Bol. Soc. Mat. Mex. (2021) 27:69
https://doi.org/10.1007/s40590-021-00383-7

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

The sharp bound of the third Hankel determinant for functions of bounded turning

Bogumiła Kowalczyk[1] · Adam Lecko[1]

Received: 20 June 2021 / Accepted: 23 August 2021 / Published online: 4 September 2021

Abstract

We find the sharp bound for the third Hankel determinant

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for analytic functions f with $a_n := f^{(n)}(0)/n!,\ n \in \mathbb{N},\ a_1 := 1$, such that

$$\operatorname{Re} f'(z) > 0, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Keywords Univalent function · Function of bounded turning · Hankel determinant · Carathéodory function · Coefficient

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the subclass of functions f normalized by $f(0) = 0, f'(0) = 1$, i.e., of the form

✉ Adam Lecko
alecko@matman.uwm.edu.pl

Bogumiła Kowalczyk
b.kowalczyk@matman.uwm.edu.pl

1 Department of Complex Analysis, Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn, ul. Słoneczna 54, 10-710 Olsztyn, Poland

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \ z \in \mathbb{D}, \tag{1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions. We denote by $\mathcal{P}'$ the subfamily of $\mathcal{A}$ of all functions f such that

$$\mathrm{Re} f'(z) > 0, \quad z \in \mathbb{D}. \tag{2}$$

Functions in $\mathcal{P}'$ are called functions of bounded turning (cf. [6, Vol. I, p. 101]), and in recent times many authors have denoted this class of functions by $\mathcal{R}$.

It is well-known [1] (cf. [6, Vol. I, p. 88]) that $\mathcal{P}' \subset \mathcal{S}$, and is a fundamental subfamily of univalent functions which has been extensively studied by many authors, e.g., [17, 18].

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of function $f \in \mathcal{A}$ of the form (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

General results for Hankel determinants of any degree with their applications can be found in [4, 20, 21, 23]. For a subfamily $\mathcal{F}$ of $\mathcal{A}$, q and n, computing the upper bound of $H_{q,n}(f)$ represents an interesting and important problem. Hayman [7] examined the second Hankel determinant of mean for univalent functions. Recently, many authors examined the second Hankel determinant $H_{2,2}(f) = a_2 a_4 - a_3^2$ (see e.g., [5, 15] with further references). The problem of finding sharp estimating of the third Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_1 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2) \tag{3}$$

is technically much more difficult, and few sharp bounds have been obtained. However, sharp bounds of $|H_{3,1}(f)|$ have been obtained for convex functions [12], starlike functions of order 1/2 [14], and functions $f \in \mathcal{A}$ which satisfy the condition $\mathrm{Re} f(z)/z > \alpha$, $z \in \mathbb{D}$, in the case $\alpha = 0$ and $\alpha = 1/2$ [11], and functions $f \in \mathcal{A}$ such that $|(z/f(z))^2 - 1| < 1$ for $z \in \mathbb{D}$ [19].

When $f \in \mathcal{P}'$ Janteng et al. [9] showed that $|H_{3,1}(f)| \le 439/540 = 0.81296\ldots$ which is not sharp, and originally proved in [2]. The proof used (3) and sharp estimates of expressions contained in (3), namely $|a_2 a_4 - a_3^2| \le 4/9$ found in [8], $|a_2 a_3 - a_4| \le 1/2$ in [2] and reproved in [9], $|a_3 - a_2^2| \le 2/3$ cited from [2] (see also [10], where this result was proved earlier) and $|a_n| \le 2/n$ found in [18]. We also note that proofs in [2] contained some gaps, and new proofs were obtained in [9]. The aforementioned estimate of $|H_{3,1}(f)|$ in $\mathcal{P}'$ was improved in [24], where the author shown that $|H_{3,1}(f)| \le 41/60 = 0.683\ldots$, which is also not sharp, and discussed $H_{3,1}(f)$ for subclasses $\mathcal{P}'^{(2)}$ and $\mathcal{P}'^{(3)}$ of $\mathcal{P}'$ consisting of 2-fold and 3-fold symmetric functions, respectively, and obtained the following sharp bounds: $|H_{3,1}(f)| \le 2\sqrt{6}/45 = 0.108\ldots$ and $|H_{3,1}(f)| \le 1/4$ for f in $\mathcal{P}'^{(2)}$ and $\mathcal{P}'^{(3)}$, respectively.

In this paper, we show that $|H_{3,1}(f)| \le 1/4$ for $f \in \mathcal{P}'$ and that the inequality is sharp.

Since the class $\mathcal{P}'$ can be represented using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{4}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{P}'$ can be expressed as coefficients of functions in $\mathcal{P}$. We then obtain the upper bound of $|H_{3,1}(f)|$, basing our analysis on the well-known formulas of coefficient c_2 (e.g., [22, p. 166]), the formula c_3 due to Libera and Zlotkiewicz [16, 17], and the formula for c_4 recently found in [13].

2 Main result

The basis for proof of the main result is the following lemma. It contains the well known formula for c_2 (e.g., [22, p. 166]). the formula for c_3 due to Libera and Zlotkiewicz [16, 17] and the formula for c_4 found in [13].

Lemma 1 *If $p \in \mathcal{P}$ is of the form (4) with $c_1 \ge 0$, then*

$$c_1 = 2\zeta_1, \tag{5}$$

$$c_2 = 2\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2, \tag{6}$$

$$c_3 = 2\zeta_1^3 + 4(1-\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2 + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3 \tag{7}$$

and

$$\begin{aligned} c_4 = {} & 2\zeta_1^4 + 2(1-\zeta_1^2)\zeta_2\left(\zeta_1^2\zeta_2^2 - 3\zeta_1^2\zeta_2 + 3\zeta_1^2 + \zeta_2\right) \\ & + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\left(2\zeta_1 - 2\zeta_1\zeta_2 - \overline{\zeta_2}\zeta_3\right) \\ & + 2(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned} \tag{8}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$.

We now prove the main theorem of this paper.

Theorem 1

$$\max\{|H_{3,1}(f)| : f \in \mathcal{P}'\} = \frac{1}{4} \tag{9}$$

with extreme function $f_0 \in \mathcal{P}'$ given by

$$f_0'(z) := \frac{1 - z^3}{1 + z^3}, \quad z \in \mathbb{D}. \tag{10}$$

Proof Let $f \in \mathcal{P}'$ and be given by (1). Then by (2),

$$f'(z) = p(z), \quad z \in \mathbb{D}, \tag{11}$$

for some function $p \in \mathcal{P}$ of the form (4). Since both the classes $\mathcal{P}'$ and $\mathcal{P}$ and the functional $H_{3,1}(f)$ are invariant under the rotations, we may assume that $c_1 \in [0, 2]$ ([3], see also [6, Vol. I, p. 80, Theorem 3]), i.e., in view of (5) that $\zeta_1 \in [0, 1]$.

Substituting (1) and (4) into (11) and equating coefficients, we obtain

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}c_2, \quad a_4 = \frac{1}{4}c_3, \quad a_5 = \frac{1}{5}c_4.$$

Hence, by (3), we have

$$H_{3,1}(f) = \frac{1}{2160}\left(180c_1c_2c_3 - 80c_2^3 - 135c_3^2 + 144c_2c_4 - 108c_1^2c_4\right). \tag{12}$$

Using (5)–(8) by straightforward algebraic computation we obtain

$$\begin{aligned}
180c_1c_2c_3 =& 1440\big[\zeta_1^6 + (1-\zeta_1^2)(3-\zeta_2)\zeta_1^4\zeta_2 \\
&+ (1-\zeta_1^2)^2(2-\zeta_2)\zeta_1^2\zeta_2^2 + (1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1^3\zeta_3 \\
&+ (1-\zeta_1^2)^2(1-|\zeta_2|^2)\zeta_1\zeta_2\zeta_3\big], \\
-80c_2^3 =& -640\left[\zeta_1^6 + 3(1-\zeta_1^2)\zeta_1^4\zeta_2 + 3(1-\zeta_1^2)^2\zeta_1^2\zeta_2^2 + (1-\zeta_1^2)^3\zeta_2^3\right], \\
-135c_3^2 =& -540\Big[\zeta_1^6 + 2(1-\zeta_1^2)(2-\zeta_2)\zeta_1^4\zeta_2 + (1-\zeta_1^2)^2(2-\zeta_2)^2\zeta_1^2\zeta_2^2 \\
&+ 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1^3\zeta_3 + 2(1-\zeta_1^2)^2(1-|\zeta_2|^2)(2-\zeta_2)\zeta_1\zeta_2\zeta_3 \\
&+ (1-\zeta_1^2)^2(1-|\zeta_2|^2)^2\zeta_3^2\Big]
\end{aligned}$$

and

$$\begin{aligned}
&144c_2c_4 - 108c_1^2c_4\\
&\quad = 288\big[-\zeta_1^6 + (1-\zeta_1^2)(-\zeta_1^2\zeta_2 - \zeta_1^2\zeta_2^3 + 3\zeta_1^2\zeta_2^2 - \zeta_2^2)\zeta_1^2\\
&\qquad + 2(1-\zeta_1^2)^2(3\zeta_1^2\zeta_2^2 + \zeta_1^2\zeta_2^4 - 3\zeta_1^2\zeta_2^3 + \zeta_2^3)\\
&\qquad - (1-\zeta_1^2)(1-|\zeta_2|^2)\big(2\zeta_1\zeta_3 - 2\zeta_1\zeta_2\zeta_3 - \overline{\zeta_2}\zeta_3^2\big)\zeta_1^2\\
&\qquad + 2(1-\zeta_1^2)^2(1-|\zeta_2|^2)(2\zeta_1\zeta_2\zeta_3 - 2\zeta_1\zeta_2^2\zeta_3 - |\zeta_2|^2\zeta_3^2)\\
&\qquad - (1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_1^2\zeta_4\\
&\qquad + 2(1-\zeta_1^2)^2(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_2\zeta_4\big].
\end{aligned}$$

Substituting the above expression into (12), we obtain

$$H_{3,1}(f) = \frac{1}{540}\left[\gamma_1(\zeta_1,\zeta_2) + \gamma_2(\zeta_1,\zeta_2)\zeta_3 + \gamma_3(\zeta_1,\zeta_2)\zeta_3^2 + \gamma_4(\zeta_1,\zeta_2,\zeta_3)\zeta_4\right], \tag{13}$$

where for $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$,

$$\begin{aligned}
\gamma_1(\zeta_1,\zeta_2) :=& -7\zeta_1^6 - 12(1-\zeta_1^2)\zeta_1^4\zeta_2 + 6(1-\zeta_1^2)(10-\zeta_1^2)\zeta_1^2\zeta_2^2\\
&+ 4(1-\zeta_1^2)(5\zeta_1^4 - 19\zeta_1^2 - 4)\zeta_2^3 + 9(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4,\\
\gamma_2(\zeta_1,\zeta_2) :=& -18(1-\zeta_1^2)(1-|\zeta_2|^2)\big[3\zeta_1^3 - (2\zeta_1^2+6)\zeta_1\zeta_2 + (1-\zeta_1^2)\zeta_1\zeta_2^2\big],\\
\gamma_3(\zeta_1,\zeta_2) :=& \, 9(1-\zeta_1^2)(1-|\zeta_2|^2)\Big[8\zeta_1^2\overline{\zeta_2} - (1-\zeta_1^2)(|\zeta_2|^2+15)\Big]
\end{aligned}$$

and

$$\gamma_4(\zeta_1,\zeta_2,\zeta_3) := 72(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\big[-\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2\big].$$

Since $|\zeta_4| \le 1$, from (13) we obtain

$$\begin{aligned}
|H_{3,1}(f)| \le& \frac{1}{540}\big[|\gamma_1(\zeta_1,\zeta_2)| + |\gamma_2(\zeta_1,\zeta_2)||\zeta_3|\\
&+ |\gamma_3(\zeta_1,\zeta_2)||\zeta_3|^2 + |\gamma_4(\zeta_1,\zeta_2,\zeta_3)|\big]\\
\le& \frac{1}{540}\max\{\Phi(x,y,u) : (x,y,u) \in D := [0,1]^3\},
\end{aligned}$$

where

$$\Phi(x,y,u) := \varphi_1(x,y) + \varphi_4(x,y) + \varphi_2(x,y)u + [\varphi_3(x,y) - \varphi_4(x,y)]u^2,$$

with

$$\begin{aligned}\varphi_1(x,y) :=& 7x^6 + 12(1-x^2)x^4y + 6(1-x^2)(10-x^2)x^2y^2\\ &+ 4(1-x^2)(5x^2+1)(4-x^2)y^3 + 9(1-x^2)^2x^2y^4,\\ \varphi_2(x,y) :=& 18x\left[3x^2 + 2(3+x^2)y + (1-x^2)y^2\right](1-x^2)(1-y^2),\\ \varphi_3(x,y) :=& 9\left[8x^2y + (1-x^2)(y^2+15)\right](1-x^2)(1-y^2),\end{aligned}$$

and

$$\varphi_4(x,y) := 72[x^2 + 2(1-x^2)y](1-x^2)(1-y^2).$$

We will now show that

$$\max\{\Phi(x,y,u) : (x,y,u) \in D\} = 135.$$

A. On the vertices of D, we have

$$\begin{aligned}&\Phi(0,0,0) = 0, \quad \Phi(0,0,1) = 135, \quad \Phi(0,1,0) = \Phi(0,1,1) = 16,\\ &\Phi(1,0,0) = \Phi(1,0,1) = \Phi(1,1,0) = \Phi(1,1,1) = 7.\end{aligned}$$

B. We next consider the edges of D.

1. $y = 0,\ u = 0$. Then

$$\begin{aligned}\Phi(x,0,0) &= 7x^6 - 72x^4 + 72x^2 \le \Phi(x_1,0,0)\\ &= \frac{96}{49}\left(17\sqrt{102} - 162\right) \approx 18.988, \quad x \in (0,1),\end{aligned}$$

where

$$x_1 := \sqrt{\frac{2}{7}(12 - \sqrt{102})} \approx 0.737.$$

2. $y = 1,\ u = 0$. Then,

$$\begin{aligned}\Phi(x,1,0) &= 30x^6 - 168x^4 + 129x^2 + 16 \le \Phi(x_2,1,0)\\ &= \frac{1}{225}\left(923\sqrt{1846} - 30028\right) \approx 42.795, \quad x \in (0,1),\end{aligned}$$

where

$$x_2 := \sqrt{\frac{56 - \sqrt{1846}}{30}} \approx 0.659.$$

3. $x = 0,\ u = 0$. Then,

$$\Phi(0,y,0) = -128y^3 + 144y \le \Phi\left(0, \frac{\sqrt{6}}{4}, 0\right) = 24\sqrt{6} \approx 58.788, \quad y \in (0,1).$$

4. $x = 1,\ u = 0$. Then,

$$\Phi(1, y, 0) = 7, \quad y \in [0, 1].$$

5. $y = 0,\ u = 1$. Then,

$$\Phi(x, 0, 1) = 7x^6 - 54x^5 + 135x^4 + 54x^3 - 270x^2 + 135 \le 135, \quad x \in (0, 1),$$

which is equivalent to

$$7x^4 - 54x^3 + 135x^2 + 54x - 270 \le 0, \quad x \in [0, 1],$$

and is easily seen to be true.

6. $y = 1,\ u = 1$. Since $\Phi(x, 1, 1) = \Phi(x, 1, 0)$ for $x \in (0, 1)$, which reduces to case 2.

7. $x = 0,\ u = 1$. Then,

$$\Phi(0, y, 1) = -9y^4 + 16y^3 - 126y^2 + 135 \le 135, \quad y \in (0, 1),$$

which is equivalent to

$$y^2(-9y^2 + 16y - 126) \le 0, \quad y \in (0, 1),$$

and again is evidently true.

8. $x = 1,\ u = 1$. Then,

$$\Phi(1, y, 1) = 7, \quad y \in (0, 1).$$

9. $x = 0,\ y = 0$. Then,

$$\Phi(0, 0, u) = 135u^2 \le 135, \quad u \in (0, 1).$$

10. $x = 0,\ y = 1$. Then,

$$\Phi(0, 1, u) = 16, \quad u \in (0, 1).$$

11. $x = 1,\ y = 0$. Then,

$$\Phi(1, 0, u) = 7, \quad u \in (0, 1).$$

12. $x = 1,\ y = 1$. Then,

$$\Phi(1, 1, u) = 7, \quad u \in (0, 1).$$

C. We consider now the faces of D.

1. $x = 1$. Then,

$$\Phi(1, y, u) = 7, \quad y, u \in (0, 1).$$

2. $x = 0$. Then,

$$\Phi(0, y, u) = 114y - 128y^3 + 9(1 - y^2)(1 - y)(15 - y)u^2 =: G_1(y, u), \quad y, u \in (0, 1).$$

Clearly G_1 has no critical point in $(0, 1) \times (0, 1)$.

3. $y = 1$. Since $\Phi(x,1,u) = \Phi(x,1,0)$ for $x, u \in (0,1)$, this case reduces to the case B.2.

4. $y = 0$. Then,

$$\Phi(x,0,u) = 7x^6 - 72x^4 + 72x^2 + 54x^3(1-x^2)u + 9(1-x^2)(15-23x^2)u^2$$
$$=: G_2(x,u), \quad x, u \in (0,1).$$

We have

$$\frac{\partial G_2}{\partial u} = 54x^3(1-x^2) + 18(1-x^2)(15-23x^2)u, \quad x, u \in (0,1).$$

Clearly $\partial G_2/\partial u \neq 0$ for $x = \sqrt{15/23}$. For $x \neq \sqrt{15/23}$ we see that $\partial G_2/\partial u = 0$ if, and only if,

$$u = \frac{3x^3}{23x^2 - 15} =: u(x).$$

Note that $u(x) \in (0,1)$ if, and only if, $23x^2 - 15 > 0$ and $3x^3 - 23x^2 + 15 < 0$, i.e., if, and only if, $x \in (x_3, 1)$, where $x_3 \approx 0.857$. Since

$$\frac{\partial G_2}{\partial x}(x, u(x)) = 0$$

if, and only if,

$$1840x^8 - 27360x^6 + 46176x^4 - 27360x^2 + 5400 = 0$$

and the last equation has no root in $(x_3, 1)$ (all real roots are the following: $x \approx \pm 0.68045$ and $x \approx \pm 3.60968$)), we conclude that G_2 has no critical point.

5. $u = 0$. We have

$$\begin{aligned}\Phi(x,y,0) = &\, 7x^6 - 72x^4 + 72x^2 + (-12x^6 + 156x^4 - 288x^2 + 144)y \\ &+ (6x^6 + 6x^4 - 12x^2)y^2 + (20x^6 - 240x^4 + 348x^2 - 128)y^3 \\ &+ (9x^6 - 18x^4 - 9x^2)y^4 =: G_3(x,y), \quad x, y \in (0,1).\end{aligned}$$

Now set $t := x^2 \in (0,1)$. Then $\partial G_3/\partial y = 0$ if, and only if,

$$(y+1)^2(1-3y)t^2 + (-12 - 2y + 55y^2 + 3y^3)t + 12 - 32y^2 = 0, \tag{14}$$

and $\partial G_3/\partial t = 0$ if, and only if,

$$\begin{aligned}&(7 - 12y + 6y^2 + 20y^3 + 9y^4)t^2 + (-48 + 104y - 4y^2 - 160y^3 - 12y^4)t \\ &+ 24 - 96y - 4y^2 + 116y^3 + 3y^4 = 0.\end{aligned} \tag{15}$$

(a) Let $y = 1/3$. Then, the Eq. (14) reduces to $t = 76/58 > 1$.

(b) Suppose that $y \neq 1/3$. Then by (14),

$$\Delta := 9y^6 - 54y^5 + 2373y^4 - 276y^3 - 948y^2 + 96y + 96 > 0$$

for $y \in (0,1) \setminus \{1/3\}$ (all real roots of $\Delta = 0$ are: $y \approx -0.5108$ and $y \approx -0.3278$).
Let

$$t_{1,2} := \frac{12 + 2y - 55y^2 - 3y^3 \mp \sqrt{\Delta}}{2(y+1)^2(1-3y)}.$$

(i) Observe that $t_1 > 1$. Indeed, for $0<y<1/3$ and $1/3<y<1$ this inequality is equivalent to the obviously true inequalities

$$(y+1)^2(1-3y)(72y^2 - 12y + 4) > 0$$

and

$$(y+1)^2(1-3y)(72y^2 - 12y + 4) < 0,$$

respectively.

(ii) Since $12 + 2y - 55y^2 - 3y^3 > 0$ for $0<y<1/3$, then $t_2 > 0$. Note that the inequality $t_2 > 0$ is false for $y \in (1/3, y_1)$, where $y_1 \approx 0.479$ is a unique positive solution of $12 + 2y - 55y^2 - 3y^3 = 0$. For $y \in (y_1, 1)$ the inequality $t_2 > 0$ is equivalent to

$$(y+1)^2(3y-1)(8y^2 - 3) > 0,$$

which is true for $y \in (\sqrt{6}/4, 1)$.

Further, the inequality $t_2 < 1$ which is equivalent to $\sqrt{\Delta} < -10 - 4y + 45y^2 - 3y^3$ is evidently false for $y \in (0, 1/3)$, and is true when $y \in (1/3, y_2)$, where $y_2 \approx 0.528$ is a unique root in (0, 1) of the equation $-10 - 4y + 45y^2 - 3y^3 = 0$. For $y \in (y_2, 1)$ the inequality $t_2 < 1$ is equivalent to

$$(y+1)^2(3y-1)(18y^2 - 3y + 1) > 0,$$

which is also true.

Summarising $t_2 \in (0,1)$ if, and only if, $y \in (\sqrt{6}/4, 1)$, and substituting t_2 into (15) we obtain

$$\begin{aligned}&(7 - 12y + 6y^2 + 20y^3 + 9y^4)t_2^2 + (-48 + 104y + 4y^2 - 160y^3 - 12y^4)t_2 \\ &\quad + 24 - 96y - 4y^2 + 116y^3 + 3y^4 = 0,\end{aligned} \tag{16}$$

which is equivalent to

$$
\begin{aligned}
L(y) &:= 54y^{10} + 180y^9 + 4194y^8 + 30672y^7 + 27630y^6 - 46356y^5 \\
&\quad + 13754y^4 + 4360y^3 - 7000y^2 + 1008y + 624 \\
&= (-18y^7 + 30y^6 + 600y^5 + 660y^4 - 702y^3 + 394y^2 - 44y - 72)\sqrt{\Delta} =: R(y).
\end{aligned} \tag{17}
$$

Note that $L(y) > 0$ for $y \in (y_3, 1)$ and $R(y) > 0$ for $y \in (y_4, 1)$, where $y_3 \approx 0.5782$ and $y_4 \approx 0.5555$ are roots of L and $R/\sqrt{\Delta}$, respectively. Therefore both sides of (17) are positive for $y \in (\sqrt{6}/4, 1)$. Thus (17) is equivalent to

$$
\begin{aligned}
L^2(y) - R^2(y) &= (y+1)^4(3y-1)^2(5184y^{13} - 34560y^{12} + 791424y^{11} \\
&\quad + 6098976y^{10} + 3934080y^9 - 14940000y^8 - 1216512y^7 + 6150528y^6 \\
&\quad - 4316544y^5 + 2653696y^4 - 1009152y^3 + 743424y^2 - 64512y - 108288) = 0.
\end{aligned} \tag{18}
$$

A numerical computation shows that all real solution of the above equation are as follow:

$$
\begin{aligned}
&y = -1, \quad y = \frac{1}{3}, \quad y \approx -4.2848, \quad y \approx -2.3680, \quad y \approx -1.0721, \\
&y \approx -0.2646, \quad y \approx 0.5478, \quad y \approx 0.5613, \quad y \approx 1.0977.
\end{aligned}
$$

Thus, the Eq. (18), so (16) has no solution in $(\sqrt{6}/4, 1)$, and so , G_3 has no critical point.

6. $u = 1$. We have

$$
\begin{aligned}
\Phi(x, y, 1) &= 7x^6 - 54x^5 + 135x^4 + 54x^3 - 270x^2 + 135 \\
&\quad + (-12x^6 - 36x^5 - 60x^4 - 72x^3 + 72x^2 + 108x)y \\
&\quad + (6x^6 + 72x^5 - 192x^4 - 90x^3 + 312x^2 + 18x - 126)y^2 \\
&\quad + (20x^6 + 36x^5 - 24x^4 + 72x^3 - 12x^2 - 108x + 16)y^3 \\
&\quad + (9x^6 - 18x^5 - 27x^4 + 36x^3 + 27x^2 - 18x - 9)y^4 =: G_4(x, y)
\end{aligned}
$$

for $x, y \in (0, 1)$. Then, $\partial G_4/\partial x = 0$ if, and only if,

$$\begin{aligned} &7x^5 - 45x^4 + 90x^3 + 27x^2 - 90x \\ &\quad + (-12x^5 - 30x^4 - 40x^3 - 36x^2 + 24x + 108)y \\ &\quad + (6x^5 + 60x^4 - 128x^3 - 45x^2 + 104x + 3)y^2 \\ &\quad + (20x^5 + 30x^4 - 16x^3 + 36x^2 - 4x - 18)y^3 \\ &\quad + (9x^5 - 15x^4 - 18x^3 + 18x^2 + 9x - 3)y^4 = 0 \end{aligned} \tag{19}$$

and $\partial G_4/\partial y = 0$ if, and only if,

$$\begin{aligned} &-(1-x^2)\left[-x^4 - 3x^3 - 6x^2 - 9x + (x^4 + 12x^3 - 31x^2 - 3x + 21)y \right. \\ &\quad \left. +(5x^4 + 9x^3 - x^2 + 27x - 4)y^2 + (3x^4 - 6x^3 - 6x^2 + 6x + 3)y^3\right] = 0. \end{aligned} \tag{20}$$

Another numerical computation shows that all real solutions of the system of Eqs. (19) and (20) are as follow:

$$\begin{aligned} &(x = 0, y = 0), \quad (x \approx -0.9436, y \approx 0.9093), \quad (x \approx -0.9005, y \approx 0.2632), \\ &(x \approx -0.7232, y \approx -18.8987), \quad (x \approx -0.4941, y \approx 0.9093), \\ &(x \approx 1.2834, y \approx 16.4015), \quad (x \approx 5.4293, y \approx -3.4718, \\ &(x \approx 6.1058, y \approx -3.0055), \quad (x \approx 8.9567, y \approx 0.3360). \end{aligned}$$

Therefore, G_4 has no critical point.

D. It remains to consider the interior of D, i.e., $(0,1)^3$. We have

$$\begin{aligned} \Phi(x,y,u) = {} & \varphi_1(x,y) + \varphi_4(x,y) + \varphi_2(x,y)u + [\varphi_3(x,y) - \varphi_4(x,y)]u^2 \\ & 7x^6 + 12x^2(1-x^2)y + 6x^2(1-x^2)(10-x^2)y^2 \\ & + 4(1-x^2)(1+5x^2)(4-x^2)y^3 + 9x^2(1-x^2)^2y^4 \\ & + 72\left[x^2 + 2(1-x^2)y\right](1-x^2)(1-y^2) \\ & + 18x\left[3x^2 + (6+2x^2)y + (1-x^2)y^2\right](1-x^2)(1-y^2)u \\ & + 9\left[15 - 23x^2 + (24x^2 - 16)y + (1-x^2)y^2\right](1-x^2)(1-y^2)u^2 \end{aligned}$$

for $(x,y,u) \in (0,1)^3$.

(a) Suppose that $\varphi_3(x,y) = \varphi_4(x,y)$ for $x, y \in (0,1)$. Since $\varphi_1(x,y) + \varphi_4(x,y) \geq 0$ and $\varphi_2(x,y) \geq 0$ for $x, y \in (0,1)$, we have

$$\begin{aligned}\Phi(x,y,u) &= \varphi_1(x,y)+\varphi_4(x,y)+\varphi_2(x,y)u\\ &\le \varphi_1(x,y)+\varphi_4(x,y)+\varphi_2(x,y)\\ &\le \max\{\Phi(x,y,1): x,y\in[0,1],\ \varphi_3=\varphi_4\}\\ &\le \max\{\Phi(x,y,1): x,y\in[0,1]\}\le 135.\end{aligned}$$

The last inequality follows from A.3, B.5-8 and C.6.

(b) Suppose that $\varphi_3(x,y)\neq\varphi_4(x,y)$ for $x,y\in(0,1)$. Then $\partial\Phi/\partial u=0$ if, and only if,

$$u=\frac{-x[3x^2+(6+2x^2)y+(1-x^2)y^2]}{15-23x^2+(24x^2-16)y+(1-x^2)y^2}=: u(x,y)$$

for $(x,y)\in(0,1)^2$ such that $\varphi_3(x,y)\neq\varphi_4(x,y)$. A numerical computation shows that all real and complex solutions of the system of equations $\partial\Phi/\partial x(x,y,u(x,y))=0$ and $\partial\Phi/\partial y(x,y,u(x,y))=0$ are the following:

$$\begin{aligned}&(x\approx\pm7.3296, y\approx-3.9586),\quad (x\approx\pm3.9891, y\approx-2.2430),\\ &(x\approx\pm0.7783, y\approx-0.9349),\quad (x\approx\pm2.4991, y\approx1.1601),\\ &(x\approx 4.0332\cdot10^{14}, y\approx108.2175),\quad (x\approx\pm2.6637\mathrm{i}, y\approx0.4482),\\ &(x\approx0.6713\mathrm{i}, y\approx0.5624),\quad (x\approx56781.0229\mathrm{i}, y\approx10.8430),\\ &(x\approx\pm2.5455\cdot10^{6}\mathrm{i}, y\approx16.3752).\end{aligned}$$

Therefore, Φ has no critical point.

Summarising from parts A to C (9) follows.

To see that (9) is sharp consider the function $f_0\in\mathcal{A}$ given by (10) which belongs $\mathcal{P}'$, with $a_2=a_3=a_5=0$ and $a_4=-1/2$, which completes the proof. □

Acknowledgements The authors thank the referees for their careful reading and improvements.

Funding Not applicable.

Availability of data and material Not applicable. The manuscript has no associated data.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alexander, J.W.: Functions which map the interior of the unit circle upon simple regions. Ann. Math. **17**(1), 12–22 (1915)
2. Babalola, K.O.: On $H_3(1)$ Hankel determinant for some classes of univalent functions. Inequal. Theory Appl. **6**, 1–7 (2010)
3. Carathéodory, C.: Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen. Math. Ann. **64**, 95–115 (1907)
4. Carlitz, L.: Hankel determinants and Bernoulli numbers. Tohoku Math. J. **5**, 272–276 (1954)
5. Cho, N.E., Kowalczyk, B., Kwon, O.S., Lecko, A., Sim, Y.J.: The bounds of some determinants for starlike functions of order alpha. Bull. Malays. Math. Sci. Soc. **41**, 523–535 (2018)
6. Goodman, A.W.: Univalent Functions. Mariner, Tampa (1983)
7. Hayman, W.K.: On the second Hankel determinant of mean univalent functions. Proc. Lond. Math. Soc. **18**(3), 77–94 (1968)
8. Janteng, A., Halim, S.A., Darus, M.: Coefficient inequality for a function whose derivative has a positive real part. J. Inequal. Pure Appl. Math. **7**(2), 1–5 (2006). (**Art. 50**)
9. Janteng, A., Maharana, S., Prajapat, J.K.: Third order Hankel determinant for certain univalent functions. J. Korean Math. Soc. **52**(6), 1–5 (2015)
10. Kanas, A., Lecko, A.: On the Fekete-Szegö problem and the domain of convexity for a certain class of univalent functions. Folia Sci. Univ. Tech. Resov. **73**, 49–57 (1990)
11. Kowalczyk, B., Lecko, A., Lecko, M., Sim, Y.J.: The sharp bound of the third Hankel determinant for some classes of analytic functions. Bull Korean Math. Soc. **55**(6), 1859–1868 (2018)
12. Kowalczyk, B., Lecko, A., Sim, Y.J.: The sharp bound of the Hankel determinant of the third kind for convex functions. Bull. Aust. Math. Soc. **97**, 435–445 (2018)
13. Kwon, O.S., Lecko, A., Sim, Y.J.: On the fourth coefficient of functions in the Carathéodory class. Comput. Methods Funct. Theory **18**, 307–314 (2018)
14. Lecko, A., Sim, Y.J., Śmiarowska, B.: The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. Complex Anal. Oper. Theory **13**, 2231–2238 (2019)
15. Lee, S.K., Ravichandran, V., Supramanian, S.: Bound for the second Hankel determinant of certain univalent functions. J. Inequal. Appl. **2013**(281), 1–17 (2013)
16. Libera, R.J., Zlotkiewicz, E.J.: Early coefficients of the inverse of a regular convex function. Proc. Am. Math. Soc. **85**(2), 225–230 (1982)
17. Libera, R.J., Zlotkiewicz, E.J.: Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$. Proc. Am. Math. Soc. **87**(2), 251–257 (1983)
18. MacGregor, T.H.: Functions whose derivative has a positive real part. Trans. Am. Math. Soc. **104**(4), 532–537 (1962)
19. Obradović, M., Tuneski, N.: Some properties of the class $\mathcal{U}$. Ann. Univ. Mariae Curie-Skłodowska Sect. A **73**, 49–56 (2019)
20. Pommerenke, C.: On the coefficients and Hankel determinants of univalent functions. J. Lond. Math. Soc. **41**, 111–122 (1966)
21. Pommerenke, C.: On the Hankel determinants of Univalent functions. Mathematika **14**, 108–112 (1967)
22. Pommerenke, C.: Univalent Functions. Vandenhoeck & Ruprecht, Göttingen (1975)
23. Schoenberg, I.J.: On the maxima of certain Hankel determinants and the zeros of the classical orthogonal polynomials. Indag. Math. **21**, 282–290 (1959)
24. Zaprawa, P.: Determinants, third Hankel, for subclasses of univalent functions. Mediterr. J. Math. **14**(1), 1–10 (2017). (**Art. 19**)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.