



Lipschitz constants and quadruple symmetrization by Möbius transformations

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Abstract

Due to the invariance properties of cross-ratio, Möbius transformations are often used to map a set of points or other geometric object into a symmetric position to simplify a problem studied. However, when the points are mapped under a Möbius transformation, the distortion of the Euclidean geometry is rarely considered. Here, we identify several cases where the distortion caused by this symmetrization can be measured in terms of the Lipschitz constant of the Möbius transformation in the Euclidean or the chordal metric.

Keywords Hyperbolic geometry · Lipschitz constants · Möbius transformations · Symmetrization

Mathematics Subject Classification Primary 51M10 · Secondary 51M16

1 Introduction

Symmetry is an important property considered by many mathematicians working in various areas of research, including classical analysis, geometry, and solutions to extremal problems [3, 4, 6, 7]. Problem-specific methods have been introduced with the purpose of transforming objects into symmetric ones [3]. Another use of symmetry is to reduce some complicated problem to its special case which exhibits symmetry and is therefore easier to solve than the original question.

For quadruples of points on the complex plane, a symmetrization can be performed by means of a Möbius transformation. The use of Möbius transformations is in fact a standard tool in classical function theory. However, while there are several metrics, such as the hyperbolic metric, that are invariant with respect to the Möbius transformations, these mappings affect the Euclidean distances between the complex points considered. Our aim is to analyze "cost" of the symmetrization due to distortion under Möbius transformations, measured in terms of Lipschitz constants. These

Lipschitz constants can then be formulated with respect to Euclidean, hyperbolic, or chordal geometry.

We consider symmetrization of quadruples of points in several cases, under various constraints of the location of the points of the quadruple $\{a, b, c, d\}$. In the most general case, we find a Möbius transformation h mapping the quadruple onto the quadruple $\{-1, -y, y, 1\}$ symmetric with respect to the origin. For any such Möbius transformation h , it seems natural to call $h^{-1}(0)$ a *Möbius center of the quadruple*. We give an explicit formula for such a Möbius center.

Theorem 1.1 *Let h be the Möbius transformation of $\overline{\mathbb{R}^2}$ that maps distinct points $a, b, c, d \in \mathbb{R}^2$ onto $-1, y, -y, 1$, respectively. If a, b, c, d are collinear and $|a - b| = |c - d|$, then $h^{-1}(0) = (a + d)/2$. Otherwise,*

$$h^{-1}(0) = \frac{(b - c)(a + d)y + (b + c)(a - d)}{2((b - c)y + a - d)},$$

where y is as in Lemma 4.2.

The structure of this paper is as follows. In Sect. 3, we study a certain chordal isometry and related Lipschitz constants of the chordal metric. In Sect. 4, we find a Möbius transformation for the symmetrization of any four distinct points on the complex plane and prove Theorem 1.1. In Sect. 5, we introduce three normalization methods for a pair of points in the unit disk and, in Sect. 6, we consider the symmetrization of quadruples of points on the unit circle.

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2 Preliminaries

Let us first introduce the notations used in this paper. For a dimension $n \geq 0$, e_n is the n th unit vector of the real space \mathbb{R}^n , \mathbb{B}^n is the unit ball, and S^{n-1} is the unit sphere. For a point $x \in \mathbb{R}^n$ and a radius r , let $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ and $S^{n-1}(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}$. For any point $x \in \mathbb{C}$, denote its complex conjugate by \bar{x} . For two distinct points $x, y \in \mathbb{R}^n$, let $L(x, y)$ be the Euclidean line passing through them. Denote the extended real space by $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

The intersection point of two non-parallel lines $L(a, b)$ and $L(c, d)$ is given by [11, Ex. 4.3(1), p. 57 & p. 373]

$$\text{LIS}[a, b, c, d] = \frac{(\bar{a}b - a\bar{b})(c - d) - (\bar{c}d - c\bar{d})(a - b)}{(\bar{a} - \bar{b})(c - d) - (\bar{c} - \bar{d})(a - b)}. \tag{2.1}$$

If the four points a, b, c, d are on the unit circle, then we have $\bar{z} = 1/z$ for $z \in \{a, b, c, d\}$ and the formula (2.1) can be simplified as in the following proposition.

Proposition 2.2 [12, p.13] *If a, b, c, d are four distinct complex points on the unit circle S^1 chosen so that the lines $L(a, b)$ and $L(c, d)$ are non-parallel, then the complex conjugate \bar{f} of the intersection point f of the lines $L[a, c]$ and $L[b, d]$ is*

$$\bar{f} = \frac{a + c - b - d}{ac - bd}.$$

Proposition 2.3 *Let $a, b \in \mathbb{C}$ with $|a| \neq |b|, |a||b| \neq 1$. Then*

- (1) $\text{LIS}[a, b, -1/\bar{a}, -1/\bar{b}] = \frac{b(1 + |a|^2) - a(1 + |b|^2)}{|a|^2 - |b|^2},$
- (2) $\text{LIS}[a, b, 1/\bar{a}, 1/\bar{b}] = \frac{a(1 - |b|^2) - b(1 - |a|^2)}{|a|^2 - |b|^2},$
- (3) $\text{LIS}[a, 1/\bar{b}, b, 1/\bar{a}] = \frac{a(1 - |b|^2) + b(1 - |a|^2)}{1 - |a|^2|b|^2},$
- (4) $\text{LIS}[a, -1/\bar{b}, b, -1/\bar{a}] = \frac{a(1 + |b|^2) + b(1 + |a|^2)}{1 - |a|^2|b|^2}.$

Proof The proof follows from (2.1). □

Proposition 2.4 *Let $a, b \in \mathbb{B}^2$ be points non-collinear with 0 and $c = \text{LIS}[a, b, 0, i(a - b)]$. Then $L[a, b] \cap S^1 = \{a_1, b_1\}$ where*

$$a_1 = c - i \frac{c}{|c|} \sqrt{1 - |c|^2}, \quad b_1 = c + i \frac{c}{|c|} \sqrt{1 - |c|^2},$$

and a_1, a, b, b_1 are ordered in such a way that $|a_1 - a| < |a_1 - b|$.

Proof The proof follows from (2.1). □

2.5 Möbius transformations. [11, Ex. 3.2, pp. 25-26 & Def. 3.6, p. 27] For any $t \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$, the hyperplane perpendicular to the vector u and at distance $t/|u|$ from the origin is

$$P(u, t) = \{x \in \mathbb{R}^n \mid x \cdot u = t\} \cup \{\infty\},$$

where \cdot denotes the dot product. The reflection in this hyperplane is defined by the function $h : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$,

$$h(x) = x - 2(x \cdot u - t) \frac{u}{|u|^2}, \quad h(\infty) = \infty.$$

Define then the inversion in $S^{n-1}(v, r)$ as $g : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$,

$$g(x) = v + \frac{r^2(x - v)}{|x - v|^2}, \quad g(v) = \infty, \quad g(\infty) = v.$$

A Möbius transformation is a function $f : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ that can be defined as a function composition $f = h_1 \circ \dots \circ h_m$ with an integer $m \geq 1$ so that each $h_j, j = 1, \dots, m$, is either a reflection in some hyperplane or an inversion in a sphere. The Möbius transformation is called sense-preserving when m here is even and sense-reversing when m is odd. By Liouville's theorem, any conformal mapping with a domain in $\mathbb{R}^n, n \geq 3$, is a Möbius transformation in $\overline{\mathbb{R}^n}$ restricted to this domain [11, Rmk 3.44, p. 47].

In the two-dimensional case, the expression of a sense-preserving Möbius transformation can be written as

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d, z \in \mathbb{C}, \quad ad - bc \neq 0. \tag{2.6}$$

The expression of the sense-reversing Möbius transformation can be obtained by replacing z by \bar{z} in (2.6). The special Möbius transformation

$$T_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{B}^2 \setminus \{0\}, \tag{2.7}$$

maps the unit disk \mathbb{B}^2 onto itself with $T_a(a) = 0, T_a(\pm a/|a|) = \pm a/|a|$.

Ahlfors [1] gave the following factorization for $T_a, a \in \mathbb{B}^2 \setminus \{0\}$, as a composition of an inversion followed by a reflection which is given here in the complex notation. Denote $a^* = a/|a|^2$. Let $r = \sqrt{|a|^{-2} - 1}$ and σ_a be the inversion in the circle $S^1(a^*, r)$ orthogonal to the unit circle S^1 . Then $\sigma_a(a) = 0$ and $\sigma_a(a^*) = \infty$. Let p_a be the reflection over the line through 0 and perpendicular to a . Then both σ_a and p_a are sense-reversing transformations and their composite transformation T_a in (2.7) is a sense-preserving Möbius

transformation with the decomposition

$$T_a(z) = p_a(\sigma_a(z)), \sigma_a(z) = \frac{a(1 - (\bar{z}/\bar{a}))}{1 - a\bar{z}}, p_a(z) = -\frac{a}{\bar{a}}\bar{z}.$$

Clearly, p_a is a Euclidean isometry and hence the information about distance distortion under T_a is due to the mapping σ_a . Moreover, for $a, b \in \mathbb{B}^2 \setminus \{0\}$, $|a| \neq |b|$, a Möbius transformation $T_{a,b} : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ with $T_{a,b}(a) = b$ is given by

$$T_{a,b}(z) = \frac{b\bar{c}z - 1}{\bar{b}z - c}, \quad c = \frac{a - b + ab(\bar{a} - \bar{b})}{|a|^2 - |b|^2}. \tag{2.8}$$

The above simple formula for the point c follows from Proposition 2.3(2) for the intersection of two lines $c = \text{LIS}[a, b, a^*, b^*]$. For $a, b \in \mathbb{B}^2 \setminus \{0\}$, the Möbius transformation $T_{a,b}$ is a reflection over the line $L(0, b)$ composed with the inversion in $S^1(c, r)$ where $r = \sqrt{|c|^2 - 1}$ so that $S^1(c, r)$ is orthogonal to the unit circle. See Fig. 1.

Theorem 2.9 *Let T_a and $T_{a,b}$ be the above Möbius transformations (2.8). Then*

$$\text{Lip}(T_a) = \frac{1 + |a|}{1 - |a|}. \tag{2.10}$$

and

$$\text{Lip}(T_{a,b}) = \frac{|c| + 1}{|c| - 1}, \quad c = \frac{a - b + ab(\bar{a} - \bar{b})}{|a|^2 - |b|^2}, \tag{2.11}$$

where the Lipschitz constant are defined for the Euclidean metric.

Proof The formula (2.10) was proved in [5, p.43]. The second part follows from (2.8). \square

2.12 Hyperbolic geometry. Denote hyperbolic sine, cosine and tangent by sh, ch, and th, respectively. The hyperbolic metric $\rho_{\mathbb{B}^n}$ in the unit ball \mathbb{B}^n is defined by

$$\text{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}}, \tag{2.13}$$

and, if $n = 2$, this is equivalent to

$$\text{th} \frac{\rho_{\mathbb{B}^2}(x, y)}{2} = \frac{|x - y|}{|1 - x\bar{y}|} = \frac{|x - y|}{A[x, y]}, \tag{2.14}$$

where $A[x, y]$ is the Ahlfors bracket [2, 7.37]

$$A[x, y] = |1 - x\bar{y}| = \sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}. \tag{2.15}$$

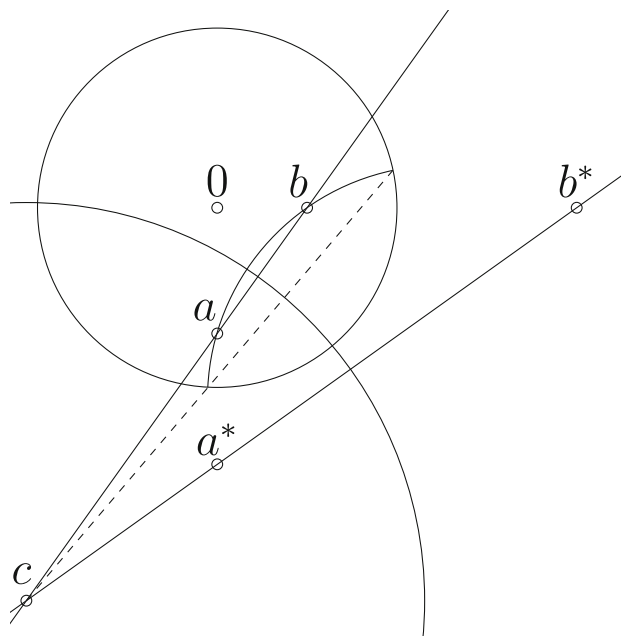


Fig. 1 The points a, b, a^*, b^* when $a = -0.7i$ and $b = 0.5$, the intersection point $c = \text{LIS}[a, b, a^*, b^*]$ of the lines $L(a, b)$ and $L(a^*, b^*)$, the unit circle, the circle $S^1(c, r)$ for $r = \sqrt{|c|^2 - 1}$, and the hyperbolic line $J^*[a, b]$. The end points of $J^*[a, b]$ are collinear with c , as denoted with dashed line. The Möbius transformation $T_{a,b}$ is a reflection over the line $L(0, b)$ composed with the inversion in the circle $S^1(c, r)$

We denote the hyperbolic line through $x, y \in \mathbb{B}^2$ by $J^*[x, y]$. If x, y are collinear with the origin, the line $J^*[x, y]$ is a diameter of the unit disk and, otherwise, an arc of the circle that passes through x, y and is orthogonal to the unit circle [5]. For $x, y \in \mathbb{B}^2 \setminus \{0\}$, the end points of the hyperbolic line $J^*[x, y]$ can be expressed as $ep(x, y)$ and $ep(y, x)$ where

$$ep(x, y) = T_{-y}(T_y(x)/|T_y(x)|). \tag{2.16}$$

Define then the absolute ratio of any four distinct points $a, b, c, d \in \mathbb{R}^n$ as [5]

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|}. \tag{2.17}$$

The hyperbolic metric fulfills

$$\rho_{\mathbb{B}^2}(x, y) = \log |ep(x, y), x, y, ep(y, x)|, \tag{2.18}$$

which shows that the hyperbolic metric is invariant under Möbius transformations of the unit disk onto itself.

Theorem 2.19 [13, Thm 1.4, p. 126] *For given $x, y \in \mathbb{B}^2$, the hyperbolic midpoint $z \in \mathbb{B}^2$ with $\rho_{\mathbb{B}^2}(x, z) = \rho_{\mathbb{B}^2}(y, z) = \rho_{\mathbb{B}^2}(x, y)/2$ is given by*

$$z = \frac{y(1 - |x|^2) + x(1 - |y|^2)}{1 - |x|^2|y|^2 + A[x, y]\sqrt{(1 - |x|^2)(1 - |y|^2)}}. \tag{2.20}$$

2.21 Chordal geometry. Define the *stereographic projection* $\pi : \mathbb{R}^2 \rightarrow S^2(\frac{1}{2}e_3, \frac{1}{2})$, $e_3 = (0, 0, 1)$, as [11, (3.4), p. 28]

$$\pi(x) = e_3 + \frac{x - e_3}{|x - e_3|^2}, \quad x \in \mathbb{R}^2; \quad \pi(\infty) = e_3. \tag{2.22}$$

The mapping π is now the restriction to $\overline{\mathbb{R}^2}$ of the inversion in $S^2(e_3, 1)$ and, since $f^{-1} = f$ for every inversion f , π maps the ‘‘Riemann sphere’’ $S^2(\frac{1}{2}e_3, \frac{1}{2})$ onto $\overline{\mathbb{R}^2}$. By using the stereographic projection π , we can define the *spherical (chordal) metric* q in $\overline{\mathbb{R}^2}$ as [11, (3.5), p. 29]

$$q(x, y) = |\pi(x) - \pi(y)|; \quad x, y \in \overline{\mathbb{R}^2}, \tag{2.23}$$

which can be extended to the n -dimensional case so that [11, (3.6), p. 29]

$$\begin{cases} q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}; & x, y \in \mathbb{R}^n, \\ q(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}; & x \in \mathbb{R}^n. \end{cases} \tag{2.24}$$

For $x \in \mathbb{R}^n$ and $r > 0$, denote the chordal ball by $B_q(x, r) = \{y \in \mathbb{R}^n : q(x, y) < r\}$.

Trivially, t_a is a Möbius transformation and it satisfies $t_a(a) = 0$. In fact, we have the following formula [11, (B.9), p. 459]

$$t_a(z) = \frac{z - a}{1 + \bar{a}z}.$$

Furthermore, we have [11, (B.10), p. 459]

$$t_a(z) = -t_{-a}(-z).$$

Theorem 3.3 [8, Thm 5.12] For $a, b \in \mathbb{R}^2$, the chordal midpoint m is given by

$$m = \frac{a(1 + |b|^2) + b(1 + |a|^2)}{|1 + \bar{a}b|\sqrt{(1 + |a|^2)(1 + |b|^2)} - |ab|^2 + 1}.$$

We see from the above formula that, if $a, b \in \mathbb{B}^2$, their chordal midpoint $m \in \mathbb{B}^2$.

Corollary 3.4 For $a, b \in \mathbb{R}^2$, let m be their chordal midpoint as in Theorem 3.3. Then the chordal isometry $t_m : \overline{\mathbb{R}^2} \rightarrow \overline{\mathbb{R}^2}$ has the formula

$$t_m(z) = \frac{(|1 + \bar{a}b|\sqrt{(1 + |a|^2)(1 + |b|^2)} + (1 - |ab|^2))z - ab(\bar{a} + \bar{b}) - (a + b)}{(\bar{a}b(a + b) + (\bar{a} + \bar{b}))z + |1 + \bar{a}b|\sqrt{(1 + |a|^2)(1 + |b|^2)} + 1 - |ab|^2}.$$

3 Lipschitz constants in chordal metric

In this section, we introduce an isometry t_m of the chordal metric. Note that, while the mapping T_a is an isometry for the hyperbolic metric, it is not a chordal isometry. However, our computer tests suggest that it has the following Lipschitz constant.

Conjecture 3.1 For $a \in \mathbb{B}^n$, the Lipschitz constant of the mapping T_a in the chordal metric is

$$Lip(T_a|\overline{\mathbb{R}^n}) \equiv \sup \left\{ \frac{q(T_a(x), T_a(y))}{q(x, y)} : x, y \in \overline{\mathbb{R}^n} \right\} = \frac{1 + |a|}{1 - |a|}.$$

It is well-known that $(1 + |a|)/(1 - |a|)$ is also the Lipschitz constant $Lip(T_a|\mathbb{B}^n)$ for the Euclidean metric, as stated in Theorem 2.9.

Let us then define the chordal isometry. Recall the notation a^* for $a \in \mathbb{R}^2 \setminus \{0\}$ and p_a from Subsection 2.5. Let s_a be the inversion in the circle $S^1(-a^*, \sqrt{1 + |a|^{-2}})$. Now, we can define a chordal isometry as [11, (3.23), p. 40]

$$t_a = p_a \circ s_a. \tag{3.2}$$

Furthermore, we have

$$\begin{aligned} t_m(a) &= \frac{a|1 + \bar{a}b|\sqrt{1 + |b|^2} - b(1 + \bar{a}b)\sqrt{1 + |a|^2}}{|1 + \bar{a}b|\sqrt{1 + |b|^2} + (1 + \bar{a}b)\sqrt{1 + |a|^2}}, \\ t_m(b) &= \frac{b|1 + \bar{a}b|\sqrt{1 + |a|^2} - a(1 + \bar{a}b)\sqrt{1 + |b|^2}}{|1 + \bar{a}b|\sqrt{1 + |a|^2} + (1 + \bar{a}b)\sqrt{1 + |b|^2}}. \end{aligned}$$

Proof The result follows by inputting m as in Theorem 3.3 into $t_m(z) = (z - m)/(1 + \bar{m}z)$. □

4 Möbius symmetrization

In this section, we prove a few results related to using a Möbius transformation for the symmetrization of quadruples of points on the complex plane.

Lemma 4.1 [11, Lemma 3.17, p. 34] Let f be a Möbius transformation of $\overline{\mathbb{R}^n}$ that maps a quadruple $0, e_1, x$, and ∞ of distinct points onto the quadruple $-e_1, y, -y$, and e_1 , respectively, such that $|y| \leq 1$. Then

$$|y| = \frac{|x - e_1|}{1 + |x| + t} \quad \text{and} \quad |y + e_1|^2 = \frac{|y - e_1|^2}{|x|} = \frac{4}{1 + |x| + t},$$

where $t = \sqrt{(1 + |x|)^2 - |x - e_1|^2}$.

Lemma 4.2 For four distinct points $a, b, c, d \in \mathbb{R}^2$, let h be the Möbius transformation of $\overline{\mathbb{R}^2}$ that maps a, b, c, d onto $-1, y, -y, 1$, respectively, for some point $y \in \mathbb{R}^2$. If a, b, c, d are collinear and $|a - b| = |c - d|$, then

$$h(z) = -\frac{2}{a-d}z + \frac{a+d}{a-d}.$$

Otherwise,

$$h(z) = \frac{pz + q}{z + s} \text{ with } p = -\frac{(b-c)y + a - d}{a - b - c + d}, \quad q = \frac{(b-c)(a+d)y + (b+c)(a-d)}{2(a-b-c+d)},$$

$$s = -\frac{(b+c)(a-d)y + (b-c)(a+d)}{2((a-d)y + (b-c))},$$

where

$$y = \frac{-k_1 \pm \sqrt{k_1^2 - k_0^2}}{k_0} \text{ with } k_0 = (b-c)(a-d), \quad k_1 = (a-c)(b-d) + (a-b)(c-d).$$

The two possible solutions for y above are reciprocal numbers so we can choose y so that $|y| < 1$ in this case.

Proof First, if a, b, c, d are collinear and satisfy $|a - b| = |c - d|$, then h is a linear map and can be written $h(z) = uz + v$. Since

$$h(a) = ua + v = -1, \quad h(d) = ud + v = 1,$$

we have

$$v = -\frac{(a+d)u}{2}, \quad u = -\frac{2}{a-d}.$$

Thus, the first part of the lemma follows.

Next, let us consider the other case. We may assume that $h(z) = \frac{pz + q}{z + s}$ since h is clearly not a linear map. From the conditions $h(a) = -1, h(b) = y, h(c) = -y, h(d) = 1$, we obtain

$$ap + q + s + a = 0, \tag{4.3}$$

$$-(s + b)y + bp + q = 0, \tag{4.4}$$

$$(s + c)y + cp + q = 0, \tag{4.5}$$

$$dp + q - s - d = 0. \tag{4.6}$$

Calculating (4.3)+(4.6), (4.3)-(4.6), (4.4)+(4.5), and (4.4)-(4.5) results in the following equations, respectively:

$$(a + d)p + 2q + a - d = 0, \tag{4.7}$$

$$(a - d)p + 2s + a + d = 0, \tag{4.8}$$

$$(-b + c)y + (b + c)p + 2q = 0, \tag{4.9}$$

$$(-2s - b - c)y + (b - c)p = 0. \tag{4.10}$$

By eliminating q from (4.7) and (4.9), we have

$$Y_1 \equiv (b - c)y + (a - b - c + d)p + a - d = 0$$

$$\Leftrightarrow p = -\frac{(b - c)y + a - d}{a - b - c + d}.$$

Similarly, by eliminating p from (4.7) and (4.9), we have

$$(b - c)(a + d)y - 2(a - b - c + d)q + (b + c)(a - d) = 0,$$

$$\Leftrightarrow q = \frac{(b - c)(a + d)y + (b + c)(a - d)}{2(a - b - c + d)}.$$

By eliminating s from (4.8) and (4.10), we have

$$Y_2 \equiv ((a - d)p + a - b - c + d)y + (b - c)p = 0.$$

Elimination of p from (4.8) and (4.10) gives us

$$(2(a - d)s + (b + c)(a - d))y + 2(b - c)s + (b - c)(a + d) = 0$$

$$\Leftrightarrow s = -\frac{(b + c)(a - d)y + (b - c)(a + d)}{2((a - d)y + (b - c))}.$$

Finally, by eliminating p from $Y_1 = 0$ and $Y_2 = 0$, we have

$$(b - c)(a - d)y^2 + 2((a - c)(b - d) + (a - b)(c - d))y + (b - c)(a - d) = 0,$$

which is equivalent to $k_0y^2 + 2k_1y + k_0 = 0$ where k_0 and k_1 are as in the lemma. The latter part of our lemma follows from this. \square

The Möbius transformation of the former lemma is a very useful mapping as it can be used for symmetrization. Namely, with this transformation, we can map the arbitrary quadruple of points a, b, c, d onto the points $-1, y, -y, 1$ that are clearly symmetric with respect to the origin. However, as the origin is not preserved under the mapping h , this raises the question about the location of the Möbius center $h^{-1}(0)$ in terms of a, b, c, d . We can now prove Theorem 1.1 from Introduction.

4.11 Proof of Theorem 1.1 The mapping h is as in Lemma 4.2. If a, b, c, d are collinear and $|a - b| = |c - d|$, then it is a linear map and $h^{-1}(0)$ is the midpoint of a and d .

Consider then the other case. Denote $w = h^{-1}(0)$. Now, for p, q, s as in Lemma 4.2,

$$\begin{aligned} \frac{pw + q}{w + s} = 0 &\Leftrightarrow w = -\frac{q}{p} \\ &= \frac{(b - c)(a + d)y + (b + c)(a - d)}{2((b - c)y + a - d)}. \end{aligned}$$

□

5 Three normalizations of point pairs

Mathematical proofs involving distances are often simplified if the configurations studied exhibit some kind of symmetry or special structure. We study here, for a given pair of distinct points a, b in $\mathbb{B}^2 \setminus \{0\}$, three natural normalizations:

- (1) $a = -b$,
- (2) $a, b, 0$ are collinear,
- (3) $|a| = |b|$.

If the points are in general position, then they can be mapped onto each of the above special positions by means of a Möbius transformation. The cost is that the geometry is changed in each case: we measure the cost in terms of the Lipschitz constant of the Möbius transformation.

A simple way to achieve the normalization (2) above is to relabel the points a and b such that $|a| \leq |b|$ and to use the mapping T_a . By Theorem 2.9 the cost of normalization can be immediately obtained. In our discussion below, the plan is to carry out the normalization with a smaller cost, if possible.

Theorem 5.1 *Consider a given pair of distinct points a, b in $\mathbb{B}^2 \setminus \{0\}$. Let k be the point on the hyperbolic line $J^*[a, b]$ that is closest to the origin. In other words, $k = 0$ if $a, b, 0$ are collinear and, if $a, b, 0$ are not collinear, then*

$$k = \frac{o_{ab}}{t}(t - \sqrt{t^2 - 1}), \quad t = |o_{ab}|,$$

where let o_{ab} be the center of the circle through a and b , orthogonal to the unit circle. Let h_{ab} be the hyperbolic midpoint of a and b . Then there are Möbius transformations $m_j : \mathbb{B}^2 \rightarrow \mathbb{B}^2, j = 1, 2, 3$ with the following properties: $m_1(a) = -m_1(b)$ with

$$Lip(m_1) \leq Lip(T_{h_{ab}}) = \frac{1 + |h_{ab}|}{1 - |h_{ab}|}, \tag{5.2}$$

$m_2(a), m_2(b), 0$ are collinear and

$$Lip(m_2) \leq Lip(T_k) = \frac{1 + |k|}{1 - |k|}, \tag{5.3}$$

and $|m_3(a)| = |m_3(b)|$ and

$$Lip(m_3) \leq Lip(T_{h_{ab},k}), \tag{5.4}$$

where $T_{h_{ab},k}$ is the mapping defined in (2.8).

Proof We can choose for instance the mappings $m_1 = T_{h_{ab}}, m_2 = T_k$, and $m_3 = T_{h_{ab},k}$ in which case the following inequalities of the Lipschitz constants $Lip(m_j)$ presented in the theorem hold with equality. □

For all $a, b \in \mathbb{B}^2 \setminus \{0\}, |k| \leq |h_{ab}|$ where h_{ab} and k are as in Theorem 5.1. Namely, the hyperbolic point h_{ab} trivially is on the hyperbolic line $J^*[a, b]$, but, unlike k, h_{ab} is not necessarily the closest point to the origin on this line. It follows from the inequality $|k| \leq |h_{ab}|$ that the Lipschitz constants $Lip(T_{h_{ab}})$ and $Lip(T_k)$ in Theorem 5.1 satisfy

$$Lip(T_k) \leq Lip(T_{h_{ab}}) \tag{5.5}$$

for all $a, b \in \mathbb{B}^2$. However, a similar inequality cannot be presented for the Lipschitz constant $Lip(T_{h_{ab},k})$. Namely, for $a = 0.3$ and $b = 0.1i$,

$$Lip(T_{h_{ab}}) \approx 1.379, \quad Lip(T_k) \approx 1.207, \quad Lip(T_{h_{ab},k}) \approx 1.301,$$

but, for $a = 0.3$ and $b = 0.1i$,

$$Lip(T_{h_{ab}}) \approx 1.508, \quad Lip(T_k) \approx 1.508, \quad Lip(T_{h_{ab},k}) \approx 2.029.$$

In other words, $Lip(T_{h_{ab},k})$ is sometimes less than the other two Lipschitz constants and sometimes greater than them.

6 Quadruples of points on the unit circle

Consider four distinct points a, b, c, d on the unit circle in the positive order. We can symmetrize the quadruple in two different ways using a Möbius transformation of the unit disk:

- (1) After the symmetrization, the new quadruple is symmetric with respect to a diameter of the unit disk.
- (2) After the symmetrization, the points of the new quadruple form the vertices of a rectangle.

We will next show how to construct the corresponding Möbius transformations and estimate the cost of symmetrization, the Lipschitz constant of the transformation.

Theorem 6.1 [13] (1) Let

$$w_1 = LIS[a, b, c, d], \quad w_2 = LIS[a, c, b, d], \quad w_3 = LIS[a, d, b, c].$$

Fig. 2 The Möbius transformation T_k maps the nearest point k on the hyperbolic line $J^*[a, b]$ through $a = -0.9i$ and $b = 0.5 - 0.3i$ to the origin so that the images of a and b are collinear with the origin

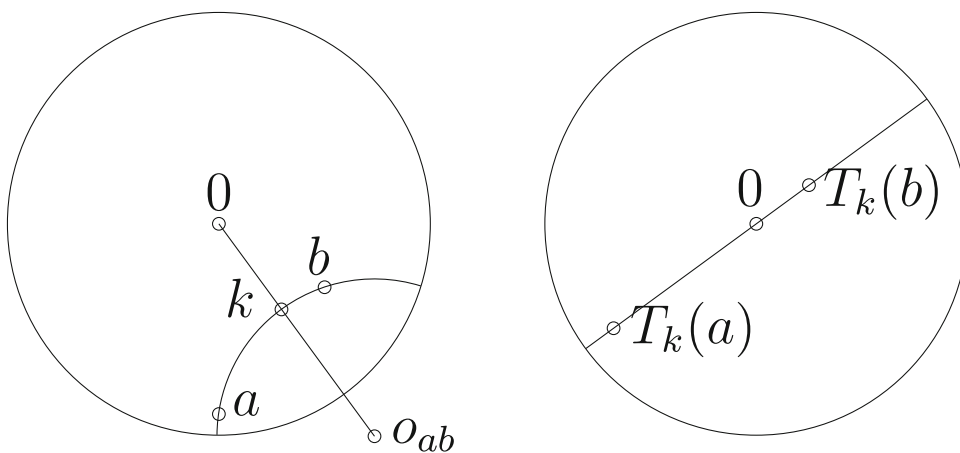
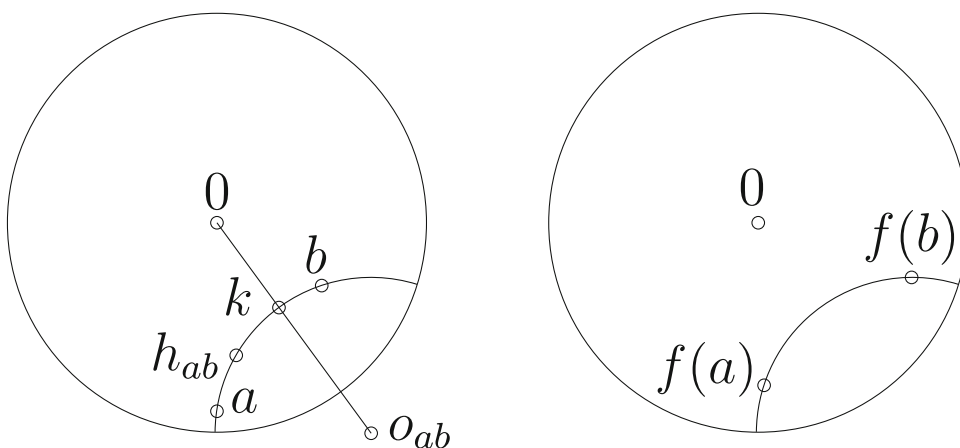


Fig. 3 The Möbius transformation $f = T_{h_{ab}, k}$ maps the hyperbolic midpoint h_{ab} of a and b to the nearest point k on the hyperbolic line $J^*[a, b]$ through $a = -0.9i$ and $b = 0.5 - 0.3i$ so that the images of a and b are at the same distance from the origin



Then the point w_2 is the orthocenter of the triangle with vertices at the points $0, w_1, w_3$.

(2) The point of intersection of the hyperbolic lines $J^*[a, c]$ and $J^*[b, d]$ is given by

$$w_4 = \frac{(ac - bd) \pm \sqrt{(a - b)(b - c)(c - d)(d - a)}}{a - b + c - d},$$

where the sign "+" or "-" in front of the square root is chosen such that $|w_4| < 1$.

Theorem 6.2 Let a, b, c, d be four distinct points on the unit circle in the positive order. Let w_4 be the point in Theorem 6.1(2) and w_5 be the midpoint of the orthogonal arc with center at $w_1 = \text{LIS}[a, b, c, d]$. There exists a Möbius transformation $m_4 : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ such that $m_4(a), m_4(b), m_4(c), m_4(d)$ are symmetric with respect to a line through 0 and

$$\text{Lip}(m_4) = \text{Lip}(T_{w_4, w_5}). \tag{6.3}$$

Moreover, if $m_5 = T_{w_4}$, then $m_5(a) = -m_5(c), m_5(b) = -m_5(d)$.

For a given point $w_1 \in \mathbb{R}^2 \setminus \mathbb{B}^2$, a circle $S^1(r, w_1), r > 0$, is orthogonal to the unit circle if and only if its radius r is $\sqrt{|w_1|^2 - 1}$. Consequently, the point w_5 in Theorem 6.2 has the formula

$$w_5 = \frac{w_1}{|w_1|} (|w_1| - r), \quad \sqrt{|w_1|^2 - 1}. \tag{6.4}$$

The intersection point $w_1 = \text{LIS}[a, b, c, d]$ can be found with Proposition 2.2.

If $a = e^{1i}, b = e^{3i}, c = e^{4i}$, and $d = e^{5i}$, we have

$$\text{Lip}(T_{w_4, w_5}) \approx 1.496, \quad \text{Lip}(T_{w_4}) \approx 1.833,$$

and, if $a = e^{6i}, b = e^{6.1i}, c = e^{6.13i}$, and $d = e^{6.15i}$, we have

$$\text{Lip}(T_{w_4, w_5}) \approx 449.5, \quad \text{Lip}(T_{w_4}) \approx 60.00.$$

Consequently, neither of the mappings in Theorem 6.2 is always better than the other in terms of its Lipschitz constant. Computational tests suggest that T_{w_4} produces a smaller Lipschitz constant if the points a, b, c, d are close to each other, while T_{w_4, w_5} is better when they are spread more evenly

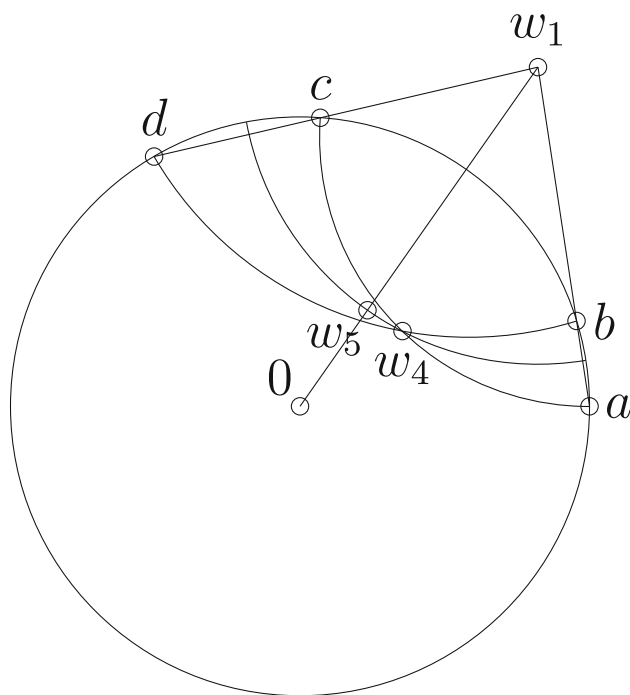


Fig. 4 The points w_4 and w_5 defining the Möbius transformation T_{w_4, w_5} of Theorem 6.2 when $a = 1$, $b = e^{0.3i}$, $c = e^{1.5i}$, and $d = e^{2.1i}$

around the whole unit circle. This observation could be also concluded from the geometric meaning of the points w_4 and w_5 .

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Declarations

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