Classes of harmonic functions defined by convolution

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Abstract

The object of the present paper is to investigate classes of harmonic functions defined by convolution. Some necessary and sufficient conditions, topological properties, radii of convexity and starlikeness, as well as extreme points for the classes are considered.

Keywords Harmonic functions · Convolution · Subordination · Extreme points · Starlike functions

Mathematics Subject Classification Primary 30C55 · Secondary 30C45

1 Introduction

Let $\mathcal{H}$ denote the class of complex harmonic functions in $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) := \{z \in \mathbb{C} : |z| < r\}$, and let $\mathcal{H}_0$ denote the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} (a_n z^n + \overline{b_n z^n}) \quad (z \in \mathbb{U}). \quad (1)$$

By $\mathcal{S}_{\mathcal{H}}$ we denote the class of functions $f \in \mathcal{H}_0$, which are univalent and sense preserving in $\mathbb{U}$.

We say that a function $f \in \mathcal{S}_{\mathcal{H}}$ is harmonic starlike in $\mathbb{U}(r)$ if f maps the circle $\partial\mathbb{U}(r)$ onto a closed curve that is starlike with respect to the origin, i.e., $\frac{\partial}{\partial t} (\arg f(re^{it})) > 0$ ($0 \leq t \leq 2\pi$) or equivalently

$$\operatorname{Re} \frac{D_{\mathcal{H}} f(z)}{f(z)} > 0 \quad (|z| = r),$$

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where

$$D_{\mathcal{H}} f(z) := z f'_z(z) - \bar{z} f'_{\bar{z}}(z) \quad (z \in \mathbb{U}).$$

Let us consider the function $\varphi \in \mathcal{H}$ of the form

$$\varphi = u + \bar{v}, \quad u(z) = z + \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=0}^{\infty} v_n z^n \quad (z \in \mathbb{U}) \quad (2)$$

with

$$|u_n| \geq 1, \quad |v_n| \geq 1 \quad (n = 2, 3, \dots). \quad (3)$$

We say that a function $f \in \mathcal{H}$ of the form (1) is correlated with the function φ , if

$$u_n a_n = -|u_n| |a_n|, \quad v_n b_n = |v_n| |b_n| \quad (n = 2, 3, \dots).$$

Let $|\eta| = 1$. We say that a function $f \in \mathcal{H}$ of the form (1) is η -correlated with the function φ , if it is correlated with the function

$$\varphi_{\eta}(z) := \eta^{-1} \varphi(\eta z) \quad (z \in \mathbb{U}).$$

In particular, functions correlated with the function

$$\varphi(z) = 2\operatorname{Re} \frac{z}{1-z} = \sum_{n=1}^{\infty} (z^n + \bar{z}^n) \quad (z \in \mathbb{U}), \quad (4)$$

are the functions with negative coefficients introduced by Silverman [34]. Also, functions η -correlated with the function φ of the form (4) are the functions with varying coefficients defined by Jahangiri and Silverman [21] (see also [17]). These functions are intensively investigated by many authors (for example, see [6–23, 27–34]).

We say that a function $f : \mathbb{U} \rightarrow \mathbb{C}$ is subordinate to a function $F : \mathbb{U} \rightarrow \mathbb{C}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0) = 0$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathbb{U}).$$

In particular, if F is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec F(z) \iff [f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U})].$$

Let A, B be real parameters with $B > \max\{0, A\}$ and let φ be given by (2). We denote by $\mathcal{W}_{\mathcal{H}}(\varphi; A, B)$ the class of functions $f \in \mathcal{S}_{\mathcal{H}}$ such that

$$\frac{D_{\mathcal{H}}(\varphi * f)}{\varphi * f} \prec \frac{1 + Az}{1 + Bz}, \quad (5)$$

where $f * \varphi$ denote the convolution of f and φ , defined by

$$(\varphi * f)(z) = \sum_{n=0}^{\infty} a_n u_n z^n + \sum_{n=1}^{\infty} \overline{b_n v_n} \bar{z}^n \quad (z \in \mathbb{U}).$$

Also, by $\mathcal{V}_{\mathcal{H}}(\varphi; A, B)$ we denote the class of functions $f \in \mathcal{S}_{\mathcal{H}}$ such that

$$\frac{(\varphi * f)(z)}{z} \prec \frac{1 + Az}{1 + Bz}. \quad (6)$$

Let $\mathcal{T}^{\eta}(\varphi)$ denote the class of functions $f \in \mathcal{H}$ of coefficients η -correlated with the function φ , i.e.,

$$u_n a_n = -|u_n| |a_n| \eta^{1-n}, \quad \overline{v_n b_n} = |v_n| |b_n| \eta^{1+n} \quad (n = 2, 3, \dots). \quad (7)$$

Moreover, let us define

$$\begin{aligned} \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B) &:= \mathcal{T}^{\eta}(\varphi) \cap \mathcal{W}_{\mathcal{H}}(\varphi; A, B), \quad \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B) := \mathcal{T}^{\eta}(\varphi) \cap \mathcal{V}_{\mathcal{H}}(\varphi; A, B), \\ \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; \alpha) &:= \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; 2\alpha - 1, B), \quad \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; \alpha) := \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; 2\alpha - 1, B). \end{aligned}$$

In the paper, we obtain some analytic criteria, radii of convexity and starlikeness for defined classes of functions. Using extreme points theory, we obtain coefficients estimates, distortion theorems and integral mean inequalities. Some applications of the obtained results are also considered.

2 Analytic criteria

Let $\mathcal{V} \subset \mathcal{H}$, $\mathbb{U}_0 := \mathbb{U} \setminus \{0\}$. Motivated by Ruscheweyh [31] we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^* := \left\{ f \in \mathcal{S}_{\mathcal{H}} : \bigwedge_{q \in \mathcal{V}} (f * q)(z) \neq 0 \quad (z \in \mathbb{U}_0) \right\}.$$

First, we show that the defined classes of functions can be presented as dual sets.

Theorem 1

$$\mathcal{W}_{\mathcal{H}}(\varphi; A, B) = \{\varphi * \psi_{\xi} : |\xi| = 1\}^*,$$

where

$$\psi_{\xi}(z) := z \frac{(B - A)\xi + (1 + A\xi)z}{(1 - z)^2} - \bar{z} \frac{2 + (A + B)\xi - (1 + A\xi)\bar{z}}{(1 - \bar{z})^2} \quad (z \in \mathbb{U}). \quad (8)$$

Proof Let $f \in \mathcal{S}_{\mathcal{H}}$. Then $f \in \mathcal{W}_{\mathcal{H}}(\varphi; A, B)$ if and only if it satisfies (5) or equivalently

$$\frac{D_{\mathcal{H}}(\varphi * f)}{\varphi * f} \neq \frac{1 + A\xi}{1 + B\xi} \quad (\xi \in \mathbb{C}, \quad |\xi| = 1). \quad (9)$$

Since $f = h + \bar{g}$ for some analytic functions h, g , and

$$D_{\mathcal{H}}h(z) = h(z) * \frac{z}{(1-z)^2}, \quad h(z) = h(z) * \frac{z}{1-z},$$

we have

$$\begin{aligned} & 0 \neq (1 + B\xi) D_{\mathcal{H}}(\varphi * f)(z) - (1 + A\xi)(\varphi * f)(z) \\ &= (1 + B\xi) D_{\mathcal{H}}(h * u)(z) - (1 + A\xi)(h * u)(z) \\ &\quad - \left[(1 + B\xi) \overline{D_{\mathcal{H}}(g * v)(z)} + (1 + A\xi) \overline{(g * v)(z)} \right] \\ &= (h * u)(z) * \left(\frac{(1 + B\xi)z}{(1-z)^2} - \frac{(1 + A\xi)z}{1-z} \right) \\ &\quad - \overline{(g * v)(z)} * \left(\frac{(1 + B\xi)\bar{z}}{(1-\bar{z})^2} + \frac{(1 + A\xi)\bar{z}}{1-\bar{z}} \right) \\ &= f(z) * (\varphi * \psi_{\xi})(z) \end{aligned}$$

Thus, $f \in \mathcal{W}_{\mathcal{H}}(\varphi; A, B)$ if and only if $f(z) * (\varphi * \psi_{\xi})(z) \neq 0$ for $z \in \mathbb{U}_0$, $|\xi| = 1$, i.e., $\mathcal{W}_{\mathcal{H}}(\varphi; A, B) = \{\varphi * \psi_{\xi} : |\xi| = 1\}^*$. $\square$

Similarly as Theorem 1 we prove the following theorem.

Theorem 2

$$\mathcal{V}_{\mathcal{H}}(\varphi; A, B) = \{\varphi * \delta_{\xi} : |\xi| = 1\}^*,$$

where

$$\delta_{\xi}(z) := z \frac{(B-A)\xi + (1+A\xi)z}{1-z} + \bar{z} \frac{1+B\xi}{1-\bar{z}} \quad (z \in \mathbb{U}). \quad (10)$$

Theorem 3 Let $f \in \mathcal{H}$ be of the form (1) and

$$\sum_{n=2}^{\infty} (|\gamma_n| |a_n| + |\delta_n| |b_n|) \leq B - A, \quad (11)$$

where

$$\gamma_n = u_n (n(1+B) - (1+A)), \quad \delta_n = v_n (n(1+B) + (1+A)). \quad (12)$$

Then $f \in \mathcal{W}_{\mathcal{H}}(\varphi; A, B)$.

Proof Let $f \in \mathcal{H}$ be a function of the form (1). By a theorem of Lewy [25], a necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{U}$ is that

$$|h'(z)| > |g'(z)| \quad (z \in \mathbb{U}). \quad (13)$$

By (3) we obtain

$$\frac{|\gamma_n|}{B-A} \geq n, \quad \frac{|\delta_n|}{B-A} \geq n \quad (n = 2, 3, \dots) \quad (14)$$

and, by (11) we get

$$\sum_{n=2}^{\infty} (n |a_n| + n |b_n|) \leq 1. \quad (15)$$

Moreover,

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^n - \sum_{n=2}^{\infty} n |b_n| |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (n |a_n| + n |b_n|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|) \geq 1 - |z| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Thus, by (13) the function f is locally univalent and sense preserving in $\mathbb{U}$. Also, for $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$, we have

$$\left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| = \left| \sum_{k=1}^n z_1^{k-1} z_2^{n-k} \right| \leq \sum_{k=1}^n |z_1|^{k-1} |z_2|^{n-k} < n \quad (n = 2, 3, \dots).$$

Hence, by (15) we get

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &\geq \left| z_1 - z_2 - \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n) \right| - \left| \sum_{n=2}^{\infty} \overline{b_n} (\overline{z_1^n - z_2^n}) \right| \\ &\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| |z_1^n - z_2^n| - \sum_{n=2}^{\infty} |b_n| |z_1^n - z_2^n| \\ &= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} (|a_n| + |b_n|) \left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| \right) > 0, \end{aligned}$$

which gives $f(z_1) \neq f(z_2)$ and, in consequence, $f \in \mathcal{S}_{\mathcal{H}}$. Thus, we need only prove that the function f satisfies (4), i.e., there exists a complex-valued function ω , $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$) such that

$$\frac{D_{\mathcal{H}}(f * \varphi)(z)}{(f * \varphi)(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{U}),$$

or equivalently

$$\left| \frac{D_{\mathcal{H}}(f * \varphi)(z) - (f * \varphi)(z)}{B D_{\mathcal{H}}(f * \varphi)(z) - A (f * \varphi)(z)} \right| < 1 \quad (z \in \mathbb{U}). \quad (16)$$

Therefore, it is suffice to prove that

$$|D_{\mathcal{H}}(f * \varphi)(z) - (f * \varphi)(z)| - |BD_{\mathcal{H}}(f * \varphi)(z) - A(f * \varphi)(z)| < 0 \quad (z \in \mathbb{U} \setminus \{0\}).$$

Indeed, letting $|z| = r$ ($0 < r < 1$) we have

$$\begin{aligned} & |D_{\mathcal{H}}(f * \varphi)(z) - (f * \varphi)(z)| - |BD_{\mathcal{H}}(f * \varphi)(z) - A(f * \varphi)(z)| \\ &= \left| \sum_{n=2}^{\infty} (n-1) u_n a_n z^n - \sum_{n=2}^{\infty} (n+1) \bar{v}_n \bar{b}_n \bar{z}^n \right| \\ &\quad - \left| (B-A)z + \sum_{n=2}^{\infty} (Bn-A) u_n a_n z^n - \sum_{n=2}^{\infty} (Bn+A) \bar{v}_n \bar{b}_n \bar{z}^n \right| \\ &\leq \sum_{n=2}^{\infty} (n-1) |u_n a_n| r^n + \sum_{n=2}^{\infty} (n+1) |v_n b_n| r^n - (B-A)r \\ &\quad + \sum_{n=2}^{\infty} (Bn-A) |u_n a_n| r^n + \sum_{n=2}^{\infty} (Bn+A) |v_n b_n| r^n \\ &\leq r \left\{ \sum_{n=2}^{\infty} (|\gamma_n| |a_n| + |\delta_n| |b_n|) r^{n-1} - (B-A) \right\} < 0. \end{aligned}$$

whence $f \in \mathcal{W}_{\mathcal{H}}(\varphi; A, B)$. □

Now, we show that the condition (11) is also a sufficient condition for a function $f \in \mathcal{T}^{\eta}(\varphi)$ to be in the class $\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$.

Theorem 4 *Let $f \in \mathcal{H}_0$ be a function of the form (1). Then $f \in \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ if and only if the condition (11) holds true.*

Proof In view of Theorem 3, we have to prove “only if” part, i.e., each function from the class $\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ satisfies the condition (11). If $f \in \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$, then it is of the form (1) with (7) and satisfies (16) or equivalently

$$\left| \frac{\sum_{n=2}^{\infty} \left\{ (n-1) \eta^{1-n} |u_n| |a_n| z^n + (n+1) \eta^{1+n} |v_n| |b_n| \bar{z}^n \right\}}{(B-A)z - \sum_{n=2}^{\infty} \left\{ (Bn-A) \eta^{1-n} |u_n| |a_n| z^n + (Bn+A) \eta^{1-n} |v_n| |b_n| \bar{z}^n \right\}} \right| < 1 \quad (z \in \mathbb{U}).$$

Thus, letting $z = r\eta$ ($0 \leq r < 1$), we have

$$\frac{\sum_{n=2}^{\infty} \left\{ (n-1) |u_n| |a_n| + (n+1) |v_n| |b_n| r^{n-1} \right\}}{(B-A) - \sum_{n=2}^{\infty} \left\{ (Bn-A) |u_n| |a_n| + (Bn+A) |v_n| |b_n| r^{n-1} \right\}} < 1. \quad (17)$$

It is clear that the denominator of the left-hand side cannot vanish for $r \in (0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in (0, 1)$. Thus, by (17) we have

$$\sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|) r^{n-1} < B - A \quad (0 \leq r < 1). \quad (18)$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|)$ is a nondecreasing sequence. Moreover, by (18) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|) = \lim_{n \rightarrow \infty} S_n \leq B - A,$$

which yields the assertion (11). $\square$

The following result may be proved in the same way as Theorem 4.

Theorem 5 *Let $f \in \mathcal{H}_0$ be a function of the form (1). Then $f \in \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ if and only if*

$$\sum_{n=2}^{\infty} (|u_n| |a_n| + |v_n| |b_n|) \leq \frac{B - A}{1 + B}.$$

By Theorems 4 and 5 we have the following corollary.

Corollary 1 *Let $a = \frac{1+A}{1+B}$ and*

$$\begin{aligned} \phi(z) &= z + \sum_{n=k}^{\infty} \left(\frac{1}{n-a} z^n + \frac{1}{n+a} \bar{z}^n \right) \quad (z \in \mathbb{U}), \\ \omega(z) &= z + \sum_{n=k}^{\infty} ((n-a) z^n + (n+a) \bar{z}^n) \quad (z \in \mathbb{U}). \end{aligned}$$

Then

$$\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B) = \mathcal{W}_{\mathcal{T}}^{\eta}(\varphi * \phi; A, B), \quad \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B) = \mathcal{V}_{\mathcal{T}}^{\eta}(\varphi * \omega; A, B).$$

In particular,

$$\mathcal{W}_{\mathcal{T}}^{\eta}(\varphi; -1, B) = \mathcal{V}_{\mathcal{T}}^{\eta}(D_{\mathcal{H}}(\varphi); -1, B).$$

3 Extreme points

Let us consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A functions $f \in \mathcal{F}$ is called an extreme point of $\mathcal{F}$ if the condition

$$f = \gamma f_1 + (1 - \gamma) f_2 \quad (f_1, f_2 \in \mathcal{F}, 0 < \gamma < 1)$$

implies $f_1 = f_2 = f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is locally uniformly bounded if for each r , $0 < r < 1$, there is a real constant $M = M(r)$ so that

$$|f(z)| \leq M \quad (f \in \mathcal{F}, |z| \leq r).$$

We say that a class $\mathcal{F}$ is convex if

$$\gamma f + (1 - \gamma)g \in \mathcal{F} \quad (f, g \in \mathcal{F}, 0 \leq \gamma \leq 1).$$

Moreover, we define the closed convex hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{\text{co}}\mathcal{F}$.

A real-valued functional $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$ is called convex on a convex class $\mathcal{F} \subset \mathcal{H}$ if

$$\mathcal{J}(\gamma f + (1 - \gamma)g) \leq \gamma \mathcal{J}(f) + (1 - \gamma)\mathcal{J}(g) \quad (f, g \in \mathcal{F}, 0 \leq \gamma \leq 1).$$

The Krein–Milman theorem (see [24]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 1 [12, pp. 45] *Let $\mathcal{F}$ be a non-empty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max \{\mathcal{J}(f) : f \in \mathcal{F}\} = \max \{\mathcal{J}(f) : f \in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem (see [26]) implies the following lemma.

Lemma 2 *A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 6 *The class $\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ is convex and compact subset of $\mathcal{H}$.*

Proof Let $f_k \in \mathcal{H}$, $k \in \mathbb{N}$, be functions of the form

$$f_k(z) = \sum_{n=0}^{\infty} a_{k,n} z^n + \sum_{n=1}^{\infty} \overline{b_{k,n} z^n} \quad (z \in \mathbb{U}). \quad (19)$$

Let $f_1, f_2 \in \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ be functions of the form (19), $0 \leq \gamma \leq 1$. Since

$$\gamma f_1(z) + (1 - \gamma)f_2(z) = z + \sum_{n=2}^{\infty} \left\{ (\gamma a_{1,n} + (1 - \gamma)a_{2,n}) z^n + \overline{(\gamma b_{1,n} + (1 - \gamma)b_{2,n}) z^n} \right\},$$

and by Theorem 4 we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \{ |\gamma_n| |\gamma a_{1,n} + (1-\gamma) a_{2,n}| + |\delta_n| |\gamma b_{1,n} + (1-\gamma) b_{2,n} z^n| \} \\ & \leq \gamma \sum_{n=2}^{\infty} \{ |\gamma_n a_{1,n}| + |\delta_n b_{1,n}| \} + (1-\gamma) \sum_{n=2}^{\infty} \{ |\gamma_n a_{2,n}| + |\delta_n b_{2,n}| \} \\ & \leq \gamma (B-A) + (1-\gamma) (B-A) = B-A, \end{aligned}$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\mathcal{W}_T^\eta(\varphi; A, B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{W}_T^\eta(\varphi; A, B)$, $|z| \leq r$, $0 < r < 1$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \leq r + \sum_{n=2}^{\infty} (\gamma_n |a_n| + \delta_n |b_n|) \leq r + (B-A). \quad (20)$$

Thus, we conclude that the class $\mathcal{W}_T^\eta(\varphi; A, B)$ is locally uniformly bounded. By Lemma 2, we only need to show that it is closed, i.e., if $f_k \in \mathcal{W}_T^\eta(\varphi; A, B)$ ($k \in \mathbb{N}$) and $f_k \rightarrow f$, then $f \in \mathcal{W}_T^\eta(\varphi; A, B)$. Let f_k and f be given by (19) and (1), respectively. Using Theorem 4 we have

$$\sum_{n=2}^{\infty} (|\gamma_n a_{k,n}| + |\delta_n b_{k,n}|) \leq B-A \quad (k \in \mathbb{N}). \quad (21)$$

Since $f_k \rightarrow f$, we conclude that $|a_{k,n}| \rightarrow |a_n|$ and $|b_{k,n}| \rightarrow |b_n|$ as $k \rightarrow \infty$ ($n \in \mathbb{N}$). The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|)$ is a nondecreasing sequence. Moreover, by (21) it is bounded by $B-A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty} (|\gamma_n a_n| + |\delta_n b_n|) = \lim_{n \rightarrow \infty} S_n \leq B-A.$$

This gives the condition (11), and, in consequence, $f \in \mathcal{W}_T^\eta(\varphi; A, B)$, which completes the proof. $\square$

Theorem 7

$$E\mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B) = \{h_n : n \in \mathbb{N}\} \cup \{g_n : n \in \{2, 3, \dots\}\},$$

where $h_1(z) = z$ and

$$h_n(z) = z - \frac{B-A}{\eta^{n-1}\gamma_n} z^n, \quad g_n(z) = z + \frac{B-A}{\eta^{-n-1}\delta_n} \bar{z}^n \quad (z \in \mathbb{U}). \quad (22)$$

Proof Suppose that $0 < \gamma < 1$ and

$$g_n = \gamma f_1 + (1 - \gamma) f_2,$$

where $f_1, f_2 \in \mathcal{W}_T^\eta(\varphi; A, B)$ are functions of the form (19). Then, by (11) we have $|b_{1,n}| = |b_{2,n}| = \frac{B-A}{|\delta_n|}$, and, in consequence, $a_{1,k} = a_{2,k} = 0$ for $k \in \{2, 3, \dots\}$ and $b_{1,k} = b_{2,k} = 0$ for $k \in \{2, 3, \dots\} \setminus \{n\}$. It follows that $g_n = f_1 = f_2$, and consequently $g_n \in E\mathcal{W}_T^\eta(\varphi; A, B)$. Similarly, we verify that the functions h_n of the form (22) are the extreme points of the class $\mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$. Now, suppose that a function f belongs to the set $E\mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$ and f is not of the form (22). Then there exists $m \in \{2, 3, \dots\}$ such that

$$0 < |a_m| < \frac{B-A}{|\gamma_m|} \quad \text{or} \quad 0 < |b_m| < \frac{B-A}{|\delta_m|}.$$

If $0 < |a_m| < \frac{B-A}{|\gamma_m|}$, then putting

$$\gamma = \frac{|\gamma_m a_m|}{B-A}, \quad \varphi = \frac{1}{1-\gamma} (f - \gamma h_m),$$

we have that $0 < \gamma < 1$, $h_m, \varphi \in \mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$, $h_m \neq \varphi$ and

$$f = \gamma h_m + (1 - \gamma) \varphi.$$

Thus, $f \notin E\mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$. Similarly, if $0 < |b_m| < \frac{B-A}{|\delta_m|}$, then putting

$$\gamma = \frac{|\delta_m b_m|}{B-A}, \quad \phi = \frac{1}{1-\gamma} (f - \gamma g_m),$$

we have that $0 < \gamma < 1$, $g_m, \phi \in \mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$, $g_m \neq \phi$ and

$$f = \gamma g_m + (1 - \gamma) \phi.$$

It follows that $f \notin E\mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$, and the proof is completed. $\square$

4 Radii of starlikeness and convexity

We say that a function $f \in \mathcal{H}_0$ is starlike of order α in $\mathbb{U}(r)$ if

$$\frac{\partial}{\partial t} \left(\arg f(\rho e^{it}) \right) > \alpha, \quad 0 \leq t \leq 2\pi, \quad 0 < \rho < r < 1. \quad (23)$$

Analogously, we say that a function $f \in \mathcal{H}_0$ is convex of order α in $\mathbb{U}(r)$ if

$$\frac{\partial}{\partial t} \left(\arg \frac{\partial}{\partial t} f(\rho e^{it}) \right) > \alpha, \quad 0 \leq t \leq 2\pi, \quad 0 < \rho < r < 1.$$

It is easy to verify that for a function $f \in \mathcal{T}(\varphi)$ the condition (23) is equivalent to the following

$$\operatorname{Re} \frac{D_{\mathcal{H}} f(z)}{f(z)} > \alpha \quad (z \in \mathbb{U}(r))$$

or equivalently

$$\left| \frac{D_{\mathcal{H}} f(z) - (1 + \alpha) f(z)}{D_{\mathcal{H}} f(z) + (1 - \alpha) f(z)} \right| < 1 \quad (z \in \mathbb{U}(r)) \quad (24)$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{H}_0$. We define the radius of starlikeness $R_{\alpha}^*(\mathcal{B})$ and the radius of convexity $R_{\alpha}^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$R_{\alpha}^*(\mathcal{B}) := \inf_{f \in \mathcal{B}} (\sup \{r \in (0, 1] : f \text{ is starlike of order } \alpha \text{ in } \mathbb{U}(r)\}),$$

$$R_{\alpha}^c(\mathcal{B}) := \inf_{f \in \mathcal{B}} (\sup \{r \in (0, 1] : f \text{ is convex of order } \alpha \text{ in } \mathbb{U}(r)\}).$$

Theorem 8 *The radius of starlikeness of order α for the class $\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ is given by*

$$R_{\alpha}^*(\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)) = \inf_{n \geq 2} \left(\frac{1 - \alpha}{B - A} \min \left\{ \frac{|\gamma_n|}{n - \alpha}, \frac{|\delta_n|}{n + \alpha} \right\} \right)^{\frac{1}{n-1}}, \quad (25)$$

where γ_n and δ_n are defined by (12).

Proof Let $f \in \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ be of the form (1) with (7). Then, for $|z| = r < 1$ we have

$$\begin{aligned} \left| \frac{D_{\mathcal{H}} f(z) - (1 + \alpha) f(z)}{D_{\mathcal{H}} f(z) + (1 - \alpha) f(z)} \right| &= \left| \frac{-\alpha z + \sum_{n=2}^{\infty} ((n - 1 - \alpha) |a_n| z^n - (n + 1 + \alpha) |b_n| \bar{z}^n)}{(2 - \alpha) z + \sum_{n=2}^{\infty} ((n + 1 - \alpha) |a_n| z^n - (n - 1 + \alpha) |b_n| \bar{z}^n)} \right| \\ &\leq \frac{\alpha + \sum_{n=2}^{\infty} ((n - 1 - \alpha) |a_n| + (n + 1 + \alpha) |b_n|) r^{n-1}}{2 - \alpha - \sum_{n=2}^{\infty} ((n + 1 - \alpha) |a_n| + (n - 1 + \alpha) |b_n|) r^{n-1}}. \end{aligned}$$

Thus, the condition (24) is true if and only if

$$\sum_{n=2}^{\infty} \left(\frac{n - \alpha}{1 - \alpha} |a_n| + \frac{n + \alpha}{1 - \alpha} |b_n| \right) r^{n-1} \leq 1, \quad (26)$$

By Theorem 1, we have

$$\sum_{n=2}^{\infty} \left(\frac{|\gamma_n|}{B-A} |a_n| + \frac{|\delta_n|}{B-A} |b_n| \right) \leq 1, \quad (27)$$

where γ_n and δ_n are defined by (12). Thus, the condition (26) is true if

$$\frac{n-\alpha}{1-\alpha} r^{n-1} \leq \frac{|\gamma_n|}{B-A}, \quad \frac{n+\alpha}{1-\alpha} r^{n-1} \leq \frac{|\delta_n|}{B-A} \quad (n = 2, 3, \dots),$$

that is, if

$$r \leq \left(\frac{1-\alpha}{B-A} \min \left\{ \frac{|\gamma_n|}{n-\alpha}, \frac{|\delta_n|}{n+\alpha} \right\} \right)^{\frac{1}{n-1}} \quad (n = 2, 3, \dots).$$

It follows that the function f is starlike of order α in the disk $U(r^*)$, where r^*

$$r^* := \inf_{n \geq 2} \left(\frac{1-\alpha}{B-A} \min \left\{ \frac{|\gamma_n|}{n-\alpha}, \frac{|\delta_n|}{n+\alpha} \right\} \right)^{\frac{1}{n-1}}.$$

The functions h_n, g_n of the form (22) realize equality in (27), and the radius r^* cannot be larger. Thus we have (25). $\square$

The following result may be proved in much the same way as Theorem 8.

Theorem 9 *The radius of convexity of order α for the class $\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ is given by*

$$R_{\alpha}^c(\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)) = \inf_{n \geq 2} \left(\frac{1-\alpha}{B-A} \min \left\{ \frac{|\gamma_n|}{n(n-\alpha)}, \frac{|\delta_n|}{n(n+\alpha)} \right\} \right)^{\frac{1}{n-1}},$$

where γ_n and δ_n are defined by (12).

5 Applications

It is clear that if the class $\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \sum_{n=1}^{\infty} \gamma_n = 1, \gamma_n \geq 0 \ (n \in \mathbb{N}) \right\}. \quad (28)$$

Thus, by Theorem 8 we have the following corollary.

Corollary 2

$$\mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B) = \left\{ \sum_{n=1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=1}^{\infty} (\gamma_n + \delta_n) = 1, \delta_1 = 0, \gamma_n, \delta_n \geq 0 \ (n \in \mathbb{N}) \right\},$$

where h_n, g_n are defined by (22).

For each fixed value of $n \in \mathbb{N}$, $z \in \mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_n|, \quad {}_-\mathcal{J}(f) = |b_n|, \quad \mathcal{J}(f) = |f(z)|, \quad \mathcal{J}(f) = |D_{\mathcal{H}}f(z)| \quad (f \in \mathcal{H}). \quad (29)$$

Moreover, for $\gamma \geq 1$, $0 < r < 1$, the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\gamma d\theta \right)^{1/\gamma} \quad (f \in \mathcal{H}) \quad (30)$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma 1 and Theorem 8 we have the following corollaries.

Corollary 3 Let $f \in \mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$ be a function of the form (1). Then

$$|a_n| \leq \frac{B-A}{|\gamma_n|}, \quad |b_n| \leq \frac{B-A}{|\delta_n|} \quad (n = 2, 3, \dots), \quad (31)$$

where γ_n, δ_n are defined by (12). The result is sharp. The functions h_n, g_n of the form (22) are the extremal functions.

Corollary 4 Let $f \in \mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$, $|z| = r < 1$. Then

$$\begin{aligned} r - \frac{B-A}{|u_2|(1+2B-A)} r^2 &\leq |f(z)| \leq r + \frac{B-A}{|u_2|(1+2B-A)} r^2, \\ r - \frac{2(B-A)}{|u_2|(1+2B-A)} r^2 &\leq |D_{\mathcal{H}}f(z)| \leq r + \frac{2(B-A)}{|u_2|(1+2B-A)} r^2, \end{aligned}$$

where u_2 is defined by (2). The result is sharp. The function h_2 of the form (22) is the extremal function.

Corollary 5 Let $0 < r < 1$, $\gamma \geq 1$. If $f \in \mathcal{W}_{\mathcal{H}}^\eta(\varphi; A, B)$, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\gamma d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |h_2(re^{i\theta})|^\gamma d\theta, \\ \frac{1}{2\pi} \int_0^{2\pi} |D_{\mathcal{H}}f(z)|^\gamma d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |D_{\mathcal{H}}h_2(re^{i\theta})|^\gamma d\theta, \end{aligned}$$

where h_2 is the function defined by (22).

The following covering result follows from Corollary 4.

Corollary 6 If $f \in \mathcal{W}_{\mathcal{H}}^{\eta}(\varphi; A, B)$, then $\mathbb{U}(r) \subset f(\mathbb{U})$, where

$$r = 1 - \frac{B - A}{|u_2|(1 + 2B - A)}.$$

Using Corollary 1 and the results above we obtain corollaries listed below.

Corollary 7

$$R_{\alpha}^*(\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)) = \inf_{n \geq 2} \left(\frac{1+B}{B-A} \min \left\{ \frac{1-\alpha}{n-\alpha} n |u_n|, \frac{1-\alpha}{n+\alpha} n |v_n| \right\} \right)^{\frac{1}{n-1}},$$

$$R_{\alpha}^c(\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)) = \inf_{n \geq 2} \left(\frac{1+B}{B-A} \min \left\{ \frac{1-\alpha}{n-\alpha} |u_n|, \frac{1-\alpha}{n+\alpha} |v_n| \right\} \right)^{\frac{1}{n-1}},$$

where u_n and v_n are defined by (2).

Corollary 8 The class $\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ is convex and compact subset of $\mathcal{H}$. Moreover,

$$E\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B) = \{h_n : n \in \mathbb{N}\} \cup \{g_n : n \in \{2, 3, \dots\}\}$$

and

$$\mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B) = \left\{ \sum_{n=1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=1}^{\infty} (\gamma_n + \delta_n) = 1, \delta_1 = 0, \gamma_n, \delta_n \geq 0 \ (n \in \mathbb{N}) \right\},$$

where $h_1(z) = z$ and

$$h_n(z) = z - \frac{(B-A)\eta^{1-n}}{(1+B)u_n} z^n, \quad g_n(z) = z + \frac{(B-A)\eta^{1+n}}{(1+B)v_n} \bar{z}^n \quad (z \in \mathbb{U}). \quad (32)$$

Corollary 9 Let $f \in \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)$ be a function of the form (1). Then us assume (3). If $f \in \mathcal{V}_{\mathcal{H}}^{\eta}(\varphi; A, B)$, then $\mathbb{U}(r) \subset f(\mathbb{U})$, where

$$|a_n| \leq \frac{B-A}{(1+B)|u_n|}, \quad |b_n| \leq \frac{B-A}{(1+B)|v_n|} \quad (n = 2, 3, \dots),$$

$$r - \frac{B-A}{(1+B)|u_2|} r^2 \leq |f(z)| \leq r + \frac{B-A}{(1+B)|u_2|} r^2 \quad (|z| = r < 1),$$

$$r - \frac{2(B-A)}{(1+B)|u_2|} r^2 \leq |D_{\mathcal{H}}f(z)| \leq r + \frac{2(B-A)}{(1+B)|u_2|} r^2 \quad (|z| = r < 1),$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |h_2(re^{i\theta})|^{\lambda} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} f(re^{i\theta}) \right|^\gamma d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} h_2(re^{i\theta}) \right|^\gamma d\theta,$$

where u_n and v_n are defined by (2). The results are sharp. The functions h_n, g_n of the form (32) are the extremal functions.

Corollary 10 Let

$$r = 1 - \frac{B - A}{(1 + B) |u_2|}.$$

6 Concluding remarks

We conclude the paper by observing that, in view of the subordination relation (5) and (6), choosing the function φ and the parameters A, B we can consider new and also well-known classes of functions. In particular, the classes

$$\mathcal{W}_{\mathcal{H}}^\eta(\varphi; \alpha) := \mathcal{W}_{\mathcal{H}}^\eta(\varphi; 2\alpha - 1, B), \quad \mathcal{W}_{\mathcal{H}}^\eta(\varphi; \alpha) := \mathcal{W}_{\mathcal{H}}^\eta(\varphi; 2\alpha - 1, B)$$

contain functions $f \in \mathcal{S}_H$ which satisfy the conditions

$$\begin{aligned} \operatorname{Re} \frac{D_{\mathcal{H}}(\varphi * f)(z)}{(\varphi * f)(z)} &> \alpha \quad (z \in \mathbb{U}), \\ \operatorname{Re} \frac{(\varphi * f)(z)}{z} &> \alpha \quad (z \in \mathbb{U}), \end{aligned}$$

respectively.

The considered classes are defined using the convolution $\varphi * f$ or equivalently by the linear operator

$$J_\varphi : \mathcal{H} \rightarrow \mathcal{H}, \quad J_\varphi(f) = \varphi * f.$$

By choosing the function φ , we can obtain a lot of important linear operators, and in consequence new and also well-known classes of functions.

Let $q, s \in \mathbb{N}_0, q \leq s + 1$. For complex parameters $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$, where $\beta_j \neq 0, -1, -2, \dots$, we define the generalized hypergeometric function by

$$F_{q,s}(z) = {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \quad (z \in \mathbb{U}), \quad (33)$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ , by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}) \end{cases}.$$

Corresponding to the generalized hypergeometric function, Dziok and Srivastava [16] introduced a linear operator on the class of analytic functions, defined by the convolution:

$$H_{q,s}f(z) := (zF_{q,s}(z)) * f(z) \quad (z \in \mathbb{U}).$$

Al-Kharsani and Al-Khal [6] (see also [14]) consider a harmonic generalization of the operator $H_{q,s}$. Let $H_{\mathcal{H}}^{s,\tau} : \mathcal{H} \rightarrow \mathcal{H}$ denote the operator defined by

$$H_{\mathcal{H}}^{s,\tau} f := H_{s+1,s}h + \tau \overline{H_{s+1,s}g} \quad (f = h + \bar{g} \in \mathcal{H}, |\tau| = 1).$$

Let $\mathcal{S}_{\mathcal{H}}^{s,\tau}(A, B)$ denote the class of functions $f \in \mathcal{S}_{\mathcal{H}}$ such that

$$\frac{H_{\mathcal{H}}^{s+1,-\tau} f(z)}{H_{\mathcal{H}}^{s,\tau} f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (\alpha_{s+2} = 2, \beta_{s+1} = 1).$$

Also, by $\mathcal{R}_{\mathcal{H}}^{s,\tau}(A, B)$ we denote the class of functions $f \in \mathcal{S}_{\mathcal{H}}$ such that

$$\frac{H_{\mathcal{H}}^{s,\tau} f(z)}{z} \prec \frac{1 + Az}{1 + Bz}.$$

It is clear that

$$\mathcal{S}_{\mathcal{H}}^{s,\tau}(A, B) = \mathcal{W}_{\mathcal{H}}(\varphi_{\tau}; A, B), \quad \mathcal{R}_{\mathcal{H}}^{s,\tau}(A, B) = \mathcal{V}_{\mathcal{H}}(\varphi_{\tau}; A, B),$$

where

$$\varphi_{\tau}(z) = zF_{s+1,s}(z) + \tau \overline{zF_{s+1,s}(z)} \quad (z \in \mathbb{U}).$$

In particular, if we put $s = \beta_1 = \alpha_2 = \beta_2 = 1, \alpha_1 = 1 + n \in \mathbb{N}, \tau = (-1)^n$, then we obtain the classes

$$\mathcal{S}_{\mathcal{H}}^n(A, B) := \mathcal{S}_{\mathcal{H}}^{1,(-1)^n}(A, B), \quad \mathcal{R}_{\mathcal{H}}^n(A, B) := \mathcal{R}_{\mathcal{H}}^{1,(-1)^n}(A, B)$$

related to the harmonic Ruscheweyh derivatives $\mathcal{D}_{\mathcal{H}}^n(f) := H_{\mathcal{H}}^{1,\tau}(f)$ (see [15]). If we put $s = n \in \mathbb{N}_0, \tau = (-1)^n, \alpha_k = 2, \beta_k = 1$ ($k = 1, 2, \dots, n$), then we obtain the classes

$$\mathcal{V}_{\mathcal{H}}^n(A, B) := \mathcal{S}_{\mathcal{H}}^{n,(-1)^n}(A, B), \quad \mathcal{W}_{\mathcal{H}}^n(A, B) := \mathcal{R}_{\mathcal{H}}^{n,(-1)^n}(A, B)$$

related to the harmonic Sălăgean operator $J_{\mathcal{H}}^n(f) := H_{\mathcal{H}}^{n,\tau}(f)$ (see [7, 11, 17, 37]). The classes $\mathcal{S}_{\mathcal{H}}(A, B) := \mathcal{S}_{\mathcal{H}}^0(A, B) = \mathcal{V}_{\mathcal{H}}^0(A, B)$, $\mathcal{K}_{\mathcal{H}}(A, B) := \mathcal{S}_{\mathcal{H}}^1(A, B) = \mathcal{V}_{\mathcal{H}}^1(A, B)$, and $\mathcal{R}_{\mathcal{H}}(A, B) := \mathcal{R}_{\mathcal{H}}^1(A, B) = \mathcal{W}_{\mathcal{H}}^1(A, B)$ are defined in [12] (see also [17]).

The classes $\mathcal{S}_{\mathcal{H}}^n(A, B)$ and $\mathcal{R}_{\mathcal{H}}^n(A, B)$ are related to harmonic starlike functions, harmonic convex functions and harmonic Janowski functions. The classes $\mathcal{S}_{\mathcal{H}}(\alpha) := \mathcal{S}_{\mathcal{H}}^0(2\alpha - 1, 1)$ and $\mathcal{K}_{\mathcal{H}}(\alpha) := \mathcal{S}_{\mathcal{H}}^1(2\alpha - 1, 1)$ are investigated by Jahangiri [19] (see also [3, 30]). They are the classes of starlike and convex functions of order α , respectively. The classes $\mathcal{N}_{\mathcal{H}}(\alpha) := \mathcal{R}_{\mathcal{H}}^1(2\alpha - 1, 1)$ and $\mathcal{R}_{\mathcal{H}}(\alpha) := \mathcal{R}_{\mathcal{H}}^2(2\alpha - 1, 1)$ are studied in [1] (see also [23]). Finally, the classes $\mathcal{S}_{\mathcal{H}}^* := \mathcal{S}_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}} := \mathcal{K}_{\mathcal{H}}(0)$ are the classes of functions which are starlike and convex in $\mathbb{U}(r)$, respectively, for

all $r \in (0, 1)$ (see [2]). We should notice, that the classes $\mathcal{S}(A, B) := \mathcal{S}_{\mathcal{H}}(A, B) \cap \mathcal{A}$ and $\mathcal{R}(A, B) := \mathcal{R}_{\mathcal{H}}(A, B) \cap \mathcal{A}$ are introduced by Janowski [22].

The class $\mathcal{W}_{\mathcal{H}}(\varphi; A, B)$ generalize also classes of starlike functions of complex order. The class $\mathcal{CS}_{\mathcal{H}}(\gamma) := \mathcal{S}_{\mathcal{H}}(2; 1 - 2\gamma, 1)$ ($\gamma \in \mathbb{C} \setminus \{0\}$) was defined by Yalçın and Öztürk [36]. In particular, if we put $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$, then we obtain the class $\mathcal{RS}_{\mathcal{H}}(\alpha, \eta) := \mathcal{S}_{\mathcal{H}}\left(2; \frac{2\alpha-1+e^{i\eta}}{1+e^{i\eta}}, 1\right)$ studied by Yalçın et al. [37]. It is the class of functions $f \in \mathcal{H}_0$ such that

$$\operatorname{Re} \left\{ \left(1 + e^{i\eta}\right) \frac{D_{\mathcal{H}}f(z)}{f(z)} - e^{i\eta} \right\} > \alpha \quad (z \in \mathbb{U}, \eta \in \mathbb{R}).$$

Thus, using the obtained results to the classes defined above we can obtain new and also well-known results (see for example [4–16, 19–23, 27–37]).

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