Global behavior of a third order difference equation with quadratic term

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Abstract

In this paper, we solve and study the global behavior of the admissible solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{-ax_{n-1} + bx_{n-2}}, \quad n = 0, 1, \dots,$$

where $a, b > 0$ and the initial values x_{-2}, x_{-1}, x_0 are real numbers.

Keywords Difference equation · Forbidden set · Invariant set · Convergence

Mathematics Subject Classification 39A20

1 Introduction

In [11], the author obtained the solutions and determined the forbidden sets of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \quad n = 0, 1, \dots,$$

and

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-2}}, \quad n = 0, 1, \dots,$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.

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In [3], the author determined the forbidden set and discussed the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{ax_n x_{n-2}}{-bx_n + cx_{n-3}}, \quad n = 0, 1, \dots,$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

He also in [12] determined the forbidden sets and discussed the global behaviors of solutions of the two difference equations

$$x_{n+1} = \frac{ax_n x_{n-1}}{\pm bx_{n-1} + cx_{n-2}}, \quad n = 0, 1, \dots,$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

Recently, there has been a great interest in studying properties of nonlinear and rational difference equations (see [1, 4–10, 12–37] and the references therein).

In this paper, we study the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{-ax_{n-1} + bx_{n-2}}, \quad n = 0, 1, \dots, \quad (1)$$

where $a, b > 0$ and the initial values x_{-2}, x_{-1}, x_0 are real numbers.

The transformation

$$y_n = \frac{x_{n-1}}{x_n}, \quad \text{with } y_{-1} = \frac{x_{-2}}{x_{-1}}, \quad y_0 = \frac{x_{-1}}{x_0} \quad (2)$$

reduces Eq. (1) into the difference equation

$$y_{n+1} = -\frac{a}{y_{n-1}} + b, \quad n = 0, 1, \dots \quad (3)$$

The solution of (3) depends on the relation between a and b . By solving Eq. (3) and after some calculations, the solution of Eq. (1) can be obtained.

2 Case $b^2 > 4a$

In this section, we assume that $b^2 > 4a$. Suppose that

$$\theta_j = \frac{\lambda_+^j - \lambda_-^j}{\sqrt{b^2 - 4a}},$$

where $\lambda_- = \frac{b}{2} - \frac{\sqrt{b^2 - 4a}}{2}$ and $\lambda_+ = \frac{b}{2} + \frac{\sqrt{b^2 - 4a}}{2}$, $j = 0, 1, \dots$

We give the following Lemma without proof:

Lemma 2.1 *The following identities are true:*

(1)

$$-a\theta_j + b\theta_{j+1} = \theta_{j+2}, \quad j = 0, 1, \dots$$

(2)

$$\begin{aligned} & -a(-ax_0\theta_j + x_{-1}\theta_{j+1}) + b(-ax_0\theta_{j+1} + x_{-1}\theta_{j+2}) \\ & = -ax_0\theta_{j+2} + x_{-1}\theta_{j+3}, \quad j = 0, 1, \dots \end{aligned}$$

(3)

$$\begin{aligned} & -a(-ax_{-1}\theta_j + x_{-2}\theta_{j+1}) + b(-ax_{-1}\theta_{j+1} + x_{-2}\theta_{j+2}) \\ & = -ax_{-1}\theta_{j+2} + x_{-2}\theta_{j+3}, \quad j = 0, 1, \dots \end{aligned}$$

Theorem 2.2 Let $\{x_n\}_{n=-2}^\infty$ be an admissible solution of Equation (1). Then

$$x_n = \begin{cases} \frac{v}{\gamma_0\left(\frac{n-1}{2}\right)\gamma_{-1}\left(\frac{n+1}{2}\right)}, & n = 1, 3, \dots, \\ \frac{v}{\gamma_0\left(\frac{n}{2}\right)\gamma_{-1}\left(\frac{n}{2}\right)}, & n = 2, 4, \dots, \end{cases} \quad (4)$$

where $v = x_0x_{-1}x_{-2}$, $\gamma_{-i}(n) = -ax_{-i}\theta_n + x_{-i-1}\theta_{n+1}$, $i = 0, 1$ and $j = 0, 1, \dots$

Proof We can write the given solution (4) as

$$\begin{aligned} x_{2m+1} &= \frac{v}{\gamma_0(m)\gamma_{-1}(m+1)}, \\ x_{2m+2} &= \frac{v}{\gamma_0(m+1)\gamma_{-1}(m+1)}. \end{aligned} \quad (5)$$

When $m = 0$,

$$\begin{aligned} x_1 &= \frac{v}{(-ax_0\theta_0 + x_{-1}\theta_1)(-ax_{-1}\theta_1 + x_{-2}\theta_2)} \\ &= \frac{x_0x_{-1}x_{-2}}{(x_{-1})(-ax_{-1} + bx_{-2})} = \frac{x_0x_{-2}}{-ax_{-1} + bx_{-2}} \\ &= x_1. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= \frac{v}{(-ax_0\theta_1 + x_{-1}\theta_2)(-ax_{-1}\theta_1 + x_{-2}\theta_2)} \\ &= \frac{x_0x_{-1}x_{-2}}{(-ax_0 + bx_{-1})(-ax_{-1} + bx_{-2})} = \frac{x_1x_{-1}}{-ax_0 + bx_{-1}} \\ &= x_2. \end{aligned}$$

Suppose that the result is true for $m > 0$.

Then

$$\begin{aligned}
& \frac{x_{2m+1}x_{2m-1}}{-ax_{2m} + bx_{2m-1}} \\
&= \frac{\frac{v}{\gamma_0(m)\gamma_{-1}(m+1)} \frac{v}{\gamma_0(m-1)\gamma_{-1}(m)}}{-a \frac{v}{\gamma_0(m)\gamma_{-1}(m)} + b \frac{v}{\gamma_0(m-1)\gamma_{-1}(m)}} \\
&= \frac{v}{\gamma_{-1}(m+1)(-a\gamma_0(m-1) + b\gamma_0(m))}.
\end{aligned}$$

Using Lemma (2.1), we get

$$\begin{aligned}
& \frac{x_{2m+1}x_{2m-1}}{-ax_{2m} + bx_{2m-1}} \\
&= \frac{v}{\gamma_{-1}(m+1)\gamma_0(m+1)} \\
&= x_{2m+2}.
\end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2}x_{2m}}{-ax_{2m+1} + bx_{2m}} = x_{2m+3}.$$

This completes the proof. $\square$

Consider the set

$$D_1 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x}{\lambda_+^2/a^2} = \frac{y}{\lambda_+/a} = z \right\}$$

and

$$D_2 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x}{\lambda_-^2/a^2} = \frac{y}{\lambda_-/a} = z \right\}.$$

Theorem 2.3 *The two sets D_1 and D_2 are invariant sets for Equation (1).*

Proof Let $(x_0, x_{-1}, x_{-2}) \in D_1$. We show that $(x_n, x_{n-1}, x_{n-2}) \in D_1$ for each $n \in \mathbb{N}$. The proof is by induction on n . The point $(x_0, x_{-1}, x_{-2}) \in D_1$ implies

$$\frac{x_0}{\lambda_+^2/a^2} = \frac{x_{-1}}{\lambda_+/a} = x_{-2}.$$

Now for $n = 1$, we have

$$x_1 = \frac{x_0x_{-2}}{-ax_{-1} + bx_{-2}} = \frac{(\lambda_+/a)x_{-1}(a/\lambda_+)x_{-1}}{-ax_{-1} + b(a/\lambda_-)x_{-1}} = \frac{\lambda_+^2}{a^2}x_{-1}.$$

Then we have

$$\frac{x_1}{1/\lambda_-^2} = \frac{x_0}{1/\lambda_-} = x_{-1}.$$

This implies that $(x_1, x_0, x_{-1}) \in D_1$.

Suppose now that $(x_n, x_{n-1}, x_{n-2}) \in D_1$. That is

$$\frac{x_n}{\lambda_+^2/a^2} = \frac{x_{n-1}}{\lambda_+/a} = x_{n-2}.$$

Then

$$x_{n+1} = \frac{x_n x_{n-2}}{-ax_{n-1} + bx_{n-2}} = \frac{((\lambda_+/a)x_{n-1}(a/\lambda_+)x_{n-1})}{-ax_{n-1} + b(a/\lambda_+)x_{n-1}} = \frac{\lambda_+^2}{a^2} x_{n-1}.$$

Then $(x_{n+1}, x_n, x_{n-1}) \in D_1$. Therefore, D_1 is an invariant set for Eq. (1).

By similar way, we can show that D_2 is an invariant set for Eq. (1).

This completes the proof. $\square$

Theorem 2.4 Assume that $\{x_n\}_{n=-2}^\infty$ is an admissible solution of Equation (1). The following statements are true:

- (1) If $b > a + 1$, then the solution $\{x_n\}_{n=-2}^\infty$ converges to zero.
- (2) If $b = a + 1$, then we have the following:
 - (a) If $b < 2$, then the solution $\{x_n\}_{n=-2}^\infty$ converges to a finite limit.
 - (b) If $b > 2$, then the solution $\{x_n\}_{n=-2}^\infty$ converges to zero.
- (3) If $b < a + 1$, then we have the following:
 - (a) If $b < 2$, then the solution $\{x_n\}_{n=-2}^\infty$ is unbounded.
 - (b) If $b > 2$, then the solution $\{x_n\}_{n=-2}^\infty$ converges to zero.

Proof We can write $\theta_j = \lambda_+^j \frac{(1 - (\frac{b-a}{\lambda_+})^j)}{\sqrt{b^2 - 4a}}$.

- (1) If $b > a + 1$, then $\lambda_+ > 1$. That is $\theta_m \rightarrow \infty$ as $m \rightarrow \infty$. Also both of $\gamma_0(m)$ and $\gamma_{-1}(m)$ are unbounded. This implies that

$$x_{2m+1} = \frac{v}{\gamma_0(m)\gamma_{-1}(m+1)} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Similarly for x_{2m+2} .

- (2) Suppose that $b = a + 1$, then either $\lambda_- = 1$ or $\lambda_+ = 1$.

- (a) If $b < 2$, then $\lambda_+ = 1$. It follows that $\theta_m \rightarrow \frac{1}{\sqrt{b^2 - 4a}}$ as $m \rightarrow \infty$. This implies that

$$x_{2m+1} \rightarrow \frac{v(b^2 - 4a)}{(-ax_0 + x_{-1})(-ax_{-1} + x_{-2})}.$$

Similarly for x_{2m+2} .

- (b) If $b > 2$, then $\lambda_- = 1$. It follows that $\theta_m \rightarrow \infty$ as $m \rightarrow \infty$. This implies that

$$x_{2m+1} = \frac{v}{\gamma_0(m)\gamma_{-1}(m+1)} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Similarly for x_{2m+2} .

- (3) The proof is similar to that in (2) and is omitted.

This completes the proof. $\square$

Example (1)

Figure 1 shows that, if $a = 1.1, b = 2.2$ ($b > a + 1$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ converges to zero.

Example (2)

Figure 2 shows that, if $a = 0.7, b = 1.7$ ($b = a + 1$ and $b < 2$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ converges to

$$\frac{v(b^2 - 4a)}{(-ax_0 + x_{-1})(-ax_{-1} + x_{-2})} \simeq 0.06081.$$

Example (3)

Figure 3 shows that, if $a = 0.2, b = 1$ ($b < a + 1$ and $b < 2$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ is unbounded.

Example (4)

Figure 4 shows that, if $a = 3, b = 3.7$ ($b < a + 1$ and $b > 2$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ converges to zero.

Fig. 1 $x_{n+1} = \frac{x_n x_{n-2}}{-1.1x_{n-1} + 2.2x_{n-2}}$

Fig. 2 $x_{n+1} = \frac{x_n x_{n-2}}{-0.7x_{n-1} + 1.7x_{n-2}}$

3 Case $b^2 = 4a$

During this section, we assume that $b^2 = 4a$.

Theorem 3.1 Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1). Then

$$x_n = \begin{cases} \left(\frac{2}{b}\right)^n \frac{4v}{\mu_0\left(\frac{n-1}{2}\right)\mu_{-1}\left(\frac{n+1}{2}\right)}, & n = 1, 3, \dots, \\ \left(\frac{2}{b}\right)^n \frac{4v}{\mu_0\left(\frac{n}{2}\right)\mu_{-1}\left(\frac{n}{2}\right)}, & n = 2, 4, \dots, \end{cases} \quad (6)$$

where $v = x_0 x_{-1} x_{-2}$, $\mu_{-i}(n) = -bnx_{-i} + 2(1+n)x_{-i-1}$, $i = 0, 1$ and $j = 0, 1, \dots$

Proof It is sufficient to write the formula (6) as

Fig. 3 $x_{n+1} = \frac{x_n x_{n-2}}{-0.2x_{n-1} + x_{n-2}}$

Fig. 4 $x_{n+1} = \frac{x_n x_{n-2}}{-3x_{n-1} + 3.7x_{n-2}}$

$$\begin{aligned} x_{2m+1} &= \left(\frac{2}{b}\right)^{2m+1} \frac{4v}{\mu_0(m)\mu_{-1}(m+1)}, \\ x_{2m+2} &= \left(\frac{2}{b}\right)^{2m+2} \frac{4v}{\mu_0(m+1)\mu_{-1}(m+1)}, \end{aligned} \quad (7)$$

as well as noting that

$$\mu_{-i}(m-1) + \mu_{-i}(m+1) = 2\mu_{-i}(m), \quad i = 0, 1.$$

□

Theorem 3.2 Assume that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution of Equation (1). The following statements are true:

- (1) If $b \geq 2$, then the solution $\{x_n\}_{n=-2}^{\infty}$ converges to zero.
- (2) If $b < 2$, then the solution $\{x_n\}_{n=-2}^{\infty}$ is unbounded.

Proof Clear that both of $\mu_0(m)$ and $\mu_{-1}(m)$ are unbounded. Then (1) is satisfied directly. When $b < 2$, then $\left(\frac{2}{b}\right)^m \rightarrow \infty$ as $m \rightarrow \infty$.

Then using formulas (7) we can write

$$|x_{2m+1}| = \left| \left(\frac{2}{b}\right)^{2m+1} \frac{4v}{m^2(-bx_0 + 2(\frac{1}{m} + 1)x_{-1})(-b(1 + \frac{1}{m})x_{-1} + 2(\frac{2}{m} + 1)x_{-2})} \right|.$$

By applying L'Hospital's rule, we get $|x_{2m+1}| \rightarrow \infty$ as $m \rightarrow \infty$.

Similarly for $|x_{2m+2}|$.

This completes the proof. □

Example (5)

Figure 5 shows that, if $a = 0.64$, $b = 1.6$ ($b^2 = 4a$ and $b < 2$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3$, $x_{-1} = -1$ and $x_0 = 2$ is unbounded.

Example (6)

Figure 6 shows that, if $a = 2.25, b = 3$ ($b^2 = 4a$ and $b > 2$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ converges to zero.

4 Case $b^2 < 4a$

In this section, we study the final case when $b^2 < 4a$ and provide the forbidden set of Eq. (1).

Theorem 4.1 *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1). Then*

$$x_n = \begin{cases} \frac{\sin^2 \alpha}{a^{\frac{n}{2}}} \frac{v}{\beta_0\left(\frac{n-1}{2}\right)\beta_{-1}\left(\frac{n+1}{2}\right)}, & n = 1, 3, \dots, \\ \frac{\sin^2 \alpha}{a^{\frac{n}{2}}} \frac{v}{\beta_0\left(\frac{n}{2}\right)\beta_{-1}\left(\frac{n}{2}\right)}, & n = 2, 4, \dots, \end{cases} \quad (8)$$

where $v = x_0 x_{-1} x_{-2}$, $\beta_{-i}(n) = \sqrt{a} x_{-i} \sin n\alpha - x_{-i-1} \sin(n+1)\alpha$,
 $\alpha = \tan^{-1} \frac{\sqrt{4a-b^2}}{b} \in]0, \frac{\pi}{2}[$ and $i = 0, 1$.

Proof We can write the given solution (8) as

$$\begin{aligned} x_{2m+1} &= \frac{\sin^2 \alpha}{a^{m+\frac{1}{2}}} \frac{v}{\beta_0(m)\beta_{-1}(m+1)}, \\ x_{2m+2} &= \frac{\sin^2 \alpha}{a^{m+1}} \frac{v}{\beta_0(m+1)\beta_{-1}(m+1)}. \end{aligned} \quad (9)$$

When $m = 0$,

Fig. 5 $x_{n+1} = \frac{x_n x_{n-2}}{-0.64 x_{n-1} + 1.6 x_{n-2}}$

Fig. 6 $x_{n+1} = \frac{x_n x_{n-2}}{-2.25x_{n-1} + 3x_{n-2}}$

$$\begin{aligned}
 x_1 &= \frac{\sin^2 \alpha}{\sqrt{a}} \frac{v}{\beta_0(0)\beta_{-1}(1)} \\
 &= \frac{\sin^2 \alpha}{\sqrt{a}} \frac{v}{(-x_{-1} \sin \alpha)(\sqrt{a}x_{-1} \sin \alpha - x_{-2} \sin 2\alpha)} \\
 &= \frac{1}{\sqrt{a}} \frac{v}{(-x_{-1})\left(\sqrt{a}x_{-1} - x_{-2} \frac{b}{\sqrt{a}}\right)} = \frac{x_0 x_{-2}}{-ax_{-1} + bx_{-2}} \\
 &= x_1.
 \end{aligned}$$

Similarly

$$\begin{aligned}
 x_2 &= \frac{\sin^2 \alpha}{a} \frac{v}{\beta_0(1)\beta_{-1}(1)} \\
 &= \frac{\sin^2 \alpha}{a} \frac{v}{(\sqrt{a}x_0 \sin \alpha - x_{-1} \sin 2\alpha)(\sqrt{a}x_{-1} \sin \alpha - x_{-2} \sin 2\alpha)} \\
 &= \frac{1}{a} \frac{v}{\left(\sqrt{a}x_0 - x_{-1} \frac{b}{\sqrt{a}}\right)\left(\sqrt{a}x_{-1} - x_{-2} \frac{b}{\sqrt{a}}\right)} \\
 &= \frac{v}{(-ax_0 + bx_{-1})(-ax_{-1} + bx_{-2})} \\
 &= \frac{x_1 x_{-1}}{-ax_0 + bx_{-1}} = x_2.
 \end{aligned}$$

Suppose that the result is true for $m > 0$.

Then

$$\begin{aligned}
& \frac{x_{2m+1}x_{2m-1}}{-ax_{2m} + bx_{2m-1}} \\
&= \frac{\left(\frac{\sin^2 \alpha}{a^{m+\frac{1}{2}} \beta_0(m) \beta_{-1}(m+1)}\right) \left(\frac{\sin^2 \alpha}{a^{m-\frac{1}{2}} \beta_0(m-1) \beta_{-1}(m)}\right)}{-a \frac{\sin^2 \alpha}{a^m \beta_0(m) \beta_{-1}(m)} + b \frac{\sin^2 \alpha}{a^{m-\frac{1}{2}} \beta_0(m-1) \beta_{-1}(m)}} \\
&= \frac{(\sin^2 \alpha) v}{a^{m+\frac{1}{2}} \beta_{-1}(m+1) (-\sqrt{a} \beta_0(m-1) + b \beta_0(m))}.
\end{aligned}$$

Using the identity

$$\sqrt{a} \beta_0(m-1) + \sqrt{a} \beta_0(m+1) = b \beta_0(m)$$

we get

$$\begin{aligned}
& \frac{x_{2m+1}x_{2m-1}}{-ax_{2m} + bx_{2m-1}} \\
&= \frac{\sin^2 \alpha}{a^{m+1}} \frac{v}{\beta_0(m+1) \beta_{-1}(m+1)} \\
&= x_{2m+2}.
\end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2}x_{2m}}{-ax_{2m+1} + bx_{2m}} = x_{2m+3}.$$

This completes the proof. $\square$

In the following result, we show the existence of periodic solutions for Eq. (1).

Theorem 4.2 Assume that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution of Equation (1) and let $a = 1$. If $\alpha = \frac{l}{k} \pi$ is a rational multiple of π (with $0 < l < \frac{k}{2}$), then $\{x_n\}_{n=-2}^{\infty}$ is periodic with prime period $2k$.

Proof Suppose that $a = 1$. Using formula (9), we can write

$$x_{2m+2k+1} = \sin^2 \alpha \frac{v}{\beta_0(m+k) \beta_{-1}(m+1+k)} \quad (10)$$

But for each $i = 0, 1$ we have

$$\begin{aligned}
\beta_{-i}(m+k) &= x_{-i} \sin(m+k)\alpha - x_{-i-1} \sin(m+1+k)\alpha \\
&= x_{-i} \sin(m\alpha + k\alpha) - x_{-i-1} \sin((m+1)\alpha + k\alpha) \\
&= x_{-i} \sin(m\alpha + l\pi) - x_{-i-1} \sin((m+1)\alpha + l\pi) \\
&= (-1)^l (x_{-i} \sin m\alpha - x_{-i-1} \sin(m+1)\alpha) \\
&= (-1)^l \beta_{-i}(m).
\end{aligned} \quad (11)$$

Then

$$\begin{aligned}
 x_{2m+2k+1} &= \sin^2 \alpha \frac{v}{\beta_0(m+k)\beta_{-1}(m+1+k)} \\
 &= \sin^2 \alpha \frac{v}{(-1)^{2l}\beta_0(m)\beta_{-1}(m+1)} \\
 &= x_{2m+1}.
 \end{aligned} \tag{12}$$

Similarly, we can show that $x_{2m+2k+2} = x_{2m+2}$.

This completes the proof. $\square$

Theorem 4.3 Assume that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution of Equation (1). The following statements are true:

- (1) If $a < 1$, then the solution $\{x_n\}_{n=-2}^{\infty}$ is unbounded.
- (2) If $a > 1$, then the solution $\{x_n\}_{n=-2}^{\infty}$ converges to zero.

Proof The proof is a direct consequence using formulas (9). $\square$

Example (7)

Figure 7 shows that, if $a = 1, b = \sqrt{\frac{3-\sqrt{5}}{2}}$ ($b^2 < 4a$ and $\alpha = \frac{2\pi}{5}$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ is periodic with prime period 10.

Example (8)

Figure 8 shows that, if $a = 1, b = \sqrt{3}$ ($b^2 < 4a$ and $\alpha = \frac{\pi}{6}$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) with $x_{-2} = 3, x_{-1} = -1$ and $x_0 = 2$ is periodic with prime period 12.

In the remainder of this section, we give the forbidden set of Eq. (1) (which depends on the relation between a and b).

Clear that, if $x_0 = 0$ and $x_{-1}x_{-2} \neq 0$, then x_3 is undefined. If $x_{-1} = 0$ and $x_0x_{-2} \neq 0$, then x_5 is undefined. If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then x_4 is undefined.

Therefore, the point (x_0, x_{-1}, x_{-2}) with $x_0x_{-1}x_{-2} = 0$ belongs to the forbidden set of Eq. (1).

The following result gives the forbidden sets of Eq. (1) for all values of $a > 0$ and $b > 0$.

Theorem 4.4 The following statements are true:

- (1) If $b^2 > 4a$, then the forbidden set of equation (1) is

$$\begin{aligned}
 F_1 &= \bigcup_{i=0}^2 \{(u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-i} = 0\} \\
 &\cup \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_0 = \frac{u_{-1}}{a} \frac{\theta_{m+1}}{\theta_m} \right\} \\
 &\cup \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-1} = \frac{u_{-2}}{a} \frac{\theta_{m+1}}{\theta_m} \right\}.
 \end{aligned}$$

Fig. 7 $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + \sqrt{\frac{3-\sqrt{5}}{2} x_{n-2}}}$

Fig. 8 $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + \sqrt{3} x_{n-2}}$

(2) If $b^2 = 4a$, then the forbidden set of equation (1) is

$$F_2 = \bigcup_{i=0}^2 \{(u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-i} = 0\} \cup \\ \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_0 = 2 \frac{1+m}{m} u_{-1} \right\} \\ \cup \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-1} = 2 \frac{1+m}{m} u_{-2} \right\}.$$

(3) If $b^2 > 4a$, then the forbidden set of equation (1) is

$$F_3 = \bigcup_{i=0}^2 \{(u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-i} = 0\} \cup \\ \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_0 = \frac{\sin(m+1)\alpha}{\sqrt{a} \sin m\alpha} u_{-1} \right\} \\ \cup \bigcup_{m=1}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}) \in \mathbb{R}^3 : u_{-1} = \frac{\sin(m+1)\alpha}{\sqrt{a} \sin m\alpha} u_{-2} \right\}.$$

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards.

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