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ORIGINAL ARTICLE

Henkin–Ramirez kernels

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Abstract

I will survey Henkin–Ramirez reproducing kernels in the weakly pseudoconvex case. The literature is enormous and I will not try to be complete.

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Contents

1 Support surfaces

Let $\Omega \subset \mathbb{C}^n$. Let $\zeta \in \partial\Omega$, $z \in U(\overline{\Omega})$.

Suppose $\Phi(\zeta, z) : \partial\Omega \times U(\overline{\Omega}) \to \mathbb{C}$ is a smooth function which is holomorphic in z.

For Enrique Ramírez de Arellano on the 52 anniversary (a Mayan Calendar Round) of the Henkin-Ramirez kernel.

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We write $\Phi_\zeta(z) = \Phi(\zeta, z)$

Then $\Sigma_\zeta = \{z; \Phi_\zeta(z) = 0\}$ is called a support hyper surface if $\Sigma_\zeta \cap \overline{\Omega} = \{\zeta\}$.

This can be done locally for strongly pseudo-convex domains and gives locally defined Φ.

To get globally defined Φ, one can use the $\overline{\partial}$ techniques (which does not require supnorm estimates for $\overline{\partial}$) of Hörmander.

Henkin and Ramirez used these, satisfying optimal estimates as far as tangency:

$$\Re(\Phi_\zeta)(z) \geq c\|\zeta - z\|^2$$

for z in $\overline{\Omega}$, $c > 0$.

These support functions are used in integral formulas where $\frac{1}{\Phi}$ appears in the integrals. It is important to have as much control as possible on how close Φ is to 0 inside the domain.

2 A key difficulty

A key difficulty appears in the following simple example:

$$\Omega = \{r = u + v^2 + x^2 < 0\}$$

in $\mathbb{C}^2$ where $z = x + iy$ and $w = u + iv$. The natural Φ for $\zeta = 0$ is given from the complex tangent plane: $\Phi_0(z, w) = w$. This, however, is very bad because $\Phi = 0$ on the line $w = 0, x = 0$, i.e., the y-axis, which lies in the boundary. Hence, for points inside the domain, but close to the y axis, $1/\Phi$ is too large. In fact, the kernels will not be integrable.

This is overcome with a trick:

$$x^2 = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x^2 - y^2) = \frac{1}{2}|z|^2 + \frac{1}{2}\Re(z^2)$$

Hence, we can write the domain as

$$r = u + v^2 + \frac{1}{2}|z|^2 + \frac{1}{2}\Re(z^2) + \cdots$$

Since $u = \Re(w)$, we get

$$r = \Re\left(w + \frac{1}{2}z^2\right) + v^2 + \frac{1}{2}|z|^2 + \cdots$$

Now we change coordinates, $z' = z$, $w' = w + \frac{1}{2}z^2$

so we get $r = \Re(w') + (v')^2 + \frac{1}{2}|z'|^2 + \cdots$

Now we can set $\Phi_0(z', w') = w'$

Then on the boundary, we get $\Re(\Phi_0) \leq (v')^2 + \frac{1}{2}|z'|^2 \cdots$ which is optimal.

3 The Henkin–Ramirez reproducing kernel

The Henkin–Ramirez reproducing kernel was developed to handle strongly pseudoconvex domains.

We recall the definition of pseudoconvex domains $\Omega = \{r < 0\}$. The complex tangent space at $p \in \partial\Omega$ is given by $\{t = (t_1, \ldots, t_n); \sum \frac{\partial r}{\partial z_j} t_j = 0\}$.

The Levi form $L(p, t)$ is $\sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j$.

The strongly pseudoconvex domains are those for which $L(p, t) > 0$ for all complex tangent vectors $t \neq 0$.

For Ω strongly pseudoconvex, Henkin and Ramirez found a holomorphic reproducing kernel using Φ as a major building block.

This generalizes the classical Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

from one complex variable. The Φ in this case is $\zeta - z$.

4 The weakly pseudoconvex case

The next step is to handle more general pseudoconvex domains.

The weakly pseudoconvex domains are those for which $L(p, t) \geq 0$.

Among weakly pseudoconvex domains, one can have infinitely flat domains. Those are difficult to analyze. An intermediary case are those of finite type:

If $p \in \partial\Omega$, and γ is a complex curve through p, then one can define the order of tangency $T(p, \gamma)$ with $\partial\Omega$ of γ at p.

We say that the set Ω is of finite type if there is a uniform upper bound on $T(p, \gamma)$ as p varies over the boundary and γ varies over all curves.

This definition goes back to D'Angelo [6].

5 The Kohn–Nirenberg example

A serious difficulty in carrying the Henkin–Ramirez reproducing kernel over to the weakly pseudoconvex case was discovered by Kohn and Nirenberg [16].

A nonessential adjustment was given in:

Fornæss [12].

$$\Omega_t = \{(z, w) \in \mathbb{C}^2; \Re(w) + |zw|^2 + |z|^6 + t|z|^2\Re(z^4) < 0\}, 1 < t < 9/5.$$

These domains are pseudoconvex. But there exists no complex curve γ through the origin so that $\gamma \setminus \{0\}$ lies outside Ω_t.

In other words, there is no smooth family $\Phi(\zeta, z)$ of support surfaces as was used in the Henkin–Ramirez reproducing kernel in the strongly pseudoconvex case.

6 The convex case

Then the question is the convex case.

For the convex case, Φ can be constructed easily just using the complex tangent plane. However, to generalize the Henkin–Ramirez reproducing kernel, these are not good enough in general.

One needs control of tangency of the support surfaces. The problem is with $\frac{1}{\phi}$ in the kernel.

7 Diederich–Wiegerinck F kernel on ellipsoids

Diederich et al. [10].

Here we investigated the case of some simple convex domains of finite type.

$$\Omega = \left\{ z = x + iy \in \mathbb{C}^N; r = \sum x_j^{2n_j} + y_j^{2m_j} < 1 \right\}$$

The case when $n_j = m_j$ was done already a decade earlier by Range [17].

The main new difficulty then arises when some of the n_j and m_j are different.

The simplest support surface $\Phi = 0$ is given by the complex tangent plane. However, the complex tangent plane has order of tangency $2n_j$ in the x_j direction and $2m_j$ in the y_j direction. When these are different, we get the same difficulty as we had earlier for the example $u + v^2 + x^2 + \cdots < 0$ in $\mathbb{C}^2$. There the trick was to change coordinates so that the order of tangency of $\Phi = 0$ is 2 in both the x and y-direction.

The idea used generalizes:

Let us say we have $r = u + x^6 + \cdots < 0$.

We need to have the same order of tangency in the x and y direction. We can write $r = u + x^6 - \epsilon\Re(z^6) + \epsilon\Re(z^6) + \cdots$

or

$$\begin{aligned} r &= \Re(w + \epsilon z^6) + x^6 - \epsilon\Re(x + iy)^6 + \cdots \\ &= \Re(w + \epsilon z^6) + x^6 - \epsilon(x^6 - 4x^4y^2 + 4x^2y^4 - y^6) + \cdots \\ &\geq \Re w' + x^6(1 - \epsilon) + \epsilon y^6 - 4\epsilon x^2y^4 + \cdots \\ &\geq \Re w' + (1 - 82\epsilon)x^6 + \frac{8}{9}\epsilon y^6 + \cdots \\ &\geq \Re w' + b|z|^6 + \cdots \end{aligned}$$

So when $w' = 0$, we have $r \geq b|z|^6 + \cdots$ making the hypersurface $w' = 0$ tangent to order 6 in all directions.

One has the following explicit formula [10], $\zeta_k = \eta_k + i\xi_k$:

$$\Phi(\zeta, z) = \sum_{j=1}^{N} r_j(z_j - \zeta_j)$$
$$-\gamma \sum_{k=1}^{N} \left[\left(\eta_k^{2m_k-2} - \xi_k^{2n_k-2} \right) (z_k - \zeta_k)^2 + (z_k - \zeta_k)^{2m_k} \right]$$

8 Convex domains of finite type

The next step was to generalize the case of ellipsoids to general convex domains of finite type. The first $\overline{\partial}$ result was obtained by Anne Cumenge.

Cumenge [4].

Cumenge [5].

Later the result was also proved by a different method:

Diederich et al. [7].

The latter was based on the construction of support surfaces:

Diederich and Fornæss [8].

Let Ω be a bounded convex domain with $\mathcal{C}^\infty$ boundary of finite type in $\mathbb{C}^n$.

Let $\zeta \in \partial\Omega$. We make a coordinate change which moves ζ to the origin and which rotates coordinates so that the unit exterior normal points in the positive $\Re(w_1)$ direction. The coordinate change is denoted by $w = \ell_\zeta(z)$. It has the form $w = \ell_\zeta(z) = \Psi(\zeta)(z - \zeta)$ where the unitary matrix $\Psi(\zeta)$ depends smoothly on ζ at least locally.

Let ρ be a smooth convex defining function for Ω. Define $r_\zeta(w) := \rho(\ell_\zeta^{-1}(w))$ which then for each given ζ is a defining function for the image of Ω in the new coordinates. We then get explicit formulas for the support functions $\Phi_\zeta(w)$.

$$\Phi_\zeta(w) := 3w_1 + Kw_1^2 - c\sum_{j=2}^{m} M^{2^j} \sigma_j \sum_{|\alpha|=j, \alpha_1=0} \frac{1}{\alpha!} \frac{\partial^j r_\zeta}{\partial^\alpha w}(0) w^\alpha$$

The uniform constant $M > 0$ is large enough, $c > 0$ is small enough. We use the notation $\sigma_j = 1$ for j a multiple of 4, $\sigma_j = -1$ for $j + 2$ a multiple of 4 and 0 otherwise.

We then let $\Phi(\zeta, z) = \Phi_\zeta(\ell_\zeta(z))$.

We note that the formula contains terms of all orders up to m. We will discuss this issue: Say Ω is $u + x^6 + \cdots$, so 0 is of type 6. The type changes along the boundary. Near a point of type 6, the domain might be given as $u + \epsilon x^2 + x^6$ where $\epsilon \to 0$ when one approaches a point of type 6. One needs formulas for Φ which vary nicely. And the domain might also be of the form $u + \epsilon x^2 - \delta x^5 + x^6$ where both ϵ and $\delta \to 0$. So terms of order 5 need to be included. So this is the reason why the above formula for $\Phi_\zeta(w)$ is so complicated.

We also need estimates on how close the zero set of S is to the boundary.

Theorem 8.1 (Theorem 2.3 in [8]) *Let n_ζ be the unit normal vector to $\partial\Omega$ at $\zeta \in \partial\Omega$. Let v be a complex tangent vector to $\partial\Omega$ at ζ. Define*

$$a_{\alpha\beta}(\zeta, v) := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha \partial\overline{\lambda}^\beta}\rho(\zeta + \lambda v)|_{\lambda=0}.$$

For points z of the from $z = \zeta + \mu n_\zeta + \lambda v$ with $\mu, \lambda \in \mathbb{C}$, we have

$$\Re(\Phi(z,\zeta)) \le \frac{\Re(\mu)}{2} - \frac{K}{2}(\Im(\mu))^2 - c\hat{c}\sum_{j=2}^{m}\sum_{\alpha+\beta=j}|a_{\alpha\beta}(\zeta, v)|\lambda|^j$$

where $\hat{c}$ is a constant not depending on ζ or v.

A main tool in the proof is the following Lemma 3.1 in [8]:

Lemma 8.2 *Let $j \ge 2$ be an integer. Then there is an $\epsilon_0(j) > 0$, such that for any homogeneous convex polynomial*

$$P_j(z) = \sum_{k+\ell=j} a_{k,\ell} z^k \overline{z}^\ell$$

in one complex variable and any $0 < \epsilon < \epsilon_0(j)$ we have the following inequality:

$$P_j(z) + \sigma_j \epsilon \Re(a_{j,0} z^j) \ge \frac{\epsilon |z|^j}{2^{j+1}} \sum |a_{k,\ell}|, \ \forall z \in \mathbb{C}.$$

Proof of this Lemma is again based on the Bruna et al. [3].

This is about convex functions in one real variable: For positive integers m and $0 < T < \infty$, let $C(m, T)$ denote the set of real polynomials $P(t) = \sum_{j=0}^m a_j t^j$ on $\mathbb{R}$ such that

- $a_0 = a_1 = 0$
- P is convex ($P''(t) \ge 0$) for $0 \le t \le T$.

Lemma 8.3 (Bruna–Nagel–Wainger Lemma) *there is a constant C_m independent of T so that if $P \in C(m, T)$, $P(t) = \sum_{j=2}^m a_j t^j$, then*

$$P(t) \ge C_m \sum_{j=2}^{m} |a_j| t^j, 0 \le t \le T.$$

9 Lineally convex of finite type

A slightly more general condition than convex is lineally convex.

We define first lineally convex domains: Ω is a smooth bounded domain. We say that Ω is lineally convex if for every $p \in \partial\Omega$ the complex tangent plane only touches the closure of Ω at the point p.

There is an important characterization of lineally convex, the Behnke–Peschl condition: Behnke and Peschl [2].

Let r be any defining function of a domain in $\mathbb{C}^n$: Set $L(p,t)$ to be the Levi form for a point p in the boundary and a complex tangent vector t. Similarly, let

$$H(p,t) = \sum_{i,j} \frac{\partial r^2}{\partial z_i \partial z_j} t^i t^j$$

The Behnke–Peschl condition is that lineal convexity is equivalent to the condition $\Re H(p,t) + L(p,t) \geq 0$. (The proof of equivalence was completed by Kiselman [15].)

In addition, we assume that the domains are of finite type.

Then the conclusion in the convex case still holds:

There exist local smooth families of support surfaces given by a $\Phi(z,\zeta)$ which is holomorphic in z for each fixed ζ in the boundary of Ω. Moreover, the hypersurfaces $\Sigma_\zeta = \{z; \Phi(z,\zeta) = 0\}$ touch the boundary according to the type of the point as for the convex case.

This was proved in Diederich and Fornæss [9].

10 Reproducing kernels for the Kohn–Nirenberg domains in $\mathbb{C}^2$

Fornæss and Wiegerinck [13].

As observed earlier, the Kohn–Nirenberg domain Ω is not convex, in fact not even lineally convex. So the above reproducing kernels don't apply.

I state the result first, then explain the notation.

Theorem 10.1 *Let f be a continuous function on $\partial\Omega_R$, then*

$$C[f](w) := \sum_{j=1}^{3} \int_{F_j} f(\zeta) K_j(w,\zeta) \mathrm{d}\zeta$$

is a holomorphic function on Ω_R. Moreover, if $f \in A(\Omega_R) = \mathcal{C}(\overline{\Omega}_R) \cap H(\Omega_R)$, then $C[f] = f$.

Explanation of notation:

$$\Omega = \{w = (w_1, w_2) \in \mathbb{C}^2; \Re(w_2) + P(w_1) < 0\},\ \Sigma = \{w \in \partial\Omega; w_1 \neq 0\}.$$

Here P is a real valued homogeneous polynomial with $\Delta P > 0$ when $w_1 \neq 0$. Choose $R > 0$ and let $\Omega_R = \Omega \cap \{\|w\| < R\}$. We consider the following boundary parts of $\partial\Omega_R$.

- $F_1 = \overline{\partial\Omega \cap B(0,R)}$
- $F_2 = \overline{\partial B(0,R) \cap \Omega}$
- $F_3 = F_1 \cap F_2$

This kind of decomposition was used already by Range and Siu [19]

- $K_1(w,\zeta) = (\psi_1(w,\zeta)\overline{\partial}_\zeta \psi_2 - \psi_2(w,\zeta)\overline{\partial}_\zeta \psi_1) \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2$
- $K_2(w,\zeta) = (\psi_1^2(w,\zeta)\overline{\partial}_\zeta \psi_2^2 - \psi_2^2(w,\zeta)\overline{\partial}_\zeta \psi_1^2) \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2$
- $K_3(w,\zeta) = (\psi_1\psi_2^2 - \psi_1^2\psi_2)\mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2$

Here $\psi = (\psi_1, \psi_2)$ is a Leray map for Σ, i.e., $1 = (w_1 - \zeta_1)\psi_1 + (w_2 - \zeta_2)\psi_2$ and $\psi_i^2(w,\zeta)$ is a Leray map for $\partial B(0, R)$.
$\psi_i^2(w,\zeta) = -\frac{\tilde{\zeta}_i}{(w\cdot\zeta - R^2)}$ where $w \cdot \zeta = \sum_i w_i\tilde{\zeta}_i$, $w \in B(0, R)$, $\zeta \in \partial B(0, R)$
The $\tilde{x}$ coordinates:

- $\tilde{w}_1 = w_1$
- $\tilde{w}_2 = \tilde{w}_2(\zeta_1,) = w_2 + \alpha(\theta_1)w_1^{2k} - (\epsilon/M)G_1(w)$

For $\zeta \in \Sigma, \theta_1 := \arg(\zeta_1)$. Fix ζ_1, and approximate $P(w_1)$ near ζ_1. There exists a unique number $\alpha = \alpha(\theta_1)$ so that $P(w_1) = \Re(\alpha w_1^{2k}) + \mathcal{O}(|w_1 - \zeta_1|^2)$
Now define, for ζ_1 fixed:

$$G_1(w_1) = G\zeta_1, w_1) = w_1^{2k}(w_1 - \zeta_1)^2 e^{-i(2k+2)\theta_1}$$

This gives a local support surface valid in a sector around ζ_1. From this, one gets a local Leray map, and then to get the Leray map, one needs to do a patching.

And then one needs careful estimating to show that one actually gets a reproducing kernel.

11 Concluding remarks

Problem 11.1 *If Ω is an arbitrary finite type pseudoconvex domain in $\mathbb{C}^2$, does there exists a Henkin–Ramirez reproducing kernel for holomorphic functions?*

Problem 11.2 *Same question as above in higher dimension.*

Remark 11.3 It seems that maybe finding Henkin–Ramirez reproducing kernels for weakly pseudoconvex domains is more difficult than solving $\overline{\partial}u = f$ in sup-norm. This was my impression after first solving $\overline{\partial}u = f$ for the Kohn–Nirenberg example and then observing that the technique did not give a reproducing kernel (so this became a separate research project with Wiegerinck).

Also Range [18] solved $\overline{\partial}u = f$, for all finite type pseudoconvex domains in $\mathbb{C}^2$ and [14], Grundmeier–Simon–Stensønes did recently the same in $\mathbb{C}^3$, Sup-norm estimates for $\overline{\partial}u = f$ in $\mathbb{C}^3$, Pure and Applied Mathematics Quarterly (to appear).
Note that the result in $\mathbb{C}^2$ was obtained by different methods by Fefferman and Kohn [11].

A similar remark concerns peak functions for finite type pseudoconvex domains in $\mathbb{C}^2$. Those were found by Bedford and Fornæss [1] and was easier than solving $\overline{\partial}u = f$. However, in higher dimension, the peak function problem remains unsolved. Which one is most difficult? Peak functions or Henkin–Ramirez reproducing kernels on finite type domains.

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