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Views on level ℓ curves, K3 surfaces and Fano threefolds

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Abstract
An analogue of the Mukai map $m_g : \mathcal{P}_g \to \mathcal{M}_g$ is studied for the moduli $\mathcal{R}_{g,\ell}$ of genus g curves C with a level ℓ structure. Let $\mathcal{P}^{\perp}_{g,\ell}$ be the moduli space of 4-tuples $(S, \mathcal{L}, \mathcal{E}, C)$ so that $(S, \mathcal{L})$ is a polarized K3 surface of genus g, $\mathcal{E}$ is orthogonal to $\mathcal{L}$ in Pic S and defines a standard degree ℓ K3 cyclic cover of S, $C \in |\mathcal{L}|$. We say that $(S, \mathcal{L}, \mathcal{E})$ is a level ℓ K3 surface. These exist for $\ell \leq 8$ and their families are known. We define a level ℓ Mukai map $r_{g,\ell} : \mathcal{P}^{\perp}_{g,\ell} \to \mathcal{R}_{g,\ell}$, induced by the assignment of $(S, \mathcal{L}, \mathcal{E}, C)$ to $(C, \mathcal{E} \otimes \mathcal{O}_C)$. We investigate a curious possible analogy between m_g and $r_{g,\ell}$, that is, the failure of the maximal rank of $r_{g,\ell}$ for $g = g_\ell \pm 1$, where g_ℓ is the value of g such that $\dim \mathcal{P}^{\perp}_{g,\ell} = \dim \mathcal{R}_{g,\ell}$. This is proven here for $\ell = 3$. As a related open problem we discuss Fano threefolds whose hyperplane sections are level ℓ K3 surfaces and their classification.

1 Introduction

Our aim is to convince the reader, showing a program and new results, of the interest represented by some complex projective varieties whose curvilinear sections are canonical curves C of genus g, endowed with a distinguished nonzero ℓ-torsion element $\eta \in$ Pic C. Often one says that (C, η) is a level ℓ curve of genus g, cfr. [7]. Fixing (g, ℓ) the moduli space of these pairs is integral, quasi projective and denoted by $\mathcal{R}_{g,\ell}$.

To enter further in the matter let us mention two other names from the title: $K3$ surface and Fano threefold. The $K3$ surfaces S we consider are very special: they admit a non split cyclic cover of degree ℓ, still birational to a K3 surface. This is defined by a line bundle $\mathcal{O}_S(E) := \mathcal{E}$ such that $h^0(\mathcal{O}_S(\ell E)) = 1$ and $h^0(\mathcal{O}_S(mE)) = 0$ for $m < \ell$. The study of these surfaces stems from Nikulin's classification of K3 surfaces with an order ℓ symplectic

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automorphism and the classification implies $\ell \leq 8$, [24]. Since then several foundational results, in use here, did follow, cfr. [13–16,26].

Now let $\mathcal{L} \in \operatorname{Pic} S$ be a genus g polarization orthogonal to $\mathcal{E}$. Let $\eta := \mathcal{O}_C(E)$, where $C \in |\mathcal{L}|$ is smooth, then it turns out that (C, η) is a level ℓ curve. We say that the triple $(S, \mathcal{L}, \mathcal{E})$ is a level ℓ K3 surface of genus g, see definition (3.1) for some precision. Fixing ℓ the moduli of these triples are reducible for infinitely many values of g. However a distinguished irreducible component exists for every g, namely the moduli space of triples $(S, \mathcal{L}, \mathcal{E})$ such that Pic S is the sum of $\mathbb{Z}\mathcal{L}$ and its orthogonal lattice. We denote it by

$$\mathcal{F}^{\perp}_{g,\ell}. \tag{1}$$

Finally we come to the moduli space $\mathcal{P}^{\perp}_{g,\ell}$ of 4-tuples $(S, \mathcal{L}, \mathcal{E}, C)$ such that $C \in |\mathcal{L}|$ and $(S, \mathcal{L}, \mathcal{E})$ defines a point in $\mathcal{F}^{\perp}_{g,\ell}$. Such a space is strictly related with the first topic considered in our paper. To introduce it let us define the level ℓ Mukai map. This is the rational map

$$r_{g,\ell} : \mathcal{P}^{\perp}_{g,\ell} \to \mathcal{R}_{g,\ell}, \tag{2}$$

assigning the moduli point of the 4-tuple $(S, \mathcal{L}, \mathcal{E}, C)$ to the moduli point of the pair (C, η), where η is $\mathcal{O}_C(E)$. Let $\mathcal{P}_g$ be the moduli space of triples $(S, \mathcal{L}, C)$, where $(S, \mathcal{L})$ is a polarized K3 surface of genus g and $C \in |\mathcal{L}|$, then the previous name is motivated by the well known Mukai map

$$m_g : \mathcal{P}_g \to \mathcal{M}_g, \tag{3}$$

assigning the moduli point of the triple $(S, \mathcal{L}, C)$ to the moduli point of the curve C. Some famous connections between canonical curves of genus g, K3 surfaces and Fano threefolds are well represented by m_g and, in particular, by a curious variation of its rank. We recall that a rational map $f : X \to Y$ of integral varieties has maximal rank if $\dim f(X) = \min\{\dim X, \dim Y\}$.

Considering m_g we recall that $\dim \mathcal{P}_g = 19 + g$ and $\dim \mathcal{M}_g = 3g - 3$, therefore $\dim \mathcal{P}_g = \dim \mathcal{M}_g$ iff $g = 11$. Now m_{11} is birational but, curiously, m_g fails to be of maximal rank precisely before and after this transition value, that is, for $g = 11 \pm 1$. For the rest m_g is dominant for $g \leq 9$ and generically injective for $g \geq 13$. As is well known this anomaly is due to the presence behind the scene of some Fano varieties, whose curvilinear sections are general canonical curves of genus 11 ± 1, cfr. [8,22,23,25].

A task of this paper is to point out the same possible anomalies for the level ℓ Mukai maps $r_{g,\ell}$. The case $\ell = 2$ has already been done and it is an experimental origin to this work. If $\ell = 2$ we have $\dim \mathcal{P}^{\perp}_{g,2} = \dim \mathcal{R}_{g,2}$ for $g = 7$. Then $r_{g,2}$ fails to be of maximal rank for $g = 7 \pm 1$ and is birational for $g = 7$, [11,19,27]. The 'Fano varieties behind the scene' for $g = 8$ and $g = 6$ are addressed or revisited in Sect. 7.

In Sect. 5 we summarize the question for each ℓ. Let g_ℓ be the unique value of g such that $\dim \mathcal{P}^{\perp}_{g,\ell} = \dim \mathcal{R}_{g,\ell}$, for $l = 2, 3, 4, 5, 6, 7, 8$ we respectively have:

$$g_\ell = 7\ ,\ 5\ ,\ 4\ ,\ 3\ ,\ 2\ ,\ 2\ ,\ 2. \tag{4}$$

In this paper we present the following theorem, solving the question for $\ell = 3$.

Theorem 1.1 *Let $r_{g,3} : \mathcal{P}^{\perp}_{g,3} \to \mathcal{R}_{g,3}$ be the level 3 Mukai map then:*

(1) $r_{4,3}$ *has not maximal rank,*
(2) $r_{5,3}$ *is birational,*
(3) $r_{6,3}$ *has not maximal rank.*

The image of $r_{4,3}$ is contained in a divisor of $\mathcal{R}_{4,3}$, parametrizing pairs (C, η) such that the multiplication map $\mu : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^{\otimes 2})$ is not an isomorphism. This case seems interestingly related to the G_2-variety, see [23] and Sect. 7.

The proof of (3) is sketched here and it will appear elsewhere. The image of $r_{6,3}$ parametrizes pairs (C, η), where C is a curvilinear section of a suitable Gushel–Mukai threefold singular along a rational normal sextic curve, see Sect. 7.

Let $(S, \mathcal{L}, \mathcal{E})$ be a level ℓ K3 surface of genus g and $\phi : S \to \mathbb{P}^g$ the morphism defined by $\mathcal{L}$, we assume for simplicity that ϕ is birational onto $\overline{S} := \phi(S)$. Then we close this introduction with few lines addressing the classification of Fano threefolds

$$\overline{X} \subset \mathbb{P}^{g+1}$$

whose general hyperplane sections are projective models $\overline{S}$ as above. The problem sounds similar to that of classifying threefolds $T \subset \mathbb{P}^g$ whose hyperplane sections are Enriques surfaces, that is, Enriques–Fano threefolds. It seems however quite neglected.

Some examples of threefolds $\overline{X}$ appear in this paper, most are normal and Sing $\overline{X}$ is a curve. Moreover $\overline{X}$ admits a cyclic cover $\pi : \tilde{X} \to \overline{X}$, branched exactly on Sing $\overline{X}$. A basic notion of level ℓ polarized projective variety $(X, \mathcal{L}, \mathcal{E})$ is introduced in the next section, since it is useful in the cases we want to consider.

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2 Some preliminaries

In what follows X is a smooth, irreducible complex projective variety and $\mathcal{L}$ is a big and nef line bundle on X, we say that $(X, \mathcal{L})$ is a polarized projective variety. On the other hand we are interested, along this paper, in some families of cyclic coverings

$$\pi : \tilde{X} \to X. \tag{5}$$

Then we fix our conventions about, [10], [21, I p.242]. By definition π is a finite morphism of degree $\ell \geq 2$ and it is the quotient map of the action of an automorphism of order ℓ of $\tilde{X}$. We assume that $\tilde{X}$ is normal, up to composing π with the normalization map. Hence $\tilde{X}$ is reduced with irreducible connected components. Starting from π, we briefly review the recipe for its construction. Notice that $\pi_* \mathcal{O}_{\tilde{X}} \cong \mathcal{A}$, where

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{E}^{-1} \oplus \cdots \oplus \mathcal{E}^{-\ell+1} \tag{6}$$

and $\mathcal{E} \in \operatorname{Pic} X$. Assume $\tilde{X}$ is connected and hence irreducible. Then π defines the field extension $\pi^* : k(X) \to k(\tilde{X})$ and its trace map induces the exact sequence

$$0 \to \mathcal{E}^{-\ell} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_B \to 0, \tag{7}$$

for some $s \in H^0(\mathcal{E}^\ell)$. The multiplication by s defines a structure of $\mathcal{O}_X$-Algebra on $\mathcal{A}$. We have $\tilde{X} = \operatorname{Spec} \mathcal{A}$, moreover π factors through the projection $u : \mathbb{P}(\mathcal{A}) \to X$. The branch divisor of π is $\operatorname{div}(s)$ and will be denoted by B. For B we fix the notation

$$B = m_1 B_1 + \cdots + m_r B_r, \tag{8}$$

where $B_1, \ldots, B_r$ are prime divisors. Conversely, a pair $(\mathcal{E}, B)$ such that $B \in |\mathcal{E}^\ell|$ defines on $\mathcal{A}$ an $\mathcal{O}_X$- Algebra structure as above and a cyclic cover π. Notice that the condition $g.c.d.(\ell, m_1, \ldots, m_r) = 1$ implies the irreducibility of $\tilde{X}$.

Now let C be a reduced curve and $\eta \in \operatorname{Pic} C$ a nontrivial ℓ-torsion element. Then (C, η) uniquely defines, using a nonzero vector $s \in H^0(\eta^\ell)$, a nonramified cyclic cover

$$\pi : \tilde{C} \to C,$$

which is nontrivial. To give a pair (C, π) is equivalent to give a singular level ℓ curve (C, η). Now recall that a curve $C \subset X$ is mobile if moves in an irreducible algebraic family covering X, with integral general member. In the Néron–Severi group $N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the mobile classes of such curves generate an important convex cone, [5, 1.3 (vi)], [21, II p. 307]. Finally we introduce the following definition.

Definition 2.1 Let $\mathcal{E} \in \operatorname{Pic} X$, the pair $(X, \mathcal{E})$ is a level ℓ structure on X if:
- $|\mathcal{E}^\ell| \neq \emptyset$ and a general $B \in |\mathcal{E}^\ell|$ defines an integral cyclic cover,
- there exists a mobile curve C in X such that $CB = 0$.

Assume $\dim X = 1$ then X is the smooth, integral curve C and $\mathcal{E}$ is a line bundle of degree 0 such that $\mathcal{E}^\ell \cong \mathcal{O}_C$. Moreover we are assuming that the cover $\pi : \tilde{C} \to C$ defined by $\mathcal{E}$ is integral. Hence $\mathcal{E}$ is a nontrivial ℓ-torsion element. Then, for curves, the definition is the traditional one. In higher dimension the next property is clear.

Proposition 2.1 *Let $(X, \mathcal{E})$ be a level ℓ structure on X and $C \subset X$ a mobile curve such that $CE = 0$, where $\mathcal{O}_X(E) \cong \mathcal{E}$. Then $\mathcal{O}_C(E)$ is an ℓ-torsion element of* $\operatorname{Pic} C$.

Proof Consider $D \in |\mathcal{E}^\ell|$. Since C is movable we can assume that C is not a component of D. Then $C \cap D$ is empty because $CE = 0$. This implies that $\mathcal{E}^\ell \otimes \mathcal{O}_C \cong \mathcal{O}_C(D) \cong \mathcal{O}_C$. □

Remark 2.1 Nevertheless we may have a trivial $\mathcal{O}_C(E)$ even when $\mathcal{E}$ is not, and even generically when C moves in its family. This is obvious if C is smooth and rational. Furthermore consider a curve F and the projection $p : F \times X \to X$. Then $(F \times X, p^*\mathcal{E})$ is a level ℓ-structure on $F \times X$ and $p^*\mathcal{E}$ is trivial on the mobile curve $p^*(x)$, $x \in X$.

Then, to address the concrete topics of our paper, we turn to polarized pairs $(X, \mathcal{L})$ and we denote by d the dimension of X. We assume that $|\mathcal{L}^m|$ is globally generated for $m >> 0$ and observe that a general complete intersection of $d - 1$ elements of $|\mathcal{L}^m|$ is a smooth, integral mobile curve, which moves in an irreducible family $\mathcal{C}_m$ of transversal complete intersections in X.

Proposition 2.2 *Let X, $\mathcal{L}$, $\mathcal{E}$ be as above. Assume $CE = 0$, where $C \in \mathcal{C}_m$ and $\mathcal{O}_X(E) \cong \mathcal{E}$. Then $\mathcal{O}_C(E)$ is a nontrivial ℓ-torsion element of* $\operatorname{Pic} C$, *moreover*

$$h^0(\mathcal{O}_X(kE)) = 0, \quad k \not\equiv 0 \mod \ell.$$

Proof By induction on $d = \dim X$. Let $d = 1$ then $X = C$ and $\{C\} = \mathcal{C}_m$. Since $\mathcal{E}$ defines an integral cover, the statement follows. Let $d \geq 2$ and $C = D_1 \cdot \dots \cdot D_{d-1}$, where $D_1, \dots, D_{d-1} \in |\mathcal{L}^m|$, then a general D in the linear system generated by $D_1 \cdots D_{d-1}$ is smooth. $\mathcal{O}_D(D)$ is nef, big and globally generated. Let $\pi : \tilde{X} \to X$ be the cyclic cover, branched on B, since C is mobile and $CB = 0$ we can assume $C \cap B = \emptyset$. Now let $f : X \to \mathbb{P}^n$ be the morphism defined by $|D|$, then f is generically finite onto its image and the same is true for $f \circ \pi : \tilde{X} \to \mathbb{P}^n$. Then $\tilde{C} = \pi^{-1}(C)$ is connected by the connectedness theorem and $\mathcal{O}_C(E)$ is non trivial of ℓ-torsion in $\operatorname{Pic} C$. Moreover $(D, \mathcal{O}_D(E))$ is a level ℓ structure and the second statement follows by induction on d. □

Keeping this notation we finally come to the following definition.

Definition 2.2 A level ℓ polarized variety is a triple $(X, \mathcal{L}, \mathcal{E})$ such that $(X, \mathcal{E})$ is a level ℓ structure on X and $CE = 0$, where $C \in \mathcal{C}_m$.

Actually the triples $(X, \mathcal{L}, \mathcal{E})$ we will consider always satisfy the additional property: $|\mathcal{L}|$ is base point free and defines a birational morphism onto its image

$$f : X \to \mathbb{P}^n. \tag{9}$$

Hence we assume $C = H_1 \cap \cdots \cap H_{d-1} \in \mathcal{C}_1$, where $H_1 \cdots H_{d-1} \in |f^*\mathcal{O}_{\mathbb{P}^n}(1)|$. So C shows the distinguished line bundles $\eta_C := \mathcal{E} \otimes \mathcal{O}_C$ and $\mathcal{L}_C := \mathcal{L} \otimes \mathcal{O}_C$ and these lead us to the varieties we are interested in. For these $\mathcal{L}_C$ is the canonical sheaf ω_C. For the triples considered, we will also have that the restriction $r : H^0(\mathcal{L}) \to H^0(\omega_C)$ is surjective and that $\overline{X} := f(X)$ is normal. So we are going to deal with projective varieties $\overline{X}$ whose curvilinear sections are canonical curves C, endowed with the étale cover defined by η_C. This includes K3 surfaces and Fano threefolds with a prescribed level ℓ structure.

3 Level ℓ K3 surfaces

We begin discussing the families of level ℓ polarized $K3$ surfaces $(S, \mathcal{L}, \mathcal{E})$ and the chances that $C \in |\mathcal{L}|$ be a curve with general moduli. We say that $C^2 = 2g - 2$ is the degree of $(S, \mathcal{L})$ and g its genus. As usual the moduli space of $(S, \mathcal{L})$ is denoted by

$$\mathcal{F}_g, \tag{10}$$

it is an integral quasi projective variety of dimension 19. Let $[S, \mathcal{L}] \in \mathcal{F}_g$ be a general point, we recall that then $\operatorname{Pic} S \cong \mathbb{Z}\mathcal{L}$ and $|\mathcal{L}|$ defines an embedding

$$f : S \to \mathbb{P}^g \tag{11}$$

for $g \geq 3$. Coming to level ℓ structures $(S, \mathcal{L}, \mathcal{E})$, these properties are no longer satisfied, as we are going to recall. We fix our notation as follows, the map

$$\pi' : \tilde{S}' \to S \tag{12}$$

is the covering morphism defined by $\mathcal{E}$. As already established its branch divisor is

$$B = m_1 B_1 + \cdots + m_r B_r,$$

where $B_1, \ldots, B_r$ are the irreducible components of $\operatorname{Supp} B$. Of course, since $\operatorname{Pic} S$ has no torsion, B is not zero. We fix the following convention:

- r is the number of irreducible components of $\operatorname{Supp} B$,
- t is the number of its connected components.

Moreover we set

$$B_1 + \cdots + B_r = B_{\mathrm{red}} = N_1 + \cdots + N_t, \tag{13}$$

where $N_1 \cdots N_t$ denote the connected components of $\operatorname{Supp} B$. Notice that $CB_i = 0$ for $i = 1 \cdots r$. Indeed C is integral and $\dim |C| \geq 1$ so that $CB_i \geq 0$. Since $B \in |\ell E|$ then $CB = 0$ and this implies $CB_i = 0$. Then, applying the Hodge Index Theorem, B_i is an integral curve on S with $B_i^2 < 0$. Hence $B_i^2 = -2$ and B_i is $\mathbb{P}^1$. The same argument applies to N_j which is a reduced connected curve of arithmetic genus 0. In particular each N_j is contracted by f to a quadratic singularity and $\operatorname{Pic} S$ is not isomorphic to $\mathbb{Z}$.

It is not difficult to see that the Kodaira dimension of $\tilde{S}'$ is zero, moreover, with some elaboration, one has the following property, cfr. [14,24].

Proposition 3.1 *Either $\tilde{S}'$ is birational to a K3 surface or to an abelian surface.*

Definition 3.1 Let $(S, \mathcal{L}, \mathcal{E})$ be a level ℓ K3 surface, we say that:

(1) $(S, \mathcal{L}, \mathcal{E})$ is of K3 type if $\tilde{S}'$ is birational to a K3 surface,
(2) $(S, \mathcal{L}, \mathcal{E})$ is of abelian type if $\tilde{S}'$ is birational to an abelian surface.

Case (2) is scarcely interesting for our purposes. We aim indeed to use the curves $C \in |\mathcal{L}|$ in order to parametrize the moduli space $\mathcal{R}_{g,\ell}$ of level ℓ curves in low genus. But in case (2) C has not enough moduli for $g \geq 3$.

We assume since now that $(S, \mathcal{L}, \mathcal{E})$ is a level ℓ K3 surface of K3 type. Then, to ameliorate the expositon, we just say with some abuse that $(S, \mathcal{L}, \mathcal{E})$ is a level ℓ K3 surface. We say that two triples $(S_n, \mathcal{L}_n, \mathcal{E}_n)$, $(n = 1, 2)$, are isomorphic if there exists a biregular map $\beta : S_1 \to S_2$ such that $\beta^*\mathcal{L}_2 \cong \mathcal{L}_1$ and $\beta^*\mathcal{E}_2 \cong \mathcal{E}_1$, $i = 1, 2$.

As mentioned the classification of these triples is due to Nikulin and originates from his paper [24]. The part of interest here is the classification of pairs $(\tilde{S}, G)$, where $\tilde{S}$ is a K3 surface and G is a finite group of symplectic automorphisms of $\tilde{S}$. There exist 14 classes of pairs $(\tilde{S}, G)$ such that G is commutative and G is $\mathbb{Z}/\ell\mathbb{Z}$ exactly for $2 \leq \ell \leq 8$. After the classification, several papers addressed the description of the moduli and the projective models of these K3 surfaces. It is due to mention here [13–16,26].

The triple $(S, \mathcal{L}, \mathcal{E})$ determines an associated triple $(\tilde{S}, \tilde{\mathcal{L}}, \gamma)$, where $\gamma \in \operatorname{Aut} \tilde{S}$ is a symplectic automorphisms of order ℓ and $(\tilde{S}, \tilde{\mathcal{L}})$ is a polarized K3 surface of degree $\ell(2g-2)$. We have indeed $B_{\mathsf{red}} = N_1 + \cdots + N_t$, where the summands are the connected components and -2-curves. Let $\nu : S \to \overline{S}$ be their contraction morphism, then the Cartesian square

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\pi'} & S \\ {\scriptstyle \nu'}\downarrow & & \downarrow{\scriptstyle \nu} \\ \tilde{S} & \xrightarrow{\pi} & \overline{S} \end{array} \tag{14}$$

is the Stein factorization of $\nu \circ \pi'$. In it ν' is a birational morphism. Let $G \subset \operatorname{Aut} \tilde{S}'$ be the group whose quotient map is π'. As we will see $\pi'^* H^0(\mathcal{L}(-E))$ sits in $H^0(\tilde{\mathcal{L}})$ as an eigenspace of the natural representation of G and defines a generator γ of G. Moreover π is the quotient map of the induced action of G on $\tilde{S}$. Conversely, starting from π and the minimal desingularization ν, π' is reconstructed from the fibre product $\pi \times_{\overline{S}} \nu$.

In order to describe the rational singularities occurring in $\operatorname{Sing} \overline{S}$ we use the notation

$$\mathsf{T} := n_1\mathsf{T}_1 + \cdots + n_s\mathsf{T}_s, \tag{15}$$

where T_j is the singularity type and n_j the number of points of type T_j in $\operatorname{Sing} \overline{S}$.

Theorem 3.2 *Let $(S, \mathcal{E}, \mathcal{L})$ be a level ℓ K3 surface of genus g, then one has $2 \leq \ell \leq 8$ and $(S, \mathcal{E})$ satisfies one of the following conditions:*

(1) $\ell = 2$. *One has $t = 8$, $r = 8$ and $\mathsf{T} = 8\mathsf{A}_1$.*
(2) $\ell = 3$. *One has $t = 6$, $r = 12$ and $\mathsf{T} = 6\mathsf{A}_2$.*
(3) $\ell = 4$. *One has $t = 6$, $r = 14$ and $\mathsf{T} = 4\mathsf{A}_3 + 2\mathsf{A}_1$.*
(4) $\ell = 5$. *One has $t = 4$, $r = 16$ and $\mathsf{T} = 4\mathsf{A}_4$.*
(5) $\ell = 6$. *One has $t = 6$, $r = 16$ and $\mathsf{T} = 2\mathsf{A}_5 + 2\mathsf{A}_2 + 2\mathsf{A}_1$.*
(6) $\ell = 7$. *One has $t = 3$, $r = 18$ and $\mathsf{T} = 3\mathsf{A}_6$.*
(7) $\ell = 8$. *One has $t = 4$, $r = 18$ and $\mathsf{T} = 2\mathsf{A}_7 + \mathsf{A}_3 + \mathsf{A}_1$.*

See [24]. It is also useful to observe that always one has

$$E^2 = \frac{B^2}{\ell^2} = -4. \tag{16}$$

Now, in view of the concrete applications in this paper, we mention some relevant properties of the structure of Pic S and of the moduli of the above triples.

Definition 3.2 $\mathcal{F}_{g,\ell}$ is the moduli space of level ℓ K3 surfaces of genus g.

As in the case of $(S, \mathcal{L})$, the construction of $\mathcal{F}_{g,\ell}$ relies on the usual notion of lattice polarized variety, see [3,9,18,24] for this K3 case. In particular, for every $g \geq 2$, $\mathcal{F}_{g,\ell}$ has a standard irreducible component to be constructed as follows. We may have

$$\mathbb{Z}[\mathcal{L}] \oplus \mathbb{M}_S \subseteq \text{Pic } S, \tag{17}$$

where the sum is orthogonal. Moreover $\mathbb{M}_S$ has rank r and it is generated by the classes $[B_1], \ldots, [B_r], [E]$, with $\mathcal{E} \cong \mathcal{O}_S(E)$, so that the relation $\ell[E] - [B] = 0$ is satisfied in Pic S. We can see the inclusion as the image of a primitive embedding of lattices

$$\upsilon : \mathbb{Z}c \oplus \mathbb{M}_\ell \to \text{Pic } S, \tag{18}$$

where $\upsilon(c) := [\mathcal{L}]$ and $\upsilon(\mathbb{M}_\ell) = \mathbb{M}_S$. The lattice $\mathbb{M}_\ell$ is given with the set of generators $\{e, b_1, \ldots, b_r\}$ so that $\upsilon(e) = [E]$, $\upsilon(b_1) = [B_1], \ldots, \upsilon(b_r) = [B_r]$. Notice also that

$$c^2 = 2g - 2\ ,\ e^2 = -4\ ,\ b_1^2 = \cdots = b_r^2 = -2, \tag{19}$$

cfr. [24]. Fixing these data, the moduli space of triples $(S, \mathcal{L}, \mathcal{E})$ endowed with an embedding υ, can be constructed as a moduli space of lattice polarized K3 surfaces (S, υ). In our case S is M-polarized with $M := \mathbb{Z}c \oplus \mathbb{M}_\ell$ and the induced embedding $M \subset L := H^2(S, \mathbb{Z})$ is unique up to isometries, [24]. Then the moduli space is constructed as quotient of the period domain of these surfaces S. In particular its dimension is $19 - r$, [9, Section 4.1 and Theorem 1.4.8], [4, Section 2.4 and Proposition 2.6]. Moreover a unique irreducible component of it is the closure of the moduli points of pairs (S, υ) such that

$$\text{Pic } S = \mathbb{Z}[\mathcal{L}] \oplus \mathbb{M}_S. \tag{20}$$

In this case we will say that $(S, \mathcal{L}, \mathcal{E})$ is a standard triple of genus g and level ℓ. Let us fix our notation:

Definition 3.3 $\mathcal{F}^{\perp}_{g,\ell}$ is the moduli space of standard triples of genus g and level ℓ.

$\mathcal{F}^{\perp}_{g,\ell}$ exists for any $g \geq 2$ and $\ell = 2 \cdots 8$. Fixing ℓ, $\mathcal{F}^{\perp}_{g,\ell}$ is the unique irreducible component of $\mathcal{F}_{g,\ell}$ along a proper countable set of values $g \in \mathbb{N}$.

Remark 3.1 Let $(S, \mathcal{L}, \mathcal{E})$ be a non standard triple and $C \in |\mathcal{L}|$. Then, at least experimentally for $\ell = 2$, C is never general in moduli for $g \geq 4$. This is true even when the parameter count makes that possible in low genus, see [20]. The situation is quite different for standard triples. This paper studies indeed the modular properties of C in this case: standard behavior or peculiarities of C.

4 A standard projective model

Given a standard triple $(S, \mathcal{L}, \mathcal{E})$, let us construct a projective realization of S useful to our purposes. Consider $C \in |\mathcal{L}|$ such that $C \cap B = \emptyset$ and $\tilde{C}' = \pi'^* C$. Then the curve $\tilde{C} = \nu'_* \tilde{C}'$ is biregular to $\tilde{C}'$ via the contraction $\nu' : \tilde{S}' \to \tilde{S}$ and the linear map

$$\nu'_* : H^0(\mathcal{O}_{\tilde{S}'}(\tilde{C}')) \to H^0(\mathcal{O}_{\tilde{S}}(\tilde{C})) \tag{21}$$

is an isomorphism, we identify the two spaces under it. Then, using $\tilde{C}$, it is easy to remind of the action of the group $\mathbb{Z}/\ell\mathbb{Z}$ on this space and of its eigenspaces. Let

$$0 \to \mathcal{O}_{\tilde{S}'} \to \mathcal{O}_{\tilde{S}'}(\tilde{C}') \to \omega_{\tilde{C}} \to 0 \tag{22}$$

be the standard exact sequence, then $\mathbb{Z}/\ell\mathbb{Z}$ acts on its associated long exact sequence

$$0 \to H^0(\mathcal{O}_{\tilde{S}'}) \to H^0(\mathcal{O}_{\tilde{S}'}(\tilde{C}')) \to H^0(\omega_{\tilde{C}}) \to 0.$$

As is well known the $\mathbb{Z}/\ell\mathbb{Z}$-decomposition of $H^0(\omega_{\tilde{C}})$ is as follows

$$H^0(\omega_{\tilde{C}}) = \bigoplus_{k=1\cdots\ell-1} \pi'^* H^0(\omega_C \otimes \eta^{-k}) \bigoplus \pi'^* H^0(\omega_C). \tag{23}$$

and this implies that $H^0(\mathcal{O}_{\tilde{S}}(\tilde{C}'))$ decomposes as

$$H^0(\mathcal{O}_{\tilde{S}}(\tilde{C}')) = \bigoplus_{k=1\ldots\ell-1} \pi'^* H^0(\mathcal{O}_S(H_k)) \bigoplus \pi'^* H^0(\mathcal{O}_S(C)), \tag{24}$$

where $\mathcal{O}_S(H_1) \ldots \mathcal{O}_S(H_{\ell-1}) \in \operatorname{Pic} S$ and $\mathcal{O}_C(H_k) \cong \omega_C \otimes \eta^{\otimes -k}$, up to reindexing. Since $\tilde{C}$ has genus $\tilde{g} = g + (\ell-1)(g-1)$ it follows $\dim H^0(\mathcal{O}_{\tilde{S}}(\tilde{C})) = g + 1 + (\ell-1)(g-1)$. In particular the above decomposition immediately implies that

$$\dim H^0(\mathcal{O}_S(H_k)) = \dim H^0(\omega_C \otimes \eta^{-k}) = g - 1, \quad k = 1 \cdots \ell - 1. \tag{25}$$

In what follows, it is also useful to recall the mentioned fact that $E^2 = -4$.

Lemma 4.1 *It holds $h^i(\mathcal{O}_S(E)) = h^i(\mathcal{O}_S(-E)) = 0$, for $i \geq 0$.*

Proof By assumption E is not effective. The same is true for $-E$, since $\ell E \sim B$ and $B > 0$. This implies $h^0(\mathcal{O}_S(E)) = 0$ and $h^2(\mathcal{O}_S(E)) = h^0(\mathcal{O}_S(-E)) = 0$. Since $E^2 = -4$ we have $\chi(\mathcal{O}_S(E)) = 0$ and then $h^1(\mathcal{O}_S(E)) = 0$. The same argument applies to $-E$. □

Now we consider the line bundle $\mathcal{O}_S(C - E)$ and the standard exact sequence

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S(C - E) \to \mathcal{O}_C(C - E) \to 0.$$

Lemma 4.2 *Let $g \geq 2$ then the associated long exact sequence is*

$$0 \to H^0(\mathcal{O}_S(C - E)) \to H^0(\omega_C \otimes \eta^{-1}) \to 0,$$

in particular it follows $\dim |C - E| = g - 2$ *and* $h^i(\mathcal{O}_S(C - E)) = 0,\ i \geq 1$.

Proof By the previous lemma $h^i(\mathcal{O}_S(E)) = h^i(\mathcal{O}_S(-E)) = 0$, for $i \geq 0$. Moreover we have $h^0(\omega_C \otimes \eta^{-1}) = g - 1$ and $h^1(\omega_C \otimes \eta^{-1}) = 0$. Then the statement follows. □

Now we observe that the pull-back by π' defines a linear embedding

$$\pi'^{*} : H^0(\mathcal{O}_S(C-E)) \to H^0(\mathcal{O}_{\tilde{S}'}(\tilde{C}')).$$

We have indeed $\mathcal{O}_{\tilde{S}'}(\tilde{C}') \otimes \pi'^{*}\mathcal{O}_S(E-C) \cong \mathcal{O}_{\tilde{S}'}(\pi'^{*}E)$ and finally

$$h^0(\mathcal{O}_{\tilde{S}'}(\pi'^{*}E)) = h^0(\pi'_{*}\mathcal{O}_{\tilde{S}'}(\pi'^{*}E)) = h^0(\mathcal{A}(E)) = 1, \tag{26}$$

with $\mathcal{A} = \mathcal{O}_S \oplus \mathcal{O}_S(-E) \oplus \cdots \oplus \mathcal{O}_S((1-\ell)E)$. The equality defines, up to a nonzero constant factor, the linear embedding π'^{*}. Then Im π'^{*} is the $\mathbb{Z}/\ell\mathbb{Z}$-invariant space

$$\pi'^{*}H^0(\mathcal{O}_S(C-E)).$$

Proposition 4.3 *Let* $g \geq 3$ *and* Pic $S \cong \mathbb{Z}c \oplus \mathbb{M}_\ell$*, then* $|C-E|$ *is base point free.*

Proof Since S is a K3 surface, it suffices to prove that $|C-E|$ has no fixed component. Let F be an integral fixed component of $|C-E|$, set $f = F \cdot C$ for a general C. Then f is a fixed divisor of $|\omega_C \otimes \eta^{-1}|$. Applying Riemann-Roch to C it follows dim $|\eta(f)| = \deg f - 1$. Since $g \geq 3$ then $\deg f \leq 2$. Hence F is a line, a conic or $FC = 0$. We have $F \sim xC + \sum y_j B_j + zE$ in Pic S. Assume $\deg f > 0$ then $0 < CF = (2g-2)x \leq 2$ with $x \in \mathbb{Z}$: a contradiction for $g \geq 3$. Let $CF = 0$ then $F^2 = -2$ by the Hodge Index Theorem and F is a $\mathbb{P}^1$ contracted by $f_{|C|} : S \to \mathbb{P}^g$. By Lemma 4.2, $h^0(C-E) = g-1 = (C-E)^2/2 + 2$. Let M be the moving part of the linear system $|C-E|$, then dim $|M| \geq 1$ and $MF \geq 0$. Moreover we have $C - E \sim M + kF + R$, where R is a curve not containing F and $k \geq 1$. Let $G \in |M+F|$ be general then G contains F: otherwise the curve kF could'nt be a component of the element $G + (k-1)F + R \in |C-E|$. Hence F is a fixed component of $|M+F|$. Now observe that $MF \geq 0$ and then consider the standard exact sequence

$$0 \to \mathcal{O}_S(M) \to \mathcal{O}_S(M+F) \to \mathcal{O}_F(M) \to 0.$$

We claim that, passing to the associated long exact sequence, it follows

$$\chi(\mathcal{O}_S(M)) = \chi(\mathcal{O}_S(M+F))$$

and $\chi(\mathcal{O}_F(M)) = 0$. Since $F = \mathbb{P}^1$ this implies $MF < 0$: a contradiction. To prove the claim consider a smooth $D \in |M|$. Then either D is integral of genus $g-2$ and $h^1(\mathcal{O}_S(M)) = 0$ or $M \sim (g-2)N$ and N is a smooth integral elliptic curve. Via Serre duality we have $h^2(\mathcal{O}_S(M)) = h^2(\mathcal{O}_S(M+F)) = 0$. Moreover $MF \geq 0$ implies $h^1(\mathcal{O}_F(M)) = 0$. Then, in the former case, $h^1(\mathcal{O}_S(M)) = 0$ implies $h^1(\mathcal{O}_S(M+F)) = 0$ and the claim follows. In the latter case replace M by N. Then the equality and the same contradiction follow by the same type of arguments. □

Now we introduce a second linear system associated with E. At first let us set

$$B_{\mathrm{red}} := B_1 + \cdots + B_r, \tag{27}$$

where the summands are the irreducible components of Supp B. Then we recall that

$$E = \frac{1}{\ell}(m_1 B_1 + \cdots + m_r B_r), \quad \text{with } m_1 \cdots m_r \in [1 \cdots \ell - 1].$$

Definition 4.1 Set $\mathring{E} = B_{\mathrm{red}} - E = \frac{1}{\ell}(\mathring{m}_1 B_1 + \cdots + \mathring{m}_r B_r)$, where $\mathring{m}_i := \ell - m_i$.

Let us denote by n_i the coefficients of the curves B_i in $-\ell E$. Then $n_i \equiv \mathring{m}_i \mod \ell$. More precisely, E is a generator of $\mathbb{Z}/\ell\mathbb{Z} = \langle B_i, E\rangle/\langle B_i\rangle$ and $\mathring{E}$ is its opposite in $\mathbb{Z}/\ell\mathbb{Z}$; in particular it is a different generator of the same group. Hence $\mathring{\mathcal{E}} := \mathcal{O}_S(\mathring{E})$ is a level ℓ structure, with the same properties of $\mathcal{E}$. We notice that $\mathring{E}$ defines a cover $\mathring{\pi}' : \tilde{S}' \to S$ so that $\mathring{\pi}' = \pi' \circ a$ and $a^\ell = id_{\tilde{S}'}$. Then we define

$$|H| := |C - E|, \quad \mathring{H} := |C - \mathring{E}|. \tag{28}$$

The rational maps associated with these linear systems respectively will be

$$p : S \to \mathbb{P}, \quad \mathring{p} : S \to \mathring{\mathbb{P}}, \tag{29}$$

where $\mathbb{P} := |H|^*$ and $\mathring{\mathbb{P}} := |\mathring{H}|^*$ are the projective space $\mathbb{P}^{g-2}$. Let ι be the inclusion

$$\mathbb{P} \times \mathring{\mathbb{P}} \subset \mathbb{P}^{(g-1)^2-1} \tag{30}$$

defined by the Segre embedding, we set $f := \iota \circ (p \times \mathring{p})$ and fix the notation

$$f : S \to \mathbb{P} \times \mathring{\mathbb{P}} \subset \mathbb{P}^{(g-1)^2-1}. \tag{31}$$

Definition 4.2 The morphism f is the main projective model of $(S, \mathcal{L}, \mathcal{E})$.

The next two remarks are simple but relevant in order to discuss f, (the second one follows by a direct computation of $E \cdot \mathring{E}$, where the class E is explicitly given in [24]):

(1) $f^*\mathcal{O}_{\mathbb{P}^{(g-1)^2-1}}(1) \cong \mathcal{O}_S(H + \mathring{H}) \cong \mathcal{O}_S(2C - B_{red})$,
(2) $H\mathring{H} = 2g + 2 - t$.

Proposition 4.4 *The divisors* $[H - \mathring{H}]$ *and* $[\mathring{H} - H]$ *are not effective classes for* $\ell \geq 3$ *and*

$$h^1(\mathcal{O}_S(H - \mathring{H})) = h^1(\mathcal{O}_S(\mathring{H} - H)) = 6 - t. \tag{32}$$

Proof We have $H(H - \mathring{H}) = \mathring{H}(\mathring{H} - H) = t - 8$. Since the general elements of $|H|$ and $|\mathring{H}|$ are irreducible curves, the first statement follows for $\ell \geq 3$ because then $t \leq 6$. The second statement just follows from Riemann-Roch. □

Now let us consider, for a general $C \in |\mathcal{L}|$, the standard exact sequence

$$0 \to \mathcal{O}_S(C - B_{red}) \to \mathcal{O}_S(2C - B_{red}) \to \mathcal{O}_C(2C - B_{red}) \to 0. \tag{33}$$

Since C is smooth and disjoint from B_{red}, then $\mathcal{O}_C(-B_{red})$ is trivial and $|2C - B_{red}|$ cuts on C a linear system of bicanonical divisors. Moreover we know that both $|H|$ and $|\mathring{H}|$ are base point free. Hence the same is true for $|H + \mathring{H}| = |2C - B_{\mathsf{red}}|$. Notice that

$$(2C - B_{\mathsf{red}})^2 = 8(g - 1) - 2t,$$

which is ≥ 0 for $g \geq 3$ and any of the prescribed values of t, ℓ. Actually the zero value is only reached in the known situation $g = 3, \ell = 2$. Hence we assume $g \geq 4$ for $\ell = 2$. Then a general $D \in |H + \mathring{H}|$ is a smooth integral curve such that $D^2 > 0$. As is well known, this implies $h^i(\mathcal{O}_S(H + \mathring{H})) = 0$ for $i \geq 1$ and the next property follows.

Proposition 4.5 *Let* g *be as above then* $\dim |2C - B_{\mathsf{red}}| = 4g - t - 3$ *and the long exact sequence associated with the exact sequence* (33) *is as follows:*

$$0 \to H^0(\mathcal{O}_S(C - B_{red})) \to H^0(\mathcal{O}_S(2C - B_{red})) \to H^0(\omega_C^{\otimes 2}) \to H^1(\mathcal{O}_S(C - B_{red})) \to 0.$$

The linear system $|C - B_{\mathrm{red}}|$ also deserves some observations. Since we are dealing with a general standard triple $(S, \mathcal{L}, \mathcal{E})$, we know that $|C|$ defines a morphism

$$f_{|C|} : S \to \mathbb{P}^g$$

which is the contraction $\nu : S \to \overline{S}$, composed with the embedding $\overline{S} \subset \mathbb{P}^g$ defined by $|\nu_* C|$. Since a general C is disjoint from B, $|\nu_* C|$ is a linear system of Cartier divisors. Let $\mathcal{I}_{\mathrm{Sing}\,\overline{S}}$ be the ideal sheaf of $\mathrm{Sing}\,\overline{S}$, it is clear that the natural map

$$f^*_{|C|} : H^0(\mathcal{I}_{\mathrm{Sing}\,\overline{S}}(1)) \to H^0(\mathcal{O}_S(C - B_{\mathrm{red}}))$$

is an isomorphism. Then, considering the above exact sequence (33), we have

$$h^0(\mathcal{O}_S(C - B_{\mathrm{red}})) - h^1(\mathcal{O}_S(C - B_{\mathrm{red}})) = \chi(\mathcal{O}_S(2C - B_{\mathrm{red}})) - \chi(\omega_C^{\otimes 2}) = g + 1 - t. \tag{34}$$

This implies the next property.

Proposition 4.6 *It holds $h^1(\mathcal{O}_S(C - B_{red})) = 0$ if and only if $h^0(\mathcal{O}_S(C - B_{red})) = g + 1 - t$, that is, the points of* $\mathrm{Sing}\,\overline{S}$ *are linearly independent in $\mathbb{P}^g$.*

On the other hand consider the commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & H^0(\mathcal{O}_S(C - B_{\mathrm{red}})) \\
 & & \downarrow \\
H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H + \mathring{H})) \\
{\scriptstyle \rho_H \otimes \rho_{\mathring{H}}} \downarrow & & {\scriptstyle \rho_C} \downarrow \\
H^0(\omega_C \otimes \eta^{-1}) \otimes H^0(\omega_C \otimes \eta) & \xrightarrow{\mu_C} & H^0(\omega_C^{\otimes 2}) \\
 & & \downarrow \\
 & & H^1(\mathcal{O}_S(C - B_{\mathrm{red}})) \\
 & & \downarrow \\
 & & 0
\end{array} \tag{35}$$

where μ_S and μ_C are the multiplication maps and the vertical arrows are the restriction maps. It follows from Lemma (4.2) that $\rho_H \otimes \rho_{\mathring{H}}$ is an isomorphism. The next property is clear.

Proposition 4.7 *If μ_C is surjective then $h^1(\mathcal{O}_S(C - B_{red})) = 0$ i.e. ρ_C is surjective.*

Since $\chi(\mathcal{O}_S(C - B_{\mathrm{red}}) = g + 1 - t$ let us point out that μ_C is not surjective if

$$g < t - 1. \tag{36}$$

We do not further investigate the diagram, for our applications these results suffice.

5 Views on the Mukai maps in level ℓ

In this section we only put in large the picture we have outlined in the introduction. This picture concerns the maps in (3) and (2), that is, the Mukai map

$$m_g : \mathcal{P}_g \to \mathcal{M}_g$$

and the level ℓ Mukai maps

$$r_{g,\ell} : \mathcal{P}^{\perp}_{g,\ell} \to \mathcal{R}_{g,\ell}.$$

These maps, and the involved moduli spaces, have been previously considered. We recall that the points of $\mathcal{P}_g$ are the elements $[S, \mathcal{L}, C]$ such that $[S, \mathcal{L}] \in \mathcal{F}_g$ and $C \in |\mathcal{L}|$. The Mukai map m_g is the natural forgetful map. We have

(1) m_g is dominant for $g \leq 9$,
(2) m_g is not dominant for $g = 10$,
(3) m_g is birational for $g = 11$,
(4) m_g has 1-dimensional fibre for $g = 12$.
(5) m_g is generically injective for $g \geq 13$.

Thus m_g has not maximal rank for $g = 10, 12$. It is indeed known that a general $[C] \in m_{10}(\mathcal{P}_{10})$ is a linear section C of the G_2 variety $W \subset \mathbb{P}^{13}$, [23]. Hence the family of 2-dimensional linear sections of W through C is a $\mathbb{P}^3$. It turns out from this fact that the fibre of m_{10} at $[C]$ is 3-dimensional. Then $m_{10}(\mathcal{P}_{10})$ has codimension 1. Genus 12 Fano threefolds play a similar role, then a general fibre of m_{12} is a rational curve.

In this perspective, asking about the connections between the moduli space $\mathcal{F}^{\perp}_{g,\ell}$, of level ℓ K3 surfaces of genus g, and $\mathcal{R}_{g,\ell}$ is, as observed, natural. For a general point $[S, \mathcal{L}, \mathcal{E}] \in \mathcal{F}^{\perp}_{g,\ell}$ one can ask if (C, η), with $C \in |\mathcal{L}|$ and $\eta = \mathcal{E} \otimes \mathcal{O}_C$, defines a general point of $\mathcal{R}_{g,\ell}$. More precisely recall that $\mathcal{P}^{\perp}_{g,\ell}$ is the moduli space of 4-tuples $(S, \mathcal{L}, \mathcal{E}, C)$ such that $[S, \mathcal{L}, \mathcal{E}] \in \mathcal{F}^{\perp}_{g,\ell}$ and $C \in |\mathcal{L}|$. The level ℓ Mukai map $r_{g,\ell} : \mathcal{P}^{\perp}_{g,\ell} \to \mathcal{R}_{g,\ell}$ is the morphism sending $[S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}^{\perp}_{g,\ell}$ to the point $[C, \eta_C] \in \mathcal{R}_{g,\ell}$, where η_C is $\mathcal{E} \otimes \mathcal{O}_C$. About the possible dominance of the map $r_{g,\ell}$ we have:

(1) $3g - 3 = \dim \mathcal{R}_{g,2} \leq \dim \mathcal{P}^{\perp}_{g,2} = 11 + g$ iff $g \leq 7$.
(2) $3g - 3 = \dim \mathcal{R}_{g,3} \leq \dim \mathcal{P}^{\perp}_{g,3} = 7 + g$ iff $g \leq 5$.
(3) $3g - 3 = \dim \mathcal{R}_{g,4} \leq \dim \mathcal{P}^{\perp}_{g,4} = 5 + g$ iff $g \leq 4$.
(4) $3g - 3 = \dim \mathcal{R}_{g,5} \leq \dim \mathcal{P}^{\perp}_{g,5} = 3 + g$ iff $g \leq 3$.
(5) $3g - 3 = \dim \mathcal{R}_{g,6} \leq \dim \mathcal{P}^{\perp}_{g,6} = 3 + g$ iff $g \leq 3$.
(6) $3g - 3 = \dim \mathcal{R}_{g,7} \leq \dim \mathcal{P}^{\perp}_{g,7} = 1 + g$ iff $g \leq 2$.
(7) $3g - 3 = \dim \mathcal{R}_{g,8} \leq \dim \mathcal{P}^{\perp}_{g,8} = 1 + g$ iff $g \leq 2$.

These issues have not been systematically considered but for $\ell = 2$. We close this expository section with a summary on what happens for $\ell = 2, 3$.

5.1 The picture for $\ell = 2$

We have $3g - 3 = \dim \mathcal{M}_g \leq \dim \mathcal{P}^{\perp}_{g,2} = 11 + g$ iff $g \leq 7$. Again, $r_{g,2}$ behaves unexpectedly near the value of transition, which is now $g = 7$.

(1) $r_{g,2}$ is dominant for $g \leq 5$,

(2) $r_{g,2}$ is not dominant for $g = 6$,
(3) $r_{g,2}$ is birational for $g = 7$,
(4) $r_{g,2}$ has not finite fibres for $g = 8$.
(5) $r_{g,2}$ is generically injective for $g \geq 9$.

These surfaces are known as (standard) Nikulin surfaces. Cases (1), (2), (3) are treated in [11,12], the remaining ones, (standard and non standard), in [19,20]. Notice that $r_{g,2}$ is not of maximal rank for $g = 6, 8$. In genus 6 the condition $C \subset S$ implies that the following multiplication map is not an isomorphism as expected:

$$\mu : \mathrm{Sym}^2 H^0(\omega_C \otimes \eta_C) \to H^0(\omega_C^{\otimes 2}). \tag{37}$$

Then (C, η_C) does not define a general point of $\mathcal{R}_{g,2}$, see [3]. We point out that, studying the two cases where $r_{g,2}$ has not maximal rank, two families of singular Fano threefolds appear. Their hyperplane sections are singular models $\overline{S}$ of general Nikulin surfaces S. The existence of these threefolds implies the failure of the maximal rank.

5.2 The picture for $\ell = 3$

We will prove that $r_{g,3}$ behaves unexpectedly near $g = 5$:

(1) $r^s_{g,3}$ is dominant for $g \leq 3$,
(2) $r^s_{g,3}$ has not maximal rank for $g = 4$,
(3) $r^s_{g,3}$ is birational for $g = 5$,
(4) $r^s_{g,3}$ has not maximal rank for $g = 6$.

Remark 5.1 The case $g \geq 7$ should be considered for further investigation, addressing the generic injectivity. The (uni)rationality of $\mathcal{R}_{g,3}$ is known, or elementary, for $g \leq 5$, cfr. [1,2,28]. We recall that $\mathcal{R}_{g,3}$ is of general type for $g \geq 12$ and of Kodaira dimension ≥ 19 for $g = 11$, [7]. Bruns proved in [6] that $\mathcal{R}_{8,3}$ is of general type. The cases $g = 6, 7, 9, 10$ and partially $g = 11$ are open.

6 The Mukai map in level 3

6.1 The case of genus 4

Let $[S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}^{\perp}_{g,\ell}$ be general and $\ell = 3$, as in Sect. 2, (35) we consider the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H + \mathring{H})) \\ \rho_H \otimes \rho_{\mathring{H}} \downarrow & & \rho_C \downarrow \\ H^0(\omega_C \otimes \eta^{-1}) \otimes H^0(\omega_C \otimes \eta) & \xrightarrow{\mu_C} & H^0(\omega_C^{\otimes 2}). \end{array} \tag{38}$$

Since $\ell = 3$ we have $t = 6$ connected components of Supp B. Then, by proposition (4.7), μ_C is not surjective if $g < t - 1 = 5$. This is obvious for $g \leq 3$. For $g = 4$ the dimension count suggests that in $\mathcal{R}_{4,3}$ the map μ_C is not surjective in codimension 1 .

Proposition 6.1 *Let $[C, \eta] \in \mathcal{R}_{4,3}$ be a general point then μ_C is surjective, moreover the locus of points such that μ_C is not surjective is an effective Cartier divisor in $\mathcal{R}_{4,3}$.*

Indeed, for $g = 4$ and $\ell = 3$, this locus turns out to be the locus $\mathcal{D}_{g,\ell}$ defined in [7, p. 77]. There, for low level $\ell \geq 3$ and for $g \leq 16$, the so defined Torsion bundle conjecture B is proven, which implies that $\mathcal{D}_{4,3}$ is an effective Cartier divisor in $\mathcal{R}_{4,3}$. Then the next theorem follows. Notice also that, for $g = 4$, theorem 1.7 of [2] implies that μ_C is an isomorphism for a general (C, η).

Theorem 6.2 *The map $r_{4,3} : \mathcal{P}^{\perp}_{4,3} \to \mathcal{R}_{4,3}$ fails to be dominant.*

Remark 6.1 The case $g = 4$ turns out to be of special interest. See the last section for a natural, presently conjectural, geometric interpretation.

6.2 The case of genus 5

Differently from the case $g \leq 4$ the multiplication map

$$\mu_C : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^{\otimes 2})$$

can be surjective for $g \geq 5$ and a general point $[C, \eta] \in \mathcal{R}_{g,3}$. This property occurs in genus $g = 5$ and makes possible the proof of the next birationality theorem.

Theorem 6.3 *The Mukai map $r_{5,3} : \mathcal{P}^{\perp}_{5,3} \to \mathcal{R}_{5,3}$ is birational.*

Before proving it we cannot avoid a long series of preliminaries. We will always assume that $[S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}^{\perp}_{5,3}$ is a general point, in particular $\mathrm{Pic}\, S \cong \mathbb{Z}c \oplus \mathbb{M}_3$. Let

$$0 \to \mathcal{O}_S(H + \mathring{H} - C) \to \mathcal{O}_S(H + \mathring{H}) \to \omega_C^{\otimes 2} \to 0 \tag{39}$$

be the standard exact sequence, at first we point out the following fact.

Proposition 6.4 *The associated long exact sequence is*

$$0 \to H^0(\mathcal{O}_S(H + \mathring{H})) \stackrel{\rho_C}{\to} H^0(\omega_C^{\otimes 2}) \to 0. \tag{40}$$

Since $H + \mathring{H} - C \sim C - B_{\mathrm{red}}$, the next lemma implies the previous statement.

Lemma 6.5 *It holds $h^i(\mathcal{O}_S(C - B_{red})) = 0$ for $i \geq 0$.*

Proof Since $C(B_{\mathrm{red}} - C) < 0$, $h^0(\mathcal{O}_S(B_{\mathrm{red}} - C)) = 0$. Hence $h^2(\mathcal{O}_S(C - B_{\mathrm{red}}))$ is zero by Serre duality. Since $(C - B_{\mathrm{red}})^2 = -4$ then $\chi(\mathcal{O}_S(C - B_{\mathrm{red}})) = 0$ and the statement follows if $h^0(\mathcal{O}_S(C - B_{\mathrm{red}})) = 0$. Assume $A \in |C - B_{\mathrm{red}}|$ then A is not connected. This follows from $\chi(\mathcal{O}_S(A)) = h^0(\mathcal{O}_S(A)) - h^1(\mathcal{O}_S(A)) = 0$ and the standard exact sequence

$$0 \to \mathcal{O}_S(-A) \to \mathcal{O}_S \to \mathcal{O}_A \to 0.$$

This implies $A = A_1 + A_2$, where A_1 is a connected component and $A_2 = A - A_1$ is a curve. We have $C(A_1 + A_2) = C(C - B_{\mathrm{red}}) = 8$ and we can choose A_1 so that $CA_1 > 0$. Assume $CA_2 = 0$ then the morphism $\phi : S \to \mathbb{P}^5$, defined by $|C|$, maps birationally $A_1 + A_2 + B_{\mathrm{red}}$ onto a degree 8 hyperplane section of $\overline{S} = \phi(S)$. This is the curve $\phi_* A_1$, singular at the points of $\phi(B_{\mathrm{red}}) = \mathrm{Sing}\, \overline{S}$. These points are the images by ϕ of the six connected components of B_{red} and are exactly six. Indeed each fibre of ϕ is connected and hence two connected components V_1, V_2 of B_{red}, contracted to the same point, are connected by an effective divisor W orthogonal to C. On the other hand, under our generality assumption, we have $\mathrm{Pic}\, S \cong \mathbb{Z}c \oplus \mathbb{M}_3$. Moreover a direct computation shows that, in the negative definite lattice $\mathbb{M}_3$, Supp W is union of irreducible components of B_{red}. Actually one computes that the only classes of irreducible (-2)-curves are the classes of $B_1 \cdots B_{12}$. This implies $W = 0$ and $V_1 = V_2$. But then $\phi_* A_1$ is not integral, because it is a hyperplane section of $\phi(S)$ with six singular points. Then there exists an irreducible component R of it such that $0 < CR < 8$. The same is obvious if $CA_2 > 0$. Since Pic $S \cong \mathbb{Z}c \oplus \mathbb{M}_3$ we have $[R] = x[C] + \sum y_i[B_i] + z[E]$, with $x, y_i, z \in \mathbb{Z}$. But this implies $0 < CR = x8 < 8$ with $x \notin \mathbb{Z}$: a contradiction. □

Proposition 6.6 *The linear systems $|H|$ and $|\mathring{H}|$ are not hyperelliptic.*

Proof Let $|H|$ be hyperelliptic, then $|H|$ defines a $2:1$ morphism $\psi : S \to \mathbb{P}^3$ onto a quadric surface $Q := \psi(S)$. As is well known the pull-back of a ruling of lines of Q defines a pencil $|F_2|$ of curves such that $F_2^2 = 0$ and $HF_2 = 2$. Moreover $|F_1| := |H - F_2|$ is a pencil of irreducible elliptic curves. The same is true for the moving part of $|F_2|$. Since $H \sim F_1 + F_2$ and $C \sim H + E$ we have $C(F_1 + F_2) = 8$ and also $CF_i \geq 2, i = 1, 2$. Let $|F|$ be the moving part of the pencil $|F_i|$ such that CF_i is minimal, then it follows $2 \leq CF \leq 4$. On the other hand we have $F \sim xC + \sum y_j B_j + zE$ in Pic S. This implies $2 \leq CF = 8x \leq 4$ and $x \notin \mathbb{Z}$: a contradiction. The same argument works for $|\mathring{H}|$. □

Lemma 6.7 *It holds $h^i(\mathcal{O}_S(2H - \mathring{H})) = h^i(\mathcal{O}_S(2\mathring{H} - H)) = 0$ for $i \geq 0$.*

Proof From $H \sim C - E$ and $\mathring{H} \sim C - \mathring{E}$ we have $2H - \mathring{H} \sim C - 2E + \mathring{E}$, moreover

$$\mathring{H}(\mathring{H} - 2H) = -8 \Rightarrow h^0(\mathcal{O}_S(\mathring{H} - 2H)) = 0 \Rightarrow h^2(\mathcal{O}_S(2H - \mathring{H})) = 0.$$

Since $(2H - \mathring{H})^2 = -4$ then $\chi(\mathcal{O}_S(2H - \mathring{H})) = 0$. Hence the statement follows for $2H - \mathring{H}$ if we prove $h^0(\mathcal{O}_S(2H - \mathring{H})) = 0$. For this we observe that the well known descriptions of E and $\mathring{E}$ are as follows. For $i = 1 \cdots 6$ consider $N_i = B_i + B_i'$, that is, the i-th connected component of $B_{\text{red}} = \sum_{i=1\cdots 6} B_i + B_i'$. Then in Pic S we have

$$[E] = \sum_{i=1\cdots 6} \frac{1}{3}[B_i + 2B_i'], \quad [\mathring{E}] = \sum_{i=1\cdots 6} \frac{1}{3}[2B_i + B_i'] \tag{41}$$

up to exchanging E with $\mathring{E}$. Since $2H - \mathring{H} \sim C - 2E + \mathring{E}$, it follows that

$$2H - \mathring{H} \sim C - \sum_{i=1\cdots 6} B_i'. \tag{42}$$

This implies that $[2H - \mathring{H}]$ is not an effective class. Indeed let $B' := B_1' + \cdots + B_6'$, observe that $(C - B')B_i = -1, i = 1 \cdots 6$. Assume $C - B' \sim F$ where F is an effective divisor. Then $FB_i = -1$ implies $B_i \subset F$ and $F = F' + B_1 + \cdots + B_6$ where F' is effective. Hence $C - B_{\text{red}} \sim F' > 0$: a contradiction to the above lemma (6.5). □

We will profit of genus 3 curves of the non hyperelliptic linear systems $|H|$ or $|\mathring{H}|$.

Lemma 6.8 *It holds $\forall D \in |H|$, $h^0(\mathcal{O}_D(\mathring{H} - H)) = 0$ and $\forall \mathring{D} \in |\mathring{H}|$, $h^0(\mathcal{O}_{\mathring{D}}(H - \mathring{H})) =$* 0.

Proof Let $D \in |H|$, once more consider the standard exact sequence

$$0 \to \mathcal{O}_S(\mathring{H} - 2H) \to \mathcal{O}_S(\mathring{H} - H) \to \mathcal{O}_D(\mathring{H} - H) \to 0$$

and its long exact sequence. We have $h^1(\mathcal{O}_S(\mathring{H} - 2H)) = h^1(\mathcal{O}_S(2H - \mathring{H})) = 0$ by the previous lemma and $h^0(\mathcal{O}_S(\mathring{H} - 2H)) = 0$ because $H(\mathring{H} - 2H) = -2$. Then it follows $h^0(\mathcal{O}_D(\mathring{H} - H)) = h^0(\mathcal{O}_S(\mathring{H} - H))$. Finally the latter is zero by Proposition (4.4). □

Let $D \in |H|$ be smooth then $\mathcal{O}_D(\mathring{H} - H) \cong \mathcal{O}_D(b)$, where $\deg b = 2$. We fix the notation b for such a divisor and the notation μ_D for the following multiplication map:

$$\mu_D : H^0(\omega_D) \otimes H^0(\omega_D(b)) \to H^0(\omega_D^{\otimes 2}(b)). \tag{43}$$

Let us also point out that $h^0(\mathcal{O}_D(b)) = 0$ by the above lemma. Moreover we fix the notation

$$\begin{aligned} &\nu_D : H^0(\mathcal{O}_S(H)) \to H^0(\omega_D), \quad \mathring{\nu}_D : H^0(\mathcal{O}_S(\mathring{H})) \to H^0(\omega_D(b)) , \\ &\rho_D : H^0(\mathcal{O}_S(H + \mathring{H})) \to H^0(\omega_D^{\otimes 2}(b)) \end{aligned} \tag{44}$$

for the natural restriction maps. Then we consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H + \mathring{H})) \\ {\scriptstyle \nu_D \otimes \mathring{\nu}_D}\downarrow & & {\scriptstyle \rho_D}\downarrow \\ H^0(\omega_D) \otimes H^0(\omega_D(b)) & \xrightarrow{\mu_D} & H^0(\omega_D^{\otimes 2}(b)). \end{array} \tag{45}$$

which is similar to our main diagram (35)

Proposition 6.9 *The vertical arrows and the horizontal arrow μ_D are surjective.*

Proof Let $p : S \to \mathbb{P}^3$ be the map defined by $|H|$, then $p|D : D \to \mathbb{P}^2 = |\omega_D|^*$ is the canonical map and $|\omega_D(b)|$ is cut on D by $|\mathcal{I}_{d|S}(3H)|$, where d is any element of $|\omega^{\otimes 2}(-b)|$ and $\mathcal{I}_{d|S}$ is its ideal sheaf. Moreover the map $p^* : |\mathcal{O}_{\mathbb{P}^2}(3)| \to |\omega_D^{\otimes 3}|$ is an isomorphism and $|\mathcal{I}_{d|S}(3H)| = p^*|\mathcal{I}_{Z|\mathbb{P}^2}(3)|$, where $Z = p_* d$ and $\mathcal{I}_{Z|\mathbb{P}^2}$ is its ideal sheaf. Hence it follows $h^0(\mathcal{I}_{Z|\mathbb{P}^2}(2)) = h^0(\omega_D^{\otimes 2}(-b)) = h^0(\mathcal{O}_D(b)) = 0$ and $h^1(\mathcal{O}_D(b)) = h^0(\mathcal{O}_D(b)) = 0$. This easily implies $h^i(\mathcal{I}_{Z|\mathbb{P}^2}(3 - i)) = 0$ for $i > 0$, that is, $\mathcal{I}_{Z|\mathbb{P}^2}$ is 3-regular. Hence, by Castelnuovo-Mumford regularity theorem, the multiplication map

$$\mu : H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{I}_{Z|\mathbb{P}^2}(3)) \to H^0(\mathcal{I}_{Z|\mathbb{P}^2}(4))) \tag{46}$$

is surjective. Now consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{p(D)|\mathbb{P}^2}(4) \to \mathcal{I}_{Z|\mathbb{P}^2}(4) \xrightarrow{\rho} \mathcal{I}_{Z|p(D)}(4) \to 0$$

and its associated long exact sequence. Since $\mathcal{I}_{p(D)|\mathbb{P}^2}(4) \cong \mathcal{O}_{\mathbb{P}^2}$ it follows that

$$h^0(\rho) : H^0(\mathcal{I}_{Z|\mathbb{P}^2}(4)) \to H^0(\omega_D^{\otimes 2}(b))$$

is surjective. On the other hand we have $\mu_D \circ \lambda = h^0(\rho) \circ \mu$, where λ is the tensor product

$$\lambda_1 \otimes \lambda_2 : H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{I}_{Z|\mathbb{P}^2}(3)) \to H^0(\omega_D) \otimes H^0(\omega_D(b))$$

of the natural isomorphisms $\lambda_1 : H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\omega_D)$ and $\lambda_2 : H^0(\mathcal{I}_{Z|\mathbb{P}^2}(3)) \to H^0(\omega_D(b))$. Since λ is an isomorphism and $h^0(\rho)$ and μ are surjective, then μ_D is surjective. The surjectivity of ρ_D follows from the vanishing of $h^1(\mathcal{O}_S(\mathring{H}))$ and the standard exact sequence

$$0 \to \mathcal{O}_S(\mathring{H}) \to \mathcal{O}_S(H + \mathring{H}) \to \omega_D^{\otimes 2}(b) \to 0.$$

Since $\omega_D^{\otimes 2}(b)$ is $\mathcal{O}_D(H + \mathring{H})$, the surjectivity of ν_D follows from the above exact sequence twisted by $-\mathring{H}$. Finally the exact sequence

$$0 \to \mathcal{O}_S(\mathring{H} - H) \to \mathcal{O}_S(\mathring{H}) \to \omega_D(b) \to 0$$

implies that $\mathring{\nu}_D$ is an isomorphism. Indeed we have $h^0(\mathcal{O}_S(\mathring{H} - H)) = h^1(\mathcal{O}_S(\mathring{H} - H)) = 0$ in its long exact sequence by (32). Hence $\nu_D \otimes \mathring{\nu}_D$ is surjective too. □

Proposition 6.10 *The map* $\mu_S : H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) \to H^0(\mathcal{O}_S(H+\mathring{H}))$ *is surjective.*

Proof Let us consider again the commutative diagram (45), that is,

$$\begin{array}{ccc} H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H+\mathring{H})) \\ {\scriptstyle \nu_D \otimes \mathring{\nu}_D}\big\downarrow & & {\scriptstyle \rho_D}\big\downarrow \\ H^0(\omega_D) \otimes H^0(\omega_D(b)) & \xrightarrow{\mu_D} & H^0(\omega_D^{\otimes 2}(b)). \end{array}$$

Counting dimensions we have $\dim \operatorname{Ker} \mu_S \geq 4$, hence it suffices to show that the equality holds. Now we know that μ_D and $\nu_D \otimes \mathring{\nu}_D$ are surjective. Let $\mathbb{K}$ be the Kernel of $\mu_D \circ (\nu_D \otimes \mathring{\nu}_D)$, then the dimension count gives $\dim \mathbb{K} = 8$ and, of course, we have $\operatorname{Ker} \mu_S \subseteq \mathbb{K}$. Therefore, to prove $\dim \operatorname{Ker} \mu_S = 4$, it suffices to produce a 4-dimensional subspace $V \subset \mathbb{K}$ such that $V \cap \operatorname{Ker} \mu_S = (0)$. To this purpose consider the space of decomposable vectors $V := \langle s \rangle \otimes H^0(\mathcal{O}_S(\mathring{H}))$, where s is nonzero and $\operatorname{div}(s) = D$. Then we have $(\nu_D \otimes \mathring{\nu}_D)(V) = (0)$ and hence $V \subset \mathbb{K}$. On the other hand let $t \in H^0(\mathcal{O}_S(\mathring{H}))$, then $\mu_S(s \otimes t) = st$ and this is zero iff $t = 0$. Hence $V \cap \operatorname{Ker} \ \mu_S = (0)$. □

Now we go back, in genus 5, to our usual diagram (35) in Sect. 2. This is

$$\begin{array}{ccc} H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\mathring{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H+\mathring{H})) \\ {\scriptstyle \rho_H \otimes \rho_{\mathring{H}}}\big\downarrow & & {\scriptstyle \rho_C}\big\downarrow \\ H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) & \xrightarrow{\mu_C} & H^0(\omega_C^{\otimes 2}). \end{array} \tag{47}$$

Proposition 6.11 $\mu_C : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^{\otimes 2})$ *is surjective.*

Proof We have already shown that μ_S and $\rho_H \otimes \rho_{\mathring{H}}$ are surjective. By (40) and its related lemma the same is true for ρ_C. Hence the surjectivity of μ_C follows. □

Let $\mathbb{P}^{15} := \mathbb{P}(H^0(\mathcal{O}_S(H))^* \otimes H^0(\mathcal{O}_S(\mathring{H}))^*)$ and let $\mathbb{P}^3 \times \mathbb{P}^3 := \iota(|H|^* \times |\mathring{H}|^*)$ be the image in $\mathbb{P}^{15}$ of the Segre embedding ι. Now we study the morphism defined in (4.2)

$$f : S \to \mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15},$$

that is, $f = \iota \circ (p \times \mathring{p})$. Since the map μ_S is surjective it follows that

$$(p \times \mathring{p})^* H^0(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1)) = H^0(\mathcal{O}_S(H+\mathring{H})). \tag{48}$$

Let $\mathbb{P}^{11} \subset \mathbb{P}^{15}$ be the linear embedding of $\mathbb{P}(\operatorname{Im}\mu_S^*)$ defined by μ_S^*, then we have

$$f(S) \subseteq \mathbb{P}^{11} \cdot (\mathbb{P}^3 \times \mathbb{P}^3) \subset \mathbb{P}^{15}, \tag{49}$$

In other words f is just the morphism defined by the complete linear system $|H + \mathring{H}|$ composed with the linear embedding $\mathbb{P}^{11} \subset \mathbb{P}^{15}$.

Proposition 6.12 *The map* $p \times \mathring{p}$ *is an embedding for a general point* $[S, \mathcal{L}, \mathcal{E}] \in \mathcal{F}^{\perp}_{5,3}$.

Proof The linear systems $|H|$ and $|\mathring{H}|$ are non hyperelliptic. Hence p, $\mathring{p}$ are generically injective and the same is true for f. In particular $f : S \to f(S)$ is biregular over $f(S) - \operatorname{Sing} f(S)$ and $\operatorname{Sing} f(S)$ is a finite set of rational double points. Let $R \subset S$ be an integral curve contracted by f then R is biregular to $\mathbb{P}^1$ but it is not B_i. Indeed R is contracted by p and $\mathring{p}$ while B_i is not, as one can directly compute. Notice also that $C \sim \frac{1}{2}(H + \mathring{H} + B_{red})$. Therefore, since $RC \geq 0$, it follows

$$RC = \frac{1}{2} \sum_{i=1...12} RB_i \geq 0$$

with $RB_i \geq 0$. Assume $RB_i = 0$ for each i, then $RC = 0$. Since the Picard group of S is $\mathbb{Z}[\mathcal{L}] \oplus \mathbb{M}_3$, R is necessarily contained in $\mathbb{M}_3 = \mathbb{Z}[\mathcal{L}]^{\perp}$. By [14] the unique (-2)-curves contained in $\mathbb{M}_3$ are the B_i's, which contradicts the fact that R cannot be a B_i. Now assume that $RB_i \geq 2$ for some B_i and consider, among the maps p and $\mathring{p}$, the one not contracting B_i, say p. Then p embeds B_i as a line. On the other hand p contracts $R \cdot B_i$, which is a divisor of degree ≥ 2 in B_i: a contradiction. This implies $RB_i = 1$ for each i. Finally consider two distinct curves as above, say B_1 and B_2, which are contracted by p. Let us also claim that $p(B_1)$ and $p(B_2)$ are distinct points for a general $(S, \mathcal{L}, \mathcal{E})$. Since $RB_1 = RB_2 = 1$ then $p(R)$ is not a point: a contradiction.

We now prove that $p(B_1) \neq p(B_2)$ for a general $(S, \mathcal{L}, \mathcal{E})$. If two curves are contracted by a map p to the same point, there is a tree of (-2)-curves connecting these curves which is contracted by p. Since p is defined by $|H|$, the (-2)-curves contracted by p are orthogonal to H in $\mathbb{Z}[\mathcal{L}] \oplus \mathbb{M}_3$, which is the Picard group of a general S. By a direct computation one observes that the negative defined lattice orthogonal to H contains exactly 12 (-2)-classes, which are $\pm B_i$ for $i = 1, \ldots, 6$. Since $B_i B_j = 0$ if $i, j \in \{1, \ldots, 6\}$ and $i \neq j$, $p(B_1) \neq p(B_2)$. □

At this point the special geometry determined by μ_S appears, we have

$$\operatorname{Ker} \mu_S = H^0(\mathcal{I}(1, 1)), \tag{50}$$

where $\mathcal{I}$ is the ideal sheaf of $\mathbb{P}^{11} \cdot (\mathbb{P}^3 \times \mathbb{P}^3)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ and $\dim \operatorname{Ker} \mu_S = 4$. Let

$$\Sigma := \mathbb{P}^{11} \cdot (\mathbb{P}^3 \times \mathbb{P}^3), \tag{51}$$

then $f(S)$ sits in $\mathbb{P}^{11}$ as a K3 surface of degree 20 and $f(S) \subseteq \Sigma$. Now assume that the intersection scheme Σ is proper, then Σ is a K3 surface of degree 20 and hence

$$f(S) = \Sigma. \tag{52}$$

Postponing its proof, we therefore assume the following claim.

Claim For a general triple $(S, \mathcal{L}, \mathcal{E})$ the intersection scheme Σ is proper. Then we prove the birationality of the Mukai map $r_{5,3} : \mathcal{P}^{\perp}_{5,3} \to \mathcal{R}_{5,3}$.

Proof (Proof of the birationality) Since $\mathcal{P}^{\perp}_{5,3}$ and $\mathcal{R}_{5,3}$ are irreducible of the same dimension, it suffices to show that $r_{5,3}$ is birational onto $\mathcal{M} := r_{5,3}(\mathcal{P}^{\perp}_{5,3})$. Let $x = [S, \mathcal{L}, \mathcal{E}, C]$ be general in $\mathcal{P}^{\perp}_{5,3}$ and $y = r_{5,3}(x)$, then $y = [C, \eta]$ with $\eta := \mathcal{E} \otimes \mathcal{O}_C$. Let $y \in \mathcal{M}$ be general, we prove that a unique $x = [S, \mathcal{L}, \mathcal{E}, C]$ exists so that $[C, \mathcal{E} \otimes \mathcal{O}_C] = y$. We already know, for a general $y = [C, \eta] \in \mathcal{M}$, the surjectivity of the multiplication map

$$\mu_C : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^{\otimes 2}),$$

because this condition is open and non empty on $\mathcal{M}$. Then, applying to μ_C the same construction applied to μ_S, one obtains

$$C \subseteq \Sigma := \mathbb{P}^{11} \cdot (\mathbb{P}^3 \times \mathbb{P}^3) \subset \mathbb{P}^{15}. \tag{53}$$

Let $V = H^0(\omega_C \otimes \eta)^*$ and $\mathring{V} = H^0(\omega_C \otimes \eta^{-1})^*$, here C is bicanonically embedded in $\mathbb{P}^{11} := \mathbb{P}(\mathrm{Im}\ \mu_C)^*$ and the inclusion is the Segre embedding $\mathbb{P}(V) \times \mathbb{P}(\mathring{V}) \subset \mathbb{P}(V \otimes \mathring{V})$. Now the properness of Σ is an open condition on $\mathcal{M}$, not empty under our claim. Then $(\Sigma, \mathcal{O}_\Sigma(1))$ is a polarized K3 surface as above. Since $y = r_{5,3}(x)$ for some $x = [S, \mathcal{L}, \mathcal{E}, C]$, the commutative diagram (47) implies that $[\Sigma, \mathcal{O}_\Sigma(1)] = [S, \mathcal{L}]$. Therefore μ_C defines a rational map, sending $y = [C, \eta] \in \mathcal{M}$ to $x \in \mathcal{P}^\perp_{5,3}$, which is inverse to $r_{5,3}$. □

Proof (Proof of the claim) Since each component of Σ has dimension ≥ 2, it suffices to construct one $\mathbb{D} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1, 1)|$ so that $\mathbb{D} \cdot \Sigma = \mathbb{D} \cdot S$. We choose the hyperplane section

$$\mathbb{D} = (P \times \mathbb{P}^3) + (\mathbb{P}^3 \times \mathring{P}), \tag{54}$$

where P and $\mathring{P}$ are general planes. Then we have $\mathbb{D} \cdot S = D + \mathring{D}$, where $D \in |H|$ and $\mathring{D} \in |\mathring{H}|$ are smooth, non hyperelliptic curves of genus 3. We show, only for D, that

$$D = \mathbb{P}^{11} \cdot (P \times \mathbb{P}^3)\ ,\ \ \mathring{D} = \mathbb{P}^{11} \cdot (\mathbb{P}^3 \times \mathring{P}). \tag{55}$$

The map $p : D \to P$ is the canonical map; we fix on P coordinates $(x) = (x_1 : x_2 : x_3)$. The map $\mathring{p} : D \to \mathbb{P}^3$ is defined by $|\omega_D(b)|$, where $\deg b = 2$ and $h^0(\mathcal{O}_D(b)) = 0$. This implies that $\omega_D(b)$ is very ample, we fix coordinates $(y) = (y_1 : \cdots : y_4)$ on $\mathbb{P}^3$. The resolution of $\mathcal{O}_{\mathring{p}(D)}(1) \cong \omega_D(b)$ is definitely well known, [17]. We have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \xrightarrow{A} \mathcal{O}^{\oplus 4}_{\mathbb{P}^3} \to \omega_D(b) \to 0, \tag{56}$$

$A = (a_{ij})$ being a 4×3 matrix of linear forms in (y). Then $\mathring{p}(D)$ is a determinantal curve defined by the cubic minors of A. In particular A has rank 3 on $\mathbb{P}^3 - \mathring{p}(D)$ and, since $\mathring{p} : D \to \mathring{p}(D)$ is biregular and $\mathring{p}(D)$ is smooth, it also follows that $\mathring{p}(D)$ is the set of points $y \in \mathbb{P}^3$ such that A has exactly rank 2. This implies that the equations $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = 0,\ i = 1 \ldots 4$, define a complete intersection $\hat{D} \subset P \times \mathbb{P}^3$ such that $\mathrm{Supp}\ \hat{D} = D$. Finally one easily computes that $\hat{D}$ and D have the same degree 10 with respect to $\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1, 1)$. This implies $\hat{D} = D$ and the claim follows. □

6.3 The case of genus 6

Theorem 6.13 *The Mukai map $r_{6,3} : \mathcal{P}^\perp_{6,3} \to \mathcal{R}_{6,3}$ has not maximal rank.*

In this paper we only sketch the proof of this theorem and its geometric motivation: see Sect. 7 and also [28]. We postpone some details to further investigation on $\mathcal{R}_{6,3}$. We conclude that the mentioned analogies are confirmed for $\ell = 3$: the Mukai maps

$$m_{11\pm 1}\ ,\ r_{7\pm 1,2}\ ,\ r_{5\pm 1,3} \tag{57}$$

have not maximal rank, while they are birational for $g = 11,\ 7,\ 5$. These maps are not dominant for $g = 10,\ 6,\ 4$ and they have positive dimensional fibre for $g = 12,\ 8,\ 6$.

7 Views on Fano threefolds with sections of level 2 or 3

We close this paper discussing some families of Fano threefolds $\overline{X} \subset \mathbb{P}^{g+1}$, whose general hyperplane sections are singular K3 surfaces $\overline{S}$ of the considered types. Then $\overline{S}$ is endowed with a degree ℓ cyclic cover $\pi : \tilde{S} \to \overline{S}$ with branch locus $\mathrm{Sing}\ \overline{S}$. Moreover its minimal desingularization $\nu : S \to \overline{S}$ fits in a standard level ℓ K3 surface $(S, \mathcal{L}, \mathcal{E})$, so that $\mathcal{L} \cong \nu^*\mathcal{O}_{\overline{S}}(1)$ and $\mathcal{E}$ induces $\pi : \tilde{S} \to \overline{S}$. We have $\ell = 2, 3$.

For some families a natural cyclic cover $\pi_{\overline{X}} : \tilde{X} \to \overline{X}$ is visible, with branch locus the curve Sing $\overline{X}$. However we do not address it here. The existence of these families implies that $r_{g,\ell}$ has not maximal rank. They correspond to the peculiar values

$$(g, \ell) = (6, 3), (6, 2), (8, 2), (4, 3). \tag{58}$$

For $\ell = 2$ these families are known, [11,19,27]. The case (6, 2) is revisited here with emphasis on a singular quadratic complex of the Grassmannian $G(2, 5)$. This implies that $r_{6,2}$ is not of maximal rank. For (6, 3) we introduce a family of Gushel - Mukai threefolds singular along a rational normal sextic curve. This is responsible for the failure of the maximal rank of $r_{6,3}$. The case (8, 2) is similar and not treated here, [27]. Finally we point out the plausible relation of the case (4, 3) to the G_2-variety.

7.1 A singular Gushel–Mukai threefold: $\ell = 3$ and $g = 6$

We sketch the geometric construction implying theorem (6.13). Let $g = 6$ and $\ell = 3$, keeping our notation we consider $p \times \mathring{p} : S \to \mathbb{P}^4 \times \mathbb{P}^4$. Then p is defined by the linear system

$$|H| = |C - \frac{1}{3} \sum_{i=1\cdots 6} (B_i + 2B_i')|, \tag{59}$$

where $B_i + B_i'$, are the connected components of B_{red}. Let $x_0 := [S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}^{\perp}_{6,3}$ be a general point, then a standard analysis shows that $p : S \to p(S)$ is the contraction of $\sum B_i$ to six points and that $p(B_i')$ is a line. Moreover we have

$$p(S) = F_0 \cap Q, \tag{60}$$

where F_0 is a cubic and Q a smooth quadric. Notice that $p|C$ is the embedding defined by $\omega_C \otimes \eta^{-1}$, since $CB_i = 0$ then $p(C) \cap \mathrm{Sing}\, p(S) = \emptyset$. Let $C' := p(C)$ and let

$$0 \to \mathcal{I}_{p(S)}(3) \to \mathcal{I}_{p(C)}(3) \to \mathcal{I}_{C'|p(S)}(3) \to 0 \tag{61}$$

be the standard exact sequence of ideal sheaves of Q, we notice the isomorphisms $\mathcal{I}_{p(S)}(3) \cong \mathcal{O}_Q$ and $p_* : H^0(\mathcal{O}_S(3H - C)) \to H^0(\mathcal{I}_{p(C)|p(S)}(3))$. This implies that

$$0 \to H^0(\mathcal{O}_Q) \to H^0(\mathcal{I}_{C'}(3)) \to H^0(\mathcal{O}_S(3H - C)) \to 0 \tag{62}$$

is its associated long exact sequence. It easily follows that C' is projectively normal. A second standard step is the remark that $\mathcal{O}_S(3H - C)$ is a genus 3 polarization of S. Now let $M \in |3H - C|$, then $p_*(C + M) \in |\mathcal{I}_{p(C)|p(S)}(3)|$ and it is cut on $p(S)$ by a cubic hypersurface. Therefore we have in Q the complete intersection scheme

$$p_*(C + M) = F_0 \cap F_\infty \cap Q, \tag{63}$$

where F_0, F_∞ are cubics. Let $S_0' = F_0 \cdot Q$ and $S_\infty' = F_\infty \cdot Q$. We consider the pencil

$$P_M = \{S_t', \ t \in \mathbb{P}^1\}, \tag{64}$$

of cubic sections of Q generated by S_0' and S_∞'. We can assume $p(S) = S_0'$, notice that a general S_t' is a possibly singular $K3$ surface, smooth along C'. Let $\sigma_t : S_t \to S_t'$ be its minimal desingularization and $C_t := \sigma_t^* C'$, then S_t is endowed with the line bundles

$$\mathcal{H}_t := \sigma_t^* \mathcal{O}_Q(1), \quad \mathcal{L}_t := \mathcal{O}_{S_t}(C_t), \ \mathcal{E}_t := \mathcal{L}_t \otimes \mathcal{H}_t^{-1}. \tag{65}$$

For $t = 0$ the fourtuple $(S_t, \mathcal{L}_t, \mathcal{E}_t, C_t)$ defines the point $x_0 = [S, \mathcal{L}, \mathcal{E}, C]$ of $\mathcal{P}^{\perp}_{6,3}$. For $t \neq 0$ we have constantly $C_t = C$. Now consider the family of fourtuples

$$\{(S_t, \mathcal{L}_t, \mathcal{E}_t, C_t),\ t \in \mathbb{P}^1\}, \tag{66}$$

then the assignment $t \to [S_t, \mathcal{L}_t] \in \mathcal{F}_6$ defines a non constant rational map $m : \mathbb{P}^1 \to \mathcal{F}_6$. Assume $(S_t, \mathcal{L}_t, \mathcal{E}_t)$ is a K3 surface of level 3 for a general t. Then m lifts to a map $\tilde{m} : \mathbb{P}^1 \to \mathcal{P}^{\perp}_{6,3}$, sending t to $[S_t, \mathcal{L}_t, \mathcal{E}_t, C_t]$, and the next statement immediately follows.

Proposition 7.1 *If $(S_t, \mathcal{L}_t, \mathcal{E}_t)$ is a K3 surface of level 3 for a general t, the curve $\tilde{m}(\mathbb{P}^1)$ is in the fibre at the point $[C, \eta]$ of the Mukai map $r_{6,3}$, which is therefore not of maximal rank.*

The assumption mentioned in the statement depends on the choice of the element M in $|3H - C|$ and in general it is not satisfied. However the assumption is satisfied choosing in $|M|$ the very special element

$$M_0 := 2A + \sum_{i=1\cdots 6} B_i, \tag{67}$$

where A is the unique element of $|C - \sum_{i=1\cdots 6}(B_i + B_i')|$. The curve A is biregular to $\mathbb{P}^1$ and $p|A$ embeds it as a rational normal quartic curve. Let $A' = p(A)$, then the base scheme of P_{M_0} is a non reduced, complete intersection curve and its 1-cycle is

$$p_*(M_0 + C) = 2A' + C'. \tag{68}$$

In other words the surfaces S_t' intersect along a contact curve A' of multiplicity two and along C'. It turns out that a general Sing S_t' consists of six nodes moving in A' and each node belongs to a line in S_t'. This can be shown using the special property that $\eta \cong \omega_{C'}(-1) \in \operatorname{Pic} C$ is of 3-torsion. Omitting further details of this construction, let us just say that M_0 defines a pencil of level 3 and genus 6 K3 surfaces as required.

To close geometrically this sketch let A be the non reduced component, supported on A', of the base curve of P_{M_0} and $\mathcal{I}_{\mathsf{A}|Q}$ its ideal sheaf. Consider the rational map

$$\phi : Q \to \mathbb{P}^7 \tag{69}$$

defined by the linear system $|\mathcal{I}_{\mathsf{A}|Q}(3)|$. Let us notice the following property.

Proposition 7.2 *The map ϕ is birational onto its image W, which is a singular Gushel–Mukai threefold whose general hyperplane sections are singular $K3$ surfaces $\overline{S}$ as above.*

Therefore W is a complete intersection of type $(1, 1, 2)$ in the Grassmannian $G(2, 5)$. We notice that Sing W is a rational normal sextic curve. This completes our sketch.

7.2 The tangential quadratic complex of $\mathbb{P}^4$: $\ell = 2$ and $g = 6$

Let $\mathbb{G}_n$ be the Plücker embedding of the Grassmannian of lines of $\mathbb{P}^n$, a quadratic complex is just a quadratic section of $\mathbb{G}_n$. Let $Q \subset \mathbb{P}^n$ be a quadric, then the family $\mathbb{T}$ of tangent lines to Q is a quadratic complex, named sometimes the tangential quadratic complex. We assume Q is smooth, then $\mathbb{T}$ is a Fano variety. Notice that Sing $\mathbb{T}$ is the Hilbert scheme of lines of Q, of codimension and multiplicity 2 in $\mathbb{T}$.

Now we assume n is even. Then $\mathbb{T}$ has a unique nontrivial quasi étale 2:1 cover

$$\pi : \tilde{\mathbb{T}} \to \mathbb{T}, \tag{70}$$

whose branch locus is $\operatorname{Sing}\mathbb{T}$. Let us describe the known map π in the case $n = 4$, since it is linked to the Mukai map $r_{6,2} : \mathcal{P}^{\perp}_{6,2} \to \mathcal{R}_6$ and its behavior. This is treated in [11]. For $n = 4$ the Hilbert scheme of lines of Q is the 2-Veronese embedding of $\mathbb{P}^3$, say

$$V \subset \mathbb{G}_4 \subset \mathbb{P}^9. \tag{71}$$

Let $t \in \mathbb{T}$, consider the pencil $\{H_p, p \in t\}$, where H_p is the polar hyperplane to Q at p. Its base locus is a plane P_t and $Q_t := P_t \cdot Q$ is a conic. Since t is tangent to Q, a standard exercise shows that $\operatorname{Sing} Q_t = t \cap Q$. This defines a smooth, integral correspondence

$$\tilde{\mathbb{T}} := \{(t, r) \in \mathbb{T} \times V \mid r \subset Q_t\}. \tag{72}$$

Notice that its projection onto $\mathbb{T}$ is a quasi étale $2:1$ cover branched on V, say

$$\pi : \tilde{\mathbb{T}} \to \mathbb{T}. \tag{73}$$

Indeed the fibre $\zeta_t := \pi^*(t)$ is the Hilbert scheme of lines of Q_t and is finite of length 2. Then ζ_t is smooth iff rank $Q_t = 2$ iff $t \notin V$ and ζ_t has multiplicity 2 iff rank $Q_t = 1$ iff $t \in V$.

Now it is well known that a general 2-dimensional linear section $\overline{S} = \mathbb{T} \cap \mathbb{P}^6$ is the model defined by $|\mathcal{L}|$ of S, where $[S, \mathcal{L}, \mathcal{E}] \in \mathcal{F}^{\perp}_6$ is general. In particular $\operatorname{Sing}\overline{S} = V \cap \mathbb{P}^6$ is an even set of 8 nodes, defining $\pi|\tilde{S}$ with $\tilde{S} = \pi^{-1}(\overline{S})$, cfr. [11,19,20]. For $\ell = 2$ and $[S, \mathcal{L}, \mathcal{E}] \in \mathcal{F}^{\perp}_g$, the surface S, or its model $\overline{S}$, is known as a standard Nikulin surface of genus g. Therefore we can say that a general 3-dimensional linear section of $\mathbb{T}$ is a Fano threefold whose hyperplane sections are standard Nikulin surfaces of genus 6. Let us denote such a section by

$$X = \mathbb{T} \cap \mathbb{P}^7, \tag{74}$$

notice that $\operatorname{Sing} X$ is a curvilinear section of V, hence an elliptic curve of degree 8.

Finally let $\mathcal{C}$ and $\overline{\mathcal{S}}$ respectively be the family of general curvilinear sections C and that of general 2-dimensional linear sections $\overline{S}$ of $\mathbb{T}$. Consider the family of pairs

$$\mathcal{P} := \{(C, \overline{S}) \in \mathcal{C} \times \overline{\mathcal{S}} \mid C \subset \overline{S}\}. \tag{75}$$

Let $(C, \overline{S}) \in \mathcal{P}$ then C is a canonical curve and $C \in |\mathcal{O}_{\overline{S}}(1)|$. Let $\nu : S \to \overline{S}$ be the desingularization then $\nu^* C \in |\mathcal{L}|$ and $\eta := \mathcal{E} \otimes \mathcal{O}_{\nu^* C}$ defines $\pi|\tilde{C}$, where $\tilde{C} = \pi^{-1}(C)$. Then the assignment of $(C, \overline{S})$ to $[S, \mathcal{L}, \mathcal{E}, \nu^* C]$ defines a dominant rational map

$$m : \mathcal{P} \to \mathcal{P}^{\perp}.$$

We already know that the Mukai map $r_{6,2}$ fails to be of maximal rank. However we can now see this fact from a geometric perspective: the existence of the Fano variety $\mathbb{T}$ and its quasi finite $2:1$ cover π. Indeed this implies that $C \in \mathcal{C}$ is contained in a higher dimensional family of sections $\overline{S}$ of $\mathbb{T}$, so that C cannot have general moduli.

More precisely the parameter space $\mathcal{C}$ is open in the Grassmannian $G(5, 9)$, hence $\dim \mathcal{C} = 24$. Moreover $\operatorname{Aut} Q \subset \operatorname{Aut}\mathbb{P}^4$ has dimension 10 and acts faithfully on $\mathcal{C}$. Then we have $\dim \mathcal{C} /\!\!/ \operatorname{Aut} Q = 14 < \dim \mathcal{R}_6 = 15$. Hence $r_{6,2}$ cannot be dominant.

Remark 7.1 Let $C \in \mathcal{C}$ then $\tilde{C} = \pi^{-1}(C)$ is a smooth, integral curve of genus 11. We have $\tilde{C} \subset \tilde{S} \subset \tilde{X} \subset \mathbb{P}^{12}$, where $\tilde{X} = \pi^{-1}(X)$ is a non prime Fano threefold of genus 11. We just mention that $\tilde{C}$ is the base locus of a pencil of hyperplane sections of $\tilde{X}$ and that the birational Mukai map $m_{11} : \mathcal{P}_{11} \to \mathcal{M}_{11}$ is not invertible at $[\tilde{C}]$.

7.3 The G_2-variety: $\ell = 3$ and $g = 4$

A geometric interpretation seems plausible and it is possibly postponed to future work. It relates to the failure of the Mukai map in genus 10. As in (14) let $\pi : \tilde{S} \to \overline{S}$ be the cover induced by $\mathcal{E}$ and $\nu : S \to \overline{S}$ the desingularization map. For a general C the map $\nu : C \to \overline{S} \setminus \operatorname{Sing} \overline{S}$ is an embedding, then we set $C := \nu(C)$. Let $\tilde{C} := \pi^{-1}(C)$ then $(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{C}))$ is a K3 surface of genus 10. This suggests that $\tilde{S}$ embeds in the G_2-variety $W \subset \mathbb{P}^{13}$ as a linear section, [23]. Now a general curvilinear section of W is not general as a genus 10 curve. In the same way, if it is a triple cover of a genus 4 curve, it seems not a general genus 4 triple cover.

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