Integration of polar-analytic functions and applications to Boas' differentiation formula and Bernstein's inequality in Mellin setting

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Dedicated to the memory of our close Friend and Colleague Professor Domenico Candeloro, who survives in our hearts

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Abstract

We establish a general version of Cauchy's integral formula and a residue theorem for polar-analytic functions, employing the new notion of logarithmic poles. As an application, a Boas-type differentiation formula in Mellin setting and a Bernstein-type inequality for polar Mellin derivatives are deduced.

Keywords Polar-analytic functions · Cauchy's integral formulae · Identity theorems · Logarithmic poles · Boas' differentiation formula

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1 Introduction

In [1], we introduced the notion of polar-analytic function as follows: Let $\mathbb{H}$ be the half-plane $\{(r, \theta) : r \in \mathbb{R}^+, \theta \in \mathbb{R}\}$, where $\mathbb{R}^+$ the set of positive real numbers, and let f be a complex-valued function defined on a neighborhood of $(r_0, \theta_0) \in \mathbb{H}$. Then the polar derivative of f at the point (r_0, θ_0) , denoted by $(D_{\text{pol}}f)(r_0, \theta_0)$, is given by the limit

$$\lim_{(r, \theta) \rightarrow (r_0, \theta_0)} \frac{f(r, \theta) - f(r_0, \theta_0)}{re^{i\theta} - r_0e^{i\theta_0}}.$$

This definition leads naturally to the classical Cauchy–Riemann equations when written in their polar form, often considered in the literature; see e.g. [5, Sec. 23, p. 68]. Polar

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analyticity provides a simple alternative for functions which are analytic on a part of the Riemann surface of the logarithm and avoids the necessity of specifying analytic branches. The main applications of this concept are in Mellin analysis and in the realm of quadrature formulae on $\mathbb{R}^+$; see [2]. It turns out to be very helpful for a foundation of Mellin analysis independent of Fourier analysis.

The corresponding notion of polar Mellin derivative has been introduced as (see [1])

$$\tilde{\Theta}_c f(r, \theta) := r e^{i\theta} (D_{\text{pol}} f)(r, \theta) + c f(r, \theta).$$

A first development of the theory of polar-analytic functions is given in the recent paper [3], in which extensions of Cauchy's integral formula and Taylor-type series have been established among other results.

In Fourier analysis and signal processing, series expansions are often obtained by methods of complex analysis with the residue theorem as a decisive tool. Since in these applications only poles occur, the residue theorem can be replaced by skillful use of Cauchy's integral formula and its extension to derivatives.

In the case of polar-analytic functions, our analogue of Cauchy's integral formula in [3] is useful in horizontal strips of width less than 2π only since otherwise it may produce additional residues as undesired artifacts; see [3, formula (10)]. Therefore the aim of this paper is to establish a further analogue of Cauchy's integral formula which is always free of artifacts and a version of the residue theorem for polar-analytic functions. As interesting applications, we establish an analogue of Boas' differentiation formula for polar Mellin derivatives and, as a consequence, a Bernstein-type inequality. For the Fourier counterparts, see [7]. An interesting connection can be found in the papers [8, 9], in which Boas-type formulae were considered in the realm of general one-parameter groups of isometries in Banach spaces.

We wish to dedicate the present article to our close friend and fine mathematician Professor Domenico Candeloro (1951–2019). Among the wide range of interests of Professor Candeloro, we quote the paper [4], in which he gave an interesting connection between stochastic processes, approximation theory and Mellin analysis. This paper was rewarded with the JMAA Ames Award in 2014. We conclude this introduction with a personal remembrance of Paul Butzer:

Mimmo, Mimmo, please help me, I am stuck. Mimmo, Mimmo, I think that I have found a new theorem, another voice noted; Mimmo looked at the proof, with the words I feel it is an easy application of a classical result. It are these calls for assistance which drew my attention to the mathematician himself, Domenico Candeloro, "Mimmo" being the familiar name his close friends call him. It is his expertise in many areas, Calculus of Variations, Measure Theory, Real Analysis, Functional Analysis, Probability Theory, which place him in a central role in Perugia, one who is always willing to help.

At a few meetings with Mimmo, during my first years at Perugia from 1990 onward and my participations in the conference of the Research Group Real Analysis and Measure Theory, conducted by Calogero Vinti and colleagues in Capri, Ischia, Maiori, and Grado, together with my friends Carlo, Gianluca and Anna Rita, Mimmo was in the company of his spouse, Prof. Doretta Vivona. He seemed happy that she carried on the conversation (in English). Both were very friendly, Mimmo having an engaging and modest personality.

2 Preliminaries

Let $\mathbb{H} := \{(r, \theta) \in \mathbb{R}^+ \times \mathbb{R}\}$ be the right half-plane and let $\mathcal{D}$ be a domain in $\mathbb{H}$, which means that $\mathcal{D}$ is open, connected and non-empty.

Definition 1 We say that $f : \mathcal{D} \rightarrow \mathbb{C}$ is *polar-analytic* on $\mathcal{D}$ with *polar derivative* $D_{\text{pol}}f$ if for any $(r_0, \theta_0) \in \mathcal{D}$ the limit

$$\lim_{(r, \theta) \rightarrow (r_0, \theta_0)} \frac{f(r, \theta) - f(r_0, \theta_0)}{re^{i\theta} - r_0e^{i\theta_0}} =: (D_{\text{pol}}f)(r_0, \theta_0)$$

exists and is the same howsoever (r, θ) approaches (r_0, θ_0) within $\mathcal{D}$.

For the properties of the polar-analytic functions and their connections with the analytic functions in classical sense, see [3]. In particular, $f = u + iv$ with real-valued functions u and v is polar-analytic on $\mathcal{D}$ if and only if u and v have continuous partial derivatives on $\mathcal{D}$ that satisfy the classical Cauchy–Riemann differential equations in polar form, and we have

$$\begin{aligned} (D_{\text{pol}}f)(r, \theta) &= e^{-i\theta} \left(\frac{\partial}{\partial r} u(r, \theta) + i \frac{\partial}{\partial r} v(r, \theta) \right) \\ &= \frac{e^{-i\theta}}{r} \left(\frac{\partial}{\partial \theta} v(r, \theta) - i \frac{\partial}{\partial \theta} u(r, \theta) \right). \end{aligned} \quad (1)$$

As we anticipated in the introduction, for a fixed real number $c \in \mathbb{R}$, we define the polar derivative of a polar-analytic function f in Mellin setting by the formula

$$\tilde{\Theta}_c f(r, \theta) := re^{i\theta} (D_{\text{pol}}f)(r, \theta) + cf(r, \theta). \quad (2)$$

For $\varphi(r) := f(r, 0)$, we have $\tilde{\Theta}_c f(r, 0) = (\Theta_c \varphi)(r) := r\varphi'(r) + c\varphi(r)$, where Θ_c is the usual Mellin differential operator; see [6].

The higher order polar Mellin derivatives may be defined through the representation formula for (usual) Mellin derivatives in terms of Stirling numbers of the second type $S_c(k, j)$ (see [6, Lemma 9]), namely

$$\tilde{\Theta}_c^k f(r, \theta) := \sum_{j=0}^k S_c(k, j) r^j e^{ij\theta} D_{\text{pol}}^j f(r, \theta) \quad (k \in \mathbb{N}, (r, \theta) \in \mathbb{H}). \quad (3)$$

3 A general Cauchy integral formula

We begin with the following theorem whose proof is a slight extension of that of [3, Theorem 6.1]. For $(r_0, \theta_0) \in \mathbb{H}$ and $\rho > 0$, the *polar disk* centered at (r_0, θ_0) with radius ρ is defined by

$$E((r_0, \theta_0), \rho) := \left\{ (r, \theta) \in \mathbb{H} : \left(\log \frac{r}{r_0} \right)^2 + (\theta - \theta_0)^2 < \rho^2 \right\}.$$

Theorem 1 Let $f : \mathcal{D} \rightarrow \mathbb{C}$ be polar-analytic on a domain $\mathcal{D} \subset \mathbb{H}$ and let $c \in \mathbb{R}$. If $(r_0, \theta_0) \in \mathcal{D}$, then there holds the expansion

$$(re^{i\theta})^c f(r, \theta) = (r_0e^{i\theta_0})^c \sum_{k=0}^{\infty} \frac{(\tilde{\Theta}_c^k f)(r_0, \theta_0)}{k!} \left(\log \frac{r}{r_0} + i(\theta - \theta_0) \right)^k, \quad (4)$$

converging uniformly on every polar disk $E((r_0, \theta_0), \rho) \subset \mathcal{D}$.

Proof Setting $A := \{z = x + iy \in \mathbb{C} : (e^x, y) \in \mathcal{D}\}$, we have shown in [3, proof of Theorem 6.1] that $g : z = x + iy \mapsto f(e^x, y)$ is analytic on A and for $z_0 := \log r_0 + i\theta_0$, we have $g^{(k)}(z_0) = (\tilde{\partial}_0^k f)(r_0, \theta_0)$. Now define $h(z) := e^{cz}g(z)$. Then h is also analytic on A and

$$h^{(k)}(z_0) = (r_0 e^{i\theta_0})^c \sum_{j=0}^k \binom{k}{j} (\tilde{\partial}_0^j f)(r_0, \theta_0) c^{k-j}.$$

It can be shown by induction on k that the sum on the right-hand side is equal to $(\tilde{\partial}_c^k f)(r_0, \theta_0)$. Since by Taylor expansion

$$e^{cz}g(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(z_0)}{k!} (z - z_0)^k,$$

we obtain (4) by substituting $z = x + iy$ with $x = \log r$ and $y = \theta$. The statement on convergence is seen as in [3]. $\square$

As a consequence of the series expansion (4), we deduce the following identity theorem for polar-analytic functions.

Theorem 2 *Let $\mathcal{D}$ be a domain in $\mathbb{H}$ and let $f : \mathcal{D} \rightarrow \mathbb{C}$ be polar-analytic. Suppose that $(r_0, \theta_0) \in \mathcal{D}$ is an accumulation point of distinct zeros of f . Then f is identically zero.*

Proof As an auxiliary result, we first prove that $(\tilde{\partial}_0^k f)(r_0, \theta_0) = 0$, for all $k \in \mathbb{N}_0$. Indeed, assume that there exists a smallest non-negative integer k_0 such that $(\tilde{\partial}_0^{k_0} f)(r_0, \theta_0) \neq 0$. Then, by Theorem 1 for $c = 0$, f has an expansion

$$f(r, \theta) = \left(\log \frac{r}{r_0} + i(\theta - \theta_0) \right)^{k_0} \times \left(\frac{(\tilde{\partial}_0^{k_0} f)(r_0, \theta_0)}{k_0!} + \sum_{k=k_0+1}^{\infty} \frac{(\tilde{\partial}_0^k f)(r_0, \theta_0)}{k!} \left(\log \frac{r}{r_0} + i(\theta - \theta_0) \right)^{k-k_0} \right)$$

converging in a polar disk $E((r_0, \theta_0), \rho)$ with $\rho > 0$. On the right-hand side, the first factor vanishes for $(r, \theta) = (r_0, \theta_0)$ only if $k_0 \geq 1$; otherwise it is identically 1. The second factor is different from zero at $(r, \theta) = (r_0, \theta_0)$. By continuity, it is also different from zero in a sufficiently small neighborhood of (r_0, θ_0) . Hence (r_0, θ_0) cannot be an accumulation point of distinct zeros of f . A contradiction!

Now, let (r^*, θ^*) be an arbitrary point in $\mathcal{D}$. It suffices to show that $f(r^*, \theta^*) = 0$. Let $\gamma : [0, 1] \rightarrow \mathcal{D}$ be a Jordan arc in $\mathcal{D}$ such that $\gamma(0) = (r_0, \theta_0)$ and $\gamma(1) = (r^*, \theta^*)$. Since the trace of γ has a positive distance from the boundary of $\mathcal{D}$, there exists a number $\rho_{\inf} > 0$ such that for each point (r, θ) on γ there is a polar disk $E((r, \theta), \rho) \subset \mathcal{D}$ with $\rho \geq \rho_{\inf}$. First consider a polar disk $E_0 := E((r_0, \theta_0), \rho_0) \subset \mathcal{D}$ with $\rho_0 \geq \rho_{\inf}$. By Theorem 1, we have a representation

$$f(r, \theta) = \sum_{k=0}^{\infty} \frac{(\tilde{\partial}_0^k f)(r_0, \theta_0)}{k!} \left(\log \frac{r}{r_0} + i(\theta - \theta_0) \right)^k$$

holding for all $(r, \theta) \in E_0$. Now, by the auxiliary result, it follows that the restriction of f to E_0 is identically zero. Thus, if $(r^*, \theta^*) \in E_0$, we have reached our aim. Otherwise, let

$(r_1, \theta_1) := \gamma(t_1)$ with $0 < t_1 \leq 1$ be a point where γ intersects the boundary of E_0 . By our previous observations, there exists a polar disk $E_1 := E((r_1, \theta_1), \rho_1) \subset \mathcal{D}$ with $\rho_1 \geq \rho_{\inf}$ such that

$$f(r, \theta) = \sum_{k=0}^{\infty} \frac{(\tilde{\Theta}_0^k f)(r_1, \theta_1)}{k!} \left(\log \frac{r}{r_1} + i(\theta - \theta_1) \right)^k$$

for all $(r, \theta) \in E_1$. Since (r_1, θ_1) , being a boundary point of E_0 , is an accumulation point of distinct zeros of f , we conclude with the help of the auxiliary result that the restriction of f to E_1 vanishes identically. If $(r^*, \theta^*) \notin E_1$, we continue this procedure with a point $(r_2, \theta_2) := \gamma(t_2)$, where $t_1 < t_2 \leq 1$, lying on the boundary of E_1 . Since $\rho_{\inf} > 0$, we arrive after a finite number of steps at a polar disk E_n , say, such that the restriction of f to E_n vanishes identically and $(r^*, \theta^*) \in E_n$. This completes the proof. $\square$

Now we state the announced Cauchy integral formula for polar-analytic functions in its general form. Here a curve will be called *regular* if it is piecewise continuously differentiable.

Theorem 3 *Let $\mathcal{D}$ be a convex domain in $\mathbb{H}$, and let $f : \mathcal{D} \rightarrow \mathbb{C}$ be polar-analytic on $\mathcal{D}$. Let γ be a positively oriented, closed, regular curve that is the boundary of a convex domain $\text{int}(\gamma) \subset \mathcal{D}$. Then, for $(r_0, \theta_0) \in \text{int}(\gamma)$, $c \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we have*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{(re^{i\theta})^{c-1} f(r, \theta) e^{i\theta}}{(\log(r/r_0) + i(\theta - \theta_0))^{k+1}} (dr + i r d\theta) = (r_0 e^{i\theta_0})^c \frac{(\tilde{\Theta}_c^k f)(r_0, \theta_0)}{k!}. \quad (5)$$

Proof As in [3, Lemma 5.1], we draw a positively oriented circle λ_{ε} of sufficiently small radius $\varepsilon > 0$ around (r_0, θ_0) and decompose $\text{int}(\gamma)$ into four parts using line segments which lie on rays starting at (r_0, θ_0) and intersecting under an angle $2\pi/3$. Denote the positively oriented boundaries of the three parts other than the disk by β_1, β_2 and β_3 . These are closed regular curves. It is geometrically evident that each β_j , lies in a convex subset of $\mathcal{D}$ on which the function

$$F : (r, \theta) \mapsto \frac{(re^{i\theta})^{c-1} f(r, \theta)}{(\log(r/r_0) + i(\theta - \theta_0))^{k+1}} \quad (6)$$

is polar-analytic. By [3, Theorem 4.1], we have

$$\int_{\beta_j} F(r, \theta) e^{i\theta} (dr + i r d\theta) = 0 \quad (j = 1, 2, 3).$$

This enables us to conclude that

$$\frac{1}{2\pi i} \int_{\gamma} F(r, \theta) e^{i\theta} (dr + i r d\theta) = \frac{1}{2\pi i} \int_{\lambda_{\varepsilon}} F(r, \theta) e^{i\theta} (dr + i r d\theta). \quad (7)$$

From Theorem 1 we know that

$$F(r, \theta) = (r_0 e^{i\theta_0})^{c-1} \sum_{\ell=0}^{\infty} \frac{(\tilde{\Theta}_{c-1}^{\ell} f)(r_0, \theta_0)}{\ell!} \left(\log \frac{r}{r_0} + i(\theta - \theta_0) \right)^{\ell-k-1}.$$

Note that on the right-hand side all terms with $\ell \geq k+1$ are polar-analytic as functions of (r, θ) . Thus

$$\frac{1}{2\pi i} \int_{\lambda_{\varepsilon}} F(r, \theta) e^{i\theta} (dr + i r d\theta) = (r_0 e^{i\theta_0})^{c-1} \sum_{\ell=0}^k \frac{(\tilde{\Theta}_{c-1}^{\ell} f)(r_0, \theta_0)}{\ell!} I_{k+1-\ell}, \quad (8)$$

where

$$I_j := \frac{1}{2\pi i} \int_{\lambda_\varepsilon} \frac{e^{i\theta} (dr + i r d\theta)}{(\log(r/r_0) + i(\theta - \theta_0))^j} \quad (j = 1, \dots, k+1).$$

Now consider the mapping $(r, \theta) \mapsto z = r e^{i\theta}$. If (r, θ) traverses λ_ε once, then, for sufficiently small $\varepsilon > 0$, the point z runs along a piecewise continuously differentiable, closed curve γ_ε , say, in $\mathbb{C}$, surrounding $z_0 = r_0 e^{i\theta_0}$ exactly once in the mathematically positive sense. Therefore,

$$I_j = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{dz}{(\log z - \log z_0)^j} = \frac{1}{2\pi i} \int_{\log \circ \gamma_\varepsilon} \frac{e^w dw}{(w - \log z_0)^j} = \frac{r_0 e^{i\theta_0}}{(j-1)!}.$$

With this result (8) becomes

$$\frac{1}{2\pi i} \int_{\lambda_\varepsilon} F(r, \theta) e^{i\theta} (dr + i r d\theta) = \frac{(r_0 e^{i\theta_0})^c}{k!} (\tilde{\Theta}_c^k f)(r_0, \theta_0).$$

The last conclusion is seen by noting that $\tilde{\Theta}_c f = \tilde{\Theta}_{c-1} f + f$ and using this relation repeatedly. This completes the proof. $\square$

4 A polar residue theorem

First we introduce some notions.

Definition 2 Let $(r_0, \theta_0) \in \mathbb{H}$ and let $\mathcal{U} \subset \mathbb{H}$ be an open neighborhood of (r_0, θ_0) .

- (i) If $f : \mathcal{U} \setminus \{(r_0, \theta_0)\} \rightarrow \mathbb{C}$ is polar-analytic, then (r_0, θ_0) will be called an *isolated singularity* of f .
- (ii) An isolated singularity (r_0, θ_0) is said to be a *logarithmic pole of order k* if $k \in \mathbb{N}$ and there exists a polar-analytic function $g : \mathcal{U} \rightarrow \mathbb{C}$ with $g(r_0, \theta_0) \neq 0$ such that

$$f(r, \theta) = \frac{g(r, \theta)}{(\log(r/r_0) + i(\theta - \theta_0))^k} \quad \text{for } (r, \theta) \in \mathcal{U} \setminus \{(r_0, \theta_0)\}. \quad (9)$$

In this case

$$(\operatorname{res}_c f)(r_0, \theta_0) := (r_0 e^{i\theta_0})^c \frac{(\tilde{\Theta}_c^{k-1} g)(r_0, \theta_0)}{(k-1)!} \quad (10)$$

will be called the *c-residue* of f at (r_0, θ_0) .

We are now in a position to formulate a residue theorem for logarithmic poles which will be suitable for many applications in Mellin analysis.

Theorem 4 Let $\mathcal{D}$ be a convex domain in $\mathbb{H}$ and let f be polar-analytic on $\mathcal{D}$ except for isolated singularities which are all logarithmic poles. Let γ be a positively oriented, closed, regular curve that is the boundary of a convex domain $\operatorname{int}(\gamma) \subset \mathcal{D}$. Suppose that no isolated singularity lies on γ while (r_j, θ_j) for $j = 1, \dots, m$ are the isolated singularities lying in $\operatorname{int}(\gamma)$. Then, for $c \in \mathbb{R}$, there holds

$$\int_{\gamma} (r e^{i\theta})^{c-1} f(r, \theta) e^{i\theta} (dr + i r d\theta) = 2\pi i \sum_{j=1}^m (\operatorname{res}_c f)(r_j, \theta_j).$$

Proof For $(x_0, y_0) \in \mathbb{R}^2$ and $\varepsilon > 0$, we consider the vertical grid

$$V(x_0, \varepsilon) := \{(r, \theta) \in \mathbb{R}^2 : \theta \in \mathbb{R}, r = x_0 + n\varepsilon, n \in \mathbb{Z}\}$$

and the horizontal grid

$$H(y_0, \varepsilon) := \{(r, \theta) \in \mathbb{R}^2 : r \in \mathbb{R}, \theta = y_0 + n\varepsilon, n \in \mathbb{Z}\}.$$

Their union constitutes a net $N((x_0, y_0), \varepsilon) := V(x_0, \varepsilon) \cup H(y_0, \varepsilon)$. The intersection $\text{int}(\gamma) \cap N((x_0, y_0), \varepsilon)$ creates a tessellation of $\text{int}(\gamma)$ into squares with edges of length ε and further convex sets whose boundary contains a piece of γ . The latter may be called boundary sets. The collection of all subsets of the tessellation shall be denoted by $\mathcal{T}$. If $\varepsilon > 0$ is sufficiently small, then the boundary sets will be free of isolated singularities and different isolated singularities in $\text{int}(\gamma)$ will lie in different squares of the tessellation. Furthermore, if (r_j, θ_j) lies in the interior of a square, then it will remain there under all sufficiently small variations of (x_0, y_0) . But if (r_j, θ_j) lies on the boundary of a square, then there exist arbitrarily small variations such that (r_j, θ_j) goes inside. We may therefore assume that each (r_j, θ_j) lies inside a square $Q_j \in \mathcal{T}$ for $j = 1, \dots, m$. For each set $P \in \mathcal{T}$, we denote by ∂P its positively oriented boundary. Note that each line segment in $\text{int}(\gamma)$ which comes from a mesh of the net belongs to the boundaries of exactly two subsets of the tessellation where it occurs with opposite orientations. Therefore

$$\int_{\gamma} (re^{i\theta})^{c-1} f(r, \theta) e^{i\theta} (dr + ir d\theta) = \sum_{j=1}^m \int_{\partial Q_j} (re^{i\theta})^{c-1} f(r, \theta) e^{i\theta} (dr + ir d\theta)$$

since by [3, Theorem 4.1] the integral along ∂P vanishes if P does not contain an isolated singularity. As all our isolated singularities are logarithmic poles, we conclude with the help of Theorem 3 and by using the notion of residue that

$$\int_{\partial Q_j} (re^{i\theta})^{c-1} f(r, \theta) e^{i\theta} (dr + ir d\theta) = 2\pi i (\text{res}_c f)(r_j, \theta_j) \quad (j = 1, \dots, m).$$

This completes the proof. $\square$

Remark 1 The proof of Theorem 4 via Theorem 3 reveals that if (9) holds with $g(r_0, \theta_0)$ being not necessarily different from zero, which implies that the order of the logarithmic pole is at most k , then formula (10) still provides the correct value of the c -residue of f at (r_0, θ_0) .

5 Boas' differentiation formula and Bernstein's inequality for polar Mellin derivatives

For $p \in [1, \infty]$, denote by $\|\cdot\|_p$ the norm of the Lebesgue space $L^p(\mathbb{R}^+)$. In Mellin analysis, the analogue of $L^p(\mathbb{R}^+)$ are the spaces X_c^p , where $c \in \mathbb{R}$, comprising all functions $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ such that $f(\cdot)(\cdot)^{c-1/p} \in L^p(\mathbb{R}^+)$ with the norm $\|f\|_{X_c^p} := \|f(\cdot)(\cdot)^{c-1/p}\|_p$. Furthermore, for $p = \infty$, we define X_c^∞ as the space of all measurable functions $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ such that $\|f\|_{X_c^\infty} := \sup_{x>0} x^c |f(x)| < \infty$.

We recall (see [1]) that the Mellin–Bernstein space $\mathcal{B}_{c,T}^p$ comprises all functions $f : \mathbb{H} \rightarrow \mathbb{C}$ with the following properties:

- (i) f is polar-analytic on $\mathbb{H}$;
- (ii) $f(\cdot, 0) \in X_c^p$;
- (iii) there exists a constant $C_f > 0$ such that $r^c |f(r, \theta)| \leq C_f e^{T|\theta|}$ for all $(r, \theta) \in \mathbb{H}$.

Next we state three useful assertions on transformations in Mellin–Bernstein spaces. They are verified by straightforward calculations. To show polar analyticity, we simply check that the Cauchy–Riemann equations in polar form are satisfied. Concerning statement (c) below, for verifying property (ii) in the definition of Mellin–Bernstein spaces, we will use Theorem 4.2 in [1].

Proposition 1 *Let $f \in \mathcal{B}_{c,T}^p$, where $p \in [1, +\infty]$, $c \in \mathbb{R}$ and $T > 0$.*

- (a) *If $t > 0$ and $g : (r, \theta) \mapsto t^c f(tr, \theta)$, then $g \in \mathcal{B}_{c,T}^p$; in particular, $\|g(\cdot, 0)\|_{X_c^p} = \|f(\cdot, 0)\|_{X_c^p}$ and $r^c |g(r, \theta)| \leq C_f e^{T|\theta|}$ for all $(r, \theta) \in \mathbb{H}$. Furthermore, $(\tilde{\Theta}_c g)(r, \theta) = t^c (\tilde{\Theta}_c f)(tr, \theta)$.*
- (b) *If $h : (r, \theta) \mapsto f(r^{1/T}, \theta/T)$, then $h \in \mathcal{B}_{c/T,1}^p$; in particular, $\|h(\cdot, 0)\|_{X_{c/T}^p} = \|f(\cdot, 0)\|_{X_c^p}$ and $r^{c/T} |h(r, \theta)| \leq C_f e^{|\theta|}$ for all $(r, \theta) \in \mathbb{H}$. Furthermore,*

$$(\tilde{\Theta}_{c/T} h)(r, \theta) = \frac{1}{T} (\tilde{\Theta}_c f) \left(r^{1/T}, \frac{\theta}{T} \right).$$

- (c) *If $\alpha \in \mathbb{R}$ and $\phi : (r, \theta) \mapsto f(r, \theta + \alpha)$, then $\phi \in \mathcal{B}_{c,T}^p$; in particular, $\|\phi(\cdot, 0)\|_{X_c^p} \leq e^{T|\alpha|} \|f(\cdot, 0)\|_{X_c^p}$ and $r^c |\phi(r, \theta)| \leq C_f e^{T(|\alpha|+|\theta|)}$ for all $(r, \theta) \in \mathbb{H}$. Furthermore, $(\tilde{\Theta}_c \phi)(r, \theta) = (\tilde{\Theta}_c f)(r, \theta + \alpha)$.*

Now we establish a formula for the polar Mellin derivatives. It is analogous to a differentiation formula of Boas for bandlimited functions in Fourier analysis. An abstract version of this result was obtained in the setting of one-parameter groups of operators in [8, § 3.3].

Theorem 5 *Let $f \in \mathcal{B}_{c,T}^p$, where $p \in [1, +\infty]$, $c \in \mathbb{R}$ and $T > 0$. Then*

$$(\tilde{\Theta}_c f)(r, \theta) = \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(2k+1)^2} e^{(k+1/2)\pi c/T} f(re^{(k+1/2)\pi/T}, \theta) \quad (11)$$

for $(r, \theta) \in \mathbb{H}$. Multiplied by r^c , the series converges absolutely and uniformly on strips of bounded width parallel to the r -axis in $\mathbb{H}$.

Proof For simplicity, we first suppose that $T = 1$. Consider the function

$$F : (r, \theta) \mapsto \frac{f(r, \theta)}{(\log r + i\theta)^2 \cos(\log r + i\theta)}.$$

It is polar-analytic on $\mathbb{H}$ except for isolated singularities at the points where the denominator vanishes. Writing $r_k := e^{(k+1/2)\pi}$ for short, we obtain the exceptional points as $(1, 0)$ and $(r_k, 0)$ for $k \in \mathbb{Z}$. The first one is a logarithmic pole of order at most two; all the others are logarithmic poles of order at most one.

For $n \in \mathbb{N}$, let $\mathcal{R}_n$ be the rectangle with vertices at $(e^{\pm n\pi}, \pm n\pi)$ and denote by $\partial \mathcal{R}_n$ its positively oriented boundary. Employing Theorem 4, we find that

$$\frac{1}{2\pi i} \int_{\partial \mathcal{R}_n} (re^{i\theta})^{c-1} F(r, \theta) e^{i\theta} (dr + i r d\theta) = (\text{res}_c F)(1, 0) + \sum_{k=-n}^{n-1} (\text{res}_c F)(r_k, 0). \quad (12)$$

Introducing $\phi(r, \theta) := f(r, \theta) (\cos(\log r + i\theta))^{-1}$, we have in view of Remark 1 that $(\operatorname{res}_c F)(1, 0) = (\tilde{\Theta}_c \phi)(1, 0) = (\tilde{\Theta}_c f)(1, 0)$. For calculating the other c -residues, we factor the cosine as

$$\cos(\log r + i\theta) = \left(\log r + i\theta - \left(k + \frac{1}{2}\right)\pi\right) \psi_k(r, \theta),$$

where ψ_k is polar-analytic in a neighborhood of $(r_k, 0)$ and

$$\psi_k(r_k, 0) = \lim_{(r, \theta) \rightarrow (r_k, 0)} \frac{\cos(\log r + i\theta)}{\log r + i\theta - \left(k + \frac{1}{2}\right)\pi} = (-1)^{k+1}.$$

Thus,

$$(\operatorname{res}_c F)(r_k, 0) = \frac{r_k^c f(r_k, 0)}{\left(k + \frac{1}{2}\right)\pi^2 \psi_k(r_k, 0)} = \frac{4}{\pi^2} \frac{(-1)^{k+1}}{(2k+1)^2} r_k^c f(r_k, 0).$$

With these values of the c -residues, we may rewrite (12) as

$$\begin{aligned} (\tilde{\Theta}_c f)(1, 0) &= \frac{4}{\pi^2} \sum_{k=-n}^{n-1} \frac{(-1)^k}{(2k+1)^2} r_k^c f(r_k, 0) \\ &\quad + \frac{1}{2\pi i} \int_{\partial \mathcal{R}_n} (re^{i\theta})^{c-1} F(r, \theta) e^{i\theta} (dr + i r d\theta). \end{aligned}$$

Noting that $|\cos(\log r + i\theta)| > e^{|\theta|}/3$ on $\partial \mathcal{R}_n$ and employing property (iii) of functions belonging to $\mathcal{B}_{c,1}^p$, we see that

$$r^{c-1} |F(r, \theta)| \leq \frac{3C_f}{r(\log^2 r + \theta^2)} \quad \text{for } (r, \theta) \in \partial \mathcal{R}_n.$$

Considering the integrals along the vertical and the horizontal line segments of $\partial \mathcal{R}_n$ separately, we find that

$$\left| \frac{1}{2\pi i} \int_{\partial \mathcal{R}_n} (re^{i\theta})^{c-1} F(r, \theta) e^{i\theta} (dr + i r d\theta) \right| \leq \frac{6C_f}{n\pi^2} \int_{-1}^1 \frac{dx}{1+x^2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$(\tilde{\Theta}_c f)(1, 0) = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} r_k^c f(r_k, 0) \quad \text{for } f \in \mathcal{B}_{c,1}^p.$$

With the Mellin translation of Proposition 1(a), for $f \in \mathcal{B}_{c,1}^p$, we deduce

$$(\tilde{\Theta}_c f)(t, 0) = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} r_k^c f(tr_k, 0) \quad (t > 0).$$

Next we want to extend this formula to $f \in \mathcal{B}_{c,T}^p$ with an arbitrary $T > 0$. For this, we note that the formula is valid for the function h defined in Proposition 1(b) provided that we replace c with c/T . Expressing h in terms of f , we obtain

$$(\tilde{\Theta}_c f)(t^{1/T}, 0) = \frac{4T}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} r_k^{c/T} f(t^{1/T} r_k^{1/T}, 0) \quad (t > 0),$$

now valid for $f \in \mathcal{B}_{c,T}^p$. In this formula, we may replace the variable t with t^T on both sides. Furthermore, we may employ Proposition 1(c) for extending the argument of the polar Mellin derivative from $(t, 0)$ to an arbitrary point $(t, \alpha) \in \mathbb{H}$. This leads us to

$$(\tilde{\Theta}_c f)(t, \alpha) = \frac{4T}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} r_k^{c/T} f(tr_k^{1/T}, \alpha)$$

for $f \in \mathcal{B}_{c,T}^p$ and all $(t, \alpha) \in \mathbb{H}$, which is identical with (11).

It remains to justify the statement on convergence. Since

$$\frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} = 1, \quad (13)$$

and by property (iii) in the definition of $\mathcal{B}_{c,T}^p$, there holds

$$t^c r_k^{c/T} \left| f(tr_k^{1/T}, \alpha) \right| \leq C_f e^{T|\alpha|} \quad (k \in \mathbb{Z}),$$

we readily see absolute and uniform convergence as asserted in the theorem. $\square$

As a consequence of Theorem 5, we can establish a Bernstein inequality for polar Mellin derivatives.

Corollary 1 *Let $f \in \mathcal{B}_{c,T}^p$, where $p \in [1, \infty]$, $c \in \mathbb{R}$ and $T > 0$. Then for any $\theta \in \mathbb{R}$, we have*

$$\|(\tilde{\Theta}_c f)(\cdot, \theta)\|_{X_c^p} \leq T \|f(\cdot, \theta)\|_{X_c^p}. \quad (14)$$

Proof Let us write $r_k = e^{(k+1/2)\pi}$, where $k \in \mathbb{Z}$, for short. Multiplying (11) by r^c , taking moduli on both sides, applying the triangular inequality on the right-hand side and using (13), we obtain

$$r^c |(\tilde{\Theta}_c f)(r, \theta)| \leq \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} r^c r_k^{c/T} \left| f(r r_k^{1/T}, \theta) \right| = T \|f(\cdot, \theta)\|_{X_c^\infty} \quad (15)$$

for any $(r, \theta) \in \mathbb{H}$. This implies (14) for $p = \infty$.

Now let $1 \leq p < \infty$ and let $R > 0$. Since $a^c \|f(a \cdot, \theta)\|_{X_c^p} = \|f(\cdot, \theta)\|_{X_c^p}$, for $a > 0$, we conclude from (15) using the triangular inequality for norms and B. Levi's theorem that

$$\begin{aligned} \left(\int_{1/R}^R r^{cp} |(\tilde{\Theta}_c f)(r, \theta)|^p \frac{dr}{r} \right)^{1/p} &\leq \frac{4T}{\pi^2} \left\| \sum_{k \in \mathbb{Z}} \frac{r_k^{c/T}}{(2k+1)^2} f(r_k^{1/T} \cdot, \theta) \right\|_{X_c^p} \\ &\leq \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{r_k^{c/T}}{(2k+1)^2} \|f(r_k^{1/T} \cdot, \theta)\|_{X_c^p} = T \|f(\cdot, \theta)\|_{X_c^p}. \end{aligned}$$

This guarantees that the X_c^p norm of $(\tilde{\Theta}_c f)(\cdot, \theta)$ exists. Letting $R \rightarrow \infty$, we obtain (14) for $p \in [1, \infty[$. $\square$

Corollary 2 *The Mellin–Bernstein space is invariant under polar Mellin differentiation, that is, if $f \in \mathcal{B}_{c,T}^p$, where $p \in [1, +\infty]$, $c \in \mathbb{R}$ and $T > 0$, then $\tilde{\Theta}_c f \in \mathcal{B}_{c,T}^p$.*

Proof We have to show that $\tilde{\Theta}_c f$ satisfies properties (i)–(iii) in the definition of Mellin–Bernstein spaces. Since for a polar-analytic function f there exist polar derivatives of arbitrary order (see [3, Theorem 5.3]), it follows immediately that $\tilde{\Theta}_c f$ is polar-analytic on $\mathbb{H}$, and so (i) is satisfied. Property (ii) is guaranteed by Corollary 1. Finally, knowing that (iii) holds for f , we deduce from (11) that

$$r^c |(\tilde{\Theta}_c f)(r, \theta)| \leq \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{C_f e^{T|\theta|}}{(2k+1)^2} = TC_f e^{T|\theta|}.$$

Hence (iii) holds with $C_{\tilde{\Theta}_c f} = TC_f$. $\square$

Combining Corollaries 1 and 2, we can apply (14) repeatedly and obtain

$$\|(\tilde{\Theta}_c^k f)(\cdot, \theta)\|_{X_c^p} \leq T^k \|f(\cdot, \theta)\|_{X_c^p} \quad (\theta \in \mathbb{R}, k \in \mathbb{N}).$$

This inequality and its special case (14) relate to [8, Theorem 3.4]; also see [9, Theorem 2.6].

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Compliance with ethical standards

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References

1. Bardaro, C., Butzer, P.L., Mantellini, I., Schmeisser, G.: A fresh approach to the Paley–Wiener theorem for Mellin transforms and the Mellin–Hardy spaces. *Math. Nachr.* **290**, 2759–2774 (2017)
2. Bardaro, C., Butzer, P.L., Mantellini, I., Schmeisser, G.: Quadrature formulae for the positive real axis in the setting of Mellin analysis: sharp error estimates in terms of the Mellin distance. *Calcolo* **55**(3), article 26 (2018)
3. Bardaro, C., Butzer, P.L., Mantellini, I., Schmeisser, G.: Development of a new concept of polar analytic functions useful in Mellin analysis. *Complex Var. Elliptic Eq.* **64**(12), 2040–2062 (2019)
4. Boccuto, A., Candeloro, D., Sambucini, A.: Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes. *J. Math. Anal. Appl.* **419**(2), 818–838 (2014)
5. Brown, J.W., Churchill, R.V.: *Complex Variables and Applications*, 8th edn. McGraw-Hill, New York (2009)
6. Butzer, P.L., Jansche, S.: A direct approach to the Mellin transform. *J. Fourier Anal. Appl.* **3**, 325–375 (1997)
7. Butzer, P.L., Schmeisser, G., Stens, R.L.: Basic relations valid for the Bernstein space B_σ^p and their extensions to functions from larger spaces with error estimates in terms of their distances from B_σ^p . *J. Fourier Anal. Appl.* **19**, 333–375 (2013)
8. Pesenson, I.Z.: Boas-type formulas and sampling in Banach spaces with applications to analysis on manifolds. In: Zayed, A.I., Schmeisser, G. (eds.) *New Perspectives on Approximation and Sampling Theory*, pp. 39–61. Springer, Cham (2014)

9. Pesenson, I.Z.: Sampling formulas for groups of operators in Banach spaces. *Sampl. Theory Signal Image Process.* **14**(1), 1–16 (2015)

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