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ORIGINAL ARTICLE

On $F_3(k,n)$-numbers of the Fibonacci type

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Abstract
In this paper, we study a generalization of Narayana's numbers and Padovan's numbers. This generalization also includes a sequence whose elements are Fibonacci numbers repeated three times. We give combinatorial interpretations and a graph interpretation of these numbers. In addition, we examine matrix generators and determine connections with Pascal's triangle.

Keywords Generalized Fibonacci numbers · Narayana's numbers · Padovan's numbers · Generating function · Pascal's triangle

Mathematics Subject Classification 11B37 · 11B39 · 11B65

1 Introduction

Integer sequences have always attracted the attention of many researchers, as number sequences find application in many other fields of science as well as in mathematics. Therefore, many generalizations and polynomials of these generalizations have been given and their properties have been studied [4–6, 9–12, 16].

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The best known integer sequence is undoubtedly the Fibonacci sequence. Fibonacci numbers F_n are defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2 \tag{1}$$

with initial conditions, $F_0 = 0$ and $F_1 = 1$. Narayana's numbers and Padovan's numbers, defined below, are not so popular.

$$N_n = N_{n-1} + N_{n-3} \text{ for } n \geq 3, \text{ with } N_0 = 0, N_1 = N_2 = 1$$
$$Pv(n) = Pv(n-2) + Pv(n-3) \text{ for } n \geq 3, \text{ with } Pv(0) = Pv(1) = Pv(2) = 1$$

But sequences formed by these numbers have also interesting properties. In this paper, we study a generalized sequence which generalize sequence of Narayana's numbers and sequence of Padovan's numbers, simultaneously. Moreover, its special cases are sequences formed by other known numbers; Fibonacci numbers and powers of 2. The aim of this study is to present, for these sequences, combinatorial interpretations, some explicit formulas, generating functions, matrix generators and determine connections with the Pascal triangle.

First we recall two types of generalizations of Fibonacci numbers used in our considerations.

First type. For an arbitrary $k \geq 2$ the nth generalized Fibonacci number is a sum of k terms. Such generalizations was studied among others by Miles [8], Er [3] end recently by Włoch and Włoch [16].

In [8], Miles defined k-Fibonacci numbers where

$$g_n = g_{(n-1)} + g_{(n-2)} + \cdots + g_{(n-k)}, \text{ for } n \geq k \geq 2$$

with $g_0 = g_1 = \cdots = g_{(k-2)} = 0$ and $g_{(k-1)} = 1$. In [3] is introduced a family of k sequences generalized Fibonacci numbers in the following way. Let $k \geq 2$, c_j, $j \in \{1, 2, \ldots, k\}$ be integers. Then for an integer $1 \leq i \leq k$, generalized Fibonacci numbers are defined as

$$f_n^i = \sum_{j=1}^{k} c_j f_{n-j}^i, \text{ for } n > 0 \tag{2}$$

with initial conditions $f_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases}$ for $1 - k \leq n \leq 0$.

In [16], it was studied Fibonacci type numbers defined recursively by the kth order linear recurrence relation.

Let $k \geq 2$, $c_i \geq 0$, $i \in \{1, \ldots, k\}$ be integers such that there are at least two positive integers c_p, c_q where $p \neq q$ and $1 \leq p, q \leq k$.

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_n f_{n-k}, \text{ for } n > 0 \tag{3}$$

with given nonnegative integers $f_{1-k}, \ldots, f_{-1}, f_0$ and there is $1 - k \leq j \leq 0$ such that $f_j > 0$.

For special values of k, c_i and f_{1-i}, $i \in \{1, 2, \ldots, k\}$ the formula (3) gives the well-known classical sequences.

Let $k \geq 2$ be an integer. Generally sequences defined recursively by the kth order linear recurrence relation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_n a_{n-k}, \text{ for } n \geq k \tag{4}$$

$b_i \in N \cup \{0\}$, $i = 1, \ldots, k$ with fixed integers $a_0, \ldots, a_{k-1}$ are named as sequences of the Fibonacci type.

The second type. For an arbitrary $k \geq 2$ the nth generalized Fibonacci number is a sum of two terms, the $(n-k)$-th term and the second chosen in such way that obtained recurrence generalizes (1). We recall some of such generalizations.

Kwaśnik and Włoch studied in [7] generalized Fibonacci numbers $F(k, n)$ defined recursively as follows:

$$F(k,n) = F(k,n-1) + F(k,n-k), \text{ for } n \geq k+1$$

with $F(k,n) = n+1$, for $n \leq k$. These numbers have many interpretations in graph theory [1].

In [2], Bednarz et al. introduced a different kind of distance generalization of Fibonacci numbers. This numbers are showed by $Fd(k, n)$ and defined recursively as follows:

$$Fd(k,n) = Fd(k,n-k+1) + Fd(k,n-k), \text{ for } n \geq k \geq 2$$

with initial conditions $Fd(k,n) = 1$ for $0 \leq n \leq k-1$.

One of the other works in this field is [15] where they introduced a new kind of distance Fibonacci numbers as follows:

$$F_2(k,n) = F_2(k,n-2) + F_2(k,n-k), \text{ for } n \geq k$$

with initial conditions $F_2(k,i) = 1$ for $i = 0, 1, 2, \ldots, k-1$.

In this paper, we will study special cases of defined above Fibonacci type numbers. Let $k \geq 1$, $n \geq 0$ be integers. (3, k)-distance Fibonacci numbers are define by the following recurrence relation;

$$F_3(k,n) = F_3(k,n-3) + F_3(k,n-k), \text{ for } n \geq \max\{3,k\} \tag{5}$$

with initial conditions $F_3(k,n) = 1$, for $n = 0, 1, 2, 3\ldots, \max\{2, k-1\}$.

Recurrence (5) of (3, k)-distance Fibonacci numbers generalizes recurrences of Narayana's numbers and Padovan's numbers.

$F_3(1, n+1) = N_n$ sequence A000930 in OEIS [13].

$F_3(2,n) = Pv(n)$ sequence A000931 in OEIS [13].

$F_3(3,n) = 2^{\lfloor \frac{n}{3} \rfloor}$ contains powers of 2 repeated 3 times, sequence A173862 in OEIS [13].

$F_3(4,n)$ sequence A079398 in OEIS [13].

$F_3(5,n)$ sequence A226503 in OEIS [13].

$F_3(6,n) = F_{\lfloor \frac{n}{3} \rfloor}$ contains Fibonacci numbers repeated 3 times, sequence A247049 in OEIS [13].

A few initial elements of these sequences for special values of k and n are included in the Table 1.

Table 1 (3, k)-distance Fibonacci numbers $F_3(k,n)$

n	0	1	2	3	4	5	6	7	8
$F_3(1,n)$	1	1	1	2	3	4	6	9	13
$F_3(2,n)$	1	1	1	2	2	3	4	5	7
$F_3(3,n)$	1	1	1	2	2	2	4	4	4
$F_3(4,n)$	1	1	1	1	2	2	2	3	4
$F_3(5,n)$	1	1	1	1	1	2	2	2	3
$F_3(6,n)$	1	1	1	1	1	1	2	2	2
$F_3(7,n)$	1	1	1	1	1	1	1	2	2

n	9	10	11	12	13	14	15	16	17	18
$F_3(1,n)$	19	28	41	60	88	129	189	277	406	595
$F_3(2,n)$	9	12	16	21	28	37	49	65	86	114
$F_3(3,n)$	8	8	8	16	16	16	32	32	32	64
$F_3(4,n)$	4	5	7	8	9	12	15	17	21	27
$F_3(5,n)$	3	4	5	5	7	8	9	12	13	16
$F_3(6,n)$	3	3	3	5	5	5	8	8	8	13
$F_3(7,n)$	2	3	3	3	4	5	5	6	8	8

For a given sequence $a(n)$, sequences $a(\lfloor \frac{n}{3} \rfloor)$ and $a(\lfloor \frac{n}{2} \rfloor)$, with repeated elements, we will call a tripled $a(n)$ sequence and a doubled $a(n)$ sequence, respectively.

It is known that tiling defined by the Fibonacci numbers cover a plane. In [15] it was shown a tiling covering of a plane by tiling defined by doubled Fibonacci sequence. We present a tiling covering of a plane by tripled Fibonacci sequence, see Fig. 1.

Similar to classical Fibonacci numbers, numbers $F_3(k,n)$ can be extended to negative integers. Let $k \geq 4$ be integer and $F_3(k,n) = 1$ for $n = 0, 1, \ldots, k-1$.

$$F_3(k,-n) = F_3(k,-n+k) - F_3(k,-n+(k-3)). \tag{6}$$

Moreover for $k = 1, 2, 3$,

$F_3(1,-n) = F_3(1,-n+3) - F_3(1,-n+2)$,

$F_3(2,-n) = F_3(2,-n+3) - F_3(2,-n+1)$ and

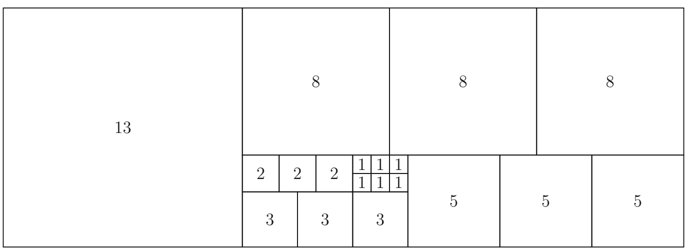


Fig. 1 A tiling interpretation of the tripled Fibonacci sequence $F_3(6,n)$f

Table 2 Numbers $F_3(k, n)$ for negative n

n	18	17	16	15	14	13	12	11	10
$F_3(1, n)$	4	− 8	1	5	− 3	− 2	3	0	− 2
$F_3(2, n)$	− 5	4	− 3	1	1	− 2	2	− 1	0
$F_3(3, n)$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$
$F_3(4, n)$	− 29	21	− 15	11	− 8	6	− 4	3	− 2
$F_3(5, n)$	− 1	4	2	− 2	− 2	1	2	0	− 1
$F_3(6, n)$	− 3	2	2	2	− 1	− 1	− 1	1	1
$F_3(7, n)$	1	− 1	− 1	− 1	0	1	1	1	0
$F_3(8, n)$	− 1	0	0	1	1	1	0	0	0

n	9	8	7	6	5	4	3	2	1	0
$F_3(1, n)$	1	1	− 1	0	1	0	0	1	1	1
$F_3(2, n)$	1	− 1	1	0	0	1	0	1	1	1
$F_3(3, n)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1
$F_3(4, n)$	2	− 1	1	0	1	0	1	1	1	1
$F_3(5, n)$	0	1	1	0	0	1	1	1	1	1
$F_3(6, n)$	1	0	0	0	1	1	1	1	1	1
$F_3(7, n)$	0	0	0	1	1	1	1	1	1	1
$F_3(8, n)$	0	0	1	1	1	1	1	1	1	1

$F_3(3, -n) = \frac{1}{2} F_3(3, -n + 3)$.

The Table 2 includes the first few elements of $F_3(k, -n)$ for special k and negative n.

2 Combinatorial interpretation

First we give a graph interpretation of $F_3(k, n)$ for integer $k \geq 4$. Let P_n, $n \geq 2$, be a path without loops and multiple edges with the vertex set $V(P_n) = \{v_1, \ldots, v_n\}$. Vertices of P_n are numbered in the natural fashion. Let consider a colouring c of vertices of P_n such that vertices v_n and v_{n-1} can be uncoloured if $k \geq 3$ or the vertex v_n can be uncoloured if $k = 2$ or all vertices are coloured if $k = 1$. Let $\{0, 1\}$ be a set of colours and $c(v_j) = i$ denote that the vertex v_j has a colour i, $0 \leq i \leq 1$.

The following recurrent procedure defines the colouring c of P_n.

Denote by A a set of uncoloured vertices of a path P_n and let $m = \min\{3, k\}$.

Until $|A| < m$, repeat following operations:

Let $v_j \in A$ be a vertex with the smallest index.

If $|A| = m$, then $c(v_j) = 0$, otherwise $c(v_j) = i$, $0 \leq i \leq 1$.

If $c(v_j) = 0$, then $c(v_{j+i}) = 0$ for $i = 1, \ldots, m - 1$.

If $c(v_j) = 1$, then $c(v_{j+i}) = 1$ for $i = 1, \ldots, \max\{3, k\} - 1$.

We can interpret obtained monochromatic paths of the length k or 3 as monochromatic scraps, so defined the above colouring c of P_n we will call a $\{P_3, P_k\}$–scrap colouring of P_n. Denote by P_3 a scrap of three vertices coloured 0 and by P_k a scrap of k vertices coloured 1. In other words the $\{P_3, P_k\}$–scrap colouring of P_n is a covering of P_n by scraps from the set $\{P_3, P_k\}$.

Note that for $k = 3$ we have two monochromatic scraps of the length 3 with different colours. To distinguish them we will write P_3 and P'_3.

Consider for example the path P_{15} and $k = 5$. Then we have the following possibilities of a $\{P_3, P_5\}$–scrap colouring:

(a) $P_5P_5P_5$ and $P_3P_3P_3P_3P_3$. So we have exactly two colouring.

(b) Colourings when the last vertex is uncoloured are $P_5P_3P_3P_3$, $P_3P_5P_3P_3$, $P_3P_3P_5P_3$, $P_3P_3P_3P_5$, what gives 4.

(c) Colourings when the last two vertices are uncoloured are $P_5P_5P_3$, $P_5P_3P_5$, $P_3P_5P_5$, so there are 3 such colourings.

Summing up, we have that there exist 9 $\{P_3, P_k\}$–scrap colourings of P_{15}.

Denote by $\alpha(n)$ the total number of $\{P_3, P_k\}$–scrap colourings of P_n.

Theorem 1 *Let $k \geq 1$, $n \geq 0$ be integers. The number of all $\{P_3, P_k\}$–scrap colourings of P_n is equal to $F_3(k, n)$.*

Proof Denote by $\alpha_k(n)$ a $\{P_3, P_k\}$–scrap colouring starting from P_k and by $\alpha_3(n)$ a $\{P_3, P_k\}$–scrap colouring starting from P_3. Consider three cases.

Case 1. $k = 1$, then all vertices are coloured.

For $n = 0$ we take $F_3(1, 0) = 1$, there is no path with 0 vertices.

For $n = 1$ we have $F_3(1, 1) = 1$. There is exactly one vertex and exactly one colouring with scrap P_1.

Analogously for $n = 2$, $F_3(1, 2) = 1$. The unique colouring is P_1P_1.

For $n = 3$ we have $F_3(1, 3) = 2$. There are colourings $P_1P_1P_1$ and P_3.

Assume that $n > 3$ and $\alpha(n) = F_3(1, n)$. Then

$$
\begin{aligned}
\alpha(n+1) &= \alpha_3(n+1) + \alpha_1(n+1) = \alpha(n+1-3) + \alpha(n+1-1) \\
&= F_3(1, n+1-3) + F_3(1, n+1-1) = F_3(1, n+1),
\end{aligned}
$$

Case 2. $k = 2$, then the vertex v_n can be uncoloured.

For $n = 0$ we take $F_3(2, 0) = 1$.

$F_3(2, 1) = 1$, the vertex v_1 is uncoloured.

$F_3(2, 2) = 1$, there is exactly one colouring with scrap P_2.

$F_3(2, 3) = 2$, there are colourings P_2 with the vertex v_3 uncoloured and P_3.

For $n > 3$ we prove analogously as for $k = 1$.

Case 3. $k = 3$, then vertices v_n and v_{n-1} can be uncoloured.

If $n = 0$, then we put $F_3(3, 0) = 1$. If $n = 1$, then the path consist of one vertex, we leave it uncoloured. So $F_3(3, 1) = 1$. Analogously for $n = 2$, $F_3(3, 2) = 1$.

If $n = 3$, then there are two colourings P_3 and P'_3, $F_3(3, 3) = 2$. Assume that $n > 3$ and $\alpha(n) = F_3(3, n)$. Then

$$\begin{aligned}\alpha(n+1) &= \alpha_3(n+1)+\alpha_3(n+1)=\alpha(n+1-3)+\alpha(n+1-3)\\ &= F_3(3,n+1-3)+F_3(3,n+1-3)=F_3(3,n+1),\end{aligned}$$

Case 4. $k>3$, then vertices v_n and v_{n-1} can be uncoloured.

If $n=0$, then we put $F_3(k,0)=1$, If $n=1$, then the path consist of one vertex, we leave it uncoloured. So $F_3(k,1)=1$. Analogously for $n=2$, $F_3(k,2)=1$.

If $3\le n\le k-1$, then there is exactly one $\{P_3,P_k\}$–scrap colouring only by scraps P_3. Thus $\alpha(n)=\alpha_3(n)=1=F_3(k,n)$.

Assume that $n\ge k$ and $\alpha(n)=F_3(k,n)$. Then

$$\begin{aligned}\alpha(n+1) &= \alpha_3(n+1)+\alpha_k(n+1)=\alpha(n+1-3)+\alpha(n+1-k)\\ &= F_3(k,n+1-3)+F_3(k,n+1-k)=F_3(k,n+1),\end{aligned}$$

which ends the proof. □

From the above graph interpretation it follow direct formulas for $F_3(k,n)$.

Theorem 2 *Let $k\ge 1$, $n\ge 0$ be integers. Then*

$$F_3(k,n+t)=\sum_{i=0}^{\lfloor\frac{n}{k}\rfloor}\binom{i+\lfloor\frac{n-ik}{3}\rfloor}{i}\ \text{where}\ t=\begin{cases}2 & \text{for } k=1,\\ 1 & \text{for } k=2,\\ 0 & \text{for } k\ge 3.\end{cases}$$

Proof If $n\le k-1$, then $\lfloor\frac{n}{k}\rfloor=0$ and

$$F_3(k,n)=\sum_{i=0}^{0}\binom{i+\lfloor\frac{n-ik}{3}\rfloor}{i}=\binom{0+\lfloor\frac{n}{3}\rfloor}{0}=1.$$

Assume that $n\ge k$. By Theorem 1, the number $F_3(k,n)$ is equal to the number of $\{P_3,P_k\}$–scrap colourings of P_n. Each $\{P_3,P_k\}$–scrap colouring consists of i monochromatic paths P_k and j monochromatic paths P_3, where $0\le i\le\lfloor\frac{n}{k}\rfloor$, $0\le j\le\lfloor\frac{n}{k}\rfloor$. Moreover, for a fixed i we have $j=\lfloor\frac{n-ik}{3}\rfloor$ and the number of $\{P_3,P_k\}$–scrap colourings is equal to $\binom{i+j}{i}=\binom{i+\lfloor\frac{n-ik}{3}\rfloor}{i}$. Thus $F_3(k,n)=\sum_{i=0}^{\lfloor\frac{n}{k}\rfloor}\binom{i+\lfloor\frac{n-ik}{3}\rfloor}{i}$. □

Now we can give an interpretation of $F_3(k,n)$ numbers for $k\ne 3$.

Let $k\ge 4$, $n\ge 3$ be integers, $X=\{1,2,3,\ldots,n\}$, and $\mathcal{Y}=\{Y_t: t\in T\}$ be the family of disjoint subsets of the set X such that each subset Y_t contains consecutive integers and satisfies the following conditions.

1. $|Y_t|\in\{3,k\}$,
2. $|X\setminus\bigcup_{t\in T}Y_t|\in\{0,1,2\}$,
3. If $p\in X\setminus\bigcup_{t\in T}Y_t$, then $p=n$ or

4. if $p, q \in X \setminus \bigcup_{t\in T} Y_t$, then $p = n$ and $q = n - 1$.

The family $\mathcal{Y}$ is called the (3, k)-decomposition with the rest or a (3, k)-decomposition family.

If $X \setminus \bigcup_{t\in T} Y_t = \emptyset$, then we have a decomposition of the set X.

Theorem 3 *Let $k \geq 1$, $k \neq 3$, $n \geq 3$ be integers. Then the number of all (3, k)-decompositions with the rest at most two of the set X is equal to $F_3(k, n)$.*

Proof We will prove for $k \geq 4$, $n \geq 3$. For $k = 1, 2$ we prove analogously. Denote by $d(n)$ the number of all (3, k)-decompositions of the set X. If $n = 3$, then we have the only one decomposition $\{\{1, 2, 3\}\}$. Thus $d(3) = 1 = F_3(k, 3)$. Analogously for $3 < n \leq k - 1$ by inspections the result follows.

Let $n \geq k$ and suppose that the equality $d(n) = F_3(k, n)$ holds for an arbitrary n. We will show that $d(n + 1) = F_3(k, n + 1)$.

Denote by $d_3(n + 1)$ the number of all (3, k)-decompositions of $X = \{1, 2, \ldots, n + 1\}$ such that $Y = \{1, 2, 3\}$, and let $d_k(n + 1)$ be the number of all (3, k)-decompositions of $X = \{1, 2, \ldots, n + 1\}$ such that $Y = \{1, 2, \ldots, k\}$. We have that $d(n + 1) = d_3(n + 1) + d_k(n + 1)$ and $d_3(n + 1) = d(n + 1 - 3)$, $d_k(n + 1) = d(n + 1 - k)$. By the induction hypothesis and the recurrence (5) we have

$d_3(n + 1) = d(n + 1 - 3)+$
$d(n + 1 - k) = F_3(k, n + 1 - 3) + F_3(k, n + 1 - k) = F_3(k, n + 1)$,
which ends the proof. □

The above interpretation of $F_3(k, n)$ leads to another interpretation of it in terms of ordered composition, with the rest at most two, of an integer. We have that $|X| = n$ and each subset Y_t, containing consecutive integers, corresponds to the addend n_t. A sum $\sum_{t\in T} n_t + n_r = n$ we call an ordered compositions, with the rest, of an integer n if

1. $n_t \in \{3, k\}$,
2. $n_r \in \{0, 1, 2\}$,
3. $\sum_{t\in T} n_t + n_r = n$ and n_r is the last addend in the sum.

3 Generating function and some identities

Theorem 4 *Let $n \geq 0$, $k \geq 1$ be integers. The generating function of $F_3(k, n)$ has the following form*

$$g(x) = \frac{1 + t}{1 - x^3 - x^k} \text{ where } t = \begin{cases} 0 \text{ for } k = 1 \\ x \text{ for } k = 2 \\ x + x^2 \text{ for } k \geq 3 \end{cases}.$$

Proof Let $g(x) = \sum_{n=0}^{\infty} F_3(k,n)x^n$. Using the recurrence (5) we have $g(x) - x^3 g(x) - x^k g(x) = 1 + t$ where $t = \begin{cases} 0 \text{ for } k = 1 \\ x \text{ for } k = 2 \\ x + x^2 \text{ for } k \geq 3. \end{cases}$ Hence $g(x) = \frac{1+t}{1-x^3-x^k}$ which ends the proof. □

From the Theorem 4, for special values of k, we obtain generating functions for Narayana numbers and Padovan numbers.

Corollary 5 *If $k = 1$, then $g(x) = \frac{1}{1-x-x^3}$ is the generating functions of Narayana numbers* [14].

If $k = 2$, then $g(x) = \frac{1+x}{1-x^2-x^3}$ is the generating functions of Padovan numbers.

The next theorem proofs that all $F_3(3k,n)$ sequences are tripled.

Theorem 6 *Let $k \geq 1$, $n \geq 0$ be integers. Then*

$$F_3(3k,3n) = F_3(3k,3n+1) = F_3(3k,3n+2). \tag{7}$$

Proof by induction on n From the definition of $F_3(k,n)$ we have that $F_3(3k,0) = \cdots = F_3(3k,3k-1) = 1$. By the formula (5), $F_3(3k,3k) = F_3(3k,3k+1) = F_3(3k,3k+2) = 2$ and next $F_3(3k,3k+3) = F_3(3k,3k+4) = F_3(3k,3k+5) = 3$.

Assume that $F_3(3k,3t) = F_3(3k,3t+1) = F_3(3k,3t+2)$ for all $t \leq n$.

We will prove

$$F_3(3k,3(n+1)) = F_3(3k,3(n+1)+1) = F_3(3k,3(n+1)+2).$$

For consecutive numbers we have

$F_3(3k,3(n+1)) = F_3(3k,3n+3)$
$= F_3(3k,3n) + F_3(3k,3n+3-3k) = F_3(3k,3n) + F_3(3k,3(n+1-k))$,

$F_3(3k,3(n+1)+1) = F_3(3k,3n+4) = F_3(3k,3n+1) + F_3(3k,3n+4-3k)$
$= F_3(3k,3n+1) + F_3(3k,3(n+1-k)+1)$,

$F_3(3k,3(n+1)+2) = F_3(3k,3n+5) = F_3(3k,3n+2) + F_3(3k,3n+5-3k)$
$= F_3(3k,3n+2) + F_3(3k,3(n+1-k)+2)$.

Using our assumption we ascertain that the above numbers are equal. □

Theorem 7 *Let $k \geq 3$, $n \geq 0$ be integers. Then*

$$\sum_{i=0}^{n} F_3(k,3i+t) = F_3(k,3n+k+t) - 1, \text{ for } 0 \leq t \leq 2 \tag{8}$$

Proof by induction on n We will prove for an arbitrary $k \geq 3$ and $t = 0$. For $k \geq 3$, $t = 1$ and $t = 2$ we prove analogously.

If $n = 0$, then $F_3(k,0) = 1 = F_3(k,3 \cdot 0 + k) - 1 = 2 - 1 = 1$.

If $n = 1$, then for $k = 3$ we have $F_3(k,0) + F_3(k,3) = 1 + 2 = F_3(k,3+3) -$

$1 = 4 - 1 = 3$ and

for $k > 3$, $F_3(k,0) + F_3(k,3) = 1 + 1 = F_3(k,3+k) - 1 = 3 - 1 = 2.$

Assume that

$$\sum_{i=0}^{n} F_3(k,3i) = F_3(k,3n+k) - 1.$$

Then

$$\begin{aligned}\sum_{i=0}^{n+1} F_3(k,3i) &= \sum_{i=0}^{n} F_3(k,3i) + F_3(k,3(n+1)) \\ &= F_3(k,3n+k) - 1 + F_3(k,3n+3) = F_3(k,3(n+1)+k) - 1.\end{aligned}$$

Thus the theorem is proved. □

4 Matrix generators

Let $Q_k = [q_{ij}]_{k\times k}$ be a square matrix. For a fixed $1 \le i \le k$ an element q_{i1} is equal to the coefficient at $F_3(k,i)$ of the right hand side of the formula (5). For $j > 1$ and an arbitrary i we have $q_{ij} = 1$ if $j = i+1$ and $q_{ij} = 0$, otherwise.

The above definition gives matrices

$$Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix},\ Q_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ldots, Q_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Moreover, we define a square matrix P_k of order k as the matrix of initial conditions

$$P_k = \begin{bmatrix} F_3(k,2k-2) & F_3(k,2k-3) & \cdots & F_3(k,k) & F_3(k,k-1) \\ F_3(k,2k-3) & F_3(k,2k-4) & \cdots & F_3(k,k-1) & F_3(k,k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_3(k,k) & F_3(k,k-1) & \cdots & F_3(k,2) & F_3(k,1) \\ F_3(k,k-1) & F_3(k,k-2) & \cdots & F_3(k,1) & F_3(k,0) \end{bmatrix}.$$

Using Laplace'a Theorem and basic properties of determinants, we get the following results.

Theorem 8 *Let $k \ge 3$ be an integer. Then*

$$\det Q_3 = 2 \text{ and } \det Q_k = (-1)^{k+1},$$
$$\det P_k = (-1)^{2k+\frac{k(k+1)}{2}-3}.$$

Theorem 9 *Let $k \geq 3$, $n \geq 1$ be integers. Then*

$$P_k Q_k^n = \begin{bmatrix} F_3(k,n+2k-2) & F_3(k,n+2k-3) & \cdots & F_3(k,n+k) & F_3(k,n+k-1) \\ F_3(k,n+2k-3) & F_3(k,n+2k-4) & \cdots & F_3(k,n+k-1) & F_3(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_3(k,n+k) & F_3(k,n+k-1) & \cdots & F_3(k,n+2) & F_3(k,n+1) \\ F_3(k,n+k-1) & F_3(k,n+k-2) & \cdots & F_3(k,n+1) & F_3(k,n) \end{bmatrix}. \tag{9}$$

Proof If $n = 1$, then by (5) and simple calculations the result immediately follows. Assume the formula (9) holds for n, we will prove it for $n+1$. Since $P_k Q_k^{n+1} = (P_k Q_k^n) Q_k$, by our assumption and by the recurrence (5) we obtain

$$A_k Q_k^{n+1} = \begin{bmatrix} F_3(k,n+2k-2) & F_3(k,n+2k-3) & \cdots & F_3(k,n+k) & F_3(k,n+k-1) \\ F_3(k,n+2k-3) & F_3(k,n+2k-4) & \cdots & F_3(k,n+k-1) & F_3(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_3(k,n+k) & F_3(k,n+k-1) & \cdots & F_3(k,n+2) & F_3(k,n+1) \\ F_3(k,n+k-1) & F_3(k,n+k-2) & \cdots & F_3(k,n+1) & F_3(k,n) \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} F_3(k,n+2k-1) & F_3(k,n+2k-2) & \cdots & F_3(k,n+k+1) & F_3(k,n+k) \\ F_3(k,n+2k-2) & F_3(k,n+2k-3) & \cdots & F_3(k,n+k) & F_3(k,n+k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_3(k,n+k+1) & F_3(k,n+k) & \cdots & F_3(k,n+3) & F_3(k,n+2) \\ F_3(k,n+k) & F_3(k,n+k-1) & \cdots & F_3(k,n+2) & F_3(k,n+1) \end{bmatrix},$$

which ends the proof. □

By Theorem 8 we get the following result.

Corollary 10 *Let $k \geq 3$, $n \geq 2$ be integers. Then*

Table 3 Distance Fibonacci numbers $F_3^i(5,n)$ and $F_3(5,n)$

n	0	1	2	3	4	5	6	7	8	
$F_3^1(5,n)$	0	0	0	0	1	0	0	1	0	
$F_3^2(5,n)$	0	0	0	1	0	0	1	0	1	
$F_3^3(5,n)$	0	0	1	0	0	1	0	1	1	
$F_3^4(5,n)$	0	1	0	0	0	0	1	0	0	
$F_3^5(5,n)$	1	0	0	0	0	1	0	0	1	
$F_3(5,n)$	1	1	1	1	1	2	2	2	3	
n	9	10	11	12	13	14	15	16	17	18
$F_3^1(5,n)$	1	1	0	2	1	1	3	1	3	4
$F_3^2(5,n)$	1	0	2	1	1	3	1	3	4	2
$F_3^3(5,n)$	0	2	1	1	3	1	3	4	2	6
$F_3^4(5,n)$	1	0	1	1	0	2	1	1	3	1
$F_3^5(5,n)$	0	1	1	0	2	1	1	3	1	3
$F_3(5,n)$	3	4	5	5	7	8	9	12	13	16

$$\det P_k Q_k^n = (-1)^{[2k+\frac{k(k+1)}{2}-3]+n(k+1)}.$$

5 Connections with the Pascal's triangle

To study connections of (3, k)-distance Fibonacci numbers with Pascal's triangle we need to consider a family of sequences given by the same recurrence as $F_3(k,n)$ with different initial conditions.

Let $k \geq 1$, $n \geq 0$ be integers and

$$F_3^i(k,n) = F_3^i(k,n-3) + F_3^i(k,n-k) \tag{10}$$

with $F_3^i(k,n) = \begin{cases} 1 \text{ if } n = k-i \\ 0 \text{ in } \text{ otherwise} \end{cases}$ for $n = 0, 1, \ldots, \max\{2,k\}$.

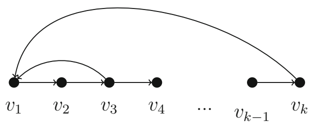
A few initial elements of these sequences for $k = 5$ and special values n are included in the Table 3.

Based on results from [16] we have

Theorem 11 *Let $k \geq 4$, $n \geq 0$, $0 \leq i \leq k-1$ be integers. Then*

$$F_3(k,n) = \sum_{i=0}^{\max\{2,k-1\}} F_3^i(k,n).$$

Fig. 2 Digraph D



Er shoved [3] that nth power of the companion matrix Q_k contains entries of sequences $F_3^i(k,n)$,

$$Q_k^n = \begin{bmatrix} F_3^1(k,n+k-1) & F_3^1(k,n+k-2) & \dots & F_3^1(k,n) \\ F_3^2(k,n+k-1) & F_3^2(k,n+k-2) & \dots & F_3^2(k,n) \\ \vdots & \vdots & \ddots & \vdots \\ F_3^k(k,n+k-1) & F_3^k(k,n+k-2) & \dots & F_3^k(k,n) \end{bmatrix}.$$

The matrix Q_k we can interpret as the adjacency matrix of a special digraph D, see the Fig. 2.

It is well known that Q_k^n contains the number off all different paths of length n between corresponding vertices in the digraph D. Namely, the entry q_{ij} is equal to the number of all paths of the length n from vertex v_i to vertex v_j in the digraph D.

Using such interpretation we can prove

Theorem 12 *Let $k \geq 4$, $n \geq 0$, $0 \leq i \leq k-1$ be integers. Then*

$$F_3^1(k,n) = \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n}} \binom{\alpha_3 + \alpha_k}{\alpha_3}, \tag{11}$$

$$F_3^j(k,n) = \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n-(3-j+1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} + \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n-(k-j+1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} \text{ for } j = 2, 3, \tag{12}$$

$$F_3^j(k,n) = \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n-(k-j+1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} \text{ for } 3 < j \leq k. \tag{13}$$

Proof For a fixed $1 \leq j \leq k$ the element $q_{j,1}$ of Q_k^n is equal to the total number of different directed paths of length n from the vertex v_j to the vertex v_1 in digraph D, see the Fig. 2. Each such path contains at the beginning a short path $P =$

$v_j - \cdots - v_1$ and next a sequence $\mathcal{C}$ of cycles of length 3 or k.

Let consider a number of such paths of length n with dependence of short path P.

Case 1. $P = v_1 - v_2 - v_3 - v_1$ or $P = v_1 - v_2 - v_3 - v_4 - \cdots - v_k - v_1$. In this case a path P form a cycle C_3 or C_k, respectively. The whole path is a sequence of cycles C_3 or C_k in random order; its length $n = 3\alpha_3 + k\alpha_k$ for all α_3, α_k satisfying the equality. Thus we have

$$\sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n}} \binom{\alpha_3 + \alpha_k}{\alpha_3} = F_3^1(k, n)$$

such paths.

Case 2. $P = v_j - v_{j+1} - \cdots - v_k - v_1$, $1 < j \leq k$.

The path P has a length $k - j + 1$ and remaining part of length $n - (k - j + 1) = 3\alpha_3 + k\alpha_k$ consists of cycles C_3 or C_k. Analogously as in the Case 1 we have

$$\sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n - (k - j + 1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3}$$

such paths.

Case 3. $P = v_j \cdots - v_3 - v_1$, $1 < j \leq 3$.

The path P has length $3 - j + 1$, the remaining part has the length $n - (3 - j + 1) = 3\alpha_3 + k\alpha_k$, and there is exactly

$$\sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n - (3 - j + 1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3}$$

such paths.

Note that for $j = 2, 3$ we have

$$\sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n - (3 - j + 1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} + \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n - (k - j + 1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} = F_3^j(k, n).$$

For $3 < j \leq k$ we have

$$\sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n - (k - j + 1)}} \binom{\alpha_3 + \alpha_k}{\alpha_3} = F_3^j(k, n).$$

□

Based on the Theorem 11 and the Theorem 12 we have

Theorem 13 *Let $k \geq 4$, $n \geq 0$ be integers. Then*

$$F_3(k,n) = \sum_{i=0}^{2} \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n-(3-i+1)}} \binom{\alpha_3+\alpha_k}{\alpha_3} + \sum_{i=0}^{k-1} \sum_{\substack{\alpha_3,\,\alpha_k \\ 3\alpha_3 + k\alpha_k = n-(k-i+1)}} \binom{\alpha_3+\alpha_k}{\alpha_3}.$$

From the Theorem 13 we can obtain binomials whose sums are equal to numbers $F_3(k,n)$. Using these binomials we can derive new formulas for $F_3(k,n)$ numbers. For a convenience we use a graphical presentation.

For example, the number $F_3(4,25)$ is a sum of $\binom{8}{7}, \binom{7}{3}, \binom{8}{8}, \binom{7}{4}, \binom{6}{0}, \binom{7}{5}, \binom{6}{1}, \binom{7}{6}, \binom{6}{2}$ and $\binom{8}{8}, \binom{7}{4}, \binom{6}{0}, \binom{7}{5}, \binom{6}{1}$; and $F_3(4,26)$ is a sum of $\binom{8}{6}, \binom{7}{2}, \binom{8}{7}, \binom{7}{3}, \binom{8}{8}, \binom{7}{4}, \binom{6}{0}, \binom{7}{5} \binom{6}{1}$ and $\binom{8}{7}, \binom{7}{3}, \binom{8}{8}, \binom{7}{4}, \binom{6}{0}$.

These binomials form a geometrical pattern, we will call it a staircase, useful for calculating $F_3(4,n)$ numbers from the Pascal Triangle.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1
\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}$$

We extending the staircase presented above up to infinity in both directions. By moving such infinite staircase one column to the left, we obtain next number $F_3(k, n)$.

In almost each step of the staircase we have two binomials adjacent. Using the basic property of binomials

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{14}$$

we immediately obtain a new simplest staircase.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}$$

Such transformations of the formula from the Theorem 13 leads to

Corollary 14 *Let* $k \geq 4$, $n \geq 0$ *be integers.*

$$F_3(4, n+3) = \sum_{i=0}^{\lfloor \frac{n+1}{3} \rfloor} sgn(n+i-2 \bmod 4) \binom{\lfloor \frac{n+i+1}{4} \rfloor}{i}. \tag{15}$$

We obtain a new formula from (15) using (14). Corresponding staircase is presented on the next Pascal's triangle.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & \underline{8} & 28 & \underline{56} & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & \underline{126} & 84 & \underline{36} & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & \underline{10} & 1 \end{bmatrix}$$

So we can write

Corollary 15 *Let $k \geq 4$, $n \geq 0$ be integers.*

$$F_3(4, n+4) = \sum_{i=0}^{\lfloor \frac{n+1}{3} \rfloor} sgn(n + i - 2 \bmod 2) \binom{\lfloor \frac{n+i+1}{4} \rfloor}{i}.$$

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