Boletín de la Sociedad Matemática Mexicana (2020) 26:349–359
https://doi.org/10.1007/s40590-019-00258-y

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

On convolution and q-calculus

Krzysztof Piejko[1] · **Janusz Sokół**[2]

Received: 28 November 2018 / Accepted: 16 August 2019 / Published online: 24 August 2019

Abstract

We consider the convolution operator

$$\mathrm{d}_\zeta f(z) = \frac{1}{z}\left\{ f(z) * \frac{z}{(1-\zeta z)(1-z)} \right\}$$

on the class of analytic functions $f(z) = z + a_2 z^2 + \cdots$, $|z| < 1$, in the complex plane, where ζ is complex, $|\zeta| \le 1$. For $\zeta = 1$, the operator becomes the derivative $f'(z)$, while for real $\zeta = q$, $0 < q < 1$ we obtain the Jackson's q-derivative $\mathrm{d}_q f(z)$.

Keywords Analytic functions · Convex functions · Starlike functions · q-calculus · q-starlike · q-derivative

Mathematics Subject Classification Primary 30C45

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. We will use the following notations:

$$\begin{cases} J_{CV}(f; z) := 1 + \frac{zf''(z)}{f'(z)}, \\ J_{ST}(f; z) := \frac{zf'(z)}{f(z)}. \end{cases} \tag{1.1}$$

✉ Janusz Sokół
jsokol@prz.edu.pl

Krzysztof Piejko
piejko@prz.edu.pl

1 Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland

2 Faculty of Mathematics and Natural Sciences, University of Rzeszów, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland

Let the function $f \in \mathcal{H}$ be univalent in the unit disc $\mathbb{D}$ with the normalization $f(0) = 0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\{J_{ST}(f; z)\} > 0 \quad \text{for all } z \in \mathbb{D}. \tag{1.2}$$

Such function f is said to be starlike in $\mathbb{D}$ with respect to $w_0 = 0$ (or briefly starlike). Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E. A function f maps $\mathbb{D}$ onto a convex domain E if and only if

$$\mathfrak{Re}\{J_{CV}(f; z)\} > 0 \quad \text{for all } z \in \mathbb{D} \tag{1.3}$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). It is well known that if an analytic function f satisfies (1.2) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{D}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathcal{S}^*$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ by $\mathcal{K}$. It is known that for $f \in \mathcal{A}$, condition (1.3) is sufficient for starlikeness of f. Also the condition

$$|J_{CV}(f; z) - 1| < 2, \quad z \in \mathbb{D},$$

is sufficient for starlikeness of f. In this paper we shall consider certain sufficient conditions for starlikeness of order $1/2$. The class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}^*(\alpha) := \{f \in \mathcal{A} : \ \mathfrak{Re}\{J_{ST}(f; z)\} > \alpha, \ z \in \mathbb{D}\}.$$

The class $\mathcal{S}^*(\alpha)$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha < 1$

$$\begin{aligned}\mathcal{K}(\alpha) : &= \{f \in \mathcal{A} : \ \mathfrak{Re}\{J_{CV}(f; z)\} > \alpha, \ z \in \mathbb{D}\} \\ &= \left\{f \in \mathbb{D} : \ zf' \in \mathcal{S}^*_\alpha\right\},\end{aligned}$$

were introduced by Robertson in [10]. It is known the old Strohhäcker result [16] that $\mathcal{K}(0) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0)$. Furthermore, note that if $f \in \mathcal{K}(\alpha)$ then $f \in \mathcal{S}^*(\delta(\alpha))$, see [17], where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha \neq 1/2, \\ \frac{1}{2\log 2} & \text{for } \alpha = 1/2. \end{cases} \tag{1.4}$$

Robertson [11] proved that if $f \in \mathcal{A}$ with $f(z)/z \neq 0$ and if there exists a k, $0 < k \leq 2$, such that

$$|J_{CV}(f; z) - 1| \leq k\, |J_{ST}(f; z)|, \quad z \in \mathbb{D},$$

then $f(z) \in \mathcal{S}^*(2/(2+k))$. In [8], it was proved that for $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$, if

$$|J_{CV}(f; z)| \leq \sqrt{2}\, |J_{ST}(f; z) + 1|, \quad z \in \mathbb{D},$$

then $f(z) \in \mathcal{S}^*$. Several more complicated sufficient conditions for starlikeness and for convexity are collected in the book [7], Chap. 5. Recall also, that if $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\} > 0, \ z \in \mathbb{D} \tag{1.5}$$

for some $g \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then f is said to be close-to-convex in $\mathbb{D}$ and denoted by $f \in \mathcal{C}$. An univalent function $f \in \mathcal{A}$ belongs to $\mathcal{C}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

On the other hand, if $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0, \ z \in \mathbb{D} \tag{1.6}$$

for some $g \in \mathcal{S}^*$ and some $\beta \in [0, 1]$, then f is said to be a Bazilevič function of type β and denoted by $f \in \mathcal{B}(\beta)$.

Jackson in [5,6] introduced and studied the q-derivative, $0 < q < 1$, as

$$\mathrm{d}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0 \tag{1.7}$$

and $d_q f(0) = f'(0)$. Then

$$\mathrm{d}_q f(z) = \frac{1}{z}\left\{z + \sum_{n=2}^{\infty}[n]_q a_n z^n\right\}, \tag{1.8}$$

where

$$[n]_q = \frac{1-q^n}{1-q} \quad n = 2, 3, \ldots.$$

Making use of q-derivative, Argawal and Sahoo in [1] introduced the class $\mathcal{S}_q^*(\alpha)$. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*(\alpha)$, $0 \leq \alpha < 1$, if

$$\left|\frac{z\mathrm{d}_q f(z)}{f(z)} - \frac{1-\alpha q}{1-q}\right| \leq \frac{1-\alpha}{1-q}, \quad z \in \mathbb{D}. \tag{1.9}$$

If $q \to 1^-$, the class $\mathcal{S}_q^*(\alpha)$ reduces to the class $\mathcal{S}^*(\alpha)$. If $\alpha = 0$, the class $\mathcal{S}_q^*(\alpha)$ coincides with the class $\mathcal{S}_q^*(0) = \mathcal{S}_q^*$, which was first introduced by Ismail et al. in [3] and was considered in [2,9,12,14].

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{D}$.

2 *q*-derivative operator on convex functions

Theorem 2.1 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all q, $0 < q < 1$, we have*

$$z\mathrm{d}_q f(z) = z + \sum_{n=2}^{\infty}[n]_q a_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

is in the class $\mathcal{S}^*\left(\frac{1-q}{2(1+q)}\right)$ of starlike univalent functions of order $\frac{1-q}{2(1+q)}$.

Proof It is easy to check that for each $q \in \mathbb{C}$, $|q| \leq 1$, $q \neq 1$, the function

$$h_q(z) = \frac{1}{1-q}\log\frac{1-qz}{1-z} = \sum_{n=1}^{\infty}\frac{1-q^n}{1-q}\frac{z^n}{n} = \sum_{n=1}^{\infty}[n]_q\frac{z^n}{n}, \quad z \in \mathbb{D}, \tag{2.2}$$

is convex univalent in $\mathbb{D}$, see also [4, Th.17,p.170]. Hence,

$$zh_q'(z) = \frac{z}{(1-qz)(1-z)} = \sum_{n=1}^{\infty}[n]_q z^n, \quad z \in \mathbb{D}, \tag{2.3}$$

is starlike univalent in $\mathbb{D}$. With a little more effort, we can find that $zh_q'(z)$ is starlike of order $\frac{1-q}{2(1+q)}$. On the other hand, we have that

$$z\mathrm{d}_q f(z) = \left\{z + \sum_{n=2}^{\infty}a_n z^n\right\} * \sum_{n=1}^{\infty}[n]_q z^n = f(z) * \frac{z}{(1-qz)(1-z)}, \tag{2.4}$$

where $*$ denotes the Hadamard product, or convolution, of power series. Hence, $z\mathrm{d}_q f(z)$ is a convolution of $f(z)$ with a starlike function of order $\frac{1-q}{2(1+q)}$. Because of the famous result [13] that $\mathcal{K} * \mathcal{S}^*(\alpha) = \mathcal{S}^*(\alpha)$, we finally obtain that the function in (2.1) is in the class $\mathcal{S}^*\left(\frac{1-q}{2(1+q)}\right)$. □

The known Alexander theorem says that

$$f \in \mathcal{K} \quad \Rightarrow \quad zf' \in \mathcal{S}^*.$$

Therefore, Theorem 2.1 is an equivalent of Alexander theorem

$$f \in \mathcal{K} \quad \Rightarrow \quad \forall 0 < q < 1: \quad z\mathrm{d}_q f(z) \in \mathcal{S}^*.$$

It is known that

$$\forall f \in \mathcal{S}^* \exists\, g \in \mathcal{K}: \quad f(z) = zg'(z).$$

A question is: Is it true that

$$\forall f \in \mathcal{S}^* \ \exists\, g \in \mathcal{K} \ \exists 0 < q < 1: \quad f(z) = z d_q g(z) \quad ?$$

In terms of the convolution, this problem we may write as: exists there a q, $0 < q < 1$, that for given $f(z) = z + a_2 z^2 + \cdots \in \mathcal{S}^*$, we have

$$z + \sum_{n=2}^{\infty} \frac{a_n}{[n]_q} z^n \in \mathcal{K} \quad ? \tag{2.5}$$

The answer on the question (2.5) is: no. Namely, for the starlike function

$$\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D},$$

$a_n = n$ and the function in (2.5) becomes

$$z + \sum_{n=2}^{\infty} \frac{n}{[n]_q} z^n,$$

which is not in the class $\mathcal{K}$ because it has the coefficients $n/[n]_q$ greater than 1, otherwise than in $\mathcal{K}$.

Because the Koebe function $f(z) = z/(1-z)^2 \in \mathcal{C}$, we have for the class of close-to-convex functions $\mathcal{C}$, (1.5), that

$$\forall f \in \mathcal{C} \ \exists\, g \in \mathcal{K} \ \exists 0 < q < 1: \quad f(z) = z d_q g(z)$$

is also false. The above facts we can write $\mathcal{S}^* \not\subset z d_q(\mathcal{K}), \mathcal{C} \not\subset z d_q(\mathcal{K})$, or equivalently $\mathcal{B}(0) \not\subset z d_q(\mathcal{K})$, $\mathcal{B}(1) \not\subset z d_q(\mathcal{K})$. A question for $\mathcal{B}(\beta)$ is: What about $\beta \in (0, 1)$? So we have the following problem.

Open Problem. There exists a β, $\beta \in (0, 1)$, such that

$$\forall f \in \mathcal{B}(\beta) \ \exists\, g \in \mathcal{K} \ \exists 0 < q < 1: \quad f(z) = z d_q g(z),$$

where $\mathcal{B}(\beta)$ is the class of Bazilevič functions of type β.

3 ζ-derivative operator

It is easy to check that if $q \to 1^-$ the function $z h_q'(z)$ (2.3) becomes the well-known Koebe function

$$z h_1'(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D}. \tag{3.1}$$

For each $f \in \mathcal{A}$, we can express its derivative in terms of the Koebe function as

$$f'(z) = \frac{1}{z}\left\{f(z) * [zh_1'(z)]\right\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-z)^2}\right\}. \tag{3.2}$$

It is a natural to consider a generalization of (3.2) for $\zeta \in \mathbb{C}$, $|\zeta| \le 1$:

$$\mathrm{d}_\zeta f(z) = \frac{1}{z}\left\{f(z) * [zh_\zeta'(z)]\right\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-\zeta z)(1-z)}\right\}. \tag{3.3}$$

For $\zeta = 1$, we have the derivative f', while for $\zeta = q$, $0 < q < 1$ we obtain the Jackson's q-derivative of f, namely $\mathrm{d}_q f$, which is defined in (1.7). Therefore, for

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_1 = 1,\ z \in \mathbb{D},$$

we have

$$\begin{aligned}
\mathrm{d}_\zeta f(z) &= \frac{1}{z}\left\{f(z) * \frac{z}{(1-\zeta z)(1-z)}\right\}\\
&= \frac{1}{z}\left\{\sum_{n=1}^{\infty} a_n z^n * \sum_{n=1}^{\infty} \frac{1-\zeta^n}{1-\zeta} z^n\right\}\\
&= \frac{1}{z}\left\{z + \sum_{n=2}^{\infty} \frac{1-\zeta^n}{1-\zeta} a_n z^n\right\}\\
&= \frac{1}{z}\left\{z + \sum_{n=2}^{\infty} [n]_\zeta a_n z^n\right\},
\end{aligned} \tag{3.4}$$

where

$$[n]_\zeta = \frac{1-\zeta^n}{1-\zeta}.$$

For these reasons, we can look on q-derivative $\mathrm{d}_q f$ as a special case of the convolution operator (3.4).

Corollary 3.1 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all ζ, $|\zeta| \le 1$, we have*

$$z\mathrm{d}_\zeta f(z) = z + \sum_{n=2}^{\infty} [n]_\zeta a_n z^n, \quad z \in \mathbb{D}, \tag{3.5}$$

is in the class $\mathcal{S}^*$ of starlike univalent functions.

Proof The proof runs in the same way as the proof of Theorem 2.1, because the function

$$zh_\zeta'(z) = \frac{z}{(1-\zeta z)(1-z)}$$

is starlike for all complex ζ, $|\zeta| < 1$. □

Theorem 3.2 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all ζ, $|\zeta| \le 1$, we have*

$$\mathfrak{Re}\left\{\frac{1}{1-\zeta^2}\frac{\mathrm{d}_\zeta f(z)}{\mathrm{d}_\zeta f(\zeta z)}\right\} > \mathfrak{Re}\left\{\frac{1+\zeta^2}{2(1-\zeta^2)}\right\}, \quad z \in \mathbb{D}, \tag{3.6}$$

or

$$\mathfrak{Re}\left\{\frac{1}{1-\zeta^2}\frac{\mathrm{d}_\zeta f(z)}{\frac{1}{1-\zeta z} * \mathrm{d}_\zeta f(z)}\right\} > \mathfrak{Re}\left\{\frac{1+\zeta^2}{2(1-\zeta^2)}\right\}, \quad z \in \mathbb{D}. \tag{3.7}$$

Proof It is known [13], [15, p.10], that if $f \in \mathcal{K}$, then for all z, v, and $w \in \mathbb{D}$, we have

$$\mathfrak{Re}\left\{\frac{z}{z-v}\frac{v-w}{z-w}\frac{f(z)-f(w)}{f(v)-f(w)} - \frac{v}{z-v}\right\} > \frac{1}{2}. \tag{3.8}$$

If we put $w = \zeta z$ and $v = \zeta^2 z$ in (3.8), then we obtain

$$\mathfrak{Re}\left\{\frac{z}{z-\zeta^2 z}\frac{\zeta^2 z-\zeta z}{z-\zeta z}\frac{f(z)-f(\zeta z)}{f(\zeta^2 z)-f(\zeta z)} - \frac{\zeta^2 z}{z-\zeta^2 z}\right\} > \frac{1}{2}.$$

Trivial calculations give

$$\mathfrak{Re}\left\{\frac{1}{1-\zeta^2}\frac{\zeta^2 z-\zeta z}{f(\zeta^2 z)-f(\zeta z)}\frac{f(z)-f(\zeta z)}{z-\zeta z} - \frac{\zeta^2}{1-\zeta^2}\right\} > \frac{1}{2}$$

or

$$\mathfrak{Re}\left\{\frac{1}{1-\zeta^2}\frac{\mathrm{d}_\zeta f(z)}{\mathrm{d}_\zeta f(\zeta z)} - \frac{\zeta^2}{1-\zeta^2}\right\} > \frac{1}{2}.$$

This establishes (3.6) and (3.7). □

Corollary 3.3 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all q, $0 < q < 1$, we have*

$$\mathfrak{Re}\left\{\frac{\mathrm{d}_q f(z)}{\mathrm{d}_q f(qz)}\right\} > \frac{1+q^2}{2}, \quad z \in \mathbb{D}$$

or

$$\mathfrak{Re}\left\{\frac{\mathrm{d}_q f(z)}{\frac{1}{1-qz} * \mathrm{d}_q f(z)}\right\} > \frac{1+q^2}{2}, \quad z \in \mathbb{D}.$$

Theorem 3.4 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all ζ, $|\zeta| \le 1$, we have*

$$\mathfrak{Re}\left\{\frac{\zeta^2}{1-\zeta^2}\frac{\mathrm{d}_\zeta f(\zeta z)}{\mathrm{d}_\zeta f(z)}\right\} < \mathfrak{Re}\left\{\frac{1+\zeta^2}{2(1-\zeta^2)}\right\}, \quad z \in \mathbb{D}, \tag{3.9}$$

or

$$\mathfrak{Re}\left\{\frac{\zeta^2}{1-\zeta^2}\frac{\frac{1}{1-\zeta z} * \mathrm{d}_\zeta f(z)}{\mathrm{d}_\zeta f(z)}\right\} < \mathfrak{Re}\left\{\frac{1+\zeta^2}{2(1-\zeta^2)}\right\}, \quad z \in \mathbb{D}. \tag{3.10}$$

Proof From (3.8), we have that for $f \in \mathcal{K}$, for all t, v, and $w \in \mathbb{D}$, we have

$$\mathfrak{Re}\left\{\frac{z}{t-v}\frac{v-w}{t-w}\frac{f(t)-f(w)}{f(v)-f(w)}-\frac{v}{t-v}\right\}>\frac{1}{2}. \tag{3.11}$$

If we put $v = z$, $w = \zeta z$ and $t = \zeta^2 z$ in (3.11), then we obtain

$$\mathfrak{Re}\left\{\frac{\zeta^2 z}{\zeta^2 z-z}\frac{z-\zeta z}{\zeta^2 z-\zeta z}\frac{f(\zeta^2 z)-f(\zeta z)}{f(z)-f(\zeta z)}-\frac{z}{\zeta^2 z-z}\right\}>\frac{1}{2}.$$

After some calculations, we obtain

$$\mathfrak{Re}\left\{\frac{\zeta^2}{\zeta^2-1}\frac{f(\zeta^2 z)-f(\zeta z)}{\zeta^2 z-\zeta z}\frac{z-\zeta z}{f(z)-f(\zeta z)}-\frac{1}{\zeta^2-1}\right\}>\frac{1}{2}$$

or

$$\mathfrak{Re}\left\{\frac{\zeta^2}{\zeta^2-1}\frac{\mathrm{d}_\zeta f(\zeta z)}{\mathrm{d}_\zeta f(z)}-\frac{1}{\zeta^2-1}\right\}>\frac{1}{2}.$$

This proves (3.9) and (3.10). □

Corollary 3.5 *If f is in the class $\mathcal{K}$ of convex univalent functions, then for all q, $0 < q < 1$, we have*

$$\mathfrak{Re}\left\{\frac{\mathrm{d}_q f(qz)}{\mathrm{d}_q f(z)}\right\}<\frac{1+q^2}{2q^2}, \quad z \in \mathbb{D},$$

or

$$\mathfrak{Re}\left\{\frac{\frac{1}{1-qz}*\mathrm{d}_q f(z)}{\mathrm{d}_q f(z)}\right\}<\frac{1+q^2}{2q^2}, \quad z \in \mathbb{D}.$$

Corollary 3.3 and Corollary 3.5 may be applied to obtain a bound for the modulus.

Corollary 3.6 *If $f(z)$ is in the class $\mathcal{K}$ of convex univalent functions, then for all q, $0 < q < 1$, we have*

$$\left|\frac{\mathrm{d}_q f(z)}{\mathrm{d}_q f(qz)}\right|>q, \quad z \in \mathbb{D}$$

4 ζ-starlike functions of order α

Definition Let f be in $\mathcal{A}$. For given ζ, $|\zeta| \le 1$, we say that f is in the class $\mathcal{S}^*(\zeta,\alpha)$ of ζ-starlike functions of order α, $\alpha \in [0, 1)$ if

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_\zeta f(z)}{f(z)}\right\}>\alpha, \quad z \in \mathbb{D}, \tag{4.1}$$

where the operator $\mathrm{d}_\zeta f$ is defined in (3.3).

Remark For $\zeta = 1$, condition (4.1) becomes

$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}, \tag{4.2}$$

and the class $\mathcal{S}^*(1, \alpha)$ becomes the well-known class of starlike functions of order α. For $\zeta \neq 1$, condition (4.1) becomes

$$\Re\mathfrak{e}\left\{\frac{f(\zeta z) - f(z)}{(\zeta - 1)z}\frac{z}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}. \tag{4.3}$$

Theorem 4.1 *If f is in the class $\mathcal{K}$ of convex univalent functions, then $f(z) \in \mathcal{S}^*(\zeta, 1/2)$ for each $|\zeta| \leq 1$.*

Proof For $\zeta = 1$, Theorem 4.1 becomes the known fact that each convex function is starlike of order $1/2$. Now, let $\zeta \neq 1$. If we put in (3.8) $v = 0$, next write $z = \zeta z$, and then, $w = z$; then, we obtain

$$\Re\mathfrak{e}\left\{\frac{-z}{\zeta z - z}\frac{f(\zeta z) - f(z)}{-f(z)}\right\} > \frac{1}{2}.$$

From (4.3), we obtain that $f \in \mathcal{S}^*(\zeta, 1/2)$ for each $|\zeta| \leq 1$. □

Theorem 4.2 *If f is in the class $\mathcal{K}$ of convex univalent functions, then*

$$\Re\mathfrak{e}\left\{\frac{1}{1-\zeta}\frac{f'(z)}{\mathrm{d}_\zeta f(z)} - \frac{1}{2}\frac{1+\zeta}{1-\zeta}\right\} \geq 0, \quad z \in \mathbb{D}, \tag{4.4}$$

for each $|\zeta| < 1$.

Proof For $v = \zeta z$, the condition (3.8) becomes

$$\Re\mathfrak{e}\left\{\frac{1}{1-\zeta}\frac{\zeta - w}{z - w}\frac{f(z) - f(w)}{f(\zeta) - f(w)} - \frac{\zeta}{1-\zeta}\right\} > \frac{1}{2}.$$

For $w \to z$, this gives

$$\Re\mathfrak{e}\left\{\frac{1}{1-\zeta}\frac{\zeta - z}{1}\frac{f'(z)}{f(\zeta) - f(z)} - \frac{\zeta}{1-\zeta}\right\} \geq \frac{1}{2},$$

and finally we get (4.4). □

Corollary 4.3 *If f is in the class $\mathcal{K}$ of convex univalent functions, then*

$$\Re\mathfrak{e}\left\{\frac{f'(z)}{\mathrm{d}_q f(z)}\right\} \geq \frac{1+q}{2}, \quad z \in \mathbb{D}, \tag{4.5}$$

for each $0 < q < 1$.

It is a natural question whether

$$S_\zeta(z) := zh'_\zeta(z) = \frac{z}{(1-\zeta z)(1-z)} \tag{4.6}$$

is in the class of ζ-starlike functions $\mathcal{S}^*(\zeta,\alpha)$ defined in (4.1), for some α.

We answer this question in the following theorem.

Theorem 4.4 *If $|\zeta| \le 1$, then the function (4.6) is in the class ζ-starlike functions $\mathcal{S}^*(\zeta,\alpha)$, where*

$$\alpha \le \alpha_0 = \frac{1+|\zeta|^2\mathfrak{Re}\{\zeta\} - |\zeta+\zeta^2|}{1-|\zeta|^4}.$$

Proof From (3.3), the function (4.6) is in the class ζ-starlike functions $\mathcal{S}^*(\zeta,\alpha)$ if and only if

$$\mathfrak{Re}\{F(\zeta,z)\} > 0 \quad z \in \mathbb{D},$$

where

$$F(\zeta,z) = \frac{S_\zeta(z) * S_\zeta(z)}{S_\zeta(z)} \quad z \in \mathbb{D}.$$

We have

$$\begin{aligned}
F(\zeta,z) &= \frac{S_\zeta(z) * S_\zeta(z)}{S_\zeta(z)} \\
&= \frac{\frac{1}{1-\zeta}\left(\frac{1}{1-z} - \frac{1}{1-\zeta z}\right) * \frac{1}{1-\zeta}\left(\frac{1}{1-z} - \frac{1}{1-\zeta z}\right)}{S_\zeta(z)} \\
&= \frac{\frac{1}{(1-\zeta)^2}\left(\frac{1}{1-z} - \frac{2}{1-\zeta z} + \frac{1}{1-\zeta^2 z}\right)}{S_\zeta(z)} \\
&= \frac{\frac{1}{(1-\zeta)^2}\frac{z\{(1-\zeta)^2+\zeta(1-\zeta)^2 z\}}{(1-z)(1-\zeta z)(1-\zeta^2 z)}}{S_\zeta(z)} \\
&= \frac{1+\zeta z}{1-\zeta^2 z}.
\end{aligned}$$

An elementary calculation shows that the function $F(\zeta,z) = \frac{1+\zeta z}{1-\zeta^2 z}$ maps the unit disc $\mathbb{D}$ on a circle with a centre $s = \frac{1+\zeta\overline{\zeta^2}}{1-|\zeta|^4}$ and a radius $r = \frac{|\zeta+\zeta^2|}{1-|\zeta|^4}$. So

$$\mathfrak{Re}\left\{\frac{1+\zeta z}{1-\zeta^2 z}\right\} > \mathfrak{Re}\{s\} - r = \frac{1+|\zeta|^2\mathfrak{Re}\{\zeta\} - |\zeta+\zeta^2|}{1-|\zeta|^4}$$

for all $|z| < 1$. Thus for all ζ, $|\zeta| \le 1$, we have

$$\mathfrak{Re}\{F(\zeta,z)\} > \alpha_0, \quad z \in \mathbb{D}.$$

□

Corollary 4.5 *If* $|\zeta| \leq 1$, *then the function* (4.6) *is in the class* ζ*-starlike functions* $\mathcal{S}^*(\zeta, 0)$.

Corollary 4.6 *For real* ζ, $0 < \zeta < 1$, *the function* (4.6) *is in the class* ζ*-starlike functions* $\mathcal{S}^*\left(\zeta, \frac{1-\zeta}{1+\zeta^2}\right)$.

References

1. Agrawal, S., Sahoo, S.K.: A generalization of starlike functions of order alpha. Hokkaido Math. J. **46**, 15–27 (2017)
2. Agrawal, S., Sahoo, S.K.: Geometric properties of basic hypergeometric functions. J. Diff. Equ. Appl. **20**, 1502–1522 (2014)
3. Ismail, M.E.H., Merkes, E., Styer, D.: A generalization of starlike functions. Complex Var. **14**, 77–84 (1990)
4. Goodman, A.W.: Univalent Functions, Vols. I and II. Mariner Publishing Co., Tampa (1983)
5. Jackson, F.H.: On q-functions and certain difference operator. Trans. R. Soc. Edinb. **46**, 253–281 (1908)
6. Jackson, F.H.: On q-definite integrals. Q. J. Pure Appl. Math. **41**, 193–203 (1910)
7. Miller, S.S., Mocanu, P.T.: Differential Subordinations, Theory and Applications. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 225. Marcel Dekker Inc., New York (2000)
8. Mocanu, P.T.: On a theorem of Robertson. Babeş-Bolyai Univ., Fac. of Math. Res. Sem., Seminar of Geometric Function Theory **5**, 77–82 (1986)
9. Raghavendar, K., Swaminathan, A.: Close-to-convexity of basic hypergeometric functions using their Taylor coefficients. J. Math. Appl. **35**, 111–125 (2012)
10. Robertson, M.S.: On the theory of univalent functions. Ann. Math. **37**, 374–408 (1936)
11. Robertson, M.S.: Certain classes of starlike functions. Mich. Math. J. **32**, 135–140 (1985)
12. Rønning, F.: A Szegö quadrature Formula Arising from q-starlike Functions: Continued Fractions and Orthogonal Functions, Theory and Applications, pp. 345–352. Marcel Dekker Inc., New York (1994)
13. Ruscheweyh, St, Sheil–Small, T.: Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture. Commun. Math. Helv. **48**, 119–135 (1973)
14. Sahoo, S.K., Sharma, N.L.: On a generalization of close-to-convex functions. Ann. Polon. Math. **113**, 93–108 (2015)
15. Schober, G.: Univalent Functions-Selected Topics. Lecture Notes in Math, vol. 478. Springer, Berlin (1975)
16. Strrohhäcker, E.: Beitrage zür Theorie der Schlichter Functionen. Math. Z. **37**, 356–380 (1933)
17. Wilken, D.R., Feng, J.: A remark on convex and starlike functions. J. Lond. Math. Soc. **21**(2), 287–290 (1980)