Fujita decomposition on families of abelian varieties

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Abstract

Let $F : \mathcal{V} \rightarrow B$ be a smooth non-isotrivial 1-dimensional family of complex polarized abelian varieties and $V_b = F^{-1}(b)$ be the general fiber. Let $\mathcal{F}^1 \subset R^1 F_* \mathbb{C}$ be the associated Hodge bundle filtration, $\mathcal{F}_b^1 = H^{1,0}(V_b)$. Under the assumption that the Fujita decomposition for $\mathcal{F}^1$ is non trivial, that is there is a non trivial flat sub-bundle $0 \neq \mathbb{U} \subset \mathcal{F}^1$, we show that V_b has non-trivial endomorphism: $\text{End}(V_b) \neq \mathbb{Z}$.

1 Introduction

Given a family of compact Kähler varieties $F : X \rightarrow B$ over an algebraic curve B , Fujita decomposition [10, 11] gives a splitting of the direct image of relative dualizing sheaf $\pi_* \omega_{X|B}$ into $\mathcal{U} \oplus \mathcal{A}$, where $\mathcal{U}$ is unitary and flat and $\mathcal{A}$ is ample. The *positive* part of the theorem had many important applications and improvements, we just mention [15, 16, 31] and [17, Corollary 3.5]. The *unitary* part was neglected until the fundamental papers of Catanese and Dettweiler [4–6]. They provided (together with a complete written proof of the decomposition) examples where the unitary part cannot be reduced by a base change to a constant factor. These results have inspired many new researches [9, 13, 14, 20, 24, 26]. This paper is a continuation of [9] where the decomposition was related to the Mumford-Tate group. Here we consider a non isotrivial family of abelian varieties, $F : \mathcal{V} \rightarrow B$, and their 1, 0 Hodge bundle $\mathcal{F}^1 = \mathcal{H}^{1,0} \subset R^1 F_* \mathbb{C}$. The Fujita decomposition for these bundles can be proved directly or by reduction to suitable family of curves. As an application of two powerful results due to Yves André and respectively to Mikhail Borovoi ([1] and [2]), we prove (see Theorem 2.3) that if the unitary part is non zero, then the abelian variety of the family $V_b = F^{-1}(b)$, must have non-trivial endomorphisms. Moreover we show that the Néron-Severi group $N(V_b)$ of V_b , that is the rational endomorphisms fixed by the Rosati involution, is non trivial (see 2.4 and 2.7).

When we have achieved these results we have been in a mixed mood. The first impression was that “this is the end” of Fujita decomposition for these families, becoming at most a paragraph of a book on the endomorphisms of abelian varieties. Now our thinking is brighter.

This article is dedicated to Fabrizio Catanese

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It might really be possible that the study of the unitary factor could be helpful. In Sect. 2 we give a small glimpse of it. We use a result of Ribet [25] on the Mumford-Tate group of abelian varieties and a geometric result of Xiao on fibration of curves [32]. We show (see Theorem (3.6)) that a unitary factor of dimension $g - 1$ in a family of Jacobian of curves of genus g is semiample. In other words it becomes trivial after a change of basis.

I would like to thank Bert van Geemen, who many years ago explained to me the basics on the Mumford-Tate group. He also pointed out a mistake in a previous version of this paper. I would also like to thank Alessandro Ghigi for the many useful conversations. It is a great pleasure to dedicate these efforts to Fabrizio, without his work I would not have even thought about these topics.

2 Fujita decomposition and Hodge classes

Let B be a smooth connected algebraic complex curve. Let $F : \mathcal{V} \rightarrow B$ be a smooth non-isotrivial 1-dimensional family of complex polarized abelian varieties of dimension g , $\dim \mathcal{V} = g + 1$, let $V_b = F^{-1}(b)$ be the general fiber. Let $\mathbb{K}$ be a commutative ring, here we will consider the cases $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$, let $R^s F_* \mathbb{K}$, $s \in \mathbb{N}$, be the local system associated to $H^s(V_b, \mathbb{K})$. The polarization gives a section ω of $R^2 F_* \mathbb{Z}$. The Hodge bundle $\mathcal{H}^1 = R^1 F_* \mathbb{C} \otimes \mathcal{O}_B$ is equipped with the sub-bundle $\mathcal{F}^1 \subset \mathcal{H}^1$, where $\mathcal{F}_b^1 = H^{1,0}(V_b)$. Scalar extension defines a natural sheaf inclusion $\mathbb{H}^1 = R^1 F_* \mathbb{C} \subset \mathcal{H}^1$. One has the Fujita decomposition:

$$\mathbb{H}^1 = (\mathcal{U} \oplus \overline{\mathcal{U}}) \oplus \mathbb{A}$$

where $(\mathcal{U} \oplus \overline{\mathcal{U}}) \otimes \mathcal{O}_B$ is the largest unitary flat bundle of $\mathcal{H}^1$, and $\mathcal{U} = \mathcal{U} \otimes \mathcal{O}_B \subset \mathcal{F}^1$.

Let k be the rank of $\mathcal{U}$. Since the family is not isotrivial then $\mathbb{A} \neq 0$ and therefore $k < g$. One has the filtration $\mathcal{A}^{1,0} \subset \mathcal{A} = \mathbb{A} \otimes \mathcal{O}_B$, where $\mathcal{F}^1 \cap \mathcal{A} = \mathcal{A}^{1,0}$, and a (non holomorphic) decomposition $\mathcal{A} = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$, $\mathcal{A}^{0,1} = \overline{\mathcal{A}^{1,0}}$. The polarization gives isomorphisms between the dual $\mathcal{U}^*$ of $\mathcal{U}$ and $\overline{\mathcal{U}} \otimes \mathcal{O}_B$ (with the anti-linear definition of the tensor) and similarly, between $\mathcal{A}^{1,0*}$, the dual of $\mathcal{A}^{1,0}$, and $\mathcal{A}^{0,1}$.

Remark 2.1 Usually the Fujita decomposition is considered only for top degree holomorphic forms. The use of the same name is motivated by the case of the Jacobians: the Fujita decomposition for a family of smooth curves is isomorphic to the decomposition of $(1, 0)$ forms of the corresponding family of Jacobians.

It has been proved in [9] that if $k > 0$ the image of the modular map $\mathcal{P} : B \rightarrow \mathcal{A}_g$ is contained in a Hodge locus, where $\mathcal{A}_g$ is the suitable moduli space of polarized abelian varieties. That is for any $b \in B$ the Mumford-Tate group $G(V_b)$ (see [21] and [12]) is smaller than $\mathbb{C}SP_g$, the conformal symplectic group. We recall that SP_g is the symplectic rational group with respect to the fixed polarization. Here we give two reasons for this:

1. The basic observation of [9]: The Hodge bundle $\mathcal{H}^k = R^k F_* \mathbb{C} \otimes \mathcal{O}_B = \bigwedge^k \mathcal{H}^1$ contains the line bundle $\mathcal{L} = \bigwedge^k \mathcal{U}$, where k is the rank of $\mathcal{U}$. After a finite base change $\pi : D \rightarrow B$ the pull-back $\mathcal{L}'$ of $\mathcal{L}$ becomes trivial [1, 7]. If $\mathcal{V}' = \pi^*(\mathcal{V})$ is the pullback family, one has that $\mathcal{L}'$ sits in the image of the map $H^k(\mathcal{V}', \mathbb{C}) \rightarrow R^k F'_* \mathbb{C}$. There is then a proper substructure of the primitive cohomology of $H^k(V_b, \mathbb{C})$ and hence $G(V_b) \neq \mathbb{C}SP_g$. We remarked in [9] that this argument extends to higher weight variations of Hodge structure.

2. Perhaps more naturally, we can consider the projector

$$p_{\mathbb{U}} : \mathbb{H}^1 \rightarrow \mathbb{U} \oplus \overline{\mathbb{U}}.$$

Over $b \in B$, it gives the projection $H^1(V_b, \mathbb{C}) \rightarrow \mathbb{U}_b \oplus \overline{\mathbb{U}}_b$, that is represented, via the decomposition $\mathcal{F}^1 = \mathcal{H}^{1,0} \rightarrow \mathbb{U} \otimes \mathcal{O}_B$, by a hermitian form

$$\Omega_{U,b} = \sqrt{-1} \sum_i^k \omega_{i,b} \wedge \overline{\omega}_{i,b},$$

for suitable $\omega_{i,b} \in \mathbb{U}_b$, of rank k , $\Omega_{U,b} \in H^{1,1}(V_b)$. Therefore $b \rightarrow \Omega_{U,b}$ defines flat section $\Omega_{\mathbb{U}}$ of $\mathcal{H}^2 = R^2F_*\mathbb{C} \otimes \mathcal{O}_B$. By construction $\Omega_{\mathbb{U}}$ is monodromy invariant, but in general not defined over $\mathbb{Q}$, as proved by Catanese and Dettweiler [4–6]. Moreover $\Omega_{\mathbb{U}}$ and the polarization ω are linearly independent. Deligne's theorem [7] shows that the image S_b of the restriction $j^* : H^2(\mathcal{V}, \mathbb{C}) \rightarrow H^2(V_b, \mathbb{C})$ contains $[\Omega_{U,b}]$ the class of $\Omega_{U,b}$ and the polarization form. Clearly the local system $\mathbb{S}$ of $R^2F_*\mathbb{C}$ defined by the image of j^* is proper subsystem defined over $\mathbb{Q}$ of rank ≥ 2 , hence $G(V_b) \neq \mathbb{C}SP_g$.

For any unitary factor $\mathbb{W} \subset \mathbb{U}$ of rank $s \leq k$ the projector $p_{\mathbb{W}} : \mathbb{H}^1 \rightarrow \mathbb{W} \oplus \overline{\mathbb{W}}$ gives raises to a 1.1 hermitian semidefinite class of rank s , $\Omega_{\mathbb{W}}$ invariant under monodromy. A priori one does not know if $\Omega_{\mathbb{W}}$ is a Hodge class, that is invariant under the Hodge group H (see [21]) of the general fiber. We recall that $G = \mathbb{C}H$, and H is the intersection of the Mumford-Tate group G with the special linear group. Nevertheless the combination of the results ([1] and [2]) due to Yves André and respectively to Mikhail Borovoi gives striking results.

Let $\text{End}(V_b)$ be the endomorphism algebra of V_b , set $\text{End}^0(V_b) = \text{End}(V_b) \otimes \mathbb{Q}$ and let $N(V_b) \subset \text{End}^0(V_b)$ be the algebra fixed by the Rosati involution, $N(V_b)$ is the space of rational 1, 1-cycles, that is the rational Néron-Severi group of V_b .

Definition 2.2 Let $\Pi_1(B, b)$ be the fundamental group of B with base point $b \in B$, $\rho : \Pi_1(B, b) \rightarrow \text{End}(H^1(V_b, \mathbb{C}))$ be the action the monodromy representation, ρ preserves the symplectic form, that is the polarization of V_b . Let M be the connected component of the Zariski closure of $\rho(\Pi_1(B, b))$.

One has that $M \subset H$ and moreover from [1] that M is a normal subgroup of the commutator subgroup $H' = (H, H)$ of H .

Theorem 2.3 Let V_b be the general fiber of $b \in B$. If $\mathbb{U}$ is not trivial, i.e. $k > 0$, then $\text{End}(V_b) \neq \mathbb{Z}$, that is $\text{End}^0(V_b) \neq \mathbb{Q}$.

Proof Let H be the Hodge group (see [21]) of the general fiber V_b of the map F and $H' = (H, H)$ be the commutator subgroup, M is a normal subgroup of H' . Assume by contradiction that $\text{End}^0(A) = \mathbb{Q}$, then by [2] the Hodge group H is $\mathbb{Q}$ -simple and then $M = H' = H$. Since $\mathbb{U}$ is not zero then $\Omega_{\mathbb{U}} \neq 0$ is invariant by $M = H$. It follows that $\Omega_{\mathbb{U}}$ is a real Hodge cycle. Since the space of the 1, 1 Hodge real cycles is $N(V_b) \otimes \mathbb{R}$ and $\langle \omega, \Omega_{\mathbb{U}} \rangle \subset N(V_b) \otimes \mathbb{R}$ we obtain that $\mathbb{Q} \neq N(V_b) \subset \text{End}^0(V_b)$ and then a contradiction. $\square$

We will give a more precise result.

Theorem 2.4 *If $\mathbb{U}$ is not trivial, then the Néron-Severi group of V_b , $N(V_b)$ is non trivial, and the form $\Omega_{\mathbb{U}}$ is a real Hodge cycle: $\Omega_{\mathbb{U}} \in N(V_b) \otimes \mathbb{R}$.*

The proof follows essentially from the mentioned papers [1,2], here we will give some detail.

Lemma 2.5 *The algebraic monodromy group M is a normal subgroup of the Hodge group H .*

Proof The group M is normal in $H' = (H, H)$. If $Z(H)$ is the center of H we have $H' \cdot Z(H) = H$, then M is normal in H . For, fix $g \in H$ and write $g = hz \in H$, $h \in H'$ and $z \in Z(H)$. If $m \in M$ we have $g^{-1}mg = z^{-1}h^{-1}mh z = h^{-1}mh \in M$. $\square$

Lemma 2.6 *The sub-representation $\mathbb{U} \oplus \overline{\mathbb{U}}$ is the largest M -unitary sub-representation of $\mathbb{H}^1$. Moreover $\mathbb{U} \oplus \overline{\mathbb{U}}$ is a representation of the Hodge group H (and of G).*

Proof The first part is the definition of the Fujita decomposition. We add some words to see that the largest unitary sub-representation has actually this expression.

Let H be a group and $M \triangleleft H$ be a normal subgroup. Let V be a complex H -finite dimensional representation. This induces a M -representation. Let $\rho : G \rightarrow GL(V)$ and $\rho_M : M \rightarrow GL(V)$ be the induced map, we write $gv = \rho(g)v$. Let $W \subset V$ be a sub-representation, we say that W is unitary if there is an hermitian form on W $\langle \cdot, \cdot \rangle$ such that $\langle mv, mw \rangle = \langle v, w \rangle \forall v, w \in W$. We remark that W is unitary $\iff \rho(M)$ is bounded in $GL(V)$, therefore if W_1 and W_2 are unitary also $W_1 + W_2$ is unitary. We can define the largest unitary M -sub-representation $W_{\max} \subset V$ of V . Let $\rho_{\max} : M \rightarrow GL(W_{\max})$ be the associated map. For any $g \in H$ $gW_{\max}$ is again a unitary M -sub-representation, in fact since M is normal, for $m \in M$ and $w \in W_{\max}$ we have: $mgw = g(g^{-1}mg)w = gm'w \in gW_{\max}$. This gives that $gW_{\max}$ is a M representation. Moreover if $\rho_{\max}(M) \subset GL(W)$ is bounded and then $\rho_{\max}(gM)$ is also bounded. This gives $gW_{\max} = W_{\max} : W_{\max}$ is a H -representation.

Now we come back to our case: H is the Hodge group, M the monodromy group, and $V = H^1$. Let $\mathbb{U}_{\max}$ be the largest unitary representation of $\mathbb{H}^1$. We first have $\mathbb{U} \oplus \overline{\mathbb{U}} \subset \mathbb{U}_{\max}$: the hermitian form on $\mathbb{U}$ is given by the polarization ω . Next we observe that a sub-representation of $\mathbb{A}$ cannot be unitary (this can be detected using the Higgs field, that is by the infinitesimal variation of the Hodge structure [19] and [29],) it follows $\mathbb{U}_{\max} \cap \mathbb{A} = \{0\}$ and therefore $\mathbb{U}_{\max} = \mathbb{U} \oplus \overline{\mathbb{U}}$.

By the above discussion $\mathbb{U} \oplus \overline{\mathbb{U}}$ is also a H -representation. $\square$

Proof (of theorem 2.4) Keep the previous notation, we have $\mathbb{L} = \mathbb{U} \oplus \overline{\mathbb{U}}$, and $\mathbb{U}$ are associated to H -representations. Then their projectors commute with the H action. It follows (see [21] and [2]) that its class $[\Omega_{\mathbb{U}}]$ belongs to $N(V_b) \otimes \mathbb{R}$. Then the rank of $N(V_b)$ is bigger than 1, it follows that $N(V_b) \neq \mathbb{Q}$. $\square$

From [2, Lemma 2], when V_b is a simple abelian variety we have a stronger result.

Theorem 2.7 *Let $F : \mathcal{V} \rightarrow B$ be a smooth non-isotrivial 1-dimensional family of complex abelian polarized varieties of dimension g . Assume that $V_b = F^{-1}(b)$ is a simple abelian variety. Let K be the center of $\text{End}^0(V_b)$ and $K_0 \subset K$ be the field fixed by the Rosati involution, let $[K_0 : \mathbb{Q}] = r$ be its degree. If $\mathbb{U} \neq 0$ then $r > 1$.*

Proof Let G be the Mumford-Tate group of V_b . Any irreducible G sub-representation of $\mathbb{A}$ is not equivalent to any G sub-representation of $\mathbb{U} \oplus \overline{\mathbb{U}}$. Set $H_{\mathbb{R}}^1 = H^1(V_b, \mathbb{R})$, let $\rho_{\mathbb{R}} : H \rightarrow \text{End}(H^1)$ and $\rho_{M\mathbb{R}} : M \rightarrow \text{End}(H_{\mathbb{R}}^1)$ be the real representations. The mentioned [2, Lemma 2] gives a decomposition

$$\mathbb{H}^1 = \bigoplus_i^r \mathbb{W}_i^{k_i}$$

where $r = [K_0 : \mathbb{Q}]$, and the $\mathbb{W}_i \neq 0$ are real irreducible representation of M , pairwise non-equivalent. Then after a rearrangement of the index we may write

$$\mathbb{U} \oplus \overline{\mathbb{U}} = \bigoplus_i^s \mathbb{W}_i^{k_i}, \quad \mathbb{A} = \bigoplus_{j=s+1}^r \mathbb{W}_j^{k_j}$$

where $0 < s < r$ and then $r > 1$. □

3 The case $k=g-1$

We keep the previous notation. Let $F : \mathcal{V} \rightarrow B$ be a smooth non-isotrivial 1-dimensional family of complex polarized abelian varieties of dimension g . Let $j : B \rightarrow B'$ be the completion of B , B' is a compact Riemann surface. Under the inclusion of the fundamental group $j_* : \pi_1(B) \rightarrow \pi_1(B')$ we let $K_{\infty} = \ker j_*$ be the kernel of j_* .

Definition 3.1 We will say that the action of the monodromy M is unipotent at ∞ if the induced representation $\rho : K_{\infty} \rightarrow \text{End}(\mathbb{H}^1)$ is unipotent.

We remark that if the action of M is unipotent at ∞ the unitary bundle $\mathbb{U}$ extends to B' . We will prove the following:

Theorem 3.2 *If F is not isotrivial, V_b is simple for a general $b \in B$, and $\text{rank}(\mathbb{U}) = g-1 > 0$, then $g \geq 8$.*

Proof We first give the following:

Lemma 3.3 *Under the hypothesis of Theorem 3.2 there is a base change $D \rightarrow B$ such that:*

1. *the monodromy is unipotent at ∞ , (see 3.1);*
2. *D is complete: if $D \xrightarrow{j} D'$ is the completion $D = D'$;*
3. *any unitary non-trivial factor $\mathbb{W}$ of the Fujita decomposition on D has rank > 1 .*

Proof It is well known (see [27]) that there is a base change $h : D \rightarrow B$ such that the action of the monodromy on $\pi^* \mathbb{H}^1$ is unipotent at ∞ . Assuming this we show that $\mathcal{V}$ extends to an abelian family over D' , let $0 \in D' \setminus D$ be a point. Since the local monodromy is trivial on $\mathbb{U}$, the compact part of the semistable limit, is an abelian variety V_0 of dimension $\geq \text{rank}(\mathbb{U}) = g-1$. In fact the elements of $(\mathbb{U} \oplus \overline{\mathbb{U}})_d$, $d \in D$, are invariant by the local monodromy action. This gives $(\mathbb{U} \oplus \overline{\mathbb{U}})_0 \subset H^1(V_0, \mathbb{C})$. If $\dim V_0 = g-1$ we get that $H^1(V_0, \mathbb{C}) = H^1(V_0, \mathbb{Q}) \otimes \mathbb{C} = (\mathbb{U} \oplus \overline{\mathbb{U}})_0$ is defined over $\mathbb{Q}$. By monodromy invariance, this would be true for any $d \in D$. Since the general fiber of the family, is simple this is impossible. Then $\dim V_0 = g$ and V_0 extends the family of abelian varieties to 0. Similarly

if $\mathbb{W} \subset \mathbb{U}$ has rank one, we get that up to a base change, $\mathbb{W}$ becomes trivial [1,7]. The fixed part theorem would give that V_d is not simple. $\square$

The statement of the theorem is invariant by a base change and from now on we assume that $F : \mathcal{V} \rightarrow B$ satisfies the three conditions of the above lemma. From Theorem 2.3 we have $K = \text{End}^0(V_b) \neq \mathbb{Q}$ that is $\dim_{\mathbb{Q}} K = e > 1$. We now show the following:

Proposition 3.4 *Assume $k = \dim \mathbb{U} = g - 1$. Let $b \in B$ be the very general point. Then the algebra $K = \text{End}^0(V_b)$ is a totally real field of degree $e = [K : \mathbb{Q}] > 1$.*

Proof Consider the action of the endomorphism on $\mathbb{A}$. The pull-back $\phi \rightarrow \phi^*$ defines a map $\psi : K \rightarrow \text{End}(\mathcal{A}^{1,0}) \cong \mathbb{C}$. The map ψ is injective, otherwise there would be endomorphisms with non trivial kernel and V_b would not be simple. It follows that K embeds in $\mathbb{C}$ and therefore K is a field.

To see that the image of ψ is contained in $\mathbb{R}$ write $K = \mathbb{Q}(\phi)$. The ϕ gives a flat section of $\text{Hom}(R^1 F_* \mathbb{C}, R^1 F_* \mathbb{C})$. Then for a local section s of $\mathcal{A}^{1,0}$ we have $\phi(s) = \lambda s$, $\lambda \in \mathbb{C}$. The λ -eigenspace of ϕ defines a flat bundle $\mathbb{V}_\lambda \subset R^1 F_* \mathbb{C} \otimes \mathcal{O}_B$ such that $\mathbb{V}_\lambda \otimes \mathcal{O}_B \supset \mathcal{A}^{1,0}$. This implies that $\mathbb{V}_\lambda \otimes \mathcal{O}_B \supset \mathcal{A} \supset \mathcal{A}^{0,1}$, otherwise since $\dim \mathcal{A}^{1,0} = 1$ we would have $\mathcal{A}^{1,0} = \mathbb{V}_\lambda \otimes \mathcal{O}_B \cap \mathcal{A}$ and $\mathcal{A}^{1,0}$ would be a flat bundle. By conjugation we have $\mathcal{A}^{0,1} \subset \mathbb{V}_{\bar{\lambda}} \otimes \mathcal{O}_B$. It follows that $\lambda = \bar{\lambda}$, and then $\psi(\phi) \in \mathbb{R}$. From the classification of the algebras of endomorphisms of abelian varieties (see [22, Chapter IV]) we get that K is a totally real field and from Theorem (2.3) that $[K : \mathbb{Q}] = e > 1$. $\square$

Write again $K = \mathbb{Q}(\phi)$, let $p_\phi(t)$ be the characteristic polynomial of ϕ and $p_{\min}(t)$ be the minimal polynomial of ϕ on $H^1(V_b, \mathbb{C})$ (or $H^1(V_b, \mathbb{Q})$). Now $p_{\min}(t)$ is irreducible, rational, it has degree e , and $p_\phi(t) = p_{\min}^k(t)$ where $ke = 2g$. The Hodge decomposition gives $\phi = \phi^{1,0} + \phi^{0,1}$, $\phi^{1,0} \in \text{End}(\mathcal{H}^{1,0})$ and $\phi^{0,1} \in \text{End}(\mathcal{H}^{0,1})$, $\phi^{0,1} = \overline{\phi^{1,0}}$. Therefore $p_\phi(t) = p_{\phi^{1,0}}(t)p_{\phi^{0,1}}(t) = p_{\phi^{1,0}}(t)\overline{p_{\phi^{1,0}}(t)}$. Then, since all the eigenvalues are real, any eigenspace of ϕ has even dimension $k = 2s$: therefore $g = es$.

The following proposition complete the proof of the Theorem 3.4.

Proposition 3.5 *Write $g = es$, s must be even and $s \geq 4$. In particular g is even and $g \geq 8$.*

Proof The ϕ eigenvector decomposition defines flat sub-local system and $\mathbb{H}^1$:

$$\mathbb{H}^1 = \oplus_i^e \mathbb{V}_{\lambda_i}, \quad H^{1,0}(V_b) = \oplus_i^e V_{\lambda_i b}. \quad (1)$$

Set

$$\mathbb{U}_i = \mathbb{V}_{\lambda_i} \cap \mathbb{U} \subset \mathbb{U},$$

and assume that $\mathbb{V}_{\lambda_1} \otimes \mathcal{O}_B \supset \mathcal{A}^{1,0}$, and then $\mathbb{A} \subset \mathbb{V}_{\lambda_1}$. Note that the rank of $\mathbb{U}_1$ is $s - 1$. Set $\mathbb{L} = \mathbb{U}_1 \oplus \overline{\mathbb{U}_1}$ we have

$$\mathbb{V}_{\lambda_1} = \mathbb{A} \oplus \mathbb{L}.$$

We see that $s > 2$. In fact if $s = 1$ $\mathbb{V}_{\lambda_i} \subset \mathbb{U}$ $i > 1$, and if $s = 2$ then $\mathbb{U}_1 = \mathbb{V}_{\lambda_1} \cap \mathbb{U} \subset \mathbb{U}$ in both cases $\mathbb{U}$ would be a rank one local subsystem, this is against our assumption 3 of 3.3. Now we show that s is even.

Assume by contradiction that $s = g/e$ is odd. In this case the theorem of Ribet [25, Theorem 1] gives that the Mumford-Tate group, $G = G(V_b)$, is given by the elements that commute with ϕ :

$$G = \{g \in \mathbb{C}Sp(2g) : g \circ \phi = \phi \circ g\}.$$

Now one has that $G \cdot \mathbb{V}_{\lambda_i} = H \cdot \mathbb{V}_{\lambda_i} = \mathbb{V}_{\lambda_i}$ and then

$$\mathbb{H}^1 = \mathbb{A} \oplus \mathbb{L} \oplus \bigoplus_{i>1}^e \mathbb{V}_{\lambda_i},$$

we find $e + 1$ not equivalent H representations, using [25, Sect. 3] or again [2, Lemma 2], we would have $[K : \mathbb{Q}] > e$ and then a contradiction. $\square$

The proof of the Theorem 3.4 is now complete. $\square$

The following result was our modest motivation:

Theorem 3.6 *Assume that $f : C \rightarrow B$ is a smooth non isotrivial family of curves of genus $g > 1$ and $F : J(C) \rightarrow B$ be the associated Jacobians family. Let $\mathbb{U}$ be the unitary factor in Fujita decomposition of $\mathcal{F}^1 \subset R^1 F_* \mathbb{C}$. If $k = g - 1$ then $g \leq 7$ and $\mathbb{U}$ has finite monodromy. That is up to a finite base change $h : D \rightarrow B$, and completion $j : D \rightarrow D'$, $j_* h^*(\mathbb{U} \otimes \mathcal{O}_D) = \mathcal{O}_{D'}^{g-1}$.*

Proof From the slope inequality of Xiao [32, Corollary 4] we get $g < 8$. If $g < 8$ $J(C)$ cannot be simple by Proposition 3.5. We may assume to have an isogeny $J(C_b) \cong \prod_{i=1}^r V_i$, where the V_i are simple. After a base change $h : D \rightarrow B$, we can make the decomposition invariant by monodromy (this can be proved by observing that the projectors $J(C_b) \rightarrow V_i$ can be represented by integers matrices with bounded norm, that is a finite set). Then we find r families of abelian varieties $F_i : \mathcal{V}_i \rightarrow D$ and $J(C) \cong \prod_{i=1}^r \mathcal{V}_i$.

Since $k = g - 1$ only one of them, say $\mathcal{V}_1$, is not trivial. Then by (3.5) $\dim \mathcal{V}_1 = 1$ and

$$\prod_{i=2}^r \mathcal{V}_i = \prod_{i=2}^r V_i \times D$$

is a trivial family of dimension $g - 1$. The unitary factor is then trivial on D and the original $\mathbb{U}$ on B has finite monodromy. $\square$

Remark 3.7 A refinement [18, Proposition 2.8] of the mentioned result of Xiao [32], shows that in the previous theorem one can put $g < 7$. We think that is very interesting to consider non isotrivial genus g fibered surface $\pi : C \rightarrow B$, where B is a projective curve, such that

$$\pi_* \omega_{C|B} = \mathcal{O}_B^{g-1} \oplus \mathcal{A}.$$

It is the case of the fibrations with relative irregularity $g - 1$. Examples are only known for $g \leq 4$ (see [33] and [23]). It has been proved (see [3, 8]) that if such a fibration of genus 5 or 6 exists, the fibers of π would be trigonal curves and when $g = 6$ of special Maroni invariant. If $g = 6$ using methods from [25, 28, 30], one could even try to prove that the relative Albanese variety is not simple. In fact the image of the (relative) Albanese map would give a surface S on abelian variety A of dimension 5. The homological class of S should not be obtained by intersections of divisors. This A would be an exotic Albanese variety (see [28]) but this is impossible for a simple abelian variety of dimension 5. Despite these efforts the existence problem is completely open.

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