{E}_n \to \hat{B})\right)^{\mathbb{Z}/2\mathbb{Z}} \implies H^{p+q}(E_n),$$

and where the differentials are given by (2.15). Now we are left with figuring out how $\mathbb{Z}/2\mathbb{Z}$ acts on $E_2^{p,q}(F_n \to \hat{E}_n \to \hat{B})$ from (2.9), which, in turn, boils down to understanding how $\mathbb{Z}/2\mathbb{Z}$ acts on $H^q(F_n)$, and on $H^p(\hat{B})$.

Let $\sigma \in \mathbb{Z}/2\mathbb{Z}$ denote the order 2 element. It is not hard to see that for $\{x, y\} \in B$, the fundamental group $\mathbb{Z}/2\mathbb{Z}$ acts on the stalks $R^q\upsilon_*\mathbb{Q}_{F_n}\Big|_{\{a,b\}} \cong H^q(F_n)$ by

$$\sigma\omega = -\omega.$$

For example, if we mark two distinct points x and y on $\mathbb{P}^1$, thinking of it as the sphere S^2, then if the Poincaré dual of ω is represented by an oriented circle that leaves x and y on different hemispheres, then σ, which is a half Dehn twist, reverses the orientation of the Poincaré dual of ω. Now looking back on the generators of $H^q(F_n)$ in (2.8), we see that

$$\sigma H^{2k}(\mathbb{F}_n) \cong \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Q}\{\omega^k \otimes \not{k}^{n-2k},\ \omega^{k-1}\not{k} \otimes \not{k}^{n-2k-1}\omega\}, & k \text{ even}, \end{cases} \tag{2.22}$$

and

$$\sigma H^{2k+1}(\mathbb{F}_n) \cong \begin{cases} \mathbb{Q}\{\omega^k \otimes \not{k}^{n-2k-1}\omega\}, & k \text{ odd}, \\ \mathbb{Q}\{\omega^k\not{k} \otimes \not{k}^{n-2k-2}\}, & k \text{ even}. \end{cases} \tag{2.23}$$

On the other hand, for the element $e \in H^2(\hat{B})$ chosen earlier, we have

$$\sigma e = -e. \tag{2.24}$$

Combining (2.22), (2.23), and (2.24), see that the spectral sequence $E_2^{p,q}(F_n \to E_n \to B)$ reads as:

$$E_2^{0,q}(F_n \to E_n \to B) \cong \begin{cases} 0, & q = 4k+2, k \geq 0 \\ \mathbb{Q}, & q = 0, 2k+1, k \geq 0 \\ \mathbb{Q}^2, & q = 4k, k \geq 1, \end{cases} \tag{2.25}$$

and

$$E_2^{2,q}(F_n \to E_n \to B) \cong \begin{cases} \mathbb{Q}^2, & q = 4k+2, k \geq 0 \\ \mathbb{Q}, & q = 2k+1, k \geq 0 \\ 0, & q = 4k, k \geq 0, \end{cases} \tag{2.26}$$

and for all other values of p, we have $E_2^{0,q}(F_n \to E_n \to B) = 0$ and where the differentials are still rank 1, wherever that makes sense (see (2.28)). Clearly, all differentials vanish on the E_3 page, and the spectral sequence on its E_3 page looks like (see (2.29)):

$$E_3^{p,q}(F_n \to E_n \to B) \cong \begin{cases} \mathbb{Q}, & \{(p,q) : p = 0, q = 4k, 4k+1, k \geq 0\} \\ & \cup\{(p,q) : p = 2, q = 4k+1, 4k+2, k \geq 0\} \\ 0, & \text{otherwise}. \end{cases} \tag{2.27}$$

Spectral sequence $E_2^{p,q}(F_n \to E_n \to B)$.

$$
\begin{tikzcd}[row sep=small]
8 & \vdots & & \vdots \\
7 & \mathbb{Q} \arrow[drr] & & \mathbb{Q} \\
6 & 0 \arrow[drr] & & \mathbb{Q}^2 \\
5 & \mathbb{Q} \arrow[drr] & & \mathbb{Q} \\
4 & \mathbb{Q}^2 \arrow[drr] & & 0 \\
3 & \mathbb{Q} \arrow[drr] & & \mathbb{Q} \\
2 & 0 \arrow[drr] & & \mathbb{Q}^2 \\
1 & \mathbb{Q} \arrow[drr] & & \mathbb{Q} \\
0 & \mathbb{Q} & & 0 \\
 & 0 & 1 & 2
\end{tikzcd}
\tag{2.28}
$$

Spectral sequence $E_3^{p,q}(F_n \to E_n \to B)$.

$$
\begin{array}{c|ccc}
10 & \vdots & & \vdots \\
9 & \mathbb{Q} & & \mathbb{Q} \\
8 & \mathbb{Q} & & 0 \\
7 & 0 & & 0 \\
6 & 0 & & \mathbb{Q} \\
5 & \mathbb{Q} & & \mathbb{Q} \\
4 & \mathbb{Q} & & 0 \\
3 & 0 & & 0 \\
2 & 0 & & \mathbb{Q} \\
1 & \mathbb{Q} & & \mathbb{Q} \\
0 & \mathbb{Q} & & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
\tag{2.29}
$$

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