On a problem of Pillai with Fibonacci numbers and powers of 3

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Abstract

Consider the sequence $\{F_n\}_{n \geq 0}$ of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In this paper, we find all integers c having at least two representations as a difference between a Fibonacci number and a power of 3.

Keywords Fibonacci numbers · Linear forms in logarithms · Baker’s method

Mathematics Subject Classification 11B39 · 11J86

1 Introduction

We consider the sequence $\{F_n\}_{n \geq 0}$ of Fibonacci numbers defined by

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

The first few terms of the Fibonacci sequence are

$$\{F_n\}_{n \geq 0} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots$$

In this paper, we are interested in studying the Diophantine equation

$$F_n - 3^m = c \tag{1}$$

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for a fixed integer c and variable integers n and m . In particular, we are interested in finding those integers c admitting at least two representations as a difference between a Fibonacci number and a power of 3. This equation is a variant of the Pillai equation

$$a^x - b^y = c \quad (2)$$

where x, y are nonnegative integers and a, b, c are fixed positive integers.

In 1936 and again in 1937, Pillai (see [16,17]) conjectured that for any given integer $c \geq 1$, the number of positive integer solutions (a, b, x, y) , with $x \geq 2$ and $y \geq 2$ to the Eq. (2) is finite. This conjecture is still open for all $c \neq 1$. The case $c = 1$ is Catalan's conjecture which was proved by Mihăilescu (see [15]). Pillai's work was an extension of the work of Herschfeld (see [12,13]), who had already studied a particular case of the problem with $(a, b) = (2, 3)$. Since then, different variants of the Pillai equation have been studied. Some recent results for the different variants of the Pillai problem involving Fibonacci numbers, Tribonacci numbers, Pell numbers and the k -generalized Fibonacci numbers with powers of 2 have been intensively studied in [3–7,10,11].

2 Main result

The main aim of this paper is to prove the following result. Since $F_1 = F_2 = 1$, we discard the situation when $n = 1$ and just count the solutions for $n \geq 2$.

Theorem 1 *The only integers c having at least two representations of the form $F_n - 3^m$ are $c \in \{-26, -6, -1, 0, 2, 4, 7, 12\}$. Furthermore, all the representations of the above integers as $F_n - 3^m$ with integers $n \geq 2$ and $m \geq 0$ are given by*

$$\begin{aligned} -26 &= F_{10} - 3^4 = F_2 - 3^3; \\ -6 &= F_8 - 3^3 = F_4 - 3^2; \\ -1 &= F_6 - 3^2 = F_3 - 3^1 \\ 0 &= F_4 - 3^1 = F_2 - 3^0; \\ 2 &= F_5 - 3^1 = F_4 - 3^0; \\ 4 &= F_7 - 3^2 = F_5 - 3^0; \\ 7 &= F_9 - 3^3 = F_6 - 3^0; \\ 12 &= F_8 - 3^2 = F_7 - 3^0. \end{aligned} \quad (3)$$

3 Auxiliary results

To prove our main result Theorem 1, we need to use several times a Baker-type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [2]. We use the one of Matveev from [14]. Matveev [14] proved the following theorem, which is one of our main tools in this paper.

Let γ be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^d (x - \gamma^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of γ . Then, the *logarithmic height* of γ is given by

$$h(\gamma) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \left(\max\{|\gamma^{(i)}|, 1\} \right) \right).$$

In particular, if $\gamma = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\gamma) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta \gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{4}$$

Theorem 2 (Matveev) *Let $\gamma_1, \dots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D , $b_1, \dots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{5}$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D) (1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \text{ for all } i = 1, \dots, t.$$

During the calculations, we get upper bounds on our variables which are too large; thus, we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 1 Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ be all the convergents of the continued fraction of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then, putting $a(M) := \max\{a_i : i = 0, 1, 2, \dots, N\}$, the inequality

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M) + 2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [8], Lemma 5a), which is itself a generalization of the result of Baker and Davenport [1]. For a real number X , we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2 Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v , and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following lemma is also useful. It is Lemma 7 in [9].

Lemma 3 (Gúzman, Luca) If $m \geq 1$, $T > (4m^2)^m$, and $T > x/(\log x)^m$, then

$$x < 2^m T (\log T)^m.$$

4 Proof of Theorem 1

Assume that there exist nonnegative integers n, m, n_1, m_1 with $\min\{n, n_1\} \geq 2$ and $\min\{m, m_1\} \geq 0$ such that $(n, m) \neq (n_1, m_1)$, and

$$F_n - 3^m = F_{n_1} - 3^{m_1}.$$

Without loss of generality, we can assume that $m \geq m_1$. If $m = m_1$, then $F_n = F_{n_1}$, so $(n, m) = (n_1, m_1)$, which gives a contradiction to our assumption. Thus, $m > m_1$. Since

$$F_n - F_{n_1} = 3^m - 3^{m_1}, \tag{6}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$.

Using the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}} \text{ for all } k \geq 0, \quad (7)$$

where $(\alpha, \beta) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ are the roots of the equation $x^2 - x - 1 = 0$, which is the characteristic equation of the Fibonacci sequence. One can easily prove by induction that

$$\alpha^{k-2} \leq F_n \leq \alpha^{k-1} \text{ for all } k \geq 1. \quad (8)$$

Using the Eq. (6), we get

$$\alpha^{n-4} \leq F_{n-2} \leq F_n - F_{n_1} = 3^m - 3^{m_1} < 3^m, \quad (9)$$

$$\alpha^{n-1} \geq F_n \geq F_n - F_{n_1} = 3^m - 3^{m_1} \geq 3^{m-1}, \quad (10)$$

from which we get that

$$1 + \left(\frac{\log 3}{\log \alpha}\right)(m-1) < n < \left(\frac{\log 3}{\log \alpha}\right)m + 4. \quad (11)$$

If $n \leq 300$, then $m \leq 127$. We ran a *Mathematica* program for $2 \leq n_1 < n \leq 300$ and $0 \leq m_1 < m \leq 127$ and found only the solutions from the list (3). From now, we assume that $n > 300$ and from (11) we have that $m > 127$. Therefore, to solve the Diophantine Eq. (1), it suffices to find an upper bound for n .

4.1 Bounding n

By substituting the Binet formula (7) in the Diophantine Eq. (1), we get

$$\begin{aligned} \left| \frac{\alpha^n}{\sqrt{5}} - 3^m \right| &= \left| \frac{\beta^n}{\sqrt{5}} + \frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} - 3^{m_1} \right| \leq \frac{\alpha^{n_1} + 2}{\sqrt{5}} + 3^{m_1} \\ &\leq \frac{2\alpha^{n_1}}{\sqrt{5}} + 3^{m_1} < 3 \max\{\alpha^{n_1}, 3^{m_1}\}. \end{aligned}$$

Multiplying through by 3^{-m} , using the relation (9) and using the fact that $\alpha < 3$, we get

$$|(\sqrt{5})^{-1} \alpha^n 3^{-m} - 1| < 3 \max \left\{ \frac{\alpha^{n_1}}{3^m}, 3^{m_1-m} \right\} < \max\{\alpha^{n_1-n+7}, 3^{m_1-m+1}\}. \quad (12)$$

For the left-hand side, we apply the result of Matveev, Theorem 2 with the following data:

$$t = 3, \quad \gamma_1 = \sqrt{5}, \quad \gamma_2 = \alpha, \quad \gamma_3 = 3, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m.$$

Throughout we work with the field $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ with $D = 2$. Since $\max\{1, n, m\} \leq 2n$, we take $B := 2n$. Furthermore, we take $A_1 := 2h(\gamma_1) = \log 5$, $A_2 := 2h(\gamma_2) = \log \alpha$, $A_3 := 2h(\gamma_1) = 2 \log 3$. We put

$$\Lambda = (\sqrt{5})^{-1} \alpha^n 3^{-m} - 1.$$

First we check that $\Lambda \neq 0$, if it were, then $\alpha^{2n} \in \mathbb{Q}$, a contradiction. Thus, $\Lambda \neq 0$. Then, by Matveev's theorem, the left-hand side of (12) is bounded as

$$\log |\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log 2n)(\log 5)(\log \alpha)(2 \log 3).$$

By comparing with (12), we get

$$\min\{(n - n_1 - 7) \log \alpha, (m - m_1 - 1) \log 3\} < 1.66 \times 10^{12} (1 + \log 2n),$$

which gives

$$\min\{(n - n_1) \log \alpha, (m - m_1) \log 3\} < 1.67 \times 10^{12} (1 + \log 2n).$$

Now, we split the argument into two cases.

Case 1. $\min\{(n - n_1) \log \alpha, (m - m_1) \log 3\} = (n - n_1) \log \alpha$.

In this case, we rewrite (6) as

$$\left| \left(\frac{\alpha^n - \alpha^{n_1}}{\sqrt{5}} \right) - 3^m \right| = \left| \left(\frac{\beta^n - \beta^{n_1}}{\sqrt{5}} \right) - 3^{m_1} \right| < 1 + 3^{m_1} \leq 3^{m_1+1},$$

which implies

$$\left| \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}} \right) \alpha^{n_1} 3^{-m} - 1 \right| < 3^{m_1-m+1}. \quad (13)$$

We put

$$\Lambda_1 = \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}} \right) \alpha^{n_1} 3^{-m} - 1.$$

To see that $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then

$$\alpha^n - \alpha^{n_1} = \sqrt{5} \cdot 3^m.$$

By conjugating the above relation in $\mathbb{K}$, we get that

$$\beta^n - \beta^{n_1} = -\sqrt{5} \cdot 3^m.$$

The absolute value of the left-hand side is at most $|\beta^n - \beta^{n_1}| \leq |\beta|^n + |\beta|^{n_1} < 2$, while the absolute value of the right-hand side is at least $|\sqrt{5} \cdot 3^m| \geq \sqrt{5} > 2$ for all $m > 127$, which is a contradiction.

We apply Theorem 2 on the left-hand side of (13) with the data

$$t = 3, \quad \gamma_1 = \frac{\alpha^{n-n_1} - 1}{\sqrt{5}}, \quad \gamma_2 = \alpha, \quad \gamma_3 = 3, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m.$$

The minimal polynomial of γ_1 divides

$$5X^2 - 5F_{n-n_1}X - ((-1)^{n-n_1} + 1 - L_{n-n_1}),$$

where $\{L_k\}_{k \geq 0}$ is the Lucas companion sequence of the Fibonacci sequence given by $L_0 = 2$, $L_1 = 1$, $L_{k+2} = 2L_{k+1} + L_k$ for all $k \geq 0$, for which the Binet formula for its general term is given by

$$L_k = \alpha^k + \beta^k \quad \text{for all } k \geq 0.$$

Thus, we obtain

$$\begin{aligned} h(\gamma_1) &\leq \frac{1}{2} \left(\log 5 + \log \left(\frac{\alpha^{n-n_1} + 1}{\sqrt{5}} \right) \right) < \frac{1}{2} \log(2\sqrt{5}\alpha^{n-n_1}) \\ &< \frac{1}{2} (n - n_1 + 2) \log \alpha < 8.4 \times 10^{11} (1 + \log 2n). \end{aligned} \quad (14)$$

So, we can take $A_1 := 1.67 \times 10^{12} (1 + \log 2n)$. Furthermore, as before, we take $A_2 := \log \alpha$ and $A_3 := 2 \log 3$. Finally, since $\max\{1, n_1, m\} \leq 2n$, we can take $B := 2n$. Then, we get

$$\begin{aligned} \log |\Lambda_1| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2) (1 + \log 2n) (16.8 \times 10^{11} (1 + \log 2n)) \\ &\quad \times (\log \alpha) (2 \log 3). \end{aligned}$$

Then,

$$\log |\Lambda_1| > -1.72 \times 10^{24} (1 + \log 2n)^2.$$

By comparing the above relation with (13), we get that

$$(m - m_1) \log 3 < 1.80 \times 10^{24} (1 + \log 2n)^2. \quad (15)$$

Case 2. $\min\{(n - n_1) \log \alpha, (m - m_1) \log 3\} = (m - m_1) \log 3$.

In this case, we rewrite (6) as

$$\left| \frac{\alpha^n}{\sqrt{5}} - (3^{m-m_1} - 1) \cdot 3^{m_1} \right| = \left| \frac{\beta^n + \alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} \right| < \frac{\alpha^{n_1} + 2}{\sqrt{5}} < \alpha^{n_1},$$

which implies that

$$\begin{aligned}
 |(\sqrt{5}(3^{m-m_1} - 1))^{-1} \alpha^n 3^{-m_1} - 1| &< \frac{\alpha^{n_1}}{3^m - 3^{m_1}} \leq \frac{3\alpha^{n_1}}{3^m} \\
 &< 3\alpha^{n_1-n+4} < \alpha^{n_1-n+7}.
 \end{aligned} \tag{16}$$

We put

$$\Lambda_2 = (\sqrt{5}(3^{m-m_1} - 1))^{-1} \alpha^n 3^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$, then $\alpha^{2n} \in \mathbb{Q}$, which is a contradiction. We again apply Theorem 2 with the following data

$$t = 3, \quad \gamma_1 = \sqrt{5}(3^{m-m_1} - 1), \quad \gamma_2 = \alpha, \quad \gamma_3 = \alpha, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m_1.$$

The minimal polynomial of γ_1 is $X^2 - 5(3^{m-m_1} - 1)^2$. Thus,

$$h(\gamma_1) = \log(\sqrt{5}(3^{m-m_1} - 1)) < (m - m_1 + 1) \log 3 < 1.25 \times 10^{12}(1 + \log 2n).$$

So, we can take $A_1 := 2.5 \times 10^{12}(1 + \log 2n)$. Further, as in the previous applications, we take $A_2 := \log \alpha$ and $A_3 := 2 \log 3$. Finally, since $\max\{1, n, m_1\} \leq 2n$, we can take $B := 2n$. Then, we get

$$\begin{aligned}
 \log |\Lambda_2| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log 2n)(2.5 \times 10^{12} \\
 &\quad \times (1 + \log 2n))(\log \alpha)(2 \log 3).
 \end{aligned}$$

Thus

$$\log |A_2| > -2.56 \times 10^{24}(1 + \log 2n)^2.$$

Now, by comparing with (16), we get that

$$(n - n_1) \log \alpha < 2.58 \times 10^{24}(1 + \log 2n)^2. \tag{17}$$

Therefore, in both Case 1 and Case 2, we have

$$\begin{aligned}
 \min\{(n - n_1) \log \alpha, (m - m_1) \log 3\} &< 1.24 \times 10^{12}(1 + \log 2n), \\
 \max\{(n - n_1) \log \alpha, (m - m_1) \log 3\} &< 2.58 \times 10^{24}(1 + \log 2n)^2.
 \end{aligned} \tag{18}$$

Finally, we rewrite the equation (6) as

$$\left| \frac{(\alpha^{n-n_1} - 1)}{\sqrt{5}} \alpha^{n_1} - (3^{m-m_1} - 1) \cdot 3^{m_1} \right| = \left| \frac{\beta^n - \beta^{n_1}}{\sqrt{5}} \right| < |\beta|^{n_1} < 1.$$

Dividing through by $3^m - 3^{m_1}$, we get

$$\left| \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}(3^{m-m_1} - 1)} \right) \alpha^{n_1} 3^{-m_1} - 1 \right| < \frac{1}{(3^m - 3^{m_1})} \leq \frac{3}{3^m} \\ \leq 3\alpha^{-(n-4)} \leq \alpha^{7-n}, \quad (19)$$

since $\alpha < 3$ and $\alpha < \alpha^{n_1}$. We again apply Theorem 2 on the left-hand side of (19) with the data

$$t = 3, \quad \gamma_1 = \frac{\alpha^{n-n_1} - 1}{\sqrt{5}(3^{m-m_1} - 1)}, \quad \gamma_2 = \alpha, \quad \gamma_3 = 3, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m_1.$$

Using the algebraic properties of the logarithmic height function, we get

$$h(\gamma_1) = h\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}(3^{m-m_1} - 1)}\right) \leq h\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}}\right) + h(3^{m-m_1} - 1) \\ < \frac{1}{2}(n - n_1 + 4) \log \alpha + (m - m_1) \log 3 < 2.80 \times 10^{24}(1 + \log 2n)^2,$$

where in the above inequalities, we used the argument from (14) as well as the bounds (18). Thus, we can take $A_1 := 5.60 \times 10^{24}(1 + \log 2n)$, and again as before $A_2 := \log \alpha$ and $A_3 := 2 \log 3$. If we put

$$\Lambda_3 = \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}(3^{m-m_1} - 1)} \right) \alpha^{n_1} 3^{-m_1} - 1,$$

we need to show that $\Lambda_3 \neq 0$. If not, $\Lambda_3 = 0$ leads to

$$\alpha^n - \alpha^{n_1} = \sqrt{5}(3^m - 3^{m_1}).$$

A contradiction is reached upon a conjugation in $\mathbb{K}$ and by taking absolute values on both sides. Thus, $\Lambda_3 \neq 0$. Applying Theorem 2 gives

$$\log |\Lambda_3| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log 2n)(5.6 \times 10^{24}(1 + \log 2n)^2) \\ \times (\log \alpha)(2 \log 3),$$

a comparison with (19) gives

$$(n - 4) < 3 \times 10^{36}(1 + \log 2n)^3,$$

or

$$2n < 6.2 \times 10^{36}(1 + \log 2n)^3. \quad (20)$$

Now by applying Lemma 3 on (20) with the data $m = 3$, $T = 6.2 \times 10^{36}$, and $x = 2n$, leads to $n < 2 \times 10^{40}$.

4.2 Reducing the bound for n

We need to reduce the above bound for n and to do so we make use of Lemma 2 several times. To begin, we return to (12) and put

$$\Gamma := n \log \alpha - m \log 3 - \log(\sqrt{5}).$$

For technical reasons, we assume that $\min\{n - n_1, m - m_1\} \geq 20$. We go back to the inequalities for Λ , Λ_1 , and Λ_2 . Since we assume that $\min\{n - n_1, m - m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|y| < 2|e^y - 1|$ holds for all $y \in (-\frac{1}{2}, \frac{1}{2})$, we get

$$|\Gamma| < 2 \max\{\alpha^{n_1-n+5}, 3^{m_1-m+1}\} \leq \max\{\alpha^{n_1-n+8}, 3^{m_1-m+2}\}.$$

Assume that $\Gamma > 0$. We then have the inequality

$$\begin{aligned} 0 < n \left(\frac{\log \alpha}{\log 3} \right) - m + \frac{\log(1/\sqrt{5})}{\log 3} &< \max \left\{ \frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}, \frac{6}{(\log 3)3^{m-m_1}} \right\} \\ &< \max\{45\alpha^{-(n-n_1)}, 8 \cdot 3^{-(m-m_1)}\}. \end{aligned}$$

We apply Lemma 2 with the data

$$\tau = \frac{\log \alpha}{\log 3}, \quad \mu = \frac{\log(1/\sqrt{5})}{\log 3}, \quad (A, B) = (45, \alpha) \text{ or } (8, 3).$$

Let $\tau = [a_0; a_1, a_2, \dots] = [0; 2, 3, 1, 1, 6, 1, 49, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 10, 3, \dots]$ be the continued fraction of τ . We choose $M := 2 \times 10^{40}$ and consider the 91th convergent

$$\frac{p}{q} = \frac{p_{91}}{q_{91}} = \frac{487624200385184167130255744232737921512174859336581}{1113251817385764505972408650620147577750763395186265}.$$

It satisfies $q = q_{91} > 6M$. Furthermore, it yields $\varepsilon > 0.4892$ and, therefore, either

$$n - n_1 \leq \frac{\log(45q/\varepsilon)}{\log \alpha} < 254, \quad \text{or} \quad m - m_1 \leq \frac{\log(8q/\varepsilon)}{\log 3} < 110.$$

In the case $\Gamma < 0$, we consider the inequality

$$\begin{aligned} m \left(\frac{\log 3}{\log \alpha} \right) - n + \frac{\log(\sqrt{5})}{\log \alpha} &< \max \left\{ \frac{\alpha^8}{\log \alpha} \alpha^{-(n-n_1)}, \frac{8}{\log \alpha} \cdot 3^{-(m-m_1)} \right\} \\ &< \max\{98\alpha^{-(n-n_1)}, 18 \cdot 3^{-(m-m_1)}\}. \end{aligned}$$

We then apply Lemma 2 with the data

$$\tau = \frac{\log 3}{\log \alpha}, \quad \mu = \frac{\log \sqrt{5}}{\log \alpha}, \quad (A, B) = (98, \alpha), \quad \text{or } (18, 3).$$

Let $\tau = [a_0; a_1, a_2, \dots] = [2; 3, 1, 1, 6, 1, 49, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 10, 3, 12, \dots]$ be the continued fraction of τ . Again, we choose $M = 2 \times 10^{40}$, and in this case we consider the 101th convergent

$$\frac{p}{q} = \frac{p_{101}}{q_{101}} = \frac{106360048375891410642967692492903700137161881169662}{56228858848524361385900581302251812795713192394033},$$

which satisfies $q = q_{101} > 6M$. Further, this yields $\varepsilon > 0.125$ and, therefore, either

$$n - n_1 \leq \frac{\log(98q/\varepsilon)}{\log \alpha} < 254, \quad \text{or} \quad m - m_1 \leq \frac{\log(18q/\varepsilon)}{\log 3} < 110.$$

These bounds agree with the bounds obtained in the case $\Gamma > 0$. As a conclusion, we have that either $n - n_1 \leq 253$ or $m - m_1 \leq 109$ whenever $\Gamma \neq 0$.

Now, we distinguish between the cases $n - n_1 \leq 253$ and $m - m_1 \leq 109$. First, we assume that $n - n_1 \leq 253$. In this case, we consider the inequality for Λ_1 , (13) and also assume that $m - m_1 \leq 20$. We put

$$\Gamma_1 = n_1 \log \alpha - m \log 3 + \log \left(\frac{\alpha^{n-n_1}}{\sqrt{5}} \right).$$

Then, the inequality (13) implies that

$$|\Gamma_1| < \frac{6}{3^{m-m_1}}.$$

If we further assume that $\Gamma_1 > 0$, we then get

$$0 < n_1 \left(\frac{\log \alpha}{\log 3} \right) - m + \frac{\log((\alpha^{n-n_1} - 1)/\sqrt{5})}{\log 3} < \frac{6}{(\log 3)3^{m-m_1}} < \frac{6}{3^{m-m_1}}.$$

Again we apply Lemma 2 with the same τ as in the case $\Gamma > 0$. We use the 91th convergent $p/q = p_{91}/q_{91}$ of τ as before. But in this case, we choose $(A, B) := (8, 3)$ and use

$$\mu_l = \frac{\log((\alpha^l - 1)/\sqrt{5})}{\log 3},$$

instead of μ for each possible value of $l := n - n_1 \in [1, 2, \dots, 253]$. We have problems at $l \in \{4, 12\}$. We discard these values for now and we will treat them later. For the remaining values of l , we get $\varepsilon > 0.0005$. Hence by Lemma 2, we get

$$m - m_1 < \frac{\log(8q/0.0005)}{\log 3} < 116.$$

Thus, $n - n_1 \leq 253$ implies that $m - m_1 \leq 115$, unless $n - n_1 \in \{4, 12\}$. A similar conclusion is reached when $\Gamma_1 < 0$ with the same two exceptions for $n - n_1 \in \{4, 12\}$. The reason we have a problem at $l \in \{4, 12\}$ is because

$$\frac{\alpha^4 - 1}{\sqrt{5}} = \alpha^2, \quad \text{and} \quad \frac{\alpha^{12} - 1}{\sqrt{5}} = 2^3 \alpha^6.$$

So, $\Gamma_1 = (n_1 + 2) \log \alpha - m \log 3$, or $(n_1 + 6) \log \alpha - (m - 3) \log 3$ when $l = 4, 12$, respectively. Thus we get that

$$\left| \tau - \frac{m}{n_1 + 2} \right| < \frac{6}{3^{m-m_1}(n_1 + 2)}, \quad \text{or} \quad \left| \tau - \frac{m-3}{n_1 + 6} \right| < \frac{6}{3^{m-m_1}(n_1 + 6)},$$

respectively. We assume that $m - m_1 > 150$. Then, $3^{m-m_1} > 8 \times (4 \times 10^{40}) > 8 \times (n_1 + 6)$; therefore

$$\frac{6}{3^{m-m_1}(n_1 + 2)} < \frac{1}{3(n_1 + 2)^2}, \quad \text{and} \quad \frac{6}{3^{m-m_1}(n_1 + 6)} < \frac{1}{3(n_1 + 6)^2}.$$

By Lemma 1, it follows that $m/(n_1 + 2)$ or $(m - 3)/(n_1 + 6)$ are convergents of τ , respectively. So, say one of $m/(n_1 + 2)$ or $(m - 3)/(n_1 + 6)$ is of the form p_k/q_k for some $k = 0, 1, 2, \dots, 92$. Here, we use that $q_{92} > 4 \times 10^{40} > n + 1 + 6$. Then

$$\frac{1}{(a_k + 2)q_k^2} < \left| \tau - \frac{p_k}{q_k} \right|.$$

Since $\max\{a_k : k = 0, 1, 2, \dots, 92\} = 140$, we get

$$\frac{1}{142q_k^2} < \frac{6}{3^{m-m_1}q_k} \quad \text{and} \quad q_k \text{ divides one of } \{n_1 + 2, n_1 + 6\}.$$

Thus, we get

$$3^{m-m_1} \leq 6 \times 142(n_1 + 6) < 6 \times 142 \times 4 \times 10^{40},$$

giving $m - m_1 \leq 92$.

Now let us turn to the case $m - m_1 \leq 109$ and we consider the inequality for Λ_2 , (16). We put

$$\Gamma_2 = n \log \alpha - m_1 \log 3 + \log(1/(\sqrt{5}(3^{m-m_1} - 1))),$$

and we also assume that $n - n_1 \geq 20$. We then have

$$|\Gamma_2| < \frac{2\alpha^8}{\alpha^{n-n_1}}.$$

We assume that Γ_2 , then we get

$$0 < n \left(\frac{\log \alpha}{\log 3} \right) - m_1 + \frac{\log(1/(\sqrt{5}(3^{m-m_1} - 1)))}{\log \alpha} < \frac{3\alpha^8}{(\log 3)\alpha^{n-n_1}} < \frac{130}{\alpha^{n-n_1}}.$$

We apply again Lemma 2 with the same τ , q , M , $(A, B) := (130, \alpha)$ and

$$\mu_l = \frac{\log(1/(\sqrt{5}(3^l - 1)))}{\log 3} \quad \text{for } k = 1, 2, \dots, 109.$$

We get $\varepsilon > 0.004$; therefore

$$n - n_1 < \frac{\log(130q/\varepsilon)}{\log \alpha} < 266.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first get that either $n - n_1 \leq 253$ or $m - m_1 \leq 109$. If $n - n_1 \leq 253$, then $m - m_1 \leq 115$, and if $m - m_1 \leq 109$ then $n - n_1 \leq 265$. Thus, we conclude that we always have $n - n_1 \leq 265$ and $m - m_1 \leq 115$.

Finally, we go to the inequality of Λ_3 , (19). We put

$$\Gamma_3 = n_1 \log \alpha - m_1 \log 3 + \log \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}(3^{m-m_1} - 1)} \right).$$

Since $n \geq 300$, the inequality (19) implies that

$$|\Gamma_3| < \frac{3}{\alpha^{n-4}} = \frac{3\alpha^4}{\alpha^n}.$$

Assuming that $\Gamma_3 > 0$, then

$$0 < n_1 \left(\frac{\log \alpha}{\log 3} \right) - m_1 + \frac{\log((\alpha^k - 1)/(\sqrt{5}(3^l - 1)))}{\log 3} < \frac{3\alpha^4}{(\log 3)\alpha^n} < \frac{20}{\alpha^n},$$

where $(k, l) := (n - n_1, m - m_1)$. We again apply Lemma 2 with the same τ , q , M , $(A, B) := (20, \alpha)$ and

$$\mu_{k,l} = \frac{\log((\alpha^k - 1)/(\sqrt{5}(3^l - 1)))}{\log 3} \quad \text{for } 1 \leq k \leq 265, \quad 1 \leq l \leq 115.$$

As before, we have a problem at $(k, l) := (4, 1), (12, 1), (8, 2)$. The cases $(k, l) := (4, 1), (12, 1)$ were treated before in the case of Γ_1 . The case $(k, l) := (8, 2)$ arises because

$$\frac{\alpha^8 - 1}{\sqrt{5}(3^2 - 1)} = \frac{3}{8}\alpha^4.$$

We, therefore, discard the cases $(k, l) := (4, 1), (12, 1), (8, 2)$ for some time. For the remaining cases, we get $\varepsilon > 0.0015$, so we obtain

$$n \leq \frac{\log(20q/\varepsilon)}{\log \alpha} < 264.$$

A similar conclusion is reached when $\Gamma_3 < 0$. Hence, $n < 300$. Now, we look at the cases $(k, l) := (4, 1), (12, 1), (8, 2)$. The cases $(k, l) := (4, 1), (12, 1)$ can be treated as before when we showed that $n - n_1 \leq 263$ implies $m - m_1 \leq 115$. The case when $(k, l) = (8, 2)$ can be dealt with in a similar way. Namely, it gives that

$$|(n_1 + 4)\tau - m_1| < \frac{20}{\alpha^n}.$$

Therefore

$$\left| \tau - \frac{m_1}{n_1 + 4} \right| < \frac{20}{(n_1 + 4)\alpha^n}. \quad (21)$$

Since $n \geq 300$, we have $\alpha^n > 2 \times 20 \times (4 \times 10^{40}) > 40(n_1 + 4)$. This shows that the right-hand side of the above inequality (21) is at most $2/(n_1 + 4)^2$. By Lemma 1, we get that $m_1/(n_1 + 4) = p_k/q_k$ for some $k = 1, 2, \dots, 92$. We then get by a similar argument as before that

$$\alpha^n < 20 \times 142 \times (4 \times 10^{40}),$$

which gives $n \leq 211$. Therefore, the conclusion is that $n < 300$ holds also in the case $(k, l) = (8, 2)$. However, this contradicts our working assumption that $n > 300$. This completes the proof of Theorem 1.

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