Fibonacci numbers in graphs with strong $(1, 1, 2)$ -kernels

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Abstract

In this paper, we study strong $(1, 1, 2)$ -kernels in graphs which generalize classical kernels, $(1, 2)$ -kernels and 2-dominating kernels, simultaneously. We give a complete characterization of trees with a strong $(1, 1, 2)$ -kernel. Moreover, we present some realization theorems and show connections between the number of strong $(1, 1, 2)$ -kernels and Fibonacci and Lucas numbers.

Keywords Domination · Independence · Kernel · Fibonacci numbers

Mathematics Subject Classification 05C20 · 05C69

1 Introduction

In general, we use the standard terminology and notation of graph theory, see [8]. Let G be an undirected, connected, simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is its number of vertices and the *size* of G is its number of edges. For a vertex $x \in V(G)$ let $d_G(x)$ denote its degree. By P_n , for $n \geq 2$, C_n and S_n for $n \geq 3$, we mean a *path*, a *cycle* and a *star* of order n , respectively. Let T_n , $n \geq 1$, be a tree of order n . In particular, a tree of order one is trivial. In any nontrivial tree, a vertex of degree 1 is a *leaf* and $L(G) = \{u \in V(G) : d_G(u) = 1\}$. A vertex adjacent to at least one leaf is a *support vertex*. Let $x \in V(G)$ be a support vertex, we define $L(x)$ to be the subset of leaves adjacent to the vertex x , and define $L[x]$ as $L[x] = L(x) \cup \{x\}$. Notice that

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$L[x] \subseteq L(G)$. Let J be a subset of $V(G)$ and $x \in V(G)$, we define $d_G(x, J)$, the distance between x and the set J , to be $d_G(x, J) = \min_{y \in J} d_G(x, y)$. The *diameter* of G is the number $\text{diam}(G) = \max_{x, y \in V(G)} d_G(x, y)$. A tree containing exactly two support vertices is called a *2-palm*. A 2-palm of diameter 3 and order $n, n \geq 4$ (also called a *double star*) with the support vertex x adjacent to n_1 leaves and the support vertex y adjacent to $n - n_1 - 2$ leaves will be denoted as $P(n, n_1)$. It is obvious, that x and y are adjacent. If $n_1 \geq 2$ and $n - n_1 \geq 4$, then such a 2-palm is denoted as $P_{2+}(n, n_1)$.

Let G, H be two disjoint graphs and let $x \in V(G), y \in V(H)$. The *local augmentation* of a graph G by a graph H is a graph denoted by $ad_{G(x,y)}(H)$ obtained from G and H by identifying vertices x and y .

A subset $S \subseteq V(G)$ is an *independent* set of G if no two vertices of S are adjacent in G . A maximum independent set will be called an α -set. A subset $D \subseteq V(G)$ is a *dominating* set of G if each vertex not belonging to D is adjacent to at least one vertex of D . A subset $J \subseteq V(G)$ being independent and dominating is a *kernel* of G .

The concept of classical kernels was introduced for digraphs in [23] by von Neumann and Morgenstern in the context of game theory. Since then, kernels have been relevant in graph theory for their relations with different problems, for example, in list colourings and perfectness. Berge was one of the pioneers in this area studying the problem of the existence of kernels in digraphs and using a kernel for solving problems in other areas of mathematics, see [7]. There are many types and generalizations of kernels, for example, (k, l) -kernels which generalize kernels in the distance sense and kernels by monochromatic paths which generalize kernels with respect to edge colouring of a graph. These types of kernels are studied mainly in digraphs, for details see papers [1, 11–13, 17, 22].

In an undirected graph, every maximal independent set is its kernel. The problem is more complicated if we add restrictions related to the domination or the independence. In [15], the authors introduced the concept of a $(1, k)$ -dominating set. A set $S \subseteq V(G)$ is called a $(1, k)$ -dominating set of G if for every vertex $v \in V(G) \setminus S$, there are two distinct vertices $u, w \in S$, such that u is adjacent to v and $d_G(v, w) \leq k$. If $k = 1$, then a $(1, 1)$ -dominating set is a 2-dominating set. If $k = 2$, then $(1, 2)$ -domination is also called *secondary domination*. Moreover, in [15], the authors also introduced a $(1, 2)$ -kernel (called an independent $(1, 2)$ -set) as a subset of vertices being independent and $(1, 2)$ -dominating simultaneously. The existence and the number of $(1, 2)$ -kernels in graph products was recently studied in [18].

In [24], Włoch introduced 2-dominating kernels in graphs using 2-domination, which is a special case of multiple domination introduced by Fink and Jacobson in [10]. Let $k \geq 1$ be an integer. A subset $D \subseteq V(G)$ is a k -dominating set of G if each vertex from $V(G) \setminus D$ has at least k neighbours in D . If $k = 1$, then a 1-dominating set is a dominating set in the classical sense, if $k = 2$, then we obtain the definition of a 2-dominating set. A subset $J \subseteq V(G)$ is a *2-dominating kernel* of G (shortly $(2-d)$ -kernel) if J is independent and 2-dominating. Some properties of $(2-d)$ -kernels were studied in [1–3] and [24]. Based on the definition of $(2-d)$ -kernel, Nagy introduced k -dominating kernels as independent and k -dominating sets. The number of k -dominating kernels were considered in [19] and later in [20]. Note that the existence of k -dominating kernels requires sufficiently large degrees of vertices,

significantly reducing the number of graphs with k -dominating kernels. Consequently, it seems to be interesting to limit requirements for degrees of vertices by considering secondary domination. This point of view leads to the concept of $(1, 1, 2)$ -kernels which was introduced in [5] and was studied in [4].

A subset $D \subseteq V(G)$ is a $(1, 1, 2)$ -dominating set of G if for each vertex $x \notin D$ there are three distinct vertices $y_1, y_2, y_3 \in D$ such that $xy_i \in E(G)$ for $i \in \{1, 2\}$ and $d_G(x, y_3) \leq 2$. A subset $J \subseteq V(G)$ is a $(1, 1, 2)$ -kernel of G if it is independent and $(1, 1, 2)$ -dominating set. Clearly every $(1, 1, 2)$ -kernel is a kernel in the classical sense, a $(1, 2)$ -kernel and a 2-dominating kernel.

Let D be a $(1, 1, 2)$ -dominating set. If for every vertex $x \notin J$, there is $y \in J$ such that $d_G(x, y) = 2$, then D is a strong $(1, 1, 2)$ -dominating set of G . A set $J \subseteq V(G)$ which is independent and a strong $(1, 1, 2)$ -dominating will be called a strong $(1, 1, 2)$ -kernel. If G has only one vertex, then we say it has a trivial strong $(1, 1, 2)$ -kernel.

In this paper, we give a complete characterization of trees with a strong $(1, 1, 2)$ -kernel. We show connections between the number of strong $(1, 1, 2)$ -kernels and Fibonacci and Lucas numbers.

2 Strong $(1, 1, 2)$ -kernels in trees

Paths and cycles are graphs without strong $(1, 1, 2)$ -kernels, so it is natural to consider first the class of trees with respect to the existence of strong $(1, 1, 2)$ -kernels. In this section, we present necessary and sufficient conditions for the existence of strong $(1, 1, 2)$ -kernels in trees using special graph operations.

Let us consider families $\mathcal{T}^{(n)}$, $n \geq 0$, of trees and in each $T \in \mathcal{T}^{(n)}$ we distinguish a subset $J_T^{(n)} \subseteq V(T)$. Let $\mathcal{T}^{(0)} = \{P_1\}$ and $J_{P_1}^{(0)} = V(P_1)$, $\mathcal{T}^{(1)} = \{P_{2^+}(t, n_1); t \geq 6\}$ and $J_{P_{2^+}}^{(1)}(t, n_1) = L(P_{2^+}(t, n_1))$, where $n_1 \geq 2, t - n_1 \geq 4$.

For $n \geq 2$ every tree $T \in \mathcal{T}^{(n)}$ is obtained using one of the following operations:

Operation $\mathcal{O}_1$. Let $T_1 \in \mathcal{T}^{(i)}, T_2 \in \mathcal{T}^{(j)}, 1 \leq i, j \leq n - 1$. Then $T \cong ad_{T_1(x,y)}(T_2)$, where $x \in L(T_1)$ and $y \in L(T_2)$. Therefore $J_T^{(n)} = J_{T_1}^{(i)} \cup J_{T_2}^{(j)}$. Operation $\mathcal{O}_2$. Let $T^* \in \mathcal{T}^{(n-1)}$. Then $T \cong ad_{T^*(x,y)}(S_p)$, where $p \geq 3$, and $y \in L(S_p)$ and $x \notin J_{T^*}^{(n-1)}$. Then $J_T^{(n)} = J_{T^*}^{(n-1)} \cup (L(S_p) \setminus \{y\})$.

Let $\mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}^{(n)}$.

Theorem 1 A tree T has a strong $(1, 1, 2)$ -kernel if and only if $T \in \mathcal{T}$.

Proof Suppose that T is a tree of order t and it has a strong $(1, 1, 2)$ -kernel J . Then, T is isomorphic to P_1 or $t \geq 6$. If T is isomorphic to P_1 , then $T \in \mathcal{T}^{(0)} \subset \mathcal{T}$. If T is isomorphic to a 2-palm $P_{2^+}(t, n_1)$, hence $T \in \mathcal{T}^{(1)} \subset \mathcal{T}$. Let $t \geq 6$ and suppose that for every tree T of order $s < t$ with a strong $(1, 1, 2)$ -kernel, $T \in \mathcal{T}$. We shall show that a tree of order t with a strong $(1, 1, 2)$ -kernel belongs to $\mathcal{T}$.

Let T be a tree of order t with a $(1, 1, 2)$ -kernel J . There exists a vertex $x \in V(T)$, such that $d_T(x) = |L(x)| + 1$ and $|L(x)| \geq 2$; otherwise, it would contradict that T has a strong $(1, 1, 2)$ -kernel. Hence, there exists a unique vertex $y \in N_T(x) \setminus L(x)$.

Clearly $L(x) \subset J$ and $x \notin J$. From the definition of a strong $(1, 1, 2)$ -kernel, there is a vertex $u \in J$, such that $d_T(x, u) = 2$. Clearly u is adjacent to y . Consequently, $y \notin J$ and there are at least two vertices from J adjacent to y . Let $J^* = J \setminus L(x)$ and $(T \setminus L[x]) \cup \{y\}$. Consider the following possibilities:

1. There exists a vertex $v \in J^*$, such that $d_T(y, v) = 2$. Since the tree T has a strong $(1, 1, 2)$ -kernel and there are at least two vertices from J^* adjacent to y , then J^* is a strong $(1, 1, 2)$ -kernel of a tree $T^* = (T \setminus L[x]) \cup \{y\}$. Then, by induction hypothesis, $T^* \in \mathcal{T}$ and by the operation $\mathcal{O}_2$, we have that $T \in \mathcal{T}$.
2. For each vertex $v \in J^*$ it holds $d_T(y, v) \neq 2$. This means that every vertex u adjacent to y , $u \neq x$ belongs to J . Moreover, there are at least two vertices $u_1, u_2 \in J^*$ adjacent to y and $d_T(y, L(x)) = 2$. Consequently, the set $N[x] \cup N[y]$ induces a 2-palm with $L(x) \cup L(y) \subset J$. Then, the subset J^* is a strong $(1, 1, 2)$ -kernel of the graph $T' = T \setminus L[x]$. It is clear that the graph T' is a forest, in which some connected components may be trivial.

Suppose that T' consists of k trees, $k \geq 1$, $T'_1, \dots, T'_k$. Then T'_p , $1 \leq p \leq k$, has a strong $(1, 1, 2)$ -kernel and, by induction hypothesis, $T'_p \in \mathcal{T}$. Consequently, using the 2-palm $N[x] \cup N[y]$ and applying operation $\mathcal{O}_1$ repeatedly, for each tree $T'_1, \dots, T'_k$ we obtain the tree T , so $T \in \mathcal{T}$.

Conversely, let $T \in \mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}^{(n)}$. Hence $T \in \mathcal{T}^{(n)}$ for some $n \geq 0$. We shall show that every tree $T \in \mathcal{T}^{(n)}$ has a strong $(1, 1, 2)$ -kernel. We prove it by induction on n . If $n \in \{0, 1\}$, then subset $J_{P_1}^{(0)}$, $J_{P_{2^+}}^{(1)}(n, n_1)$ are strong $(1, 1, 2)$ -kernels, respectively. Let $n \geq 2$ and assume that every $T^* \in \mathcal{T}^{(s)}$ for $s < n$ has a strong $(1, 1, 2)$ -kernel $J_{T^*}^{(s)}$. We prove that every $T \in \mathcal{T}^{(n)}$ has a strong $(1, 1, 2)$ -kernel. For this purpose, we consider the following possibilities:

- 2.1 $\mathcal{O}_1$ is the n th operation. Then, $T \in \mathcal{T}^{(n)}$ is obtained as $T \cong ad_{T_1(x,y)}(T_2^*)$, where $T_1 \in \mathcal{T}^{(i)}$, $T_2 \in \mathcal{T}^{(j)}$, $1 \leq i, j \leq n-1$ and $x \in L(T_1^*)$ and $y \in L(T_2^*)$. By the definition of family $\mathcal{T}$ and induction hypothesis, it immediately follows that $J_{T_1}^{(i)} \cup J_{T_2}^{(j)}$ is a strong $(1, 1, 2)$ -kernel of the tree T .
- 2.2 $\mathcal{O}_2$ is the n th operation. Then proving as in the previous case we obtain that $J_{T^*}^{(n-1)} \cup (L(S_p) \setminus \{y\})$ is a strong $(1, 1, 2)$ -kernel of the tree T .

Thus the theorem is proved. $\square$

Theorem 2 *If a tree has a strong $(1, 1, 2)$ -kernel, then this kernel is unique and it is an α -set.*

Proof Suppose that T has a strong $(1, 1, 2)$ -kernel J . Then, by Theorem 1, we have that $T \in \mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}^{(n)}$. We use induction on n . If $n = 1$, then $T \cong P_{2^+}(t, n_1)$ and $J = L(P_{2^+}(t, n_1))$ is the unique strong $(1, 1, 2)$ -kernel of T . Let $n \geq 2$ and suppose that for each $T' \in \bigcup_{0 \leq m < n} \mathcal{T}^{(m)}$ a tree T' has a unique strong $(1, 1, 2)$ -kernel. We

shall show that T has a unique strong (1, 1, 2)-kernel. Suppose on contrary that T has two strong (1, 1, 2)-kernels. Since $T \in \mathcal{T}$, so T is obtained by applying at least one of the operations $\mathcal{O}_1$ or $\mathcal{O}_2$. If T is obtained by the operation $\mathcal{O}_1$, then $T = ad_{T_1(x,y)}(T_2)$, where $x \in L(T_1), y \in L(T_2)$. By induction hypothesis $T_i, i \in \{1, 2\}$ has a unique strong (1, 1, 2)-kernel and consequently, by the definition of $\mathcal{O}_1$, T has a unique strong (1, 1, 2)-kernel, which is a contradiction. In the same way we prove if T is obtained by the operation $\mathcal{O}_2$. Moreover, by the definition of the family $\mathcal{T}$ a unique strong (1, 1, 2)-kernel is an α -set. $\square$

Let us consider the families $\tilde{\mathcal{T}}^{(n)} \subset \mathcal{T}^{(n)}, n \geq 0$, of trees and for each $T \in \tilde{\mathcal{T}}^{(n)}$ we distinguish the subset $\tilde{J}_T^{(n)} \subseteq V(T)$. Let $\tilde{T}^{(0)} = \{P_1\}$ and $\tilde{J}_T^{(0)} = V(P_1)$, $\tilde{T}^{(1)} = \{P(6, 2)\}$ and $\tilde{J}_{P(6,2)}^{(1)} = L(P(6, 2))$.

For $n \geq 2$ every tree $T \in \tilde{\mathcal{T}}^{(n)}$ is obtained using the operation $\mathcal{O}_1$, then $\tilde{J}_T^{(n)} = \tilde{J}_{T_1}^{(i)} \cup \tilde{J}_{T_2}^{(j)}$.

Theorem 3 *If T is a tree of order $n, n \geq 6$, with a strong (1, 1, 2)-kernel J , then $\sum_{v \in J} d_T(v) \leq 3\alpha(T) - n - 2$.*

The equality holds if and only if $T \in \tilde{\mathcal{T}}^{(n)}$.

Proof Since J is an α -set, then $|V(T) \setminus J| = n - \alpha$. Using the handshake lemma

$$2(n - 1) = \sum_{v \in J} d_T(v) + \sum_{u \notin J} d_T(u).$$

Moreover, for every $u \notin J, d_T(u) \geq 3$, hence

$$\begin{aligned} 2(n - 1) &= \sum_{v \in J} d_T(v) + \sum_{u \notin J} d_T(u) \geq \sum_{v \in J} d_T(v) + 3(n - \alpha(T)) \\ 2(n - 1) &\geq \sum_{v \in J} d_T(v) + 3(n - \alpha(T)) \\ 2n - 2 &\geq \sum_{v \in J} d_T(v) + 3n - 3\alpha(T). \end{aligned}$$

Hence, $\sum_{v \in J} d_T(v) \leq 3\alpha(T) - n - 2$.

If $T \in \tilde{\mathcal{T}}^{(n)}$, then by the definition of an operation $\mathcal{O}_1$, immediately follows that for every vertex $x \notin J$ there is $d_T(x) = 3$ and the equality is true.

Suppose now that $\sum_{v \in J} d_T(v) \leq 3\alpha(T) - n - 2$. Clearly, the equality holds only for the case if $d_T(u) = 3$ for all $u \notin J$. Consequently, the tree T is obtained recursively using the operation $\mathcal{O}_1$ starting with $T \cong P(6, 2)$, thus $T \in \tilde{\mathcal{T}}^{(n)}$.

Thus the theorem is proved. $\square$

3 Realization theorems

Independent sets in graphs have interesting relations with Fibonacci numbers and numbers of the Fibonacci type (i.e. numbers defined by the linear recurrence relations), see for details [6, 14].

We recall that the n th Fibonacci number F_n is defined recursively as follows: $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with $F_0 = F_1 = 1$. The Lucas numbers L_n are defined by the same recurrence relation $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. Note that Fibonacci sequence also starts with $F_0 = 0$, $F_1 = 1$. In this paper, we use initial conditions $F_0 = F_1 = 1$ based on the fundamental paper of Prodinger and Tichy, see [21]. In 1982, they initiated the huge interest in graph interpretations of Fibonacci numbers by simple observation that the total number of independent sets in paths and cycles is given by Fibonacci and Lucas numbers, respectively. For these reasons, the number of all independent sets in the graph is called the *Fibonacci number* of the graph. Interest in studying this graph parameter has multiplied by two American chemists Merrifield and Simmons. They defined the structure descriptor of a molecular graph as the number of all independent sets. Actually, the number of all independent sets in a graph is called the *Merrifield–Simmons index*, see details in the survey paper of Gutman and Wagner [14]. The results obtained by Prodinger and Tichy gave an impetus for counting independent sets in graphs, as well as for studying graph interpretations of numbers of the Fibonacci type, see [9, 25]. Graph interpretations of numbers of the Fibonacci type gave a new tool to study properties of numbers of the Fibonacci type. Using graph interpretations, we can give new identities, direct formulas and other properties. For these reasons, it is important to find a graph with a possible simple structure, so that the total number of special subsets is given by Fibonacci numbers. In this section, based on this idea, we construct a graph such that the total number of strong $(1, 1, 2)$ -kernels is equal to Fibonacci and Lucas numbers. Using this graph and the concept of strong $(1, 1, 2)$ -kernels, we give a new proof of Honsberger identity, as well as the new identity for Fibonacci numbers. In this section we also present a construction of a graph in which the smallest and the largest cardinality of a strong $(1, 1, 2)$ -kernel is equal to any two numbers a, b , ($a < b$), respectively.

Let $x \in V(G)$ be an arbitrary vertex. We define $\sigma_x(G)$ (respectively $\sigma_{-x}(G)$) to be the number of all strong $(1, 1, 2)$ -kernels in a graph G containing the vertex x (resp. not containing the vertex x) and

$$\sigma(G) = \sigma_x(G) + \sigma_{-x}(G), \quad (1)$$

is a basic rule for counting strong $(1, 1, 2)$ -kernels in G .

Let $n \geq 2$ be an integer. By $(n-1)C_4$ we mean a graph consisting of $n-1$ disjoint 4-cycles C_4^i , for $1 \leq i \leq n-1$, where $V(C_4^i) = \{x_1^i, x_2^i, x_3^i, x_4^i\}$.

For $n \geq 1$ let G_n be a graph, such that $G_1 \cong P(6, 2)$ and for $n \geq 2$, we have $V(G_n) = V((n-1)C_4) \cup V(P(6, 2))$ and $E(G_n) = E((n-1)C_4) \cup E(P(6, 2)) \cup \{x_2^j, x_4^{j+1}; j \in \{1, \dots, n-2\} \cup \{x, x_k^i\}\}$, where x is a fixed support vertex of $P(6, 2)$ and $i \in \{1, \dots, n-1\}$, $k \in \{1, \dots, 4\}$. The graph G_n is presented in Fig. 1.

Theorem 4 *Let $n \geq 1$ be an integer. Then $\sigma(G_n) = F_n$.*

Proof Proof by induction on n . If $n = 1$, then $G_1 \cong P(6, 2)$ and it has a unique strong $(1, 1, 2)$ -kernel $J = L(P(6, 2))$. Therefore $\sigma(G_1) = 1 = F_1$. If $n = 2$, then from the construction of a graph G_2 , follows that there are exactly two strong

Fig. 1 Graph G_n

$(1, 1, 2)$ -kernels $J_1 = L(P(6, 2)) \cup \{x_1^1, x_3^1\}$ and $J_2 = L(P(6, 2)) \cup \{x_2^1, x_4^1\}$. Hence, $\sigma(G_2) = 2 = F_2$, assume that $\sigma(G_m) = F_m$ for $n \geq m$. We prove that $\sigma(G_n) = F_n$. Let $J \subset V(G_n)$ be a strong $(1, 1, 2)$ -kernel of a graph G_n . Let us consider the vertex $x_1^{n-1} \in V(C_{4}^{n-1})$ and we distinguish two possibilities:

1. $x_1^{n-1} \in J$. Then, using the definition of a strong $(1, 1, 2)$ -kernel, and by the construction of G_n , the vertex $x_3^{n-1} \in J$ and vertices x_2^{n-1}, x_4^{n-1} are not in J . Since the vertex x cannot belong to any strong $(1, 1, 2)$ -kernel of G_n , then $J = J^* \cup \{x_1^{n-1}, x_3^{n-1}\}$, where J^* is an arbitrary strong $(1, 1, 2)$ -kernel of G_{n-1} . Hence, $\sigma_{x_1^{n-1}}(G_n) = \sigma(G_{n-1})$.
2. $x_1^{n-1} \notin J$. Then, vertices $x_2^{n-1}, x_4^{n-1} \in J$ by the definition of J . Moreover, the vertices $x_1^{n-2}, x_3^{n-2} \in J$ by the independence of J . Hence, from the construction of G_n , we obtain that $J = J^{**} \cup \{x_2^{n-1}, x_4^{n-1}, x_1^{n-2}, x_3^{n-2}\}$, where J^{**} is any strong $(1, 1, 2)$ -kernel of G_{n-2} . Therefore, $\sigma_{-x_1^{n-1}}(G_n) = \sigma(G_{n-2})$.

Taking into account both cases, induction hypothesis and the rule (1), we obtain that the number of all strong $(1, 1, 2)$ -kernels in G_n is equal to $\sigma(G_n) = \sigma_{x_1^{n-1}}(G_n) + \sigma_{-x_1^{n-1}}(G_n) = F_{n-1} + F_{n-2} = F_n$, for all $n \geq 1$. Thus the theorem is proved. $\square$

Now, we give a new proof of the Honsberger identity using the above graph interpretation.

Theorem 5 (Honsberger [16]) *Let $n \geq 2$ be an integer. Then $F_n = F_{k-1}F_{n-k-1} + F_kF_{n-k}$ for $1 \leq k \leq n-1$.*

Proof If $n = 2$, then $k = 1$ and the identity holds. Let $n \geq 3, 1 \leq k \leq n - 1$ and $J \subset V(G_n)$ be a strong $(1, 1, 2)$ -kernel of G_n . Then, by Theorem 4, we have that $\sigma(G_n) = F_n$. We have to consider the following cases:

1. If $x_1^k \in J$, then $x_3^k \in J$ and by independence of J it follows that $x_2^k \notin J$ and $x_4^k \notin J$. By the construction of G_n we obtain that $J = J^* \cup \{x_1^k, x_3^k\}$, where J^* is an arbitrary strong $(1, 1, 2)$ -kernel of $G_n \setminus C_4^k$. Since $L(P(6, 2)) \subset J^*$, then $\sigma_{x_1^k}(G_n) = \sigma(G_k)\sigma(G_{n-k})$.
2. If $x_1^k \notin J$, then it is clear that $x_2^k, x_4^k \in J$ and consequently it follows that $x_1^{k-1}, x_3^{k-1}, x_1^{k+1}, x_3^{k+1} \in J$. Hence $J = J^{**} \cup \{x_2^k, x_4^k, x_1^{k-1}, x_3^{k-1}, x_1^{k+1}, x_3^{k+1}\}$, where J^{**} is an arbitrary strong $(1, 1, 2)$ -kernel of $G_n \setminus \{C_4^{k-1}, C_4^k, C_4^{k+1}\}$. Since $L(P(6, 2)) \subset J^{**}$, then $\sigma_{-x_1^k}(G_n) = \sigma(G_{k-1})\sigma(G_{n-k-1})$.

From the above cases and based on the rule (1), we have that

$$\sigma(G_n) = \sigma(G_k)\sigma(G_{n-k}) + \sigma(G_{k-1})\sigma(G_{n-k-1}).$$

Using Theorem 4 we obtain that

$$F_n = F_{k-1}F_{n-k-1} + F_kF_{n-k},$$

for $1 \leq k \leq n - 1$. Thus the theorem is proved. $\square$

Theorem 6 Let $n \geq 2, p \geq 2$ be integers. Then

- (a) $F_{pn} = F_{(p-1)n}F_n + F_{(p-1)n-1}F_{n-1}$
- (b) $F_{pn+1} = F_{(p-1)n}F_{n+1} + F_{(p-1)n-1}F_n$.

Proof To prove identities (a) and (b), we use the graph G_{pn} and G_{pn+1} , respectively. First, let us consider the graph G_{pn} hence, from Theorem 4, we have that $\sigma(G_{pn}) = F_{pn}$. Let $J \subset V(G_{pn})$ be an arbitrary strong $(1, 1, 2)$ -kernel. To calculate the number $\sigma(G_{pn})$, we choose the vertex $x_1^{(p-1)-1} \in V(C_4^{(p-1)n-1})$ and distinguish the following possibilities.

If $x_1^{(p-1)-1} \in J$, then $x_2^{(p-1)n-1} \notin J, x_4^{(p-1)n-1} \notin J$ and $x_3^{(p-1)n-1} \in J$. Hence $J = J^* \cup \{x_1^{(p-1)-1}, x_3^{(p-1)n-1}\}$, where J^* is an arbitrary strong $(1, 1, 2)$ -kernel of $G_{pn} \setminus C_4^{(p-1)n-1}$. Since $L(P(6, 2)) \subset J$, so $L(P(6, 2)) \subset J^*$. From Theorem 4 this implies that

$$\sigma_{x_1^{(p-1)n-1}}(G_{pn}) = \sigma(G_{(p-1)n})\sigma(G_n) = F_{(p-1)n}F_n.$$

If $x_1^{(p-1)n-1} \notin J$, then

$$J = J' \cup \{x_2^{(p-1)n-1}, x_4^{(p-1)n-1}, x_1^{(p-1)n-2}, x_3^{(p-1)n-2}, x_1^{(p-1)n}, x_3^{(p-1)n}\},$$

where J' is an arbitrary strong $(1, 1, 2)$ -kernel of $G_{pn} \setminus \{C_4^{(p-1)n-2}, C_4^{(p-1)n-1}, C_4^{(p-1)n}\}$. Clearly $L(P(6, 2)) \subset J'$, so

$$\sigma_{-x_1^{(p-1)n-1}}(G_{pn}) = \sigma(G_{(p-1)n-1})\sigma(G_{n-1}) = F_{(p-1)n-1}F_{n-1}.$$

From the above possibilities and the rule (1), we obtain the identity (a). In the same way we prove the identity (b) considering the graph G_{pn+1} .

Thus the theorem is proved. $\square$

Note, that for $p = 2$ we obtain the well-known identities for Fibonacci numbers $F_{2n} = F_n^2 + F_{n-1}^2$ and $F_{2n+1} = F_n F_{n+1} + F_{n-1} F_n$.

Let $H_n, n \geq 1$, be a graph such that $V(H_n) = V(G_{n+1})$ and $E(H_n) = E(G_{n+1}) \cup \{x_4^1 x_2^n\}$. This graph is presented in Fig. 2.

Theorem 7 Let $n \geq 1$ be an integer. Then $\sigma(H_n) = L_n$.

Proof Let $n \geq 1$ be an integer. If $n = 1$, then $J = L(P(6, 2)) \cup \{x_1^1, x_3^1\}$ and hence $\sigma(H_1) = 1 = L_1$. If $n = 2$, then H_2 has three strong (1, 1, 2)-kernels $J_1 = L(P(6, 2)) \cup \{x_4^1, x_2^1\} \cup \{x_1^1, x_3^1\}$, $J_2 = L(P(6, 2)) \cup \{x_1^1, x_3^1\} \cup \{x_2^1, x_3^1\}$ or $J_3 = L(P(6, 2)) \cup \{x_1^1, x_3^1\} \cup \{x_2^1, x_4^1\}$. Hence $\sigma(H_2) = 3 = L_2$. Let $n \geq 3$ and suppose that $J \subset V(H_n)$ be a strong (1, 1, 2)-kernel. Let us consider a vertex $x_1^n \in V(H_n)$. We distinguish two possibilities:

1. $x_1^n \in J$. Then $x_3^n \in J$. From the construction of H_n it follows that $J = J^* \cup \{x_1^n, x_3^n\}$, where J^* is an arbitrary strong (1, 1, 2)-kernel of $H_n \setminus C_4^n$ which is isomorphic to G_n . Hence, by the Theorem 4, we obtain that $\sigma_{x_1^n}(H_n) = \sigma(G_n) = F_n$.
2. $x_1^n \notin J$ Then $x_2^n, x_4^n \in J$. Moreover, $\{x_1^{n-1}, x_3^{n-1}, x_1^1, x_3^1\} \in J$. From the construction of H_n we obtain that $J = J^{**} \cup \{x_2^n, x_4^n, x_1^{n-1}, x_3^{n-1}, x_1^1, x_3^1\}$, where J^{**} is an

Fig. 2 Graph H_n

arbitrary strong $(1, 1, 2)$ -kernel of a graph $H_n \setminus \{C_4^1, C_4^{n-1}, C_4^n\}$ which is isomorphic to a graph G_{n-2} . Therefore $\sigma_{-x_1^n}(H_n) = \sigma(G_{n-2}) = F_{n-2}$.

Finally, taking into account both cases we have $\sigma(H_n) = F_n + F_{n-2}$, for $n \geq 3$.

Using a well-known identity for Fibonacci and Lucas numbers $L_n = F_n + F_{n-2}$ and the fact that $\sigma(H_1) = 1 = L_1, \sigma(H_2) = 3 = L_2$ we obtain that $\sigma(H_n) = L_n$ for $n \geq 1$. $\square$

Let $I_{(1,1,2)}(G)$ be the largest cardinality of a strong $(1, 1, 2)$ -kernel of a graph G and let $i_{(1,1,2)}(G)$ be the smallest cardinality of this kernel.

Theorem 8 *Let a, b be positive integers and $5 \leq a < b$. There exists a graph G , such that $i_{(1,1,2)}(G) = a$ and $I_{(1,1,2)}(G) = b$.*

Proof Let a, b be integers, $5 \leq a < b$. We give a construction of a graph G such that $i_{(1,1,2)}(G) = a$ and $I_{(1,1,2)}(G) = b$. Let us consider a 2-palm $P(a+b, 2)$ of order $a+b$ with support vertices x, y . The graph G is defined in the following way $V(G) = V(P(a+b, 2))$ and $E(G) = E(P(a+b, 2)) \cup E_0$, where E_0 is a set of added edges in the 2-palm in a special way. To describe the set E_0 we introduce a 2-colouring function by distinguishing $a-2$ red vertices and $b-2$ blue. Then every blue vertex is adjacent to exactly two red vertices, assuming that none of red vertices is adjacent to all $b-2$ blue vertices. Moreover, every red vertex is joined with at least two blue vertices. Denote by B the set consisting of blue vertices and by R consisting of red vertices. Hence $B \cap R = \emptyset$ and $|B| = a-2, |R| = b-2$. By $\{w_1, w_2\}$ we denote the set of two leaves adjacent to the support vertex y . We prove that the set $J_1 = \{w_1, w_2\} \cup R$ is the smallest strong $(1, 1, 2)$ -kernel of the graph G and the set $J_2 = \{w_1, w_2\} \cup B$ is the largest strong $(1, 1, 2)$ -kernel of G . The independence of J_1 and J_2 is obvious. Now we show that J_1 and J_2 are strong $(1, 1, 2)$ -dominating sets. Vertices $x, y \notin J_1$ and $x, y \notin J_2$. The support vertex y is 2-dominated by w_1 and w_2 and $d_G(y, B) = 2$ and $d_G(y, R) = 2$. Hence y is strong $(1, 1, 2)$ -dominated by J_1 and J_2 . Analogously we can show that a vertex x is strong $(1, 1, 2)$ -dominated by J_1 and J_2 . Similarly every red vertex $r_i, i \in \{1, \dots, a-2\}$, is adjacent to two blue vertices which are in J_2 and there exists a vertex blue $b_j, j \in \{1, \dots, b-2\}$, such that $d_G(r_i, b_j) = 2$. Hence vertices from R are strong $(1, 1, 2)$ -dominated by J_2 . Analogously we can show that the vertices from B are strong $(1, 1, 2)$ -dominated by J_1 . Therefore $i_{(1,1,2)}(G) = |J_1| = |R| + |L(x)| = a$ and $I_{(1,1,2)}(G) = |J_2| = |B| + |L(x)| = b$. Thus the theorem is proved. $\square$

4 Concluding remarks

Strong $(1, 1, 2)$ -kernels are derived from k -independent dominating sets in graphs introduced and studied by Nagy in [19] who generalized $(2-d)$ -kernels by considering k -dominating sets instead of 2-dominating sets. This generalized concept is interesting, but studying the existence problems of k -independent dominating sets, we must assume that vertices have sufficiently large degrees. So, we can find more interesting results, if we consider special cases for k . For $k \geq 2$

some results were obtained, so it is natural to consider $k \geq 3$. To limit degrees of vertices, we consider the hybrid version of kernels as $(2-d)$ -kernels and additionally the secondary dominating sets assuming that the distance 2 is attained for each vertex being outside a $(1, 1, 2)$ -kernel. Note that if we omit the restriction about the distance 2, then we obtain a simple relation that every 3-independent dominating set is a $(1, 1, 2)$ -kernel. Such a point of view seems to be interesting in distinct location problems by preserving the third vertex being at the distance 2. It will also be interesting to consider a weaker version of a strong $(1, 1, 2)$ -kernel, namely a proper $(1, 1, 2)$ -kernel as a subset of vertices which is a $(1, 1, 2)$ -kernel but it is not a 3-independent dominating set. To study proper $(1, 1, 2)$ -kernels in graphs, the results obtained for strong $(1, 1, 2)$ -kernels will be helpful.

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards.

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References

1. Bednarz, P., Hernandez-Cruz, C., Włoch, I.: On the existence and number of $(2-d)$ -kernels in graphs. *Ars Comb.* **121**, 341–351 (2015)
2. Bednarz, P., Włoch, I.: On $(2-d)$ -kernels in the cartesian product of graphs. *Ann. UMCS Sec. A* **70**(2), 1–8 (2016)
3. Bednarz, P., Włoch, I.: An algorithm determining $(2-d)$ -kernels in trees. *Util. Math.* **102**, 215–222 (2017)
4. Bednarz, U.: Strong $(1,1,2)$ -kernels in the corona of graphs and some realization problems. *Iran. J. Sci. Technol. A* **44**, 401–406 (2020)
5. Bednarz, U., Włoch, I.: On strong $(1,1,2)$ -kernels in graphs. *Ars Comb.* (**in press**)
6. Bednarz, U., Włoch, I., Wołowicz-Musiał, M.: Total graph interpretation of numbers of the Fibonacci type. *J. Appl. Math.* **2015**(837917), 7 (2015)
7. Berge, C.: *Graphs and Hypergraphs*. North-Holland, Amsterdam (1973)
8. Diestel, R.: *Graph Theory*. Springer, Heidelberg (2005)
9. Dosal-Trujillo, L.A., Galeana-Sanchez, H.: The Fibonacci numbers of certain subgraphs of circulant graphs. *AKCE Int. J. Graphs Comb.* **12**(2–3), 94–103 (2015)
10. Fink, J.F., Jacobson, M.S.: *n-Domination in Graphs, Graph Theory with Applications to Algorithms and Computer Science*, pp. 283–300. Wiley-Intersci. Publ., Wiley, New York (1985)
11. Galeana-Sanchez, H., Ramirez, L.A.: Monochromatic kernel-perfectness of special classes of digraphs. *Discuss. Math. Graph Theory* **27**(3), 389–400 (2007)
12. Galeana-Sanchez, H., Sanchez-Lopez, R.: H-kernels in the D-join. *Ars Comb.* **99**, 353–377 (2011)
13. Galeana-Sanchez, H., Hernandez-Cruz, C.: On the existence of (k, l) -kernels in infinite digraphs: a survey. *Discuss. Math. Graph Theory* **34**, 431–466 (2014)

14. Gutman, I., Wagner, S.: Maxima and minima of the Hosoya index and the Merrifield–Simmons index. A survey of results and techniques. *Acta Appl. Math.* **112/3**, 323–346 (2010)
15. Hedetniemi, S.M., Hedetniemi, S.T., Knisely, J., Rall, D.F.: Secondary domination in graphs. *AKCE J. Graphs. Comb.* **5**(2), 103–115 (2008)
16. Honsberger, R.: A Second Look at the Fibonacci and Lucas Numbers, *Mathematical Gems III*. Math. Assoc. Amer., Washington, D.C. (1985)
17. Kucharska, M.: On (k, l) -kernel perfectness of special classes of digraphs. *Discuss. Math. Graph Theory* **25**, 103–119 (2005)
18. Michalski, A., Włoch, I.: On the existence and the number of independent $(1,2)$ -dominating sets in the G -join of graphs. *Appl. Math. Comput.* **377**, 125155 (2020)
19. Nagy, Z.: On the number of k -dominating independent sets. *J. Graph Theor.* **84**(4), 566–580 (2017)
20. Nagy, Z.: Generalizing Erdős, Moon and Moser’s result—the number of k -dominating independent sets. *Electron. Notes Discrete Math.* **61**, 909–915 (2017)
21. Prodinger, H., Tichy, R.F.: Fibonacci numbers of graphs. *Fibonacci Q.* **20**, 16–21 (1982)
22. Szumny, W., Włoch, A., Włoch, I.: On the existence and on the number of (k, l) -kernels in the lexicographic product of graphs. *Discrete Math.* **308**, 4616–4624 (2008)
23. von Neumann, J., Morgenstern, O.: *Theory of Games and Economic Behavior*. Princeton University Press, Princeton (1953)
24. Włoch, A.: On 2-dominating kernels in graphs. *Australas. J. Comb.* **53**, 273–284 (2012)
25. Zhao, H., Li, X.: On the Fibonacci numbers of trees. *Fibonacci Q.* **44**(1), 32 (2006)

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