Smooth covers of moduli stacks of Riemann surfaces with symmetry

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Abstract

We construct explicitly a finite cover of the moduli stack of compact Riemann surfaces with a given group of symmetries by a smooth quasi-projective variety.

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1 Introduction

For a finite group G , the locus $M_g(G)$ (in the moduli space M_g) of curves that have an effective action by G plays an important role in the study of the geometry of M_g (for example its singularities [1,2]), in the study of Shimura varieties (see e.g. [3–5] and the references therein), of totally geodesic subvarieties of M_g [6], and also in the classification of higher dimensional varieties (see e.g. [7–10]). Similar loci in the moduli space of higher dimensional varieties have been studied in [11].

To investigate the geometry of $M_g(G)$ it is more natural to introduce the moduli stack $\mathcal{M}_g(G)$ of genus g compact Riemann surfaces with an effective action by G . Then we can obtain $M_g(G)$ as the image of a finite morphism $\mathcal{M}_g(G) \rightarrow M_g$. In this paper we study some geometric properties of $\mathcal{M}_g(G)$. If $g \geq 2$, $\mathcal{M}_g(G)$ is a complex orbifold, whose connected components are in bijection with the set $\mathbb{T}$ of topological types of the G -actions. For any

Dedicated to Fabrizio Catanese on the occasion of his 70th birthday.

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$\tau \in \mathbb{T}$, let $\mathcal{M}_g(G, \tau)$ be the corresponding connected component. It is known that $\mathcal{M}_g(G, \tau)$ is isomorphic to the stack quotient $\left[\mathcal{T}_g^{\tau(G)} / C_{\Gamma_g}(\tau(G)) \right]$, where $\mathcal{T}_g^{\tau(G)}$ is the locus of points, in the Teichmüller space $\mathcal{T}_g$, that are fixed by $\tau(G)$, and $C_{\Gamma_g}(\tau(G))$ is the centralizer of $\tau(G)$ in the mapping class group Γ_g . Using the theory of level structures we define a smooth quasi-projective variety Z_τ such that $\mathcal{M}_g(G, \tau) \cong [Z_\tau / \bar{C}_\tau]$, where $\bar{C}_\tau$ is a finite group. Hence $\mathcal{M}_g(G, \tau)$ is a smooth Deligne–Mumford stack and $Z_\tau \rightarrow \mathcal{M}_g(G, \tau)$ is a finite Galois cover. The disjoint union of the Z_τ 's is a finite smooth cover of $\mathcal{M}_g(G)$, since $\mathbb{T}$ is finite (see e.g. [12]). Notice that the existence of a smooth quasi-projective variety Z and a finite flat morphism $Z \rightarrow \mathcal{M}_g(G)$ follows also from [13].

2 Moduli spaces of curves with symmetry

Throughout the article G is a finite group and g is an integer greater or equal than 2. Let $\mathcal{M}_g(G)$ be the stack, in the complex analytic category, whose objects are pairs $(\pi : \mathcal{C} \rightarrow B, \alpha)$, where $\pi : \mathcal{C} \rightarrow B$ is a family of compact Riemann surfaces of genus g and $\alpha : G \times \mathcal{C} \rightarrow \mathcal{C}$ is an effective (holomorphic) action of G on $\mathcal{C}$ such that, for any $a \in G$, $\pi \circ \alpha(a, _) = \pi$. A morphism $(\Phi, \varphi) : (\pi : \mathcal{C} \rightarrow B, \alpha) \rightarrow (\pi' : \mathcal{C}' \rightarrow B', \alpha')$ is a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & \mathcal{C}' \\ \downarrow \pi & & \downarrow \pi' \\ B & \xrightarrow{\varphi} & B' \end{array}$$

such that, for any $a \in G$, $\Phi \circ \alpha(a, _) \circ \Phi^{-1} = \alpha'(a, _)$.

Using the Teichmüller space $\mathcal{T}_g$, we are going to define a complex orbifold structure on $\mathcal{M}_g(G)$. Given a compact, connected, oriented topological surface of genus g , Σ_g , recall that a Teichmüller structure on a Riemann surface C is the isotopy class of an orientation preserving homeomorphism $f : C \rightarrow \Sigma_g$, it will be denoted with $[f]$. Two Riemann surfaces with Teichmüller structures $(C, [f])$, $(C', [f'])$ are isomorphic, if there exists an isomorphism $F : C \rightarrow C'$ such that $[f] = [f' \circ F]$. We will denote with $[C, [f]]$ the class of $(C, [f])$. Then, $\mathcal{T}_g$ is the set of isomorphism classes $[C, [f]]$ of compact Riemann surfaces of genus g with Teichmüller structures. The mapping class group of Σ_g , denoted by Γ_g , is the group of all isotopy classes of orientation preserving homeomorphisms of Σ_g . There is a natural action of Γ_g on $\mathcal{T}_g$, given by

$$[\gamma] \cdot [C, [f]] = [C, [\gamma \circ f]], \quad \forall [\gamma] \in \Gamma_g, [C, [f]] \in \mathcal{T}_g.$$

Furthermore, for any $[C, [f]] \in \mathcal{T}_g$, the homomorphism

$$\sigma_{[f]} : \text{Aut}(C) \rightarrow \Gamma_g, \quad \Phi \mapsto [f \circ \Phi \circ f^{-1}]$$

is injective and its image is the stabilizer of $[C, [f]]$ in Γ_g , that we denote by $\text{Stab}_{\Gamma_g}([C, [f]])$. We collect in the following theorem several results about the Teichmüller space, for a proof and for more details we refer to [14, 15].

Theorem 2.1 *$\mathcal{T}_g$ has a natural structure of a complex manifold which is homeomorphic to the unit ball in $\mathbb{C}^{3g-3}$. The action of Γ_g on $\mathcal{T}_g$ is holomorphic and properly discontinuous. The map*

$$\mathcal{T}_g \rightarrow \mathcal{M}_g, \quad [C, [f]] \mapsto [C],$$

to the coarse moduli space of compact Riemann surfaces of genus g yields an isomorphism $\mathcal{T}_g/\Gamma_g \cong M_g$.

Furthermore there is a universal family of Riemann surfaces of genus g with Teichmüller structure

$$\eta: \mathcal{X}_g \rightarrow \mathcal{T}_g.$$

Let α be an effective action of G on C , viewed as an injective group homomorphism $\alpha: G \rightarrow \text{Aut}(C)$. Let us choose an orientation preserving homeomorphism $f: C \rightarrow \Sigma_g$. Then we have an injective homomorphism

$$\rho := \sigma_{[f]} \circ \alpha: G \rightarrow \Gamma_g \quad (1)$$

such that $\rho(G) \subset \text{Stab}_{\Gamma_g}([C, [f]])$. Notice that, if $f': C \rightarrow \Sigma_g$ is another orientation preserving homeomorphism, we obtain a different homomorphism $\rho': G \rightarrow \Gamma_g$. However $[C, [f]]$ and $[C, [f']]$ belong to the same Γ_g -orbit, so there exists $[\gamma] \in \Gamma_g$ such that $[\gamma] \cdot [C, [f]] = [C, [f']]$ and $\rho' = [\gamma] \cdot \rho \cdot [\gamma]^{-1}$. This motivates the following definitions (that are already present in the literature, in slightly different forms, see e.g. [12] and the references therein).

Definition 2.2 Let $\alpha: G \rightarrow \text{Aut}(C)$ be an injective homomorphism. Let ρ be defined in (1). The *topological type* of the G -action α is the class of ρ in $\text{Hom}^{\text{inj}}(G, \Gamma_g)/\Gamma_g$, where $\text{Hom}^{\text{inj}}(G, \Gamma_g)$ is the set of injective group homomorphisms from G to Γ_g , $\text{Hom}^{\text{inj}}(G, \Gamma_g)/\Gamma_g$ is the quotient under the action of Γ_g by conjugation.

Definition 2.3 The *Teichmüller space of compact Riemann surfaces of genus g with G -actions* is the set

$$\mathcal{T}_g(G) = \{([C, [f]], \rho) \in \mathcal{T}_g \times \text{Hom}^{\text{inj}}(G, \Gamma_g) \mid \rho(G) \subseteq \text{Stab}_{\Gamma_g}([C, [f]])\}.$$

Notice that, if $\text{Hom}^{\text{inj}}(G, \Gamma_g) \neq \emptyset$, then $\mathcal{T}_g(G) \neq \emptyset$ by Nielsen realization problem [16] and that it carries the following action by Γ_g :

$$[\gamma] \cdot ([C, [f]], \rho) = ([\gamma] \cdot [C, [f]], [\gamma] \cdot \rho \cdot [\gamma]^{-1}), \quad (2)$$

for $[\gamma] \in \Gamma_g$ and $([C, [f]], \rho) \in \mathcal{T}_g(G)$.

Proposition 2.4 $\mathcal{T}_g(G)$ is a complex manifold. Moreover there is an object of $\mathcal{M}_g(G)$, $(\eta(G): \mathcal{X}_g(G) \rightarrow \mathcal{T}_g(G), \alpha)$, such that the associated classifying morphism $\mathcal{T}_g(G) \rightarrow \mathcal{M}_g(G)$ induces an isomorphism

$$[\mathcal{T}_g(G)/\Gamma_g] \cong \mathcal{M}_g(G),$$

where $[\mathcal{T}_g(G)/\Gamma_g]$ is the stack quotient associated to the action (2). In particular $\mathcal{M}_g(G)$ has a structure of complex orbifold in the sense of [17], [15, XII, §4].

Proof It follows directly from Definition 2.3 that

$$\mathcal{T}_g(G) = \bigsqcup_{\rho \in \text{Hom}^{\text{inj}}(G, \Gamma_g)} \mathcal{T}_g^{\rho(G)},$$

where $\mathcal{T}_g^{\rho(G)}$ is the locus of points fixed by $\rho(G)$. Since $\rho(G)$ is a finite group, $\mathcal{T}_g^{\rho(G)}$ is a complex submanifold of $\mathcal{T}_g$, so the first claim follows.

For any $\rho \in \text{Hom}^{\text{inj}}(G, \Gamma_g)$, let $\eta(\rho): \mathcal{X}_g(\rho) \rightarrow \mathcal{T}_g^{\rho(G)}$ be the restriction of the universal family. There is a natural effective action, $\alpha(\rho)$, of G on $\mathcal{X}_g(\rho)$ such that $(\eta(\rho): \mathcal{X}_g(\rho) \rightarrow$

$\mathcal{T}_g^{\rho(G)}, \alpha(\rho))$ is an object of $\mathcal{M}_g(G)$. Then we define $(\eta(G): \mathcal{X}_g(G) \rightarrow \mathcal{T}_g(G), \alpha)$ as the disjoint union of these objects.

To prove the last statement recall that the objects of $[\mathcal{T}_g(G)/\Gamma_g]$, over a base B , are pairs $(p: P \rightarrow B, f: P \rightarrow \mathcal{T}_g(G))$, where $p: P \rightarrow B$ is a principal Γ_g -bundle and f is a Γ_g -equivariant holomorphic map. Let $f^*(\eta(G))$ be the pull-back of the family $\eta(G)$. The action of Γ_g on P extends to a free action on $f^*(\eta(G))$. So $f^*(\eta(G))$ descends to a family $\pi: \mathcal{C} \rightarrow B$ over B . By construction there is a G -action, α , on $\mathcal{C}$, in such a way that $(\pi: \mathcal{C} \rightarrow B, \alpha)$ is an object of $\mathcal{M}_g(G)$. Similarly one associates to every arrow of $[\mathcal{T}_g(G)/\Gamma_g]$ an arrow of $\mathcal{M}_g(G)$ obtaining an equivalence of categories. $\square$

Furthermore we have the following result.

Theorem 2.5 *$\mathcal{M}_g(G)$ is a smooth algebraic Deligne–Mumford stack with quasi-projective coarse moduli space.*

A proof of this theorem will be given in Sect. 4, where we define a groupoid presentation of $\mathcal{M}_g(G)$, $X_1 \xrightarrow[s]{t} X_0$, with X_0 and X_1 smooth quasi-projective algebraic varieties, and such that $X_0 \rightarrow \mathcal{M}_g(G)$ is finite étale.

2.1 On the homology of $\mathcal{M}_g(G)$

Let $\mathbb{T} \subset \text{Hom}^{\text{inj}}(G, \Gamma_g)$ be a set of representatives of topological types of G -actions (see Definition 2.2). Notice that, for any $\tau \in \mathbb{T}$,

$$\text{Stab}_{\Gamma_g}(\tau) = C_{\Gamma_g}(\tau(G)),$$

where $C_{\Gamma_g}(\tau(G))$ is the centralizer of $\tau(G)$ in Γ_g . Therefore we have an homeomorphism

$$\mathcal{T}_g(G)/\Gamma_g \cong \bigsqcup_{\tau \in \mathbb{T}} \mathcal{T}_g^{\tau(G)}/C_{\Gamma_g}(\tau(G)).$$

Definition 2.6 Let $\tau \in \mathbb{T}$. The moduli space of compact Riemann surfaces of genus g with G -action of topological type τ is defined as $\mathcal{T}_g^{\tau(G)}/C_{\Gamma_g}(\tau(G))$ and will be denoted by $M_g(G, \tau)$.

We have the following theorem from [18, 19] (see also [20] and the references therein).

Theorem 2.7 *$\mathcal{T}_g^{\tau(G)}$ is bi-holomorphic to $\mathcal{T}_{g', d}$ where g' is the genus of C/G , $[C, [f]] \in \mathcal{T}_g^{\tau(G)}$, and d is the number of branch points of the quotient map $C \rightarrow C/G$. In particular $\mathcal{T}_g^{\tau(G)}$ is homeomorphic to the unit ball in $\mathbb{C}^{3g'-3+d}$.*

Let us give an interpretation of the previous theorem. Given a point $[C, [f]] \in \mathcal{T}_g^{\tau(G)}$, let $\alpha: \sigma_{[f]}^{-1} \circ \tau: G \rightarrow \text{Aut}(C)$. Then $f: (C, \alpha) \rightarrow (\Sigma_g, f \circ \alpha \circ f^{-1})$ is G -equivariant, so it yields an orientation-preserving homeomorphism of $\theta: C'/G \rightarrow \Sigma'_g/G$, where $C' \subseteq C$ and $\Sigma'_g \subseteq \Sigma_g$ are the loci of points with trivial stabilizer. The bi-holomorphism in Theorem 2.7 sends $[C, [f]]$ to $[\Sigma'_g/G, [\theta]] \in \mathcal{T}_{g', d}$.

We describe now the map $\mathcal{T}_{g', d} \rightarrow M_g(G, \tau)$, which is the composition of the inverse of the previous one, $\mathcal{T}_{g', d} \rightarrow \mathcal{T}_g^{\tau(G)}$, with the quotient $\mathcal{T}_g^{\tau(G)} \rightarrow \mathcal{T}_g^{\tau(G)}/C_{\Gamma_g}(\tau(G))$. Let $p: \Sigma_g \rightarrow \Sigma_g/G$ be the quotient map, let $y \in \Sigma'_g/G$, and let $x \in p^{-1}(y)$. Let $\mu: \pi_1(\Sigma'_g/G, y) \rightarrow$

G be the monodromy of the covering p restricted to Σ'_g . For any $[D', [\theta]] \in \mathcal{T}_{g',d}$ the composition

$$\mu \circ \theta_*: \pi_1(D', \theta^{-1}(y)) \rightarrow G$$

gives a G -cover $C' \rightarrow D'$, hence a point $[C, \alpha] \in M_g(G, \tau)$. Then the map $\mathcal{T}_{g',d} \rightarrow M_g(G, \tau)$ sends $[D', [\theta]]$ to $[C, \alpha]$.

Notice that the class of μ ,

$$[\mu] \in \text{Hom}(\pi_1(\Sigma'_g/G, y), G)/G,$$

where G acts by conjugation, does not depend from the choice of y , and that $\Gamma_{g',d}$ acts on $\text{Hom}(\pi_1(\Sigma'_g/G, y), G)/G$.

It follows from this that

$$M_g(G, \tau) \cong \mathcal{T}_{g',d} / \text{Stab}_{\Gamma_{g',d}}([\mu]).$$

In particular the homology of $M_g(G, \tau)$ with rational coefficients can be computed as the homology of the group $\text{Stab}_{\Gamma_{g',d}}([\mu])$:

$$H_n(M_g(G, \tau); \mathbb{Q}) \cong H_n(\text{Stab}_{\Gamma_{g',d}}([\mu]); \mathbb{Q}).$$

Let now $(\alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_d)$ be a geometric basis of the fundamental group $\pi_1(\Sigma_{g'} \setminus \{y_1, \dots, y_d\}, y)$ (here we follow the notation of [12]). Setting $a_i = \mu(\alpha_i)$, $b_i = \mu(\beta_i)$, $c_j = \mu(\gamma_j)$, we obtain an element

$$[(a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_d)] \in G^{2g'+d}/G,$$

where G acts diagonally by conjugation. This yields an injective correspondence between classes $[\mu]$ of monodromies and elements of $G^{2g'+d}/G$. $\Gamma_{g',d}$ acts on $G^{2g'+d}/G$ (see e.g. [12, Sec 2.]) and the homology of $\text{Stab}_{\Gamma_{g',d}}([\mu])$ is isomorphic to the equivariant homology of the orbit of $[(a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_d)]$ under the action of $\Gamma_{g',d}$. This motivates the following

Question 2.8 [21] *Let $n, d \in \mathbb{N}$. Are there constants a, b such that the dimension of $H_n^{\Gamma_{g',d}}(G^{2g'+d}/G; \mathbb{Q})$ is independent of g' , in the range $g' > an + b$?*

We proved in [12] that the previous question has an affirmative answer in the case where $n = 0$.

3 Level structures and Teichmüller structures

In this section we recall some basic results and we fix the notation, for the proofs and for more details we refer to [15, Chapter XVI]. Given two groups G_1 and G_2 we denote with $\text{Hom}(G_1, G_2)$ the set of homomorphisms from G_1 to G_2 . Notice that there is an action of G_2 on $\text{Hom}(G_1, G_2)$, which is induced by the action of G_2 on itself by conjugation $((g, h) \mapsto g^{-1}hg)$. An **exterior group homomorphism** from G_1 to G_2 is an element of the quotient set $\text{Hom}(G_1, G_2)/G_2$. For any $\varphi \in \text{Hom}(G_1, G_2)$ we denote with $\hat{\varphi}$ its class in $\text{Hom}(G_1, G_2)/G_2$. We will say that $\hat{\varphi}$ is an exterior group monomorphism (respectively epimorphism, isomorphism) if φ is a monomorphism (respectively epimorphism, isomorphism).

Let X be a path connected topological space. For any pair of points $x, y \in X$ and for any continuous path γ from x to y , let $\varphi_\gamma: \pi_1(X, x) \rightarrow \pi_1(X, y)$ be the isomorphism that sends any $[c] \in \pi_1(X, x)$ to $[\gamma^{-1} \cdot c \cdot \gamma]$, where c is a loop in X based at x and $[c]$ is its homotopy class.

Let H be a group. We will identify two exterior homomorphisms $\hat{\alpha} \in \text{Hom}(\pi_1(X, x), H)/H$ and $\hat{\beta} \in \text{Hom}(\pi_1(X, y), H)/H$ if $\hat{\alpha} = \widehat{\beta \circ \varphi_\gamma}$. Notice that this definition does not depend on the choice of γ and yields an equivalence relation on the disjoint union

$$\bigsqcup_{x \in X} \text{Hom}(\pi_1(X, x), H)/H.$$

The class of $\hat{\alpha}$ will be denoted with $[\alpha]$.

Definition 3.1 A **Teichmüller structure of level H on X** is the class $[\alpha]$ of an exterior epimorphism $\hat{\alpha} \in \text{Hom}(\pi_1(X, x), H)/H$, for $x \in X$.

Let now C and C' be two compact Riemann surfaces. Let $[\alpha]$ and $[\alpha']$ be two Teichmüller structures of level H on C and C' , respectively. We say that the pairs $(C, [\alpha])$ and $(C', [\alpha'])$ are isomorphic if there exists an isomorphism $F: C \rightarrow C'$ such that $[\alpha] = [\alpha' \circ F_*]$, where $F_*: \pi_1(C, x) \rightarrow \pi_1(C', F(x))$ is the homomorphism induced by F , $\alpha: \pi_1(C, x) \rightarrow H$ and $\alpha': \pi_1(C', F(x)) \rightarrow H$ are representatives of $[\alpha]$ and $[\alpha']$, respectively. The isomorphism class of $(C, [\alpha])$ will be denoted $[C; \alpha]$. The set of isomorphism classes of compact Riemann surfaces of genus g with Teichmüller structures of level H will be denoted ${}_H M_g$. When $g \geq 2$ it is possible to define a structure of complex analytic space on ${}_H M_g$, called the *moduli space of genus g curves with Teichmüller structure of level H* . We refer to [15] for more details.

3.1 Level structures and Teichmüller structures

Let $[\psi]$ be a Teichmüller structure of level H on Σ_g . The following map defines a morphism of complex analytic spaces:

$$t_{[\psi]}: \mathcal{T}_g \rightarrow {}_H M_g, \quad [C, [f]] \mapsto [C; \psi \circ f_*],$$

where f is a representative of $[f]$ and $\psi: \pi_1(\Sigma_g, f(x)) \rightarrow H$ is a representative of $[\psi]$.

Remark 3.2 As explained in [15], the mapping class group Γ_g acts on the set of Teichmüller structures of level H on Σ_g as follows: $[\psi] \cdot [\gamma] = [\psi \circ \gamma_*]$. Moreover every connected component of ${}_H M_g$ coincides with $t_{[\psi]}(\mathcal{T}_g)$, for some $[\psi]$, we denote $t_{[\psi]}(\mathcal{T}_g)$ by $M_g[\psi]$. Hence we have the following decomposition,

$${}_H M_g = \bigsqcup_{[\psi] \bmod \Gamma_g} M_g[\psi].$$

Let $\Lambda_{[\psi]} := \{[\gamma] \in \Gamma_g \mid [\psi] \cdot [\gamma] = [\psi]\}$. Then

$$M_g[\psi] = \mathcal{T}_g / \Lambda_{[\psi]}.$$

Furthermore, if $\Lambda_{[\psi]}$ is a normal subgroup of Γ_g and $\Gamma_g[\psi] := \Gamma_g / \Lambda_{[\psi]}$, then

$$M_g = M_g[\psi] / \Gamma_g[\psi].$$

We report the following result from [15], where a characteristic subgroup of $\pi_1(\Sigma_g, x)$ is a subgroup that is mapped to itself by every automorphism of $\pi_1(\Sigma_g, x)$.

Lemma 3.3 *If $\ker(\psi)$ is a characteristic subgroup, then $\Lambda_{[\psi]}$ is a normal subgroup of Γ_g .*

Proof Under our hypotheses any automorphism γ of $\pi_1(\Sigma_g)$ induces an automorphism $\bar{\gamma}$ of H such that $\psi \circ \gamma = \bar{\gamma} \circ \psi$. This yields a group homomorphism $\text{Out}^+(\pi_1(\Sigma_g)) \rightarrow \text{Out}(H)$, $[\gamma] \mapsto [\bar{\gamma}]$, whose kernel is $\Lambda_{[\psi]}$, under the identification of $\text{Out}^+(\pi_1(\Sigma_g))$ with Γ_g . $\square$

In the case where $\ker(\psi)$ is a characteristic subgroup we can describe the action of Γ_g on $M_g[\psi]$ as follows [15].

Lemma 3.4 *Let $\ker(\psi)$ be a characteristic subgroup of $\pi_1(\Sigma_g)$. Then Γ_g acts on $M_g[\psi]$ as follows:*

$$[\gamma] \cdot [C; \alpha] = [C; \bar{\gamma} \circ \alpha],$$

where $\bar{\gamma}$ is the automorphism of H defined in the proof of Lemma 3.3.

Let now $m \in \mathbb{Z}_{\geq 1}$ and let $\chi_m: \pi_1(\Sigma_g) \rightarrow H_1(\Sigma_g, \mathbb{Z}/m\mathbb{Z})$ be the composition of the natural morphism $\pi_1(\Sigma_g) \rightarrow H_1(\Sigma_g, \mathbb{Z})$ with the reduction modulo m . Then $H_1(\Sigma_g, \mathbb{Z}/m\mathbb{Z})$ is a strongly characteristic quotient of $\pi_1(\Sigma_g)$ (i.e. there is only one subgroup K of $\pi_1(\Sigma_g)$ such that $\pi_1(\Sigma_g)/K$ is isomorphic to $H_1(\Sigma_g, \mathbb{Z}/m\mathbb{Z})$), in particular $\ker(\chi_m)$ is a characteristic subgroup of $\pi_1(\Sigma_g)$. In this situation we use the following notation:

$$M_g[m] := M_g[\chi_m], \quad \Gamma_g[m] := \Gamma_g[\chi_m], \quad \Lambda_{[m]} := \Lambda_{[\chi_m]}, \quad t_{[m]} := t_{[\chi_m]}.$$

Furthermore we have that

$$\Gamma_g[m] \cong \text{Sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$$

and moreover $M_g[m]$ coincides with the set of isomorphism classes of pairs (C, ρ) , where $\rho: H_1(C, \mathbb{Z}/m\mathbb{Z}) \rightarrow (\mathbb{Z}/m\mathbb{Z})^{2g}$ is a symplectic isomorphism, with respect to the intersection form on $H_1(C, \mathbb{Z}/m\mathbb{Z})$ and the standard symplectic form on $(\mathbb{Z}/m\mathbb{Z})^{2g}$.

4 Smooth covers of $\mathcal{M}_g(G, \tau)$

Let $\tau: G \rightarrow \Gamma_g$ be an injective homomorphism, we denote with C_τ the centralizer of $\tau(G)$ in Γ_g , and with N_τ the normalizer of $\tau(G)$ in Γ_g . For any subgroup $H \leq \Gamma_g$, its image under the quotient morphism $\Gamma_g \rightarrow \Gamma_g[m] = \Gamma_g/\Lambda_{[m]}$ will be denoted by $\bar{H}$.

Proposition 4.1 *Let $m \geq 3$ be an integer and $Z_\tau := \mathcal{T}_g^{\tau(G)}/(\Lambda_{[m]} \cap C_\tau)$. Then Z_τ is a complex manifold and, for $M_g(G, \tau)$ defined in Definition 2.6, the quotient morphism $\mathcal{T}_g^{\tau(G)} \rightarrow M_g(G, \tau)$ induces a finite morphism*

$$Z_\tau \rightarrow M_g(G, \tau), \quad (3)$$

which gives an isomorphism between $M_g(G, \tau)$ and $Z_\tau/\bar{C}_\tau$.

Proof The first claim follows from the fact that $\Lambda_{[m]}$ acts freely on $\mathcal{T}_g$, since $m \geq 3$. To see this, let $[\gamma] \in \Lambda_{[m]}$ and let $[C, [f]] \in \mathcal{T}_g$ such that $[\gamma] \cdot [C, [f]] = [C, [f]]$. Then $[\gamma] \in \text{Aut}(C)$. Let $\bar{\gamma}_*$ be the induced automorphism of $H_1(C, \mathbb{Z}/m\mathbb{Z})$. Since $[\gamma] \in \Lambda_{[m]}$, $\bar{\gamma}_* = \text{Id}$, then $[\gamma] = \text{Id}_C$ [15, Prop. (2.8), p. 512].

The last claim follows from the fact that $\Lambda_{[m]}$ is a normal subgroup of Γ_g (Lemma 3.3) and $\bar{C}_\tau = \frac{C_\tau}{C_\tau \cap \Lambda_{[m]}}$, so

$$M_g(G, \tau) = \mathcal{T}_g^{\tau(G)}/C_\tau = Z_\tau/\bar{C}_\tau.$$

$\square$

Now we prove that Z_τ is a quasi-projective algebraic variety. Although this fact can be deduced from [20] or [2], we give here, for completeness, an elementary proof by showing that there is a finite morphism $Z_\tau \rightarrow (M_g[m])^{\overline{\tau(G)}}$, where $(M_g[m])^{\overline{\tau(G)}}$ is the set of points of $M_g[m]$ fixed by the action of $\tau(G)$. By a finite morphism of complex analytic spaces we mean a proper morphism with finite fibers.

Proposition 4.2 *Let $t_{[m]}: \mathcal{T}_g \rightarrow M_g[m]$ be the morphism defined in Sect. 3.1, $m \geq 3$. Then the following statements hold true:*

1. $t_{[m]}(\mathcal{T}_g^{\tau(G)}) = (M_g[m])^{\overline{\tau(G)}}$;
2. let $t_{[m]}$ be the restriction of $t_{[m]}$ to $\mathcal{T}_g^{\tau(G)}$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{T}_g^{\tau(G)} & \xrightarrow{\quad} & Z_\tau \\ \downarrow t_{[m]} & & \downarrow q \\ (M_g[m])^{\overline{\tau(G)}} & \xleftarrow[p]{} & Z_\tau / \left(\frac{\Lambda_{[m]} \cap N_\tau}{\Lambda_{[m]} \cap C_\tau} \right) \end{array}$$

where the horizontal arrow to the top is the quotient map by $\Lambda_{[m]} \cap C_\tau$, q is the quotient map by $\frac{\Lambda_{[m]} \cap N_\tau}{\Lambda_{[m]} \cap C_\tau}$ and p sends any equivalence class of the $(\Lambda_{[m]} \cap N_\tau)$ -action to the corresponding equivalence class of the $\Lambda_{[m]}$ -action (under the identification of $Z_\tau / \left(\frac{\Lambda_{[m]} \cap N_\tau}{\Lambda_{[m]} \cap C_\tau} \right)$ with $\frac{\mathcal{T}_g^{\tau(G)}}{\Lambda_{[m]} \cap N_\tau}$);

3. the morphism $p \circ q$ is finite, therefore Z_τ has a structure of complex quasi-projective algebraic variety and $p \circ q$ is algebraic.

Proof 1. Since $\Lambda_{[m]}$ is normal in Γ_g (Lemma 3.3), $t_{[m]}$ is G -equivariant, therefore $t_{[m]}(\mathcal{T}_g^{\tau(G)}) \subseteq (M_g[m])^{\overline{\tau(G)}}$. On the other hand, for any given $[C, \chi_m \circ f_*] \in (M_g[m])^{\overline{\tau(G)}}$ and $[\gamma] \in \tau(G)$, from the equality

$$[C, \chi_m \circ f_*] = [\gamma] \cdot [C, \chi_m \circ f_*]$$

it follows that $[C, [f]]$ and $[\gamma] \cdot [C, [f]]$ map to the same point of $M_g[m]$. So, if $[\gamma] \notin \text{Stab}_{\Gamma_g}([C, [f]])$, it would be an element of finite order of $\Lambda_{[m]}$, but this contradicts the fact that $\Lambda_{[m]}$ acts freely on $\mathcal{T}_g$.

The statement in 2) follows from the fact that $M_g[m] = t_{[m]}(\mathcal{T}_g) = \mathcal{T}_g / \Lambda_{[m]}$ and from 1). To prove 3), we show that $p \circ q$ is closed and has finite fibers. Let us first show that q has finite fibers, equivalently that $\Lambda_{[m]} \cap C_\tau$ has finite index in $\Lambda_{[m]} \cap N_\tau$. Notice that, for $\tau(G) = \{h_1, \dots, h_{|\tau(G)|}\}$, C_τ is the stabilizer of $(h_1, \dots, h_{|\tau(G)|}) \in \prod_{h \in \tau(G)} \tau(G)$ with respect to the action of N_τ given by conjugation on each factor. So $[N_\tau : C_\tau] < \infty$, hence also $[\Lambda_{[m]} \cap N_\tau : \Lambda_{[m]} \cap C_\tau] < \infty$. To see that p has finite fibers, notice that for any $[C, [\chi_m \circ f_*]] \in (M_g[m])^{\overline{\tau(G)}}$

$$p^{-1}([C, [\chi_m \circ f_*]]) = \frac{(\Lambda_{[m]} \cdot [C, [f]]) \cap \mathcal{T}_g^{\tau(G)}}{\Lambda_{[m]} \cap N_\tau},$$

and, since $\Lambda_{[m]}$ acts freely on $\mathcal{T}_g^{\tau(G)}$ (see the proof of Proposition 4.1), $p^{-1}([C, [\chi_m \circ f_*]])$ is in bijection with

$$\frac{\{\lambda \in \Lambda_{[m]} \mid \lambda^{-1} \tau(G) \lambda \subseteq \text{Stab}_{\Gamma_g}([C, [f]])\}}{\Lambda_{[m]} \cap N_\tau}. \quad (4)$$

The claim follows since the map from the quotient set in (4) to the set of subgroups of $\text{Stab}_{\Gamma_g}([C, [f]])$, induced by $\lambda \mapsto \lambda^{-1}\tau(G)\lambda$, is injective.

The fact that $p \circ q$ is closed follows from the fact that, for any closed subset $A \subseteq \mathcal{T}_g^{\tau(G)}$, the union of all $[\gamma] \cdot A$, $[\gamma] \in \Gamma_g$, is closed in $\mathcal{T}_g$ (see e.g. [20, proof of Theorem 1]). Finally, Z_τ and $p \circ q$ are algebraic by the Generalized Riemann Existence Theorem of Grauert-Remmert [22] (see [[23], Theorem 3.2, Appendix B]), Z_τ is quasi-projective since $(M_g[m])^{\overline{\tau(G)}}$ is so. $\square$

Proof of Thm. 2.5 From the proof of Proposition 2.4 we have that

$$\mathcal{M}_g(G) = \bigsqcup_{\tau \in \mathbb{T}} [\mathcal{T}_g^{\tau(G)} / C_\tau],$$

where we use the notation of Sect. 2.1. Since $\Lambda_{[m]} \cap C_\tau$ acts freely on $\mathcal{T}_g^{\tau(G)}$ the stack quotient $[\mathcal{T}_g^{\tau(G)} / (\Lambda_{[m]} \cap C_\tau)]$ is represented by the variety $Z_\tau = \mathcal{T}_g^{\tau(G)} / (\Lambda_{[m]} \cap C_\tau)$, hence

$$\mathcal{M}_g(G) = \bigsqcup_{\tau \in \mathbb{T}} [Z_\tau / \bar{C}_\tau].$$

The claim follows from the fact that the stack quotient of the algebraic variety Z_τ (Proposition 4.1) by the finite group $\bar{C}_\tau$ is a Deligne–Mumford stack (see e.g. [24, (4.6.1)]). $\square$

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