The structure of linear PF-Engel groups in characteristic zero

B. A. F. Wehrfritz¹

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Abstract

Suppose G is linear group. If G has characteristic zero, we prove that G is (polycyclic-by-finite)-Engel if and only if G has a normal series $< 1 > = T_0 \leq T_1 \leq \dots \leq T_s = T \leq G$ with s and the index $(G:T)$ finite and each T_i/T_{i-1} either polycyclic-by-finite, or G -hypercentral with $[T_i, T] \leq T_{i-1}$, or G -hypercentral, abelian and Chernikov. This is much more complex than the positive characteristic case where G is (polycyclic-by-finite)-Engel if and only if G is (polycyclic-by-finite)-by-hypercentral.

Keywords Engel conditions · Linear group · Matrix group

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If $\mathbf{X}$ is a class of groups, say that a group G is $\mathbf{X}$ -Engel if for each $g \in G$ there is an $\mathbf{X}$ -subgroup E of G such that for all x in G there is a positive integer m such that for all $n \geq m$ we have $[x, {}_n g] \in E$. Let $\mathbf{F}$ denote the class of finite groups, $\mathbf{Ch}$ the class of Chernikov groups and $\mathbf{P}$ the class of polycyclic groups, so $\mathbf{PF}$ denotes the class of polycyclic-by-finite groups.

A special case of Shumyatsky's main theorem in [4] says that a linear group G is $\mathbf{F}$ -Engel if and only if G is finite-by-hypercentral. By Theorem Ch of [7] a linear group G is $\mathbf{Ch}$ -Engel if and only if G is Chernikov-by-hypercentral and by Theorem PF of [7] a linear group G of positive characteristic is $\mathbf{PF}$ -Engel if and only if G is $\mathbf{PF}$ -by-hypercentral ($\mathbf{Ch}$ is exactly the class of linear groups satisfying the minimal condition on subgroups and $\mathbf{PF}$ is exactly the class of linear groups satisfying the maximal condition on subgroups, so for linear groups $\mathbf{Ch}$ and $\mathbf{PF}$ are fully duals of each other).

The situation for linear $\mathbf{PF}$ -Engel groups of characteristic zero is much more complex and in particular the obvious analogue of the positive characteristic case is false, see Example 2 below. In this present paper we study in detail the characteristic zero case.

Thus let G be a $\mathbf{PF}$ -Engel subgroup of $\mathrm{GL}(n, F)$ for n some positive integer and F a field of characteristic 0. Then G is soluble-by-finite by Theorem PF of [7]. Let T be a triangularizable normal subgroup of G of finite index (which always exists, e.g. use [6] 4.2) and denote the unipotent radical of T by U . Enlarging F if necessary and replacing G by a

✉ B. A. F. Wehrfritz
b.a.f.wehrfritz@qmul.ac.uk

¹ Queen Mary University of London, London E1 4NS, England, UK

conjugate, we may assume that $T \leq \text{Tr}(n, F)$. Note that U is nilpotent, T/U is abelian and G/T is finite.

Theorem *With the hypotheses and notation above G has a normal series*

$$\langle 1 \rangle = U_0 \leq V_1 \leq U_1 \leq V_2 \leq U_2 \leq \cdots \leq V_t \leq U_t = U \leq N \leq T \leq G,$$

where $t = n(n-1)/2$ and for each $i \geq 1$,

U/U_{i-1} is central in U ,

$T/C_T(U/U_{i-1})$ is finitely generated,

V/U_{i-1} is finitely generated abelian,

U/V_i is G -hypercentral and

either U/U_{i-1} is T -central or U/V_i is divisible, abelian and Chernikov.

Further N is nilpotent of class $\max\{1, n-1\} \leq n$, T/N is finitely generated abelian, G/N is polycyclic-by-finite and there exists M a normal subgroup of G such that $U \leq M$, M/U is (finitely generated abelian)-by-finite and G/M is hypercentral.

Corollary *Suppose G is linear group of characteristic zero. Then G is **PF**-Engel if and only if G has a normal series $\langle 1 \rangle = T_0 \leq T_1 \leq \cdots \leq T_s = T \leq G$ with s and the index $(G:T)$ finite and each T_i/T_{i-1}*

either polycyclic-by-finite,

or G -hypercentral with $[T_i, T] \leq T_{i-1}$,

or G -hypercentral, abelian and Chernikov.

For if G is **PF**-Engel take the series $\langle 1 \rangle = U_0 \leq V_1 \leq \cdots \leq U_t \leq M \cap T \leq T \leq G$ of the Theorem with $s = 2t + 2$. For the converse see Proposition 2 below.

We give examples showing in particular that the group G in the theorem above need not be **PF**-by-hypercentral and, unlike the positive characteristic case (see [7]), need not be nilpotent-by-finite.

Below if G is any group $\{\zeta_i(G) : 0 \leq i \leq s\}$, s some ordinal, denotes the upper central series of G with $\zeta(G) = \bigcup_i \zeta_{i \leq s}(G)$ its hypercentre.

Proposition 1 *Let F be a field of characteristic 0, A a non-trivial subgroup of the multiplicative group F^* of F and B a non-zero subgroup of the additive group F^+ of F such that $BA = B$. Suppose the split extension G of B by A is **PF**-Engel. Then G is (finitely generated abelian)-by-hypercentral. Further A is finitely generated and B is minimax. In fact B contains a finitely generated subgroup E normal in G such that $\text{rank} E = \text{rank} B$ and B/E is Chernikov and G -hypercentral.*

Proof Suppose first that $A = \langle a \rangle$ is cyclic. By hypothesis there is a finitely generated (and abelian note) subgroup E of B such that for all b in B there exists an integer $m(b) \geq 1$ such that for all $n \geq m(b)$ we have $b(a-1)^n \in E$. Subject to this we choose our E of least possible rank. Set $E_1 = \langle b(a-1)^n : n \geq m(b), b \in B \rangle \leq E$. Then E_1 is also finitely generated, so we may assume that $E = E_1$. Clearly now $E(a-1) \leq E$.

Since $a \neq 1$, so $a-1 \in F^*$ is an (additive) automorphism of F^+ . Thus $E(a-1) \cong E$ and

$$\cdots \geq E(a-1)^{-i} \geq \cdots \geq E(a-1)^{-1} \geq E \geq E(a-1) \geq \cdots \geq E(a-1)^i \geq \cdots$$

where i runs over the positive integers. Set $E_2 = \bigcup_{i \geq 0} E(a-1)^{-i}$. If $b \in B$, then for some $n \geq 1$ we have $b(a-1)^n \in E$, so $b \in E(a-1)^{-n}$. Therefore $B \leq E_2$. Further $E(a-1) \leq E$ yields that $Ea \leq E$. Also $BA = B$, so $E_3 = EA = \bigcup_{i \geq 0} Ea^{-i} \leq B$. Suppose $b \in B$ with $ba \in E$. There exists $n \geq 1$ with $b(a-1)^n \in E$. Then $(-1)^nb \in E$. If actually $b \in E_3$, then $ba^m \in E$ for some $m \geq 1$, so $(ba^{m-1})a \in E$ and thus $ba^{m-1} \in E$. A trivial induction yields that $b \in E$. This shows that $E_3 = E$ and hence E is a normal subgroup of G . Clearly $B/E \leq E_2/E$ is A -hypercentral (recall here $A = \langle a \rangle$) and therefore G/E is a hypercentral group.

Since we chose E of least rank, so $E/E(a-1)$ finite. But $E(a-1)^{-1}/E(a-1)^{-1+1}$ is isomorphic to $E/E(a-1)$ for each i . Therefore B/E is a periodic π -group for π the finite set of prime divisors of the index $(E: E(a-1))$. Also E_2 and B have finite rank equal to the rank of E and in particular B/E has finite rank. Therefore B/E is Chernikov. We have now completed the proof of the case where A is cyclic.

Now assume A is finitely generated, say $A = \langle a_1, a_2, \dots, a_t \rangle$, where we choose the generators a_i non-zero and such that for each i , a_i^{-1} equals some a_j . By the cyclic case above B is minimax and for each i there is a finitely generated subgroup E_i of B such that $E_i a_i = E_i$, $E_i(a_i - 1) \leq E_i$. B/E_i is Chernikov and $\langle a_i \rangle$ -hypercentral and $\text{rank} E_i$ is minimal with these properties and in fact equals $\text{rank} B$.

Set $E_* = E_1 + E_2 + \dots + E_t$. Since t is finite, E_* is finitely generated and there exists an integer $m \geq 1$ such that for all i and all $n \geq m$ we have $E_*(a_i - 1)^n \leq E_i \leq E_*$. Set

$$E_{**} = \langle E_* a_1^{e(1)} a_2^{e(2)} \dots a_t^{e(t)} : 0 \leq e(i) < m, 1 \leq i \leq t \rangle.$$

Then E_{**} is finitely generated. Also that $E_*(a_i - 1)^m \leq E_*$ implies $E_* a_i^m \leq \langle E_* a_i^c : 0 \leq c < m \rangle$. Consequently $E_{**} a_i \leq E_{**}$ for each i . Further $E_{**} a_i^{-1} \leq E_{**}$ since $a_i^{-1} = a_j$ for some j . Therefore E_{**} is an A submodule of B .

Now $X = G/E_{**}$ is the split extension of B/E_{**} by A and A and B are abelian. If $b \in B$, then $b(a_i - 1)^s \in E_i \leq E_* \leq E_{**}$ for some $s \geq 1$. Thus the a_i are left Engel elements of X , as clearly so are the elements of B/E_{**} . Trivially X is abelian-by-polycyclic. Therefore X is hypercentral by Gruenberg's Theorems 1.2 and 1.4 in [2]. That is, G is (finitely generated abelian)-by-hypercentral. Finally B/E_1 is Chernikov, so B/E_{**} is too and $\text{rank} E_{**} \leq \text{rank} B = \text{rank} E_1 \leq \text{rank} E_{**}$, so $\text{rank} E_{**} = \text{rank} B$. This completes the proof of the A finitely-generated case.

Finally it remains to prove that A is always finitely generated. Let J be the subring of F generated by A . Then J is a domain and B is a faithful J -module. Also B is torsion-free and, by the A cyclic case, minimax. Hence B has a free abelian subgroup B_0 of rank $r = \text{rank} B$ with B/B_0 a periodic π -group for some finite set π of primes. Let t denote the product of all the elements of π . Then the tensor product $\mathbf{Z}[t^{-1}]B$ of $\mathbf{Z}[t^{-1}]$ by B over $\mathbf{Z}$ is equal to $\mathbf{Z}[t^{-1}]B_0$ and hence is isomorphic to $\mathbf{Z}[t^{-1}]^{(r)}$, the free $\mathbf{Z}[t^{-1}]$ -module of rank r . Therefore J embeds into the endomorphism ring of this module and hence into the matrix ring $\mathbf{Z}[t^{-1}]^{r \times r}$.

There exists a finitely generated subring $R \geq \mathbf{Z}[t^{-1}]$ of the complex numbers, an element x of $\text{GL}(r, R)$ and a subgroup A_0 of A of finite index such that $(A_0)^x \leq \text{Tr}(r, R)$. (We now regard $J \supseteq A$ as a subring of $R^{r \times r}$ via the obvious embeddings $J \leq \mathbf{Z}[t^{-1}]^{r \times r} \leq R^{r \times r}$.) If $a \in A_0$ with a^x unipotent, then $(a^x - 1)^f = 0$ in $R^{r \times r}$ and hence $(a - 1)^f = 0$ in the domain J . Consequently $a = 1$ and the unipotent radical of $(A_0)^x$ is trivial. It follows that A_0 is isomorphic to a subgroup of the full diagonal group $D(r, R)$ and the latter is isomorphic to the direct product of r copies of the group U of units of R . But R is a finitely generated domain so U is finitely generated (e.g. [3]). Therefore A_0 and A are also finitely generated. This completes the proof of Proposition 1.

Lemma 1 Let $A \leq T$ be normal subgroups of the **PF**-Engel group G such that A is abelian and Chernikov, $A \leq \zeta(T)$ and G/T is finite. Then there exists an integer $r \geq 1$ such that $A(r) = \{a \in A : a^r = 1\}$ is finite with $A/A(r) \leq \zeta_\omega(G/A(r))$.

Proof Let $g_1, g_2, \dots, g_t$ be a transversal of T to G . Clearly **PF** subgroups of A are finite. Hence there exists an integer $r \geq 1$ such that for each a in A and each i there exists $n \geq 0$ such that $[a, {}_n g_i] \in A(r)$. Now $A \leq \zeta(T)$ and $A = \bigcup_{j \geq 1} A(j)$, so there is a series

$$A(r) = A_0 < A_1 < \dots < A_i < \dots < A_\omega = A,$$

of length the first infinite ordinal ω of normal subgroups of G such that each A_i/A_{i-1} is finite and T -central. Now A_i/A_{i-1} consists of right Engel elements of the (finite) split extension of A_i/A_{i-1} by G/T . Therefore $A_i/A_{i-1} \leq \zeta(G/A_{i-1})$ and the claim of the lemma follows.

Lemma 2 Let $A \leq T$ be normal subgroups of the **PF**-Engel group G such that $[A, T] = \langle 1 \rangle$ and G/T is finite. Then there is a finitely generated subgroup B of A such that B is normal in G and $A/B \leq \zeta(G/B)$.

Note that necessarily A is abelian so B is necessarily polycyclic.

Proof Let $g_1, g_2, \dots, g_t$ be a transversal of T to G . For each i there exists a finitely generated subgroup E_i of A such that for all a in A there exists $m \geq 1$ with $[a, {}_n g_i] \in E_i$ for all $n \geq m$. Set

$$B = \langle E_i^g : g \in G, 1 \leq i \leq t \rangle = \langle E_i^g : g = g_1, \dots, g_t, 1 \leq i \leq t \rangle.$$

Clearly B is finitely generated and normal in G . Now A/B consists of right Engel elements of the split extension of A/B by G/T . Consequently $A/B \leq \zeta(G/B)$, e.g. [2] Theorems 1.2 and 1.4).

Proof of the Theorem There is a central series $\langle 1 \rangle = U_0 \leq U_1 \leq \dots \leq U_t = U$ of U and normal in T , where $t = n(n-1)/2$ and such that each U_i/U_{i-1} embeds into F^+ so that the action of T on U_i/U_{i-1} is given by a subgroup of F^* and a homomorphism of it onto $T/C_T(U_i/U_{i-1})$. Thus for each i Proposition 1 is applicable. Therefore we have

a normal series $\langle 1 \rangle = U_0 \leq V_1 \leq U_1 \leq V_2 \leq U_2 \leq \dots \leq V_t \leq U_t = U$ of T such that for each i

U_i/U_{i-1} is central in U ,

$T/C_T(U_i/U_{i-1})$ is finitely generated,

V_i/U_{i-1} is finitely generated abelian,

U_i/V_i is T -hypercentral and

either U_i/U_{i-1} is T -central or U_i/V_i is divisible, abelian and Chernikov.

Now G/T is finite. Replacing each subgroup in the above series by its normal closure in G produces a new series that preserves the conditions above. Let us keep the same notation for this new series. Thus we now have the series exactly as above but with the added bonus that the terms are all normal in G .

Suppose U_i/V_i is divisible, abelian, Chernikov and T -hypercentral. Then by Lemma 1 there exists $V_i \leq W_i \leq U_i$ with W_i normal in G , W_i/V_i finite and U_i/W_i G -hypercentral. Then W_i/U_{i-1} is finitely generated abelian. Replace V_i by W_i ; that is, assume $V_i = W_i$. If

U_i/U_{i-1} is T -central then by Lemma 2 there exists $U_{i-1} \leq W_i \leq U_i$ with W_i normal in G , W_i/U_{i-1} finitely generated and U_i/W_i G -hypercentral. Again set $V_i = W_i$.

Now $T/C_T(U_i/U_{i-1})$ is finitely generated for each i . Set $N = I_i C_T(U_i/U_{i-1})$. Then $T \geq N \geq U$, the group N is nilpotent ($U_0 \leq U_1 \leq \dots \leq U_t \leq N$ is a central series of N) and normal in G , T/N is finitely generated abelian and $G/N \in \mathbf{PF}$. Also U is the unipotent radical of N . Hence N embeds into $U^* \times D$, where U^* is a unipotent linear group of degree n and characteristic zero and D is abelian (diagonalizable in fact). Therefore $U^* \times D$ is nilpotent of class $\max\{1, n-1\}$ and consequently so is N .

Finally G/U is an abelian-by-finite $\mathbf{PF}$ -Engel group, so by Lemma 2 (with $A = T$) there is a normal subgroup M_0 of G with $U \leq M_0 \leq T$, M_0/U finitely generated abelian and $T/M_0 \leq \zeta(G/M_0)$. Then there exists $M \geq M_0$ normal in G with M/M_0 finite and G/M hypercentral, see [1].

Suppose $E \leq H$ are subgroups of a group G and $g \in G$. If for each h in H there exists a positive integer $m(h)$ such that $[h, {}_ng] \in E$ for all $n \geq m(h)$, for brevity say E is a g -sink in H . (This is a slight modification of the terminology used in [4].) If this is so, note that $E_0 = \langle [h, {}_ng] : n \geq m(h), h \in H \rangle \leq E$ is also a g -sink in H and it is easy to see that $[E_0, g] \leq E_0 \leq [H, g]$. (If $a, b \in E_0$ with $[a, g]$ and $[b, g]$ in E_0 , then $[ab, g] = [a, g]^b [b, g] \in E_0$ and $[a^{-1}, g] = a[a, g]^{-1}a^{-1} \in E_0$.)

Proposition 2 *Let $\langle 1 \rangle = T_0 \leq T_1 \leq \dots \leq T_s = T \leq G$ be a normal series of the group G with s and $(G: T)$ finite and each T_i/T_{i-1}*

either polycyclic-by-finite,

or G -hypercentral with $[T_i, T] \leq T_{i-1}$,

or G -hypercentral, abelian and Chernikov.

Then G is $\mathbf{PF}$ -Engel.

Proof We induct on s ; if $s = 0$ the claim is trivial. Suppose $s \geq 1$, let $g \in G$ and assume there exists $T_1 \leq X \leq G$ with X/T_1 in $\mathbf{PF}$ and a g -sink in G/T_1 . We choose X , as we may, with $[X, g] \leq X$, so also $X^g \leq X$. If T_1 is $\mathbf{PF}$, then X is $\mathbf{PF}$ and also a g -sink in G . From now on assume that T_1 is abelian.

There is a finitely generated subgroup K of X with $T_1 K = X$. If $K = \langle k_1, k_2, \dots, k_r \rangle$, then each $[k_i, g] \in u_i K$ for some u_i in T_1 . Set $R = \langle u_1, u_2, \dots, u_r \rangle^G$, a normal subgroup of G . If $x, y \in K$ with $[x, g]$ and $[y, g]$ both in RK , then

$[xy, g] = [x, g]^y [y, g] \in (RK)^y (RK) = RK$ and $[x^{-1}, g] = x[x, g]^{-1}x^{-1} \in RK$. Therefore $[K, g] \leq RK$ and hence $[RK, g] \leq RK$.

Now K is finitely generated, $K/(K \cap T_1) \in \mathbf{PF}$ and $K \cap T_1$ is abelian. By a theorem of P. Hall (e.g. 3.10 of [6]) $K \cap T_1$ is finitely K -generated. If $[T_1, T] = \langle 1 \rangle$, since $(G: T)$ is finite, so $K \cap T_1$ and R are finitely generated (as groups note, not just as normal subgroups). Therefore $RK \in \mathbf{PF}$. If T_1 is Chernikov (and abelian), then T_1 has an ascending series of normal subgroups of G of length at most the first infinite ordinal ω and with all its factors finite. Hence here $K \cap T_1$ and R are finite. Again we obtain that $RK \in \mathbf{PF}$.

Now assume T_1 is G -hypercentral. Thus let $\langle 1 \rangle = Z_0/R \leq Z_1/R \leq \dots \leq Z_\tau/R = T_1/R$ be an ascending central series of G in T_1/R , so all the Z_σ are normal in G . If $k \in K$, then $[k, g] \in RK$, see above. Let $u \in T_1 \setminus R$ with σ the least ordinal with $u \in Z_\sigma$. Then $\sigma-1$ exists and $[uk, g] = [u, g]^k [k, g] \in Z_{\sigma-1} RK = Z_{\sigma-1} K$. Consequently there exists a positive integer n with $[uk, {}_ng] \in RK$. We have now shown that if $T_1 \in \mathbf{PF}$, then X is $\mathbf{PF}$ and a

g-sink in G and if $T_1 \notin \mathbf{PF}$, the RK is $\mathbf{PF}$ and a g-sink in X and hence in G . Therefore G is $\mathbf{PF}$ -Engel.

Example 1 Let a be the companion matrix of the polynomial $X^2 - 5X + 1$ in $GL(2, \mathbf{Z})$. The split extension G_0 of $\mathbf{Q}^{(2)}$ by $\langle a \rangle$ embeds into $GL(3, \mathbf{Q})$ and also into $Tr(2, \mathbf{C})$. Set $E = \mathbf{Z}^{(2)} \leq \mathbf{Q}^{(2)}$, $B = \sum_{i \in \mathbf{Z}} E(a-1)^{-i}$ and $G = \langle a, B \rangle \leq G_0$. (Note that $\det(a-1) = -3$, so $a-1$ is invertible.) Then G is linear of characteristic zero and polycyclic-by-hypercentral and hence $\mathbf{PF}$ -Engel, but is not $\mathbf{PF}$, is not hypercentral and is not (locally nilpotent)-by-finite.

For $E \geq E(a-1)^{-i-1} \geq E(a-1)^{-i}$ for all i and hence $B = \bigcup_{i \geq 0} E(a-1)^{-i}$. Also $Ea = E = Ea^{-1}$, so $Ba = B = Ba^{-1}$; hence B is an $\langle a \rangle$ -submodule of $\mathbf{Q}^{(2)}$ and $G = \langle a \rangle B$. Now E is normal in G and, being finitely generated abelian, polycyclic. Clearly G/E is hypercentral. Thus in particular G is $\mathbf{PF}$ -Engel.

The eigenvalues of a have infinite order (as does a of course). Thus a , and also a^r for each $r \neq 0$, have no non-zero fixed points in $\mathbf{Q}^{(2)}$ and hence none in B . It follows that $\zeta_1(G) = \langle 1 \rangle = \zeta(G)$. In particular G is not hypercentral. If $G \in \mathbf{PF}$, then B is free abelian, necessarily of rank 2, and then $(B:E)$ is finite. Hence there exists i with $E(a-1)^{-i} = E(a-1)^{-i-1}$. But then $E = E(a-1)^{-1}$ and consequently $\det(a-1) = \pm 1$. This contradiction shows that G is not polycyclic-by-finite.

If G is (locally nilpotent)-by-finite, there exists $r > 0$ with $\langle a^r \rangle B$ locally nilpotent. Locally nilpotent linear groups are hypercentral ([5] 8.2). Hence a^r has a non-trivial fixed point in B . This contradiction shows that G is not (locally nilpotent)-by-finite. Finally if $\alpha = (5 + \sqrt{21})/2 \in \mathbf{C}$, then $\alpha^2 = 5\alpha - 1$. Set $E = \mathbf{Z} \oplus \mathbf{Z}\alpha \leq \mathbf{C}^*$. Then $E\alpha^{-i+1} \leq E\alpha^{-i}$ for all i and if $B = \bigcup_{i \geq 0} E(\alpha - 1)^{-i}$, then $B\alpha = B$ and the split extension of B by $\langle \alpha \rangle$ embeds into $Tr(2, \mathbf{C})$. It is also isomorphic to the group G above. Similarly the group G_0 also embeds into $Tr(2, \mathbf{C})$. The construction of Example 1 is complete.

Example 2 Let $a = \text{diag}(1, -1, 1)$, $T = Tr_1(3, \mathbf{Z}[1/2])$ and $G = \langle a \rangle T \leq Tr(3, \mathbf{Z}[1/2]) \leq GL(3, \mathbf{Q})$. Then G is $\mathbf{PF}$ -Engel, linear of characteristic zero, but not $\mathbf{PF}$ -by-hypercentral.

For brevity represent $(x_{ij}) \in T$ by the triple (x_{21}, x_{32}, x_{31}) . Let $x = (\xi, \eta, \zeta)$ and $y = (\lambda, \mu, \nu)$ be elements of T . Then an elementary calculation in G yields that

$$[y, ax] = (-2\lambda, -2\mu, 2\lambda\mu - \mu\xi + \eta\lambda),$$

and then a simple induction yields for all $d \geq 0$ that

$$[y, {}_{d+1}ax] = (\pm 2^{d+1}\lambda, \pm 2^{d+1}\mu, \pm 2^{2d+1}\lambda\mu \pm 2^d\mu\xi \pm 2^d\eta\lambda).$$

Thus for all large enough d we have $[y, {}_{d+1}ax] \in E = Tr_1(3, \mathbf{Z})$. Note that E is polycyclic, being finitely generated nilpotent. If $g \in G$, then $y = [g, ax]$ lies in T , as does $[g, x]$ and then $[g, {}_3x] = 1$. Consequently G is $\mathbf{PF}$ -Engel.

Let N be a normal, finitely generated subgroup of T . We claim that $N \leq Z = \zeta_1(T)$. Hence suppose $x = (\xi, \eta, \nu) \in N \setminus Z$, so at least one of ξ and η is non-zero. Since T is nilpotent $N \cap Z \neq \langle 1 \rangle$, so $N \cap Z$ is infinite cyclic and NZ/N is a Prüfer 2-group. If $\xi \neq 0$, then

$$[(0, 2^{-h}, 0), x] = (0, 0, 2^{-h}\xi),$$

which does not lie in $N \cap Z$ for large enough h . Also if $\eta \neq 0$, then

$$[x, (2^{-h}, 0, 0)] = (0, 0, 2^{-h}\eta),$$

which again does not lie in $N \cap Z$ for large enough h . Thus xZ and hence no non-trivial element of NZ/Z lies in the centre of the nilpotent group T/Z . Therefore no such x exists and $N \leq Z$ as claimed.

Finally, suppose M is a normal **PF** subgroup of G . If $ax \in M$, where $x \in T$, then $[G, ax] \leq M \cap T$ and clearly $[G, ax]Z = [G, a]Z = T$. But $M \in \mathbf{PF}$, so $M \cap T$ is finitely generated. Hence $M \cap T \leq Z$ by the previous paragraph and so $Z = T$, which is false. Therefore no such ax exists, $M \leq T$ and $M \leq Z$, again by the above. But then G/M contains sections that are infinite dihedral and consequently G/M cannot be hypercentral. This completes the proof that G is not **PF**-by-hypercentral.

Remark The fact that the degree is 3 in Example 2 is significant. If G is a **PF**-Engel subgroup of $GL(2, F)$, then G is always **PF**-by-hypercentral. For if G is irreducible, then G is abelian-by-finite and Lemma 2 and the theorem of [1] apply and if G is reducible, then G is triangularizable and the Theorem applies with $t = 1$.

Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest.

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