RESEARCH

Benjamini–Schramm convergence and zeta functions

Anton Deitmar

*Correspondence:
 deitmar@uni-tuebingen.de
 Mathematisches Institut, Auf der
 Morgenstelle 10, 72076
 Tübingen, Germany

Abstract

The equivalence of Benjamini–Schramm convergence and zeta-convergence, known for graphs, is proven for sequences of compact Riemann surfaces. A program is initialized, to extend this connection to arbitrary locally homogeneous spaces.

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Introduction

Benjamini–Schramm convergence or BS-convergence of metric probability spaces X_n to a pointed metric space (X, p) means that for every radius $R > 0$ the probability of a point x having the ball $B_R(x)$ of radius R isometric with $B_R(p)$ tends to one, i.e.,

$$P_n(\{x \in X_n : B_R(x) \cong B_R(p)\}) \longrightarrow 1, \quad n \rightarrow \infty.$$

For hyperbolic surfaces, Selberg [22] introduced a geometric zeta function which counts closed geodesics. Ihara [14] established a p -adic analog of this, which later was generalized to arbitrary graphs by Hashimoto and Hori [12]. Lenz, Pogorzelski and Schmidt recently proved [17] that a sequence of graphs of bounded valency is BS-convergent to an infinite tree if and only if the corresponding Ihara zeta functions converge to the trivial one.

The present paper may be considered the starting point for a program aiming at generalizing the equivalence

$$\text{BS-convergence} \Leftrightarrow \text{zeta-convergence}$$

to arbitrary locally homogeneous spaces, i.e., double quotients $\Gamma \backslash G/K$, where G is a locally compact group, Γ, K subgroups, where K is compact and Γ discrete. In order to define BS-convergence, one needs a metric, so one assumes that the topology of G/K is induced by a proper metric. It is also necessary for this metric to be left G -invariant in order to induce a derived metric on $\Gamma \backslash G/K$. The first steps of this program are the introduction of the notions of BS- and zeta-convergence in this context. This is done in Sect. 1, extending a previous notion by the author [9]. Suitable zeta functions are present in the literature, at least for Lie groups [4, 11, 22] or p -adic groups [7, 14]. A general definition for arbitrary locally compact groups is still lacking.

The next step is to show the equivalence of BS- to zeta-convergence for compact hyperbolic surfaces. This is done in Sect. 3. In this case, the bounded valency condition of [17] is replaced by a lower bound on the injectivity radius. Note that BS-convergence means that for every bound R , the number of closed geodesics of length $\leq R$ becomes small. It could still be that closed geodesics cluster at large lengths, but the cluster moves toward ∞ . The convergence of the zeta functions, however, takes into account all geodesics at once, with means that such clustering does not happen.

In this paper, the trace formula is used to transfer the question of zeta convergence to a spectral theoretic context. Then a uniform growth estimate on Laplace eigenvalues [13] is used to derive the convergence of the zeta functions. In the first section we introduce notations and collect material from our previous paper [9]. In the second section we extend a statement of [9] in the relative case. In the third section we state and prove the main theorem, and in the fourth section we collect some further questions and projects which might come out of this paper.

1 Relative Benjamini–Schramm convergence

In this section we extend Proposition 2.4 of [9] to arbitrary locally compact groups and to the relative setting. Thus we get a very useful criterion for BS-convergence, which in [9] has been used to give a very simple and elegant proof of one of the main results of [1]. In the previous section, only convergence to a homogeneous space has been considered, in which case the base point is irrelevant. In [9] the author extended the notion of BS-convergence to a relative situation which was formulated in purely group-theoretical terms. In geometric terms it translates to the following:

Definition 1.1 (*BS-convergence without base point*) Let (X_n, d_n, P_n) be a sequence of metric probability spaces, which share a common covering X_∞ , i.e., for each $n \in \mathbb{N}$ there is a metric covering map $\pi_n : X_\infty \rightarrow X_n$, which means that π_n is surjective and for each $x \in X_n$ there exists $\varepsilon > 0$ such that for every $y \in \pi_n^{-1}(x)$ the map π_n maps the ball $B_\varepsilon(y)$ isometrically to $B_\varepsilon(x)$. Then we say that the sequence (X_n) is *BS-convergent* to X_∞ , if for every $r > 0$,

$$P_n \left(\left\{ x \in X_n : \forall_{y \in \pi_n^{-1}(x)} \pi_n|_{B_r(y)} \text{ is an isometry } B_r(y) \rightarrow B_r(x) \right\} \right) \longrightarrow 1$$

as $n \rightarrow \infty$.

Definition 1.2 In the paper [9] the following situation was considered: (Γ_n) is a sequence of lattices in a locally compact group G and Γ_∞ is a common normal subgroup. We then say that the sequence (Γ_n) *Benjamini–Schramm converges* to Γ_∞ , or *BS-converges*, written $\Gamma_n \xrightarrow{BS} \Gamma_\infty$, if for every compact set $C \subset G$ the sequence

$$P_n(\{x \in \Gamma_n \backslash G : x^{-1}(\Gamma_n \setminus \Gamma_\infty)x \cap C = \emptyset\}) \longrightarrow 1$$

as $n \rightarrow \infty$. Here P_n is the normalized Haar measure on $\Gamma_n \backslash G$.

If (Γ_n) is BS-convergent to the trivial group $\{1\}$, then we say that (Γ_n) is a *BS-sequence*.

The next proposition considerably extends Proposition 2.4 of [9].

Proposition 1.3 *Let G be a locally compact group and K a compact subgroup. Assume that the topology on $X = G/K$ is generated by a G -invariant proper metric. Let (Γ_n) be a sequence of lattices in G and let Γ_∞ be a common normal subgroup of the Γ_n .*

Consider the following statements:

- (a) *The sequence Γ_n is BS-convergent to Γ_∞ .*
- (b) *The sequence of metric probability spaces $X_n = \Gamma_n \backslash X$ is BS-convergent to $X_\infty = \Gamma_\infty \backslash X$.*

Then (a) $\Rightarrow$ (b) unconditionally.

If for every open ball $\emptyset \neq B \subset X$ and every isometry $\phi : B \rightarrow B$ there exists a uniquely determined $g \in G$ such that $\phi(x) = gx$, $x \in B$ and every Γ_n is torsion-free, then (b) $\Rightarrow$ (a) holds as well.

Note that (b) $\Rightarrow$ (a) holds for X being a symmetric space without compact factors.

Proof Throughout, we will denote the probability measures on $\Gamma_n \backslash G$ and on $\Gamma_n \backslash X$ by the same symbol P_n .

(a) $\Rightarrow$ (b): Assume that Γ_n is BS-convergent to Γ_∞ and let $r > 0$. As we encounter different metric spaces, we shall write $B_r(z, Z)$ for the open r -ball in Z around z . The space $X = G/K$ has a natural base-point $x_0 = eK$. We abbreviate $B_r = B_r(eK, X)$. Let $g \in G$ and consider the point $x = \Gamma_n gK \in X_n$. For a discrete subgroup $\Gamma \subset G$ and any set $A \subset X$ we write $\Gamma \backslash A$ for the image of A in $\Gamma \backslash X$ or, what amounts to the same, $A / \sim$, where $a \sim a'$ if and only if there exists $\gamma \in \Gamma$ with $\gamma a = a'$. We use the invariance of the metric to identify

$$B_r(\Gamma gK, \Gamma \backslash X) \cong \Gamma \backslash B_r(gK, X) = \Gamma \backslash gB_r \cong (g^{-1}\Gamma g) \backslash B_r.$$

This means that $B_r(\Gamma_n gK, X_n)$ is isometric to some $B_r(\Gamma_\infty hK, X_\infty)$ if and only if

$$(g^{-1}\Gamma_n g) \backslash B_r \cong (h^{-1}\Gamma_\infty h) \backslash B_r.$$

Let U_r be the pre-image of B_r under the projection map $G \rightarrow G/K = X$. Let $C = \overline{U_r U_r^{-1}}$. Then C is a compact subset of G . Let

$$A_n(C) = \{g \in \Gamma_n \backslash G : g^{-1}(\Gamma_n \setminus \Gamma_\infty)g \cap C = \emptyset\}.$$

Then $P_n(A_n(C))$ tends to 1 as $n \rightarrow \infty$. Let C_r denote the compact set $\overline{U_r U_r^{-1}}$. Then if $g \in A_n(C_r)$, for every $\gamma_n \in \Gamma_n$ one has

$$g^{-1}\gamma_n g \in U_r U_r^{-1} \Rightarrow \gamma_n \in \Gamma_\infty.$$

Now if $g^{-1}\Gamma_n g u K = g^{-1}\Gamma_n g v K$ for two $u, v \in U_r$, then there exists $\gamma_n \in \Gamma_n$ and $k \in K$ such that $g^{-1}\gamma_n g = vku^{-1}$ and hence $g^{-1}\gamma_n g \in U_r U_r^{-1}$ and so u and v already give the same element in $g^{-1}\Gamma_\infty g \backslash B_r$. In other words, it follows that

$$(g^{-1}\Gamma_n g) \backslash B_r = (g^{-1}\Gamma_\infty g) \backslash B_r.$$

Let $T_n(r)$ denote the set of all $x \in X_n$ such that there exists $y \in X_\infty$ with $B_r(x, X_n) \cong B_r(y, X_\infty)$. Then the above entails

$$A_n(C_r)K \subset T_n(r).$$

Hence we get $P_n(T_n(r)) \rightarrow 1$ and so (a) $\Rightarrow$ (b) is proven.

(b) $\Rightarrow$ (a): First note that under the given conditions, each Γ_n acts fixed-point-freely on X and there exists a radius $t_n > 0$ such that the projection map $p_n : X \rightarrow \Gamma_n \backslash X$ induces an isometric isomorphism $B_{t_n}(x) \rightarrow p_n(B_{t_n}(x))$ for every $x \in X$.

Let $C \subset G$ be a compact set. Then there exists some $r > 0$ such that $C \subset U_r U_r^{-1}$. Let $x_n \in T_n(r)$ and let $\phi : B_r(x_n, X_n) \rightarrow B_r(x_\infty, X_\infty)$ be an isometry. Write $x_n = \Gamma_n g_n K$ and $x_\infty = \Gamma_\infty g_\infty K$, then ϕ can be viewed as a map $g_n^{-1}\Gamma_n g_n \backslash B_r \rightarrow g_\infty^{-1}\Gamma_\infty g_\infty \backslash B_r$. If ϕ maps the origin $g_n^{-1}\Gamma_n g_n$ to some $(g_\infty^{-1}\Gamma_\infty g_\infty)g_0$, then one can replace g_∞ by $g_\infty g_0$ and ϕ will preserve origins. Next let $0 < t \leq \min(r, t_n)$. Then ϕ induces an isometry of the ball $B_t \subset X$. Then there exists $\alpha \in G$ such that, on B_t , the map ϕ is given by $z \mapsto \alpha z$. Replacing g_∞ with $g_\infty \alpha$ we arrive at

$$\phi((g_n^{-1}\Gamma_n g_n)x) = (g_\infty^{-1}\Gamma_\infty g_\infty)x, \quad x \in B_t,$$

or, what amounts to the same, the diagram

$$\begin{array}{ccc} B_t & \xrightarrow{\text{Id}} & B_t \\ \downarrow & & \downarrow \\ (g_n^{-1}\Gamma_n g_n) \backslash B_t & \xrightarrow{\phi} & (g_\infty^{-1}\Gamma_\infty g_\infty) \backslash B_t \end{array}$$

commutes. Let $0 < t \leq r$ be maximal with this property. We claim that $t = r$. If not, then there exists $z \in B_r$ with $d(z, eK) = t < r$. As ϕ is an isometry, the diagram above still commutes with B_t replaced by the closed ball

$$\overline{B}_t = \{x \in X : d(x, eK) \leq t\}.$$

Let $s = \min(t_n, r - t)$. We therefore have $\phi((g_n^{-1}\Gamma_n g_n)z) = (g_\infty^{-1}\Gamma_\infty g_\infty)z$ and as ϕ is an isometry, ϕ maps $B_s(z)$ to $B_s(z)$. Again, this isometry is induced by an element $g \in G$ and we have $gy = y$ for every y in the non-empty open set $B_s(z) \cap B_t(x)$. The uniqueness condition implies that $g = e$ the neutral element in G . This means that the diagram commutes with B_t replaced by a larger open set which contains z . As this is the case for every z , by compactness we conclude that t was not maximal. Hence the diagram

$$\begin{array}{ccc} B_r & \xrightarrow{\text{Id}} & B_r \\ \downarrow & & \downarrow \\ (g_n^{-1}\Gamma_n g_n) \backslash B_r & \xrightarrow{\phi} & (g_\infty^{-1}\Gamma_\infty g_\infty) \backslash B_r \\ & \swarrow \psi \quad \searrow \cong & \\ & (g_n^{-1}\Gamma_\infty g_n) \backslash B_r & \end{array}$$

commutes. The lower triangle implies that the natural map ψ is an isometry as well, so we end up with the commutative diagram

$$\begin{array}{ccc} B_r & \xrightarrow{\text{Id}} & B_r \\ \downarrow & & \downarrow \\ (g_n^{-1}\Gamma_n g_n) \backslash B_r & \xrightarrow{\phi} & (g_n^{-1}\Gamma_\infty g_n) \backslash B_r. \end{array}$$

Now let $\gamma_n \in \Gamma_n$ and suppose $g_n^{-1}\gamma_n g_n \in C_r$. By definition, there exist $u, v \in U_r$ with $g_n^{-1}\gamma_n g_n = uv^{-1}$, or $(g_n^{-1}\gamma_n g_n)v = u$, which means that the points vK and uK are mapped to the same point in $(g_n^{-1}\Gamma_n g_n) \backslash B_r$, so they map to the same point in $(g_n^{-1}\Gamma_\infty g_n) \backslash B_r$. This means that there exist $\gamma_\infty \in \Gamma_\infty$ and $k \in K$ such that $(g_n^{-1}\gamma_\infty g_n)v = uk$. Hence

$$\gamma_\infty^{-1}\gamma_n = g_n v k v^{-1} g_n^{-1}$$

is contained in a compact group. As Γ_n is torsion-free, it follows $\gamma_n = \gamma_\infty$ and therefore $\Gamma_n g_n \in A_n(C_r)$. We have shown

$$T_n(r) \subset A_n(C_r)K.$$

which implies $P_n(A_n(C_r)) \rightarrow 1$. $\square$

Definition 1.4 A sequence (Γ_n) of lattices is called *uniformly discrete*, if there exists a unit-neighborhood $U \subset G$ such that $x^{-1}\Gamma_n x \cap U = \{1\}$ holds for every $x \in G$.

Remark 1.5 In [9] it is shown that

$$\begin{aligned} (\Gamma_n) \text{ is BS and uniformly discrete} \\ \Rightarrow (\Gamma_n) \text{ Plancherel} \\ \Rightarrow (\Gamma_n) \text{ is BS.} \end{aligned}$$

2 Plancherel and Benjamini–Schramm sequences

Let $G = \text{SL}_2(\mathbb{R})$. Then $K = \text{SO}(2)$ is a maximal compact subgroup of G . The group G acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ via linear fractionals and this action induces an identification of $G/\{\pm 1\}$ with the group of orientation-preserving isometries of the two-dimensional hyperbolic space. We normalize the Haar measure on K to have volume 1. Next we normalize the Haar measure on G such that it induces the usual $y^{-2} dx dy$ on the upper half plane $\mathbb{H} \cong G/K$.

For a cocompact lattice $\Gamma \subset G$ the unitary representation of G , given by right translation on $L^2(\Gamma \backslash G)$ decomposes as a direct sum of irreducibles

$$L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi) \pi.$$

The multiplicities $N_\Gamma(\pi)$ are finite and are zero outside a countable subset of the unitary dual $\widehat{G}$.

Definition 2.1 The measure on $\widehat{G}$ given by

$$\mu_\Gamma = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \delta_\pi$$

is called the *spectral measure* attached to Γ .

Definition 2.2 For $f \in C_c^\infty(G)$ let $\pi(f)$ denote the operator defined by integrating the representation π against f . More precisely, for $v \in V_\pi$ we have

$$\pi(f)v = \int_G f(x)\pi(x)v \, dx,$$

where the integral is a Bochner integral. Another way to say this is that $\pi(f)$ is the uniquely defined operator such that for any two $v, w \in V_\pi$ one has

$$\langle \pi(f)v, w \rangle = \int_G f(x) \langle \pi(x)v, w \rangle \, dx.$$

For a reductive Lie group like $G = \mathrm{SL}_2(\mathbb{R})$, the operator $\pi(f)$ is known to be a trace class operator for every $\pi \in \widehat{G}$.

A sequence of cocompact lattices (Γ_n) in G is called a *Plancherel sequence*, if for every $f \in C_c^\infty(G)$ we have

$$\frac{1}{\mathrm{vol}(\Gamma_n \backslash G)} \int_{\widehat{G}} \hat{f}(\pi) \, d\mu_{\Gamma_n}(\pi) \longrightarrow f(e)$$

as $n \rightarrow \infty$, where $\hat{f}(\pi) = \mathrm{tr} \, \pi(f)$.

The unitary dual $\widehat{G}$ comes equipped with the topology of locally uniform convergence of matrix coefficients, or *Fell topology*, see [10].

The Plancherel theorem states that there is a unique Borel measure μ_{Pl} on the unitary dual $\widehat{G}$ such that for every $f \in C_c^\infty(G)$ we have $f(e) = \int_{\widehat{G}} \hat{f}(\pi) \, d\mu_{\mathrm{Pl}}(\pi)$. (For an explicit computation of the Plancherel measure for the group $G = \mathrm{SL}_2(\mathbb{R})$, see Section 11.3 of [6].) This means that the sequence (Γ_n) is Plancherel if and only if in the dual space of $C_c^\infty(G)$ one has weak- $*$ -convergence

$$\frac{1}{\mathrm{vol}(\Gamma_n \backslash G)} \mu_{\Gamma_n} \longrightarrow \mu_{\mathrm{Pl}}.$$

Remark 2.3 (a) If a sequence (Γ_n) of lattices is a Plancherel sequence, then

$$\frac{1}{\mathrm{vol}(G/\Gamma_n)} \mu_{\Gamma_n}(U) \longrightarrow \mu_{\mathrm{Pl}}(U)$$

for every relatively compact open set $U \subset \widehat{G}$, whose boundary has Plancherel measure zero. This follows from the density principle of Sauvageot [21].

(b) If (Γ_n) is a Plancherel sequence, then

$$\mathrm{vol}(\Gamma_n \backslash G) \longrightarrow \infty$$

as $n \rightarrow \infty$. This follows from the fact that the spectral measure of each Γ_n is discrete and the Plancherel measure is not.

3 The Selberg zeta function

Let (Γ_n) be a sequence of cocompact lattices in G . For simplicity, we shall assume that each Γ_n is torsion-free, which can easily be arranged as every lattice Γ contains a torsion-free sublattice.

Definition 3.1 For $\gamma \in \Gamma_n$ the *length* of γ is defined by

$$l(\gamma) = \inf_{x \in X} d(\gamma x, x).$$

It is known that for torsion-free cocompact lattices Γ , every $\gamma \in \Gamma$ satisfies $l(\gamma) > 0$. The Selberg zeta function for Γ_n is defined for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ as

$$Z_n(s) = \prod_{\gamma} \prod_{k \geq 0} \left(1 - e^{-(s+k)l(\gamma)}\right),$$

where the first product runs over all primitive conjugacy classes in Γ_n (see [6], Section 11.6). The product converges for $\operatorname{Re}(s) > 1$, and the so defined function extends holomorphically to all of $\mathbb{C}$.

Theorem 3.2 *Let (Γ_n) be a sequence of torsion-free cocompact lattices in G .*

(a) *If the sequence (Γ_n) is uniformly discrete and Plancherel, then*

$$\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n(s)}{Z_n(s)}$$

converges to zero in the set $\{\operatorname{Re}(s) > 1\}$, as n tends to infinity.

(b) *If*

$$\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n(s)}{Z_n(s)}$$

converges to zero in the set $\{\operatorname{Re}(s) > 1\}$ as $n \rightarrow \infty$, then the sequence is Plancherel.

In either case, the convergence of $\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n(s)}{Z_n(s)}$ is uniform on every set of the form $\{\operatorname{Re}(s) \geq \alpha\}$, for $\alpha > 1$.

Proof For $\operatorname{Re}(s) > 1$ we have

$$\frac{Z'_n(s)}{Z_n(s)} = \sum_{k \geq 0} \sum_{[\gamma]} \ell(\gamma_0) e^{-(s+k)\ell(\gamma)} = \sum_{[\gamma]} \ell(\gamma_0) \frac{e^{-s\ell(\gamma)}}{1 - e^{-\ell(\gamma)}},$$

where the sum runs over all conjugacy classes $[\gamma] \neq \{1\}$ in Γ_n and γ_0 is the underlying primitive of γ , i.e., $\gamma = \gamma_0^m$ for some $m \in \mathbb{N}$. Now let $\operatorname{Re}(s) \geq \alpha > 1$, then $|e^{-(s+k)\ell(\gamma)}| = e^{-(\operatorname{Re}(s)+k)\ell(\gamma)} \leq e^{-(\alpha+k)\ell(\gamma)}$ and so the addendum follows.

Let $K = \operatorname{SO}(2)$ be the standard maximal compact subgroup of G and let $\widehat{G}_K$ denote the set of all $\pi \in \widehat{G}$ such that the representation space V_π contains nonzero K -fixed vectors. Then it is known [15],

$$\widehat{G}_K = \left\{ \pi_{ir} : r \in i \left(0, \frac{1}{2}\right) \cup \mathbb{R}_{\geq 0} \right\} \cup \{\operatorname{triv}\}.$$

The map $\phi : \widehat{G}_K \rightarrow [0, \infty)$, given by

$$\begin{aligned} \phi(\operatorname{triv}) &= 0, \\ \phi(\pi_{ir}) &= \frac{1}{4} + r^2 \end{aligned}$$

is a homeomorphism. We shall from now on identify $\widehat{G}_K$ with $[0, \infty)$. For brevity, we write $\mu(f)$ instead of $\int_X f \, d\mu$ where μ is a measure on X and f a function. We shall also consider μ_n and μ_{Pl} as measures on $\widehat{G}_K \subset \widehat{G}$.

We now proceed to the proof of part (a) of the theorem. Suppose that the sequence (Γ_n) is Plancherel. By [6], Section 11, there are numbers $r_{n,j} \in \mathbb{R} \cup i(0, \frac{1}{2})$ such that

$$L^2(\Gamma_n \backslash \mathbb{H}) = L^2(\Gamma_n \backslash G)^K = \mathbb{C} \oplus \bigoplus_{j=1}^{\infty} \pi_{ir_{n,j}},$$

where $\mathbb{C}$ stands for the one-dimensional space of constant functions and π_{ir} is the induced representation (principal or complementary series) with $r \in \mathbb{R} \cup i(0, \frac{1}{2})$. Formally we set $r_{n,0} = \frac{i}{2}$. Fix $s, b \in \mathbb{C}$ with $\operatorname{Re}(s), \operatorname{Re}(b) > 1$. By Section 11.6 of [6] we can plug the function

$$h_{s,b}(r) = \frac{1}{s^2 + r^2} - \frac{1}{b^2 + r^2}$$

into the trace formula and, as in Lemma 11.6.2 of [6], we get

$$\frac{1}{s} \frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) - \frac{1}{b} \frac{Z'_n}{Z_n} \left(b + \frac{1}{2} \right) = \mu_n(H_{s,b}) - \mu_{\text{Pl}}(H_{s,b}),$$

where $H_{s,b}(\lambda) = \frac{1}{s^2 + \lambda - \frac{1}{4}} - \frac{1}{b^2 + \lambda - \frac{1}{4}}$ and $\mu_n = \mu_{\Gamma_n}$. This means

$$\begin{aligned} \frac{1}{s} \frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) &= \frac{1}{b} \frac{Z'_n}{Z_n} \left(b + \frac{1}{2} \right) - \frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \sum_{k=0}^{\infty} \left(\frac{1}{s + \frac{1}{2} + k} - \frac{1}{b + \frac{1}{2} + k} \right) \\ &\quad + 2 \sum_{j=0}^{\infty} \frac{1}{s^2 + r_{n,j}^2} - \frac{1}{b^2 + r_{n,j}^2}. \end{aligned}$$

Let D_s denote the operator $D_s(\psi)(s) = -\frac{\partial}{\partial s} \left(\frac{1}{s} \psi(s) \right)$. We get

$$D_s \left[\frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) \right] = 4s \sum_{j=0}^{\infty} \frac{1}{(s^2 + r_{n,j}^2)^2} - \frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \sum_{k=0}^{\infty} \frac{1}{(s + \frac{1}{2} + k)^2}$$

and

$$\begin{aligned} D_s^2 \left[\frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) \right] &= -\frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \left(\frac{1}{s^2} \sum_{k=0}^{\infty} \frac{1}{(s + \frac{1}{2} + k)^2} + \frac{2}{s} \sum_{k=0}^{\infty} \frac{1}{(s + \frac{1}{2} + k)^3} \right) \\ &\quad + 16s \sum_{j=0}^{\infty} \frac{1}{(s^2 + r_{n,j}^2)^3}. \end{aligned}$$

Let

$$h_s(\lambda) = \frac{1}{(s^2 + \lambda - \frac{1}{4})^3}.$$

Note that the function $r \mapsto h_s(r^2 + \frac{1}{4})$ equals $\frac{1}{4s} D_s \frac{\partial}{\partial s} h_{s,b}(r)$. Therefore the above formula says

$$D_s^2 \left[\frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) \right] = \mu_n(h_s) - \operatorname{vol}(\Gamma_n \backslash G) \mu_{\text{Pl}}(h_s).$$

For $s > \frac{1}{2}$ the continuous function h_s is positive on $[0, \infty)$.

By the trace formula, [6] Section 11.4, we have that $\sum_{j=0}^{\infty} |h_s(\lambda_{n,j})| < \infty$ for every $s > \frac{1}{2}$. Further, in [6] Section 11.3 the Plancherel measure is explicitly computed. As h_s is decreasing to the power 3, it follows that integral $\mu_{\text{Pl}}(h_s)$ is finite.

If $I \subset [0, \infty)$ is a relatively open, bounded interval, then by [21] we have that

$$\lim_{n \rightarrow \infty} \frac{1}{\text{vol}(\Gamma_n \backslash G)} \mu_n(\mathbf{1}_I) = \mu_{\text{Pl}}(\mathbf{1}_I),$$

where $\mathbf{1}_I$ is the indicator function of I . By linearity this extends to linear combinations of functions of the form $\mathbf{1}_I$. There exists a sequence $(L_k)_{k \in \mathbb{N}}$ of such linear combinations such that $0 \leq L_k \nearrow h_s$ outside a countable set S , which is of Plancherel measure zero and can also be chosen to be of μ_{Γ_n} measure zero for all n and have empty intersection with $\mathbb{N}$. We can also choose the L_k so that for each $T \in \mathbb{N}_0$ we have that

$$\Phi_k(T) = \sup_{x \in [T, T+1) \setminus S} h_s(x) - L_k(x)$$

tends to zero for $k \rightarrow \infty$.

For $T \geq 1$ let $N_n(T) = \#\{j : |\frac{1}{4} + r_{n,j}^2| < T\}$. Recall that $\lambda_{n,j} = \frac{1}{4} + r_{n,j}^2$ is the j -th Laplace eigenvalue. The sequence (Γ_n) is uniformly discrete, which means that the injectivity radii of the manifolds $\Gamma_n \backslash \mathbb{H}$ are bounded below. Therefore, by formula (1.5) of [13], there exists a constant $C > 0$ such that for every $T \geq 1$ one has

$$N_n(T) \leq C \text{vol}(\Gamma_n \backslash G) T.$$

Now, since $N_n(0) = 0$,

$$\begin{aligned} 0 \leq \mu_n(h_s - L_k) &= \sum_j h_s(\lambda_{n,j}) - L_k(\lambda_{n,j}) \\ &\leq \sum_{T=0}^{\infty} \Phi_k(T) (N_n(T+1) - N_n(T)) \\ &\leq C \text{vol}(\Gamma_n \backslash G) \sum_{T=0}^{\infty} \Phi_k(T) (2T+1). \end{aligned}$$

Since $\Phi_k(T) \leq 2h_s(T)$ and $\sum_{T=0}^{\infty} h_s(T)(2T+1) < \infty$, we can apply dominated convergence, to get that this sum tends to zero for $k \rightarrow \infty$. So let $\varepsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have that $0 \leq \mu_n(h_s) - \mu_n(L_k) < \text{vol}(\Gamma_n \backslash G)\varepsilon/3$ holds for all $n \in \mathbb{N}$ and that $|\mu_{\text{Pl}}(L_k) - \mu_{\text{Pl}}(h_s)| < \varepsilon/3$.

Fix some $k \geq k_0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has

$$\left| \frac{1}{\text{vol}(\Gamma_n \backslash G)} \mu_n(L_k) - \mu_{\text{Pl}}(L_k) \right| < \varepsilon/3.$$

And so

$$\begin{aligned} \left| \frac{1}{\text{vol}(\Gamma_n \backslash G)} \mu_n(h_s) - \mu_{\text{Pl}}(h_s) \right| &\leq \frac{1}{\text{vol}(\Gamma_n \backslash G)} |\mu_n(h_s) - \mu_n(L_k)| \\ &\quad + \left| \frac{1}{\text{vol}(\Gamma_n \backslash G)} \mu_n(L_k) - \mu_{\text{Pl}}(L_k) \right| \\ &\quad + |\mu_{\text{Pl}}(L_k) - \mu_{\text{Pl}}(h_s)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This means that $\frac{1}{\text{vol}(\Gamma_n \backslash G)} D_s^2 \frac{Z'_n}{Z_n}(s)$ converges to zero for $\text{Re}(s) > 1$. Since $\frac{Z'_n}{Z_n}(s) = \sum_{k \geq 0} \sum_{[\gamma]} \ell(\gamma_0) e^{-(s+k)\ell(\gamma)}$, the following lemma proves the ‘only if’ direction of the theorem.

Lemma 3.3 For $n, k \in \mathbb{N}$ let $a_{n,k}, b_{n,k} > 0$ be real numbers. Suppose that $L_n(s) = \sum_{k=1}^{\infty} a_{n,k} e^{-sb_{n,k}}$ converges for $\operatorname{Re}(s) > 1$ and that $D_s^2 L_n(s)$ tends to zero as $n \rightarrow \infty$. Then $L_n(s)$ also tends to zero as $n \rightarrow \infty$ for every s with $\operatorname{Re}(s) > 1$.

Proof As the sum $L_n(s)$ converges locally uniformly, by the theorem of Weierstrass, we can differentiate under the sum to get for $s > 1$ that

$$s^4 D_s^2 L_n(s) = \sum_{k=1}^{\infty} a_{n,k} (3b_{n,k}s + b_{n,k}^2 s^2 + 3) e^{-sb_{n,k}} \geq \sum_{k=1}^{\infty} a_{n,k} e^{-sb_{n,k}} = L_n(s) \geq 0.$$

Now if the former tends to zero as $n \rightarrow \infty$, then so will the latter. $\square$

(b) For the converse direction assume convergence of $\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n}{Z_n}(s)$ to zero and let $f \in C_c^\infty(G)$. As f has compact support, there exists $c > 0$ such that for every $n \in \mathbb{N}$ and every $\gamma \in \Gamma_n \setminus \{1\}$ with $l(\gamma) > c$ the conjugation orbit $\{x\gamma x^{-1} : x \in G\}$ has empty intersection with $\operatorname{supp}(f)$. Hence for such γ we have that $\mathcal{O}_\gamma(f) = 0$. Here $\mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x\gamma x^{-1}) dx$ is the orbital integral and G_γ is the centralizer of γ in G . As f is bounded and has compact support, there exists $M > 0$ such that $|\mathcal{O}_\gamma(f)| \leq M$ for all $\gamma \in \Gamma \setminus \{1\}$. Note that any cocompact lattice only contains hyperbolic elements besides the trivial one. By Theorem 9.5.1 of [6], the trace formula is valid for every $f \in C_c^\infty(G)$, i.e.,

$$\mu_n(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f),$$

where $\Gamma_\gamma = \Gamma \cap G_\gamma$. For $\gamma = 1$ we have $\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) = \operatorname{vol}(\Gamma_n \backslash G)$ and for $\gamma \neq 1$, the proof of Theorem 11.4.3 implies that $\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) = \ell(\gamma_0)$. So it follows that for given $s > 1$ we have that

$$\begin{aligned} \left| \frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \mu_n(f) - f(e) \right| &\leq \frac{M}{\operatorname{vol}(\Gamma_n \backslash G)} \sum_{[\gamma] \neq [e], \ell(\gamma) \leq c} \ell(\gamma_0) \\ &\leq \frac{M}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{1 - e^{-c}}{e^{-sc}} \sum_{[\gamma] \neq [e], \ell(\gamma) \leq c} \ell(\gamma_0) \frac{e^{-s\ell(\gamma)}}{1 - e^{-\ell(\gamma)}} \\ &= \frac{M}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{1 - e^{-c}}{e^{-sc}} \frac{Z'_n}{Z_n}(s) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. The theorem is proven. $\square$

At this point one may ask, what happens to the zeta functions at arguments $s \in \mathbb{C}$ with $\operatorname{Re}(s) \leq 1$? This question is partially answered by the following functional equation, as it implies that $\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n}{Z_n}(s)$ will converge to a nonzero limit for $n \rightarrow \infty$, if $\operatorname{Re}(s) < 0$. This makes the question of convergence in the critical strip $0 < \operatorname{Re}(s) < 1$ even more mysterious.

Proposition 3.4 For $s \in \mathbb{C} \setminus \mathbb{Z}$ and $n \in \mathbb{N}$ we have that

$$\frac{Z'_n}{Z_n}(s) + \frac{Z'_n}{Z_n}(1-s) = \left(s - \frac{1}{2}\right) \operatorname{vol}(\Gamma_n \backslash G) \cot(\pi s).$$

In particular, if $\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n}{Z_n}(s)$ converges to zero for $n \rightarrow \infty$ and $\operatorname{Re}(s) > 1$, then for $\operatorname{Re}(s) < 0$ and $s \notin \mathbb{Z}$ it will converge to $(s - \frac{1}{2}) \cot(\pi s)$.

Proof. As above, we have for $\operatorname{Re}(s) > \frac{1}{2}$ that

$$\begin{aligned} \frac{1}{s} \frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) &= \frac{1}{b} \frac{Z'_n}{Z_n} \left(b + \frac{1}{2} \right) - \frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \sum_{k=0}^{\infty} \left(\frac{1}{s + \frac{1}{2} + k} - \frac{1}{b + \frac{1}{2} + k} \right) \\ &\quad + 2 \sum_{j=0}^{\infty} \frac{1}{s^2 + r_{nj}^2} - \frac{1}{b^2 + r_{nj}^2}. \end{aligned}$$

Now the right hand side converges for all $s \in \mathbb{C} \setminus (\frac{1}{2} - \mathbb{N})$ and thus establishes an analytic continuation of the left hand side. Also, the right hand side is an even function in s with the exception of the first sum. Taking the right hand side for $s \in \mathbb{C}$ and subtracting its value at $-s$ we get

$$\frac{1}{s} \frac{Z'_n}{Z_n} \left(s + \frac{1}{2} \right) + \frac{1}{s} \frac{Z'_n}{Z_n} \left(-s + \frac{1}{2} \right) = -\frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \sum_{k=0}^{\infty} \left(\frac{1}{s - \frac{1}{2} - k} + \frac{1}{s + \frac{1}{2} + k} \right).$$

Replacing s with $s - \frac{1}{2}$ this yields

$$\begin{aligned} \frac{1}{s - \frac{1}{2}} \frac{Z'_n}{Z_n}(s) + \frac{1}{s - \frac{1}{2}} \frac{Z'_n}{Z_n}(1 - s) &= -\frac{\operatorname{vol}(\Gamma_n \backslash G)}{\pi} \sum_{k=0}^{\infty} \left(\frac{1}{s - 1 - k} + \frac{1}{s + k} \right) \\ &= \operatorname{vol}(\Gamma_n \backslash G) \cot(\pi s). \end{aligned}$$

□

4 Open questions and further projects

Convergence inside the critical strip

Let Γ_n be a sequence as in Theorem 3.2 such that $\frac{1}{\operatorname{vol}(\Gamma_n \backslash G)} \frac{Z'_n}{Z_n}(s)$ tends to zero for $\operatorname{Re}(s) > 1$ as $n \rightarrow \infty$. Then this sequence of functions also converges for $\operatorname{Re}(s) < 0$, but what happens in between, in the critical strip? It might be possible to answer this question by finding suitable test functions for the trace formula. Note here that the zeros of the zeta function, i.e., poles of $\frac{Z'_n}{Z_n}$, will accumulate on the critical line, as $n \rightarrow \infty$.

Uniform discreteness

One assertion of the main theorem was proven under the condition of uniform discreteness, or, equivalently, a lower bound on the injectivity radius. It is not clear whether that condition is necessary. It seems impossible to eliminate the injectivity radius from the eigenvalue estimates, as Theorem 8.1.2 in [3] shows. According to this theorem, for fixed genus g (and therefore fixed volume $\operatorname{vol}(\Gamma \backslash G)$), for every $\varepsilon > 0$ there exist groups Γ of genus g and $N(1 + \varepsilon)$ arbitrarily large. Therefore, the only option seems to lie in an analysis of the Teichmüller space along the lines of [19] and the references therein (although possible critical cases have been excluded in that paper).

General rank one groups

The present proof uses eigenvalue estimates for the Laplacian. For general rank one Lie groups like $\operatorname{SO}(n, 1)$ the Selberg zeta function is described by the spectrum of generalized Laplacians [2] on certain homogeneous vector bundles. An extension of the present results would therefore require an extension of the eigenvalue estimates to these bundles. One possible path is to use the group Laplacian instead, which would provide a much weaker estimate, as the dimension increases; however, it might be sufficient for the task at hand.

Higher rank

For higher-rank groups the Selberg zeta function has to be replaced by corresponding higher-rank zeta functions as in [5]. As this zeta function only collects closed geodesics which lie in an open Weyl chamber, it will be necessary to consider several zeta functions, one for each conjugacy class of non-compact Cartan subgroups. On the other hand, a simplification may arise by only considering the restriction of these several variable zeta functions to generic lines.

p -adic groups

For p -adic groups the symmetric space is replaced with the Bruhat–Tits building. The rank one case (i.e., the case of graphs) has in great generality been dealt with affirmatively in [17]. The higher-rank case will rely on the several variable zeta functions defined in [8] and otherwise face the same difficulties as in the Lie-group situation except for the fact that small radii of injectivity play no role here.

Locally compact groups

This last and most general case is highly speculative. Is it possible to give a zeta function for any uniform lattice Γ in an arbitrary locally compact group G which reflects the global geometry well enough to detect Benjamini–Schramm convergence as formulated in [9]? There are a possible top-down approach and a bottom-up approach to this problem. The top-down approach uses the data given in the trace formula to define a new type of zeta function, such that the spectral side of the trace formula yields analytic continuation. The bottom-up approach uses the known cases and the structure theory of locally compact groups given, for instance, in [23].

Non-cocompact lattices

For arithmetic congruence groups, the adelic trace formula can be used to show that certain sequences of arithmetic groups are BS, see [18, 20]. An open problem raised in these papers is the question if any sequence (Γ_n) of congruence subgroups in a given linear algebraic group G is already BS if the covolumes tend to infinity. A similar statement is known to be wrong without the congruence property.

The connection to Selberg-type zeta functions is more subtle in the non-compact situation, as the trace formula does not provide a direct link between geometric spectral data.

Quantum ergodicity

The interesting paper [16] has opened a new perspective in mixing spectral convergence and quantum ergodicity. In the paper, this is formulated for hyperbolic surfaces and, as in [9], there is room for further developments along the lines described above.

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