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RESEARCH CONTRIBUTION

An Observation About Conformal Points on Surfaces

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Abstract
We study the existence of points on a compact oriented surface at which a symmetric bilinear two-tensor field is conformal to a Riemannian metric. We give applications to the existence of conformal points of surface diffeomorphisms and vector fields.

Keywords Conformal points · Poincaré–Hopf · Line fields

Mathematics Subject Classification Primary 53C18; Secondary 57R22

1 Statement of Results

1.1 Conformal Points

Let Σ be a compact, oriented surface, possibly with non-empty boundary $\partial\Sigma$. Denote by $C_1, \ldots, C_n$ the boundary components of Σ with the induced orientation. Let $\mathrm{Sym}((T^*\Sigma)^{\otimes 2}) \to \Sigma$ be the bundle of symmetric bilinear tensors on Σ. Fix a Riemannian metric g on Σ, that is, a positive-definite section of $\mathrm{Sym}((T^*\Sigma)^{\otimes 2}) \to \Sigma$.

Definition 1.1 We say that a section h of $\mathrm{Sym}((T^*\Sigma)^{\otimes 2}) \to \Sigma$ is *conformal to g at the point $z \in \Sigma$* if there exists $c \in \mathbb{R}$ such that $h_z = cg_z$.

Motivated by [1], the goal of this note is to study the set of points

$$\mathcal{C}(g, h) \subset \Sigma$$

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at which h is conformal to g (see Theorem 1.2) in order to investigate conformal points of diffeomorphisms $F\colon \Sigma \to \Sigma$, in which case $h = F^*g$ (see Theorem 1.7 and Corollary 1.8), and of vector fields (see Corollary 1.10).

Our main observation is that $\mathcal{C}(g, h)$ is the zero-set of a section H^a in a distinguished vector bundle $E^a \to \Sigma$ over the surface, which we describe now. Let $\mathrm{End}(T\Sigma) \to \Sigma$ be the bundle of endomorphisms of $T\Sigma$ and let

$$E^a \subset \mathrm{End}(T\Sigma) \tag{1.1}$$

be the subbundle of those endomorphisms which are symmetric with respect to g and have zero trace. For all $z \in \Sigma$, an element of $R \in E^a_z$ has the matrix expression

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad a, b \in \mathbb{R},$$

with respect to a positive, orthonormal basis of $T_z\Sigma$. Thus any non-zero element $R \in E^a_z$ is, up to a positive scalar multiple, a reflection $R\colon T_z\Sigma \to T_z\Sigma$ along a line in $T_z\Sigma$. In particular, the S^1-bundle associated with E^a is the bundle of unoriented lines in $T\Sigma$. This S^1-bundle is doubly covered by the bundle of oriented lines in $T\Sigma$ which, in turn, is the unit-tangent bundle of Σ, that is, the S^1-bundle associated with $T\Sigma \to \Sigma$. The above discussion shows that E^a is an oriented plane bundle over Σ with Euler number

$$e(E^a) = 2e(T\Sigma) = 2\chi(\Sigma). \tag{1.2}$$

Given a symmetric bilinear two-tensor field h over Σ, let H be the section of $\mathrm{End}(T\Sigma)$ representing h with respect to g, namely

$$g_z(u, H_z v) = h_z(u, v), \quad \forall z \in \Sigma,\ \forall u, v \in T_z\Sigma. \tag{1.3}$$

We denote by

$$H^a := H - \frac{\mathrm{tr} H}{2} I \tag{1.4}$$

the section of E^a corresponding to the trace-free part of H. Here I is the section of $\mathrm{End}(T\Sigma)$ such that I_z is the identity of $T_z\Sigma$ for all $z \in \Sigma$.

Thus, we conclude that

$$z \in \mathcal{C}(g, h) \quad \Longleftrightarrow \quad H^a_z = 0.$$

This relationship implies that given any closed set $C \subset \Sigma$ and any field R of unoriented lines on $\Sigma \backslash C$, there is a section h such that $\mathcal{C}(g, h) = C$ and R is induced by h on $\Sigma \backslash C$, see e.g. [9, Theorem A]. Moreover, we see that, generically, h has only finitely many, non-degenerate (as zeros of H^a) conformal points, and all of them lie in the interior of Σ. If h has only finitely many (degenerate or non-degenerate) critical points, and all of them lie in the interior of Σ, then we can use the Poincaré–Hopf Theorem for unoriented line fields on oriented surfaces with boundary to algebraically count conformal points. To give the precise statement, let us introduce some notation under the assumption that $\mathcal{C}(g,h)$ is finite and $\mathcal{C}(g,h) \subset \Sigma\backslash\partial\Sigma$. For each $z \in \mathcal{C}(g,h)$, we define

$$\operatorname{ind}_{(g,h)}(z) \in \mathbb{Z}$$

as the index of z seen as a zero of the section H^a of $E^a \to \Sigma$. Thus $\operatorname{ind}_{(g,h)}(z) \in \{-1,+1\}$ if z is non-degenerate. We count the elements in $\mathcal{C}(g,h)$ algebraically via the integer

$$[\mathcal{C}(g,h)] := \sum_{z\in\mathcal{C}(g,h)} \operatorname{ind}_{(g,h)}(z) \in \mathbb{Z}. \tag{1.5}$$

Moreover, for every boundary component C_i of Σ, with $i = 1,\dots,n$, we define

$$w_i(g,h) \in \mathbb{Z} \tag{1.6}$$

as the winding number of the section $H^a|_{C_i}$ with respect to $R^i \in E^a$, where $R^i(z)$ is the reflection along the line $T_z\partial\Sigma \subset T_z\Sigma$ for $z \in C_i$.

Theorem 1.2 *Let g be a Riemannian metric on a compact, oriented surface Σ. Then the following two statements hold.*

(1) *For any symmetric bilinear two-tensor field h over Σ such that $\mathcal{C}(g,h)$ is finite and $\mathcal{C}(g,h) \subset \Sigma\backslash\partial\Sigma$, the equality*

$$[\mathcal{C}(g,h)] = 2\chi(\Sigma) + \sum_{i=1}^{n} w_i(g,h) \tag{1.7}$$

holds, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

(2) *Let $\mathcal{C} \subset \Sigma\backslash\partial\Sigma$ be a finite set of points, $\iota\colon \mathcal{C} \to \mathbb{Z}$ an arbitrary function, and $w_1,\dots,w_n \in \mathbb{Z}$ arbitrary integers satisfying*

$$\sum_{z\in\mathcal{C}} \iota(z) = 2\chi(\Sigma) + \sum_{i=1}^{n} w_i. \tag{1.8}$$

Then there exists a symmetric bilinear two-tensor field h over Σ such that $\mathcal{C} = \mathcal{C}(g,h)$, $\iota(z) = \operatorname{ind}_{(g,h)}(z)$ for all $z \in \mathcal{C}$ and $w_i = w_i(g,h)$ for all $i = 1,\dots,n$.

Remark 1.3 For the convenience of the reader, we give a short proof of Theorem 1.2 in Sect. 2 although this can be deduced from the literature. For statement (1), we refer to [8, 14, 16] which deal with the Poincaré–Hopf Theorem for oriented line fields on surfaces with boundary and to [11, III.2.2], [3, 12, 13] which deal with the Poincaré–Hopf theorem for unoriented line fields on surfaces without boundary. The version of statement (1) has been generalized from quadratic forms to symmetric totally real n-forms in [6, Section 3]. For statement (2), we refer to the Extension Theorem in [10, p. 145].

Remark 1.4 Passing to the orientation double cover, Theorem 1.2 also holds for non-orientable surfaces.

Remark 1.5 In the literature, there are several formulas for the computation of the indices $\mathrm{ind}_{(g,h)}(z)$ and the winding numbers $w_i(g, h)$, for instance [19, Theorem 1], [4, Theorem 2.1].

We discuss now two situations where the set $\mathcal{C}(g, h)$ naturally appears.

1.2 Umbilical Points of Immersed Surfaces

First, let us consider a smooth immersion $\rho\colon \Sigma \to \mathbb{R}^3$. Here we take g^ρ and h^ρ to be the first and the second fundamental form of the immersion ρ, respectively, with respect to the ambient Euclidean metric. The elements of $\mathcal{C}(g^\rho, h^\rho)$ are the so-called umbilical points, namely points at which the two principal curvatures of the immersion coincide. In this case, (1.7) yields the well-known result that $[\mathcal{C}(g^\rho, h^\rho)] = 2\chi(\Sigma)$. In general, it is natural to ask which further conditions must the points $z \in \mathcal{C}(g^\rho, h^\rho)$ and their indices satisfy besides $[\mathcal{C}(g^\rho, h^\rho)] = 2\chi(\Sigma)$. For instance, Loewner's conjecture [18] asserts that $\mathrm{ind}(z) \le 2$ for all $z \in \mathcal{C}(g^\rho, h^\rho)$. This conjecture implies Carathéodory's conjecture [7, 17], which asserts that if ρ is a convex embedding (hence $\Sigma = S^2$ and $\chi(\Sigma) = 2$), then $\mathcal{C}(g^\rho, h^\rho)$ contains at least two points. Examples where $\mathcal{C}(g^\rho, h^\rho)$ consists exactly of two points, both having index two, are ellipsoids of revolution, where the two poles are umbilical.

Remark 1.6 In [5] the notion of k-roundings of immersed m-manifolds in $\mathbb{R}^n$ has been introduced so that umbilical points correspond to $k = 2$, $m = 2$ and $n = 3$. If k, m, n satisfy a certain relation, then k-roundings are generically isolated and can algebraically be counted via an index.

1.3 Conformal Points of a Diffeomorphism

The second situation in which $\mathcal{C}(g, h)$ naturally appears is when $h = F^*g$, where $F\colon \Sigma \to \Sigma$ is any orientation-preserving diffeomorphism of Σ. In this case, $\mathcal{C}(g, F^*g)$ is the set of so-called conformal points of F (with respect to g). Assuming that $\mathcal{C}(g, F^*g)$ is finite and $\mathcal{C}(g, F^*g) \subset \Sigma \setminus \partial\Sigma$, we are going to give a formula for $w_i(g, F^*g)$ in terms of the behavior of F at the boundary. To state the result, for $i = 1, \dots, n$ let $\nu_i : C_i \to T\Sigma$ be the outward normal at the boundary component C_i and $\tau_i : C_i \to T\Sigma$ be the unit vector tangent to C_i in the positive direction. The pair (ν_i, τ_i) then forms a positive orthonormal frame for g along C_i. We trivialize $T\Sigma|_{\partial\Sigma} = \sqcup_i C_i \times \mathbb{R}^2$ using (ν_i, τ_i) at C_i, $i = 1, \dots, n$. Since F maps boundary components to boundary components (not necessarily the same) we can express $\mathrm{d}F$

in this trivialization as

$$\mathrm{d}F\big|_{C_i} =: N_i = c_i \begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix}. \tag{1.9}$$

Here $a_i, c_i : C_i \to (0, \infty)$, $b_i : C_i \to \mathbb{R}$ and (a_i, b_i) is never equal to $(1, 0)$ since F has no conformal point on C_i by assumption.

Theorem 1.7 *For all $i = 1, \ldots, n$ we have the equality*

$$w_i(g, F^*g) = w(a_i - 1, b_i), \tag{1.10}$$

where $w(a_i-1, b_i)$ is the winding number of the curve $(a_i-1, b_i)\colon C_i \cong S^1 \to \mathbb{R}^2\backslash\{0\}$ around the origin.

This formula, which will be proved in Sect. 3, allows us to compute $w_i(g, F^*g)$ if we understand the behavior of F at points on the boundary sufficiently well. A remarkable example of this phenomenon is illustrated by the next corollary.

Corollary 1.8 *If $F\colon \Sigma \to \Sigma$ is the identity on the boundary and preserves an area form on Σ, then*

$$w_i(g, F^*g) = 0, \quad \forall\, i = 1, \ldots, n. \tag{1.11}$$

It follows that for this type of diffeomorphisms

$$[\mathcal{C}(F)] = 2\chi(\Sigma), \tag{1.12}$$

that is, the number of conformal points of such an F is twice the Euler characteristic.

Proof By (1.10) the assertion is equivalent to showing $w_i(a_i - 1, b_i) = 0$. Since F is the identity at the boundary we conclude that $\mathrm{d}F \cdot \tau_i = \tau_i$ and thus $c_i = 1$ in (1.9). Since F preserves an area form, it follows that $\det N_i = 1$, which implies that $a_i = 1$ in (1.9). Therefore, the curve $(a_i - 1, b_i) = (0, b_i)$ is contained in the y-axis and does not cross 0. We conclude that its winding number around the origin $w(a_i - 1, b_i)$ vanishes.

Remark 1.9 Equation (1.12) was proved in [1], when $\Sigma = D^2$, and F satisfies some additional conditions, which hold, for instance, when F is C^1-close to the identity,

If we linearize the property of being a conformal point for a diffeomorphism at the identity of Σ, we get a corresponding condition for conformal points of vector fields on Σ. This condition is easier phrased after reinterpreting conformality in terms of complex geometry, as we explain next.

1.4 Conformal Points and Complex Structures

Let $\jmath$ be the complex structure associated with the Riemannian metric g and the orientation of Σ. In other words, $\jmath$ yields a section of $\mathrm{End}(T\Sigma)$ such that v and $\jmath_z v$ form a positive, orthogonal basis of $T_z\Sigma$ for all $z \in \Sigma$ and all $v \in T_z\Sigma\setminus\{0\}$. Thus, $\jmath_z$ has the matrix expression

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.13}$$

with respect to a positive, orthonormal basis of $T_z\Sigma$. An endomorphism $H\colon T_z\Sigma \to T_z\Sigma$ commutes with $\jmath_z$ if and only if H has the matrix expression

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a, b \in \mathbb{R},$$

in such a basis. In particular, we deduce that H is, up to a scalar multiple, a rotation matrix. Analogously, H anticommutes with $\jmath_z$ if and only if H has the matrix expression

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad a, b \in \mathbb{R},$$

in such a basis. In particular, we deduce that E^a, see (1.1), is exactly the bundle of endomorphisms anticommuting with $\jmath$. Therefore, if we denote by $E^c \to \Sigma$ the bundle of endomorphisms commuting with $\jmath$, we get the splitting

$$\mathrm{End}(T\Sigma) = E^c \oplus E^a. \tag{1.14}$$

Furthermore, as $\jmath$-complex line bundle we can write

$$E^a \cong T \otimes \overline{T^*}, \quad T := T^{(1,0)}\Sigma,$$

where $T^{(1,0)}\Sigma$ is the holomorphic tangent bundle of Σ and $\overline{T^*}$ denotes the conjugate of the dual bundle of T. With this identification, a local section of E^a is given by $\frac{\partial}{\partial z} \otimes \mathrm{d}\bar{z}$ where z is a local holomorphic coordinate compatible with $\jmath$. Thus, the Euler number of E^a as real oriented plane bundle coincides with its Chern number as complex line bundle. Using that $c_1(T) = \chi(\Sigma)$, this gives another derivation of (1.2) by computing

$$c_1(E^a) = c_1(T \otimes \overline{T^*}) = c_1(T) - c_1(T^*) = c_1(T) + c_1(T) = 2c_1(T) = 2\chi(\Sigma).$$

Finally, let us assume that z is a conformal point of an orientation-preserving diffeomorphism $F\colon \Sigma \to \Sigma$. Then,

$$(F^*g)_z = cg_z \quad \text{for some } c > 0. \tag{1.15}$$

If we denote by M the matrix representation of $\mathrm{d}_z F$ with respect to positive, orthonormal bases of $T_z\Sigma$ and $T_{F(z)}\Sigma$, then (1.15) can be rewritten as

$$M^T M = cI.$$

This condition is equivalent to saying that M is, up to a scalar multiple, a rotation matrix. Since $\jmath_z$ and $\jmath_{F(z)}$ are represented by the matrix (1.13), we see that $\mathrm{d}_z F \jmath_z = \jmath_{F(z)} \mathrm{d}_z F$. We conclude that z is a conformal point of F if and only if F is $\jmath$-holomorphic at z with respect to $\jmath$-holomorphic coordinates around z and $F(z)$.

1.5 Conformal Points of Vector Fields

Let f be a vector field on Σ. Let $F_t : \Sigma \to \Sigma$ be the time-t map of the flow of f. Suppose that $z \in \Sigma$ is a point such that $F_t(z) \in \mathcal{C}(g, (F_t)^* g)$ for all t close to zero. In particular, F_t is $\jmath$-holomorphic at z in a local $\jmath$-holomorphic chart for all small t. Taking the derivative in t at $t = 0$, we conclude that the vector field f is $\jmath$-holomorphic at z. In other words $\bar{\partial}_\jmath f$ is a section of E^a which vanishes at z. Here $\bar{\partial}_\jmath$ denotes the Cauchy–Riemann operator sending sections of the holomorphic tangent bundle $T = T^{(1,0)}\Sigma$ to sections of $T \otimes \overline{T^*} \cong E^a$, and f is identified with its image under the isomorphism

$$T\Sigma \to T^{\mathbb{C}}\Sigma \cong T^{(1,0)}\Sigma \oplus T^{(0,1)}\Sigma \to T^{(1,0)}\Sigma, \tag{1.16}$$

where $T^{\mathbb{C}}\Sigma$ is the complexification of $T\Sigma$.

Let $\mathcal{C}(\jmath, f)$ be the set of zeros of $\bar{\partial}_\jmath f$. If $\mathcal{C}(\jmath, f)$ is finite and $\mathcal{C}(\jmath, f) \subset \Sigma \backslash \partial\Sigma$, then we can associate an index $\mathrm{ind}_{(\jmath, f)}(z)$ to each $z \in \mathcal{C}(\jmath, f)$ and a winding number $w_i(\jmath, f)$ representing the relative winding number of $\bar{\partial}_\jmath f$ with respect to the canonical section R_i along C_i for every $i = 1, \dots, n$. Defining the algebraic count

$$[\mathcal{C}(\jmath, f)] := \sum_{z \in \mathcal{C}(\jmath, f)} \mathrm{ind}_{(\jmath, f)}(z),$$

we get the following consequence of Theorem 1.2(1).

Corollary 1.10 *Let $\jmath$ be a complex structure on a compact surface Σ and f a vector field on Σ such that $\mathcal{C}(\jmath, f)$ is finite and $\mathcal{C}(\jmath, f) \subset \Sigma \backslash \partial\Sigma$. Then the equation*

$$[\mathcal{C}(\jmath, f)] = 2\chi(\Sigma) + \sum_{i=1}^{n} w_i(\jmath, f)$$

holds.

1.6 An Open Question

Given any Riemannian metric g on Σ and diffeomorphism $F\colon \Sigma \to \Sigma$, it is interesting to ask which further restrictions must the points z of $\mathcal{C}(g, F^*g)$, their indices $\mathrm{ind}_{(g,F^*g)}(z)$ and the numbers $w_i(g, F^*g)$ satisfy besides Eq. (1.7). This question is related to the uniformization theorem for compact surfaces with boundary via Theorem 1.2(2). For instance, given any two metrics g and h on $\Sigma = S^2$ or $\Sigma = D^2$, we can find a diffeomorphism $F\colon \Sigma \to \Sigma$ such that F^*g and h are conformal at every point, see [15, Theorem 1]. Thus, $\mathcal{C}(g, F^*g) = \mathcal{C}(g, h)$, $\mathrm{ind}_{(g,F^*g)}(z) = \mathrm{ind}_{(g,h)}(z)$ for every z in this set, and $w_i(g, F^*g) = w_i(g, h)$ for all $i = 1, \ldots, n$. As a consequence of Theorem 1.2(2), there are no further restrictions in this case.

As an instructive example, let us construct a diffeomorphism $F\colon S^2 \to S^2$ having exactly one conformal point without resorting to the uniformization theorem. We start by defining a suitable vector field f (a section of $T^{(1,0)}\Sigma$ under the identification (1.16)) and then we will define $F := F_t$, the time-t map of f for $t > 0$ small enough. The complex manifold S^2 is obtained gluing together two charts with holomorphic local coordinates $z, w \in \mathbb{C}$ having the relationship $zw = 1$. In the z-chart, we define f via $f_1(z) := z^3\rho(|z|^2)$ where $\rho\colon [0, \infty) \to \mathbb{C}$ is any smooth curve such that $\rho'(s) \neq 0$ for all $s \in [0, \infty)$ and such that $\rho(s) = s$ for small s and $\rho(s) = -1/s$ for large s. Thus $\bar\partial f_1(z) = z^3\rho'(|z|^2)z$ is different from zero if $z \neq 0$. On the other hand, for z near zero, $\bar\partial f_1(z) = z^4$ and thus $z = 0$ is a holomorphicity point of index 4. In the w-chart, f_1 gets transformed into

$$f_2(w) := -w^2 f_1(\tfrac{1}{w}) = -\tfrac{1}{w}\rho(\tfrac{1}{|w|^2}),$$

which for w small enough becomes $f_2(w) = \frac{|w|^2}{w} = \bar w$. Thus f_2 smoothly extends at $w = 0$ and $\bar\partial f_2(0) = 1 \neq 0$. Defining f as f_2 in the w-chart, we get the desired vector field.

We now claim that the time-t flow map F_t of f has exactly one conformal point at $z = 0$, provided that $t > 0$ is small enough. Indeed, if z is close to 0, then $f_1(z) = z^4\bar z$ is real analytic and therefore the Cauchy–Kovalevskaya Theorem implies that $F_t(z)$ is real analytic and the expansion $F_t(z) = z + tz^4\bar z + O(t^2|z|^6)$ holds. Therefore, $\bar\partial F_t(z) = tz^4 + O(t^2|z|^5)$, which vanishes only for $z = 0$ when $t > 0$ is small enough. On the other hand, in any compact neighborhood of 0 in the w-chart, we have $F_t(w) = w + tf_2(w) + O(t^2)$ and $\bar\partial F_t(w) = t\bar\partial f_2(w) + O(t^2)$ which is nowhere vanishing if $t > 0$ is small enough since $|\bar\partial f_2(w)|$ is bounded away from zero.

Coming back to the case of an arbitrary surface Σ, we see that in general there are metrics g and h such that h and F^*g are not conformal at all points, no matter how we choose the diffeomorphism F. The easiest examples where this happens is when $\Sigma = \mathbb{T}^2$, or when $\Sigma = D^2$ and we require in addition the diffeomorphism F to be the identity at the boundary. For instance, on $\mathbb{T}^2$ conformal classes of metrics g are classified by lattices Γ in $\mathbb{C}$, up to Euclidean isometries and homotheties, where g is the Riemannian metric on $\mathbb{T}^2 = \mathbb{C}/\Gamma$ induced by the Euclidean metric on $\mathbb{C}$. To get an example on the disc, let us identify D^2 with the unit Euclidean disc in $\mathbb{C}$. Let g be the Euclidean metric on D^2. Recall that the group of diffeomorphisms

$\varphi\colon D^2 \to D^2$ such that g and $\varphi^* g$ are conformal at all points consists of the Möbius transformations preserving D^2. Consider $G\colon D^2 \to D^2$ to be any diffeomorphism such that $G|_{\partial D^2} \neq \varphi|_{\partial D^2}$ for all φ. Such a G surely exists since if φ is not the identity, then φ can have at most two fixed points on the boundary. If we define $h := G^* g$, then there is no diffeomorphism $F\colon D^2 \to D^2$ which is identity at the boundary and such that $F^* h$ and g are conformal at every point. Indeed, if such an F exists, then $(G \circ F)^* g = F^* G^* g = F^* h$ is conformal to g at all points, which means that $F \circ G = \varphi$ for some Möbius transformation φ preserving the disc. Since F is the identity at the boundary, this would imply that $G = \varphi$ on the boundary. A contradiction.

Thus, in the case of $\mathbb{T}^2$ and of D^2, it is meaningful to ask if there is a metric g and a diffeomorphism F (being the identity on the boundary in the case of D^2) such that $\mathcal{C}(g, F^* g)$ is empty. If one can find a vector field f (vanishing on the boundary in the case of D^2) such that $\mathcal{C}(\jmath, f) = \varnothing$, then $\mathcal{C}(g, F_t^* g) = \varnothing$ for small $t \neq 0$, as well, where F_t is the time-t map of the flow of f.

In the case of $\Sigma = \mathbb{T}^2$, we can readily find such a vector field for all conformal classes of complex structures. Indeed, let $\mathbb{T}^2 = \mathbb{C}/\Gamma$ where Γ is a lattice in $\mathbb{C}$ and let $\jmath$ be the complex structure on Σ induced by that on $\mathbb{C}$. Up to Euclidean isometries and homotheties, we can assume that Γ is generated by $1, \tau \in \mathbb{C}$, where $\tau = a + ib$ with $b > 0$. Consider the vector field which in a global holomorphic trivialization of $T^{(1,0)}\Sigma$ is written as $f(z) = \mathrm{e}^{\frac{2\pi i}{b} \operatorname{Im} z}$. Notice that f is well-defined since it is invariant under translations by 1 and τ. Moreover,

$$\bar{\partial}_\jmath f(z) = \frac{\partial}{\partial \bar{z}} \mathrm{e}^{\frac{\pi}{b}(z - \bar{z})} = -\frac{\pi}{b} f(z),$$

which is nowhere vanishing.

However, we do not know if such a vector field f exists on D^2. Since vector fields on D^2 correspond to functions in a global trivialization of $T^{(1,0)} D^2$, we have the following open question.

Question 1.11 Does there exist a smooth function $f\colon D^2 \to \mathbb{C}$ satisfying the following two conditions?

(1) $\forall z \in D^2, \quad \frac{\partial f}{\partial \bar{z}}(z) \neq 0.$
(2) $\forall z \in \partial D^2, \quad f(z) = 0.$

1.7 Plan of the Paper

Theorem 1.2 is proven in Sect. 2. Theorem 1.7 is proven in Sect. 3.

2 Proof of Theorem 1.2

We prove Theorem 1.2(1). Let h be a symmetric bilinear two-tensor field over Σ such that $\mathcal{C}(g, h)$ is finite and $\mathcal{C}(g, h) \subset \Sigma \backslash \partial\Sigma$. Recall the definition of H and H^a from (1.3) and (1.4).

If Σ has no boundary, then $[\mathcal{C}(g,h)] = e(E^a) = 2\chi(\Sigma)$ by the Poincaré–Hopf Theorem for oriented plane bundles [2, Theorem 11.17]. If Σ has boundary, let $\hat{\Sigma}$ be the closed, oriented surface that we obtain from Σ by gluing a disc $D_1,\dots,D_n$ along each boundary component $C_1,\dots,C_n$. The gluing maps $D^2 \to D_i$ have the Euclidean disc

$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$$

as domain and send the boundary ∂D^2 traversed in the positive sense to $\bar{C}_i$, that is, to C_i traversed in the negative sense. In this way, the gluing maps are positively oriented with respect to the orientation on $\hat{\Sigma}$.

We let $\hat{g}$ be any extension of g to Σ as a Riemannian metric. On the bundle $E^a|_{D_i}$ we choose a nowhere vanishing section M^i defined as the reflection along the direction of $\partial_x \in TD^2$. Let $w_{\bar{C}_i}(H^a, M^i)$ be the winding number of H^a with respect to M^i along C_i traversed in the negative direction. Then

$$\begin{aligned} w_{\bar{C}_i}(H^a, M^i) &= w_{\bar{C}_i}(H^a, R^i) + w_{\bar{C}_i}(R^i, M^i) \\ &= -w_{C_i}(H^a, R^i) + w_{\partial D^2}(R^i, M^i) = -w_i(g,h) + 2, \end{aligned}$$

where we have used that $\bar{C}_i$ is identified with ∂D^2 and that the unoriented line tangent to ∂D^2 rotates twice with respect to the horizontal unoriented line. By the Extension Theorem in [10, p. 145], it is possible to construct an extension $\hat{h}$ of h to $\hat{\Sigma}$ such that $\mathcal{C}(\hat{g},\hat{h}) = \mathcal{C}(g,h) \cup \{z_1,\dots,z_n\}$, where $z_1,\dots,z_n$ are the centers of the discs $D_1,\dots,D_n$ and

$$\mathrm{ind}_{(\hat{g},\hat{h})}(z_i) = w_{\bar{C}_i}(H^a, M^i) = 2 - w_i(g,h). \tag{2.1}$$

Therefore,

$$\begin{aligned} [\mathcal{C}(g,h)] &= [\mathcal{C}(\hat{g},\hat{h})] - \sum_{i=1}^{n} \mathrm{ind}_{(\hat{g},\hat{h})}(z_i) \\ &= 2\chi(\hat{\Sigma}) - 2n + \sum_{i=1}^{n} w_i(g,h) = 2\chi(\Sigma) + \sum_{i=1}^{n} w_i(g,h), \end{aligned}$$

where we used that $\chi(\Sigma) + n = \chi(\hat{\Sigma})$ as follows from the formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. We have thus completed the proof of Theorem 1.2(1).

Let us first prove Theorem 1.2(2) when Σ has no boundary. Let us consider an embedded closed disc D containing $\mathcal{C}$ in its interior. There is a section H^{out} of E^a which is nowhere vanishing on $\Sigma \setminus \mathring{D}$ and there is a section H^{in} which is nowhere vanishing over D. The winding number of H^{out} with respect to H^{in} along ∂D is $w(H^{\mathrm{out}}, H^{\mathrm{in}}) = 2\chi(\Sigma)$. For each $z \in \mathcal{C}$ consider an embedded closed disc D^z centered at z and contained in $\mathring{D}$. After shrinking the discs D^z we may assume that they are pairwise disjoint. Let H^z be a section of $E^a|_{D^z}$ which has just one zero at z

with index $\mathrm{ind}(z) = \iota(z)$. Thus the winding number of H^z with respect to H^{in} along ∂D^z is $w(H^z, H^{\mathrm{in}}) = \iota(z)$. Since $2\chi(\Sigma) = \sum_{z\in\mathcal{C}} \iota(z)$ by assumption, we get

$$w(H^{\mathrm{out}}, H^{\mathrm{in}}) = \sum_{z\in\mathcal{C}} w(H^z, H^{\mathrm{in}}).$$

Consider the surface

$$\tilde{\Sigma} := D \setminus \bigsqcup_{z\in\mathcal{C}} \mathring{D}^z.$$

It satisfies $\partial\tilde{\Sigma} = \partial D \sqcup (\sqcup_{z\in\mathcal{C}} \overline{\partial D^z})$. Since $w(H^{\mathrm{out}}, H^{\mathrm{in}}) - \sum_{z\in\mathcal{C}} w(H^z, H^{\mathrm{in}}) = 0$, the Extension Theorem in [10, p. 145] implies that there is a nowhere vanishing section $\tilde{H}$ of $E^a|_{\tilde{\Sigma}}$ coinciding with H^{out} on ∂D and with H^z on every ∂D^z. Thus, H^{out}, $\tilde{H}$, and all H^z glue together to yield a section H of $E^a \to \Sigma$ having the desired properties.

When Σ has boundary, we construct the closed surface $\hat{\Sigma}$ as in the proof of Theorem 1.2(1). We define $\hat{\mathcal{C}} := \mathcal{C} \cup \{z_1, \ldots, z_n\}$ and $\hat{\imath}\colon \hat{\mathcal{C}} \to \mathbb{Z}$ as the extension of ι such that $\iota(z_i) = 2 - w_i$ for all $i = 1, \ldots, n$. Applying Theorem 1.2(2) for closed surfaces to $\hat{\Sigma}$ and $\hat{\imath}$ and using (2.1) yields Theorem 1.2(2) for the case of surfaces with boundary, as well.

3 Proof of Theorem 1.7

Let C_i be a component of $\partial\Sigma$ for some $i \in \{1, \ldots, n\}$. There is $j \in \{1, \ldots, n\}$ such that $F(C_i) = C_j$. Recall that $\mathrm{d}F|_{C_i}$ is expressed by the matrix

$$N_i = c_i \begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix}$$

with respect to the positive orthonormal bases ν_i, τ_i and ν_j, τ_j.

The metric $F^*g|_{C_i}$ is represented by the endomorphism $\mathrm{d}F^T \cdot \mathrm{d}F$ via (1.3). A computation shows that the matrix representing $\mathrm{d}F^T \cdot \mathrm{d}F$ with respect to the basis ν_i, τ_i is

$$N_i^T N_i = c_i^2 Q_i, \quad \text{with } Q_i = \begin{pmatrix} a_i^2 + b_i^2 & b_i \\ b_i & 1 \end{pmatrix}.$$

We point out that the condition that (a_i, b_i) is never equal to $(1, 0)$ is equivalent to Q_i having distinct eigenvalues, since Q_i is symmetric. Let $q_i\colon C_i \to \mathbb{R}P^1 \cong \mathbb{R}/\pi\mathbb{Z}$ be the eigendirection of Q_i with larger eigenvalue. By (1.6), $w_i(g, F^*g)$ is the degree of the map $q_i\colon C_i \to \mathbb{R}/\pi\mathbb{Z}$. Therefore, our goal is to show that the degree of q_i is equal to the winding number of $(a_i - 1, b_i)\colon C_i \to \mathbb{R}^2$ around the origin. To this purpose, let us parametrize C_i in the positive direction by $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$ and, to ease notation, let us drop all the subscripts i in what follows.

We may assume without loss of generality that the curve $(a-1, b)\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ intersects the positive real axis transversely. In this case $w(a-1, b)$ counts the number of points $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ such that $(a(\theta_0)-1, b(\theta_0))$ lies on the positive real axis, namely $a(\theta_0) > 1$ and $b(\theta_0) = 0$, with sign: the intersection is counted positively if $b'(\theta_0) > 0$ and negatively if $b'(\theta_0) < 0$.

On the other hand, the degree of q is computed using a regular value $\xi \in \mathbb{R}/\pi\mathbb{Z}$ of q. Being regular means that $q'(\theta_0) \neq 0$ for all $\theta_0 \in q^{-1}(\xi)$. In this case, the degree of q counts number of points $\theta_0 \in q^{-1}(\xi)$ with sign: the point θ_0 is counted positively if $q'(\theta_0) > 0$ and negatively if $q'(\theta_0) < 0$.

Choosing $\xi = 0$, we see that $\theta_0 \in q^{-1}(0)$ if and only if $(1, 0) \in \mathbb{R}^2$ is an eigenvector of Q with eigenvalue larger than 1. This happens exactly when $b(\theta_0) = 0$ and $a(\theta_0) > 1$, that is when $(a-1, b)$ intersects the positive real axis. Therefore, we prove that 0 is a regular value of q and that $w(a-1, b)$ is the degree of q if we can show that for every such θ_0 the numbers $b'(\theta_0)$ and $q'(\theta_0)$ have the same sign.

For this purpose, let $v(\theta) = (x(\theta), y(\theta)) \in \mathbb{R}^2$ be a generator of the line $q(\theta)$ such that $v(\theta_0) = (1, 0)$ and write $\lambda(\theta)$ for the corresponding eigenvalue of $Q(\theta)$, so that $\lambda(\theta_0) = a(\theta_0)$. Then $q'(\theta_0) = y'(\theta_0)$. To compute $y'(\theta_0)$ we differentiate the vector equation $\big(Q(\theta) - \lambda(\theta)I\big)v(\theta) = 0$ at θ_0:

$$\big(Q(\theta_0) - \lambda(\theta_0)I\big)v'(\theta_0) + \big(Q'(\theta_0) - \lambda'(\theta_0)I\big)v(\theta_0) = 0.$$

Therefore, substituting the values for $Q(\theta_0)$, $\lambda(\theta_0)$ and $Q'(\theta_0)$ and taking the y-component of the vector equation, we get

$$\big(1 - a(\theta_0)\big)y'(\theta_0) + b'(\theta_0) = 0.$$

Thus,

$$q'(\theta_0) = y'(\theta_0) = \frac{b'(\theta_0)}{a(\theta_0) - 1}$$

from which we see that $q'(\theta_0)$ and $b'(\theta_0)$ have the same sign since $a(\theta_0) > 1$. This completes the proof.

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References

1. Albers, P., Tabachnikov, S.: On conformal points of area preserving maps and related topics. J. Geom. Phys. 180, Paper No. 104644, 11p. (2022)
2. Bott, Raoul, Loring, WTu.: Differential forms in algebraic topology. Graduate Texts in Mathematics, vol. 82. Springer-Verlag, New York-Berlin (1982)
3. Crowley, Diarmuid, Grant, Mark: The Poincaré-Hopf theorem for line fields revisited. J. Geom. Phys. **117**, 187–196 (2017)
4. Fontenele, Francisco, Xavier, Frederico: The index of isolated umbilics on surfaces of non-positive curvature. Enseign. Math. **61**(1–2), 139–149 (2015)
5. Toshizumi Fukui and Juan J. Nuño Ballesteros. Isolated roundings and flattenings of submanifolds in Euclidean spaces. *Tohoku Math. J. (2)*, 57(4):469–503, 2005
6. Toshizumi Fukui and Juan J. Nuño Ballesteros. Isolated singularities of binary differential equations of degree n. *Publ. Mat.*, 56(1):65–89, 2012
7. Ghomi, Mohammad, Howard, Ralph: Normal curvatures of asymptotically constant graphs and Carathéodory's conjecture. Proc. Amer. Math. Soc. **140**(12), 4323–4335 (2012)
8. Daniel H. Gottlieb. A de Moivre like formula for fixed point theory. In Brown, R.F. (Ed.) *Fixed point theory and its applications (Berkeley, CA, 1986)*, volume 72 of *Contemp. Math.*, pages 99–105. Amer. Math. Soc., Providence, RI, 1988
9. Guíñez, Víctor.: Positive quadratic differential forms and foliations with singularities on surfaces. Trans. Amer. Math. Soc. **309**(2), 477–502 (1988)
10. Guillemin, Victor, Pollack, Alan: Differential topology. Prentice-Hall Inc, Englewood Cliffs, N.J. (1974)
11. Heinz Hopf. *Differential geometry in the large*, volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 1989. Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss
12. Jänich, Klaus: Line fields with branch defects. Manuscripta Math. **48**(1–3), 297–300 (1984)
13. Markus, Lawrence: Line element fields and Lorentz structures on differentiable manifolds. Ann. of Math. **2**(62), 411–417 (1955)
14. Morse, Marston: Singular Points of Vector Fields Under General Boundary Conditions. Amer. J. Math. **51**(2), 165–178 (1929)
15. Osgood, Brad, Phillips, Ralph, Sarnak, Peter: Extremals of determinants of Laplacians. J. Funct. Anal. **80**(1), 148–211 (1988)
16. Pugh, Charles C.: A generalized Poincaré index formula. Topology **7**, 217–226 (1968)
17. Sotomayor, Jorge, Garcia, Ronaldo: Lines of curvature on surfaces, historical comments and recent developments. São Paulo J. Math. Sci. **2**(1), 99–143 (2008)
18. Titus, Charles J.: A proof of a conjecture of Loewner and of the conjecture of Caratheodory on umbilic points. Acta Math. **131**, 43–77 (1973)
19. Xavier, Frederico: An index formula for Loewner vector fields. Math. Res. Lett. **14**(5), 865–873 (2007)

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