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On the Fourier transform of rotationally invariant distributions

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Abstract
We present an extension of the Poisson–Bochner formula for the Fourier transform of rotationally invariant distributions by analytic continuation "with respect to the dimension". As application of this extension, a new derivation of the fundamental solution of the Euler–Poisson–Darboux operator is given.

Keywords Fourier transform · Poisson–Bochner formula · Euler–Poisson–Darboux operator · Analytic continuation

Mathematics Subject Classification 42B10 · 42B37 · 33C10 · 46F10 · 35A08 · 35C05 · 35L05 · 35Q05

1 Introduction and notation

If $f \in L^1(\mathbb{R}^n)$ is rotationally invariant, i.e., if $f(x) = g(|x|)$ with $g(\rho)\rho^{n-1} \in L^1((0,\infty))$, then the classical Poisson–Bochner formula expresses the Fourier transform $\mathcal{F}f \in \mathcal{C}_0(\mathbb{R}^n)$ of f by the integral

$$(\mathcal{F}f)(\xi) = (2\pi)^{n/2}|\xi|^{-n/2+1}\int_0^\infty g(\rho)\rho^{n/2}J_{n/2-1}(\rho|\xi|)\,\mathrm{d}\rho, \qquad \xi \neq 0, \tag{1.1}$$

see [36, (VII, 7; 22), p. 259], [37, Thm. 3.10, p. 158]. Our naming "Poisson–Bochner formula" is motivated by the generalization of the formula (1.1) for dimensions $n = 2, 3$ (discovered by Poisson and Cauchy, see footnote 109 in [5, p. 226]) to dimensions $n \geq 4$ by S. Bochner in [5, Satz 56, p. 186].

A generalization for functions in weighted L^1-spaces, i.e., for $g \in L^1_{\mathrm{loc}}((0,\infty))$ fulfilling

$$\int_0^\infty |g(\rho)|\rho^{n-1}(1+\rho)^{(1-n)/2}\,\mathrm{d}\rho < \infty, \tag{1.2}$$

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is given in [33, Lemma 25.1, p. 358; Engl. transl. p. 485]. (In fact, if (1.2) is satisfied, then $\mathcal{F}f$ is continuous outside the origin and formula (1.1) is valid for $\xi \neq 0$ by applying Lebesgue's theorem on dominated convergence to $\lim_{N\to\infty}\langle \phi(\xi), \mathcal{F}(f(x)\cdot Y(N-|x|))\rangle$ for $\phi \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$.) A further generalization by means of partial integration can be found in [24,25], see also [31, Ex. 1.6.13 (a), p. 102]. A limit representation of the Fourier transform of radial temperate distributions is given in [26, (2.109), p. 140].

Let us mention that, e.g., the forward fundamental solution E of the wave operator $\partial_t^2 - \Delta_n$ is given by

$$E = (2\pi)^{-n} Y(t)\mathcal{F}_x\Big(\frac{\sin(t|x|)}{|x|}\Big),$$

see Examples 2.2, 2.4 below. In this case, for $t > 0$, $g(\rho) = (2\pi)^{-n}\rho^{-1}\sin(t\rho)$ and $f(x) = g(|x|)$ is not integrable nor does g satisfy condition (1.2). In order to calculate this important Fourier transform, different approximation methods were used, compare [13, pp. 177–183], [38, p. 51], [31, Ex. 1.6.17 (a), p. 106, Ex. 2.3.6, p. 141].

The main purpose of this paper consists in generalizing formula (1.1) so as to yield a representation of the Fourier transform $\mathcal{F}S$ for arbitrary radially symmetric temperate distributions S. This is done by analytic continuation with respect to the index $\lambda = \frac{n}{2} - 1$ of the Bessel function in (1.1), see Theorems 2.1, 2.3. So in a way, we use "analytic continuation with respect to the dimension n" of the underlying space $\mathbb{R}^n$. Heuristically, this procedure goes back, at least, to A. Weinstein, comp. [39, p. 44]: "The viewpoint of spaces of 'fractional dimensions' due to Weinstein is very fruitful and led to fundamental solutions in the large of the iterated EPD-equation."

In [11, p. 8], the Bochner transform T_n is defined by

$$T_n\varphi(r) = \frac{2\pi}{r^{n/2-1}}\int_0^\infty J_{n/2-1}(2\pi r\rho)\rho^{n/2}\varphi(\rho)\,\mathrm{d}\rho$$

for suitable functions φ and $n \in \mathbb{N}$. Whereas in [16] the connection between T_n and T_{n+2} is rederived, see [33, (25.14′), Lemma 25.1′, p. 359; Engl. transl. p. 486] and [32, p. 270], and in [12,27], the general connection between T_n and T_{n+q}, $q \in \mathbb{N}$, is investigated, the present study is concerned with the analytic continuation of the function $\lambda \mapsto T_\lambda$ for *complex* λ.

In order to illustrate our method, we first apply it to the wave equation (Examples 2.2, 2.4) and then, in Sect. 3, to the Euler–Poisson–Darboux equation. In Propositions 3.2, 3.3, we derive in this way the fundamental solution E of the EPD-operator. (For the concept of fundamental solutions of linear partial differential operators with non-constant coefficients, see [36, pp. 138–142], [23, p. 29], [9, pp. 11–14].) This more complicated fundamental solution was given in [10] and verified therein by series expansion, see [3,4] for a recapitulation. Our deduction of E based on the analytic continuation of the Poisson–Bochner formula is different from that in [3,4,10] and seems to be new, comp. [2, p. 478]: "We do not know how to obtain an explicit formula (or formulas) for the inverse Fourier transform of $\tilde{F}(\xi, y; b)$ when $b \neq 0$, a problem that merits to be investigated."

Let us introduce some notation. We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{E}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}$ of "test functions", of "rapidly decreasing functions" and of C^∞ functions, respectively, see [18,20,36]. In order to display the active variable in a distribution, say $x \in \mathbb{R}^n$, we use notation as $T(x)$ or $T \in \mathcal{D}'(\mathbb{R}^n_x)$. Furthermore, we use the spaces $\mathcal{D}_{L^p}$, $\mathcal{D}'_{L^p}$, $1 \le p \le \infty$, $\mathcal{O}_M$, $\mathcal{O}'_C$, which were introduced in [36, Ch. VI, § 8, p. 199; Ch. VII, § 5, p. 243], and we set $\mathcal{S}'_r(\mathbb{R}^n) = \{S \in \mathcal{S}'(\mathbb{R}^n);\ S \text{ is radially symmetric}\}$.

For the evaluation of a distribution $T \in E'$ on a test function $\phi \in E$, we use angle brackets, i.e., $\langle \phi, T\rangle$ or, more precisely ${}_E\langle \phi, T\rangle_{E'}$. More generally, if $\phi \in E\hat{\otimes}F$ and $T \in E'$

for distribution spaces E, F, then ${}_{E\,\hat{\otimes}\,F}\langle\phi, T\rangle_{E'}$ symbolizes the vector-valued scalar product $(E \hat{\otimes} F) \times E' \to F$, see [34,35] for more information on vector-valued distributions. (In all tensor products of this study, both factors are complete and at least one of the factors is nuclear and hence $E \hat{\otimes}_\pi F = E \hat{\otimes}_\epsilon F$ and we simply write $E \hat{\otimes} F$.)

The Heaviside function is denoted by Y, see [36, p. 36], and we set

$$\chi^\mu(t) = \frac{Y(t)t^\mu}{\Gamma(\mu+1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^1_t) \quad \text{for } \mu \in \mathbb{C} \text{ with } \operatorname{Re}\mu > -1. \tag{1.3}$$

The function $\mu \mapsto \chi^\mu$ can be analytically continued in $\mathcal{S}'(\mathbb{R}^1)$ and thus yields an entire function

$$\chi : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^1) : \mu \longmapsto \chi^\mu,$$

see [18, (3.2.17), p. 73]. We write $\delta_\tau(t) \in \mathcal{D}'(\mathbb{R}^1_t),\ \tau \in \mathbb{R}$, for the delta distribution with support in τ, which is the derivative of $Y(t-\tau)$, i.e., $\langle\phi, \delta_\tau\rangle = \phi(\tau)$ for $\phi \in \mathcal{D}(\mathbb{R}^1)$. In contrast, $\delta \in \mathcal{D}'(\mathbb{R}^n)$ without any subscript stands for the delta distribution at the origin.

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive $\mathcal{C}^\infty$ function $h : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^n$ open, is defined as in [14, (7.2.4/5), p. 82] or in [31, Def. 1.2.12, p. 19], i.e.,

$$\langle\phi, h^*T\rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega Y(t-h(x))\phi(x)\,\mathrm{d}x\Big), T\Big\rangle, \qquad \phi \in \mathcal{D}(\Omega). \tag{1.4}$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int \mathrm{e}^{-\mathrm{i}\xi x}\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{S}(\mathbb{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. We write $|\mathbb{S}^{n-1}|$ for the hypersurface area $2\pi^{n/2}/\Gamma(\frac{n}{2})$ of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For $j \in \mathbb{N}$ and $w \in \mathbb{C}$, we use Pochhammer's symbol $(w)_0 = 1$, $(w)_j = w \cdot (w+1) \cdot \dots \cdot (w+j-1)$. J_λ and N_λ denote, as usual, the Bessel functions of the first and of the second kind.

2 Analytic continuation of the Poisson–Bochner formula

Let us first rewrite (1.1) in a more symmetrical fashion by the following n-dimensional integral, still under the assumptions that $f \in L^1(\mathbb{R}^n)$ and f is radially symmetric:

$$(\mathcal{F}f)(\xi) = 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right)\int_{\mathbb{R}^n} f(x)\frac{J_{n/2-1}(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^{n/2-1}}\,\mathrm{d}x. \tag{2.1}$$

We note incidentally that formula (2.1) allows a generalization (which follows, e.g., by density) for $S \in \mathcal{D}'_{L^1}(\mathbb{R}^n) \cap \mathcal{S}'_r(\mathbb{R}^n)$, i.e., for radially symmetric integrable distributions S. Then $\mathcal{F}S$ is a continuous function given by

$$(\mathcal{F}S)(\xi) = 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \cdot {}_{\mathcal{D}'_{L^1}(\mathbb{R}^n_x)}\Big\langle S(x), \frac{J_{n/2-1}(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^{n/2-1}}\Big\rangle_{\mathcal{D}_{L^\infty}(\mathbb{R}^n_x)}, \qquad \xi \in \mathbb{R}^n. \tag{2.2}$$

As can be derived from [17, p. 538], the kernel

$$K(x,\xi) = \frac{J_{n/2-1}(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^{n/2-1}} \in \mathcal{O}_M(\mathbb{R}^{2n})$$

belongs to the completed tensor product $\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$, and therefore the Fourier transform of $S \in \mathcal{S}'_r(\mathbb{R}^n)$ can be written in the form

$$(\mathcal{F}S)(\xi) = 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \cdot {}_{\mathcal{S}'(\mathbb{R}^n_x)}\langle S(x), K(x,\xi)\rangle_{\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)} \tag{2.3}$$

(by applying [35, Prop. 4, p. 41]).

Note that formula (2.3) allows to represent $\mathcal{F}_x(|x|^{-1}\sin(t|x|))$ by the $\mathcal{S}'$-valued scalar product

$$\mathcal{F}_x\Big(\frac{\sin(t|x|)}{|x|}\Big)(\xi) = 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \cdot {}_{\mathcal{S}'(\mathbb{R}^n_x)}\left\langle \frac{\sin(t|x|)}{|x|}, \frac{J_{n/2-1}(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^{n/2-1}}\right\rangle_{\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)}. \tag{2.4}$$

However, formula (2.4) cannot be evaluated for fixed ξ. In the following two theorems, we shall therefore imbed the kernel $K(x,\xi)$ into an analytic family of kernels $K_\lambda(x,\xi)$ such that $\mathcal{F}S$, $S \in \mathcal{S}'_r(\mathbb{R}^n)$, can be obtained by analytic continuation with respect to λ. Let us mention that

$$\frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda} \in \mathcal{O}_M(\mathbb{R}^{2n})$$

depends holomorphically on $\lambda \in \mathbb{C}$ (see below), but that these kernels do not belong to $\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$ and not even to $\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$ for $\lambda \in \mathbb{C}\backslash(\frac{n}{2}-\mathbb{N})$. This is the reason for the more complicated choices of K_λ below.

Theorem 2.1 *The kernel*

$$K_\lambda(x,\xi) = |\xi|^{2\lambda-n+2} \cdot \frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda} \in \mathcal{S}'(\mathbb{R}^{2n}_{x,\xi})$$

is an entire function of λ with values in $\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$. Furthermore, if $S \in \mathcal{S}'_r(\mathbb{R}^n)$, then

$$\mathcal{F}S = 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \cdot U_{n/2-1}$$

where the function

$$U : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \longmapsto U_\lambda(\xi) = {}_{\mathcal{S}'(\mathbb{R}^n_x)}\langle S(x), K_\lambda(x,\xi)\rangle_{\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)} \tag{2.5}$$

is entire.

Proof (a) Let us first show that the mapping

$$\mathbb{C} \longrightarrow \mathcal{O}_M(\mathbb{R}^{2n}_{x,\xi}) : \lambda \longmapsto \frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda} \tag{2.6}$$

is entire. From [15, 8.411.8] and using analytic continuation, we obtain the representation

$$\frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda} = \frac{1}{2^\lambda\sqrt{\pi}}\, {}_{\mathcal{E}(\mathbb{R}^1_t)}\langle \cos(t|x|\cdot|\xi|), \chi^{-1/2+\lambda}(1-t^2)\rangle_{\mathcal{E}'(\mathbb{R}^1_t)}$$

for each $(x,\xi) \in \mathbb{R}^{2n}$. (For χ^μ see Sect. 1, in particular (1.3).) Since

$$\cos(t|x|\cdot|\xi|) \in \mathcal{O}_M(\mathbb{R}^{2n+1}_{t,x,\xi}) = \mathcal{O}_M(\mathbb{R}^1_t)\,\hat{\otimes}\,\mathcal{O}_M(\mathbb{R}^{2n}_{x,\xi}),$$

see [34, Prop. 28, p. 98], and since

$$\mathbb{C} \longrightarrow \mathcal{E}'(\mathbb{R}^1_t) : \lambda \longmapsto \chi^{-1/2+\lambda}(1-t^2)$$

is entire and $\mathcal{E}' \subset \mathcal{O}'_M$, we conclude that also the mapping in (2.6) is entire, see [35, Prop. 4, p. 41].

(b) The distribution-valued function

$$F : \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > -1\} \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \longmapsto F_\lambda = |\xi|^{2\lambda-n+2} \in L^1_{\mathrm{loc}}(\mathbb{R}^n_\xi)$$

can analytically be continued to $\mathbb{C}\setminus(-\mathbb{N})$ and has simple poles for $\lambda = -k$, $k \in \mathbb{N}$, with the residues

$$\operatorname{Res}_{\lambda=-k} F_\lambda = \frac{\pi^{n/2}\Delta_n^{k-1}\delta}{2^{2(k-1)}(k-1)!\,\Gamma\left(\frac{n}{2}+k-1\right)},$$

see [30, Ex. 2.3.1, p. 41]. Therefore, the product

$$K_\lambda(x,\xi) = F_\lambda(\xi)\cdot\frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda} \in \mathcal{S}'(\mathbb{R}^n_\xi)\cdot\mathcal{O}_M(\mathbb{R}^{2n}_{x,\xi})$$

is well-defined and belongs to $\mathcal{O}_M(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$ for $\lambda \in \mathbb{C}\setminus(-\mathbb{N})$, see [34, Props. 20 bis, 28, pp. 70, 98]. Furthermore, $K_\lambda(x,\xi)$ is also holomorphic in $\lambda = -k \in -\mathbb{N}$ since

$$\frac{J_{-k}(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^{-k}} = (-1)^k(|x|\cdot|\xi|)^k\cdot J_k(|x|\cdot|\xi|)$$

vanishes of order $2k$ at $\xi = 0$ and hence its product with $\operatorname{Res}_{\lambda=-k} F_\lambda(\xi)$ vanishes.

(c) Let us next calculate the partial Fourier transform $\mathcal{F}_\xi(K_\lambda(x,\xi))$. Since the kernel $K_\lambda(x,\xi)$ belongs to $\mathcal{O}_M(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$, it is semi-regular in x, and it suffices to determine $\mathcal{F}_\xi(K_\lambda(x,\xi))$ for fixed $x \neq 0$. If $x \neq 0$, then $K_\lambda(x,\xi)$ is bounded by a constant times $|\xi|^{3/2-n+\operatorname{Re}\lambda}$ for $|\xi| \to \infty$. This implies that condition (1.2) is satisfied upon setting $g(\rho) = |x|^{-\lambda}\rho^{\lambda-n+2}J_\lambda(\rho|x|)$ and that $K_\lambda(x,\xi) \in L^1(\mathbb{R}^n_\xi) + L^2(\mathbb{R}^n_\xi)$ if $n \geq 3$ and $-1 < \operatorname{Re}\lambda < \frac{n}{2}-2$. Therefore $\mathcal{F}_\xi K_\lambda$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^n_\xi)$ for such x and λ and the Poisson–Bochner formula applies and represents $\mathcal{F}_\xi K_\lambda$ by the absolutely convergent integral in (1.1) for $\xi \neq 0$. For $x \neq 0$, $\xi \neq 0$ and λ as above, formula 6.575.1, p. 692, in [15] then yields

$$\begin{aligned}(\mathcal{F}_\xi K_\lambda)(x,\xi) &= (2\pi)^{n/2}|x|^{-\lambda}|\xi|^{-n/2+1}\int_0^\infty J_{n/2-1}(\rho|\xi|)J_\lambda(\rho|x|)\rho^{\lambda-(n/2-2)}\,\mathrm{d}\rho\\ &= 2^{\lambda+2}\pi^{n/2}|\xi|^{2-n}\chi^{n/2-\lambda-2}(|\xi|^2-|x|^2).\end{aligned}$$

(The correct parameter range for formula 6.575.1 in [15] is $-1 < \operatorname{Re}\mu < \operatorname{Re}(\nu+1)$.)

For $x \neq 0$, the function $h(\xi) = |\xi|^2 - |x|^2$ is submersive and hence the composition $\chi^{n/2-\lambda-2}(|\xi|^2-|x|^2) \in \mathcal{S}'(\mathbb{R}^n_\xi)$ is well-defined, see (1.4), and it is an entire function of λ. Since $\chi^{n/2-\lambda-2}(|\xi|^2-|x|^2)$ vanishes at $\xi = 0$, the product $|\xi|^{2-n}\chi^{n/2-\lambda-2}(|\xi|^2-|x|^2)$ is also well-defined and depends holomorphically on λ in $\mathcal{S}'(\mathbb{R}^n_\xi)$. By analytic continuation, we conclude that $\mathcal{F}_\xi K_\lambda$ is represented by the continuous function $\mathbb{R}^n_x \to \mathcal{S}'(\mathbb{R}^n_\xi)$ which, for $x \neq 0$, is given by the equation

$$\mathcal{F}_\xi K_\lambda = 2^{\lambda+2}\pi^{n/2}|\xi|^{2-n}\chi^{n/2-\lambda-2}(|\xi|^2-|x|^2). \tag{2.7}$$

If $n = 2$ or $n = 1$, then the same conclusion can be reached by proving (2.7) for $\operatorname{Re}\lambda < -\frac{3}{2}$ with the help of formula (2.2). (Note that $K_\lambda(x,\xi) \in \mathcal{D}'_{L^1}(\mathbb{R}^n_\xi)$ for $\operatorname{Re}\lambda < -\frac{3}{2}$ and fixed $x \in \mathbb{R}^n\setminus\{0\}$.) Hence (2.7) is valid for $n \in \mathbb{N}$, $x \neq 0$ and each $\lambda \in \mathbb{C}$.

(d) If $\phi \in \mathcal{S}(\mathbb{R}^n_\xi)$, then

$$\psi(x) = \langle\phi(\xi), \mathcal{F}_\xi K_\lambda\rangle = {}_{\mathcal{S}(\mathbb{R}^n_\xi)}\langle(\mathcal{F}\phi)(\xi), K_\lambda(x,\xi)\rangle_{\mathcal{O}_M(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)}$$

belongs to $\mathcal{O}_M(\mathbb{R}^n_x)$ and (2.7) implies, for $x \neq 0$,

$$\begin{aligned}\psi(x) = \langle \phi(\xi), \mathcal{F}_\xi K_\lambda \rangle &= 2^{\lambda+2}\pi^{n/2} \big\langle \phi(\xi), |\xi|^{2-n}\chi^{n/2-\lambda-2}(|\xi|^2 - |x|^2)\big\rangle \\ &= 2^{\lambda+2}\pi^{n/2}|x|^{n-2\lambda-2}\big\langle \phi(|x|\eta), |\eta|^{2-n}\chi^{n/2-\lambda-2}(|\eta|^2-1)\big\rangle .\end{aligned}$$

Because the support of the distribution $|\eta|^{2-n}\chi^{n/2-\lambda-2}(|\eta|^2-1)$ does not contain the origin $\eta = 0$, we conclude that ψ is, with all its derivatives, rapidly decreasing for $|x| \to \infty$ and hence $\psi \in \mathcal{S}(\mathbb{R}^n)$. This means that $\mathcal{F}_\xi K_\lambda$ and thus also K_λ belong to $\mathcal{S}(\mathbb{R}^n_x) \hat{\otimes} \mathcal{S}'(\mathbb{R}^n_\xi)$.

(e) We observe that $K_{n/2-1}(x,\xi) = J_{n/2-1}(|x|\cdot|\xi|)/(|x|\cdot|\xi|)^{n/2-1}$ and hence (2.3) implies $\mathcal{F}S = 2^{n/2-1}\Gamma(\frac{n}{2})U_{n/2-1}$. Finally, the map $\lambda \mapsto U_\lambda$ in (2.5) is analytic by [35, Prop. 4, p. 41]. This completes the proof. □

Example 2.2 Let us illustrate Theorem 2.1 by calculating the forward fundamental solution E of the wave operator $\partial_t^2 - \Delta_n$. We consider $E \in \mathcal{S}'(\mathbb{R}^{n+1}_{t,x})$ as the $\mathcal{C}^\infty$ mapping

$$E : [0,\infty) \longrightarrow \mathcal{S}'(\mathbb{R}^n_x) : t \longmapsto E_t(x)$$

and represent E_t by partial Fourier transform, i.e.,

$$E_t = (2\pi)^{-n}\mathcal{F}_x\Big(\frac{\sin(t|x|)}{|x|}\Big), \qquad t \geq 0,$$

compare [31, Ex. 1.6.17, p. 106].

(a) The distribution-valued function U_λ in (2.5) corresponding to $S = \sin(t|x|)/|x|$, $t > 0$ fixed, is given by

$$U_\lambda(\xi) = {}_{\mathcal{D}_{L^\infty}(\mathbb{R}^n_x)}\langle 1(x), \sin(t|x|)|\xi|^{\lambda-n+2}|x|^{-\lambda-1}J_\lambda(|x|\cdot|\xi|)\rangle_{\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_\xi)}.$$

If $\operatorname{Re}\lambda > n-1$, then

$$\xi \mapsto \sin(t|x|)|\xi|^{\lambda-n+2}|x|^{-\lambda-1}J_\lambda(|x|\cdot|\xi|)$$

is a continuous function with values in $L^1(\mathbb{R}^n_x)$. Under this assumption on λ, we therefore obtain that $U_\lambda \in \mathcal{C}(\mathbb{R}^n_\xi)$ is given by

$$U_\lambda(\xi) = |\mathbb{S}^{n-1}|\cdot|\xi|^{\lambda-n+2}\int_0^\infty \rho^{n-\lambda-2}\sin(t\rho)J_\lambda(\rho|\xi|)\,\mathrm{d}\rho. \tag{2.8}$$

(b) If $n \geq 3$ is *odd*, then [29, I, 13.11, p. 67] yields the following for $\operatorname{Re}\lambda > n-1$:

$$\begin{aligned}U_\lambda(\xi) &= |\mathbb{S}^{n-1}|\cdot(-1)^{(n-1)/2}|\xi|^{\lambda-n+2}\Big(\frac{\partial}{\partial t}\Big)^{n-2}\int_0^\infty \rho^{-\lambda}\cos(t\rho)J_\lambda(\rho|\xi|)\,\mathrm{d}\rho \\ &= |\mathbb{S}^{n-1}|\cdot(-1)^{(n-1)/2}|\xi|^{\lambda-n+2}\Big(\frac{\partial}{\partial t}\Big)^{n-2}\Big[\frac{\sqrt{\pi}}{(2|\xi|)^\lambda}\cdot\chi^{-1/2+\lambda}(|\xi|^2-t^2)\Big].\end{aligned}$$

The distributions $\chi^{-1/2+\lambda}(|\xi|^2-t^2) \in \mathcal{S}'(\mathbb{R}^n_\xi)$ depend $\mathcal{C}^\infty$ on $t > 0$ and hence the last formula holds by analytic continuation for each $\lambda \in \mathbb{C}$ and $t > 0$. This implies

$$E = (2\pi)^{-n} Y(t) \mathcal{F}_x\Big(\frac{\sin(t|x|)}{|x|}\Big) = (2\pi)^{-n} Y(t) \cdot 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \cdot U_{n/2-1}(x)$$
$$= \frac{Y(t)}{\Gamma\left(\frac{n-1}{2}\right)(-4\pi)^{(n-1)/2}}\, \partial_t^{n-2}\big[Y(|x|-t)|x|^{2-n}(|x|^2-t^2)^{(n-3)/2}\big]$$
$$= \frac{1}{(n-2)!\,|\mathbb{S}^{n-1}|}\, \partial_t^{n-2}\big[Y(t-|x|)|x|^{2-n}(t^2-|x|^2)^{(n-3)/2}\big]$$

in accordance with [31, Lemma 3.3.5, p. 218 (for $k=1$)].

(c) For *even* n and $\operatorname{Re}\lambda > n-1$ we obtain the following from (2.8):

$$U_\lambda(\xi) = |\mathbb{S}^{n-1}| \cdot (-1)^{n/2-1}|\xi|^{\lambda-n+2}\Big(\frac{\partial}{\partial t}\Big)^{n-2}\int_0^\infty \rho^{-\lambda}\sin(t\rho)J_\lambda(\rho|\xi|)\,\mathrm{d}\rho. \tag{2.9}$$

The integral in (2.9) is absolutely convergent for $\operatorname{Re}\lambda > 1$ and yields a continuous function of t and ξ depending analytically on λ. However, this integral is more complicated than the one in the case of odd n (see [15, Eq. 6.699.1, p. 747]), and we proceed differently. By applying once more [35, Prop. 4, p. 41], we obtain from Theorem 2.1 also a formula for the partial Fourier transform, i.e., the distribution-valued function

$$U : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^{m+n}_{t,\xi}) : \lambda \longmapsto U_\lambda(t,\xi) = {}_{\mathcal{S}'(\mathbb{R}^m_t)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_x)}\langle S(t,x), K_\lambda(x,\xi)\rangle_{\mathcal{S}(\mathbb{R}^n_x)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_\xi)}$$

is entire for $S(t,x) \in \mathcal{S}'(\mathbb{R}^{m+n}_{t,x})$ and $(\mathcal{F}_x S)(t,\xi) = 2^{n/2-1}\Gamma(\frac{n}{2}) \cdot U_{n/2-1}$ if S is radially symmetric with respect to x. Hence, putting $m=1$, $S(t,x) = (2\pi)^{-n}Y(t)\sin(t|x|)/|x|$ and assuming n even with $n \geq 6$ we can insert $\lambda = \frac{n}{2}-1$ into (2.9), and we obtain by analytic continuation

$$E = (2\pi)^{-n/2}Y(t)\cdot\partial_t^{n-2}\Big[|\xi|^{-n/2+1}\int_0^\infty \rho^{-n/2+1}J_{-n/2+1}(\rho|\xi|)\sin(t\rho)\,\mathrm{d}\rho\Big].$$

(Note that $J_{-k}(s) = (-1)^k J_k(s)$ for $k \in \mathbb{N}$ and $s \in \mathbb{R}$. Let us also mention that the last formula can be deduced as well for $n=2$ or $n=4$ upon using a further differentiation with respect to t.)

In order to evaluate the last integral, let us assume $\xi \neq 0$ and consider the analytic distribution-valued function

$$T : \big\{\nu \in \mathbb{C};\ \operatorname{Re}\nu > -\tfrac{1}{2}\big\} \longrightarrow \mathcal{S}'(\mathbb{R}^1_\rho) : \nu \longmapsto T_\nu(\rho) = Y(\rho)\rho^\nu J_\nu(\rho|\xi|).$$

By means of the series expansion of the Bessel function, we infer that T can be analytically continued to $\mathbb{C}\backslash(-\frac{1}{2}-\mathbb{N}_0)$ having simple poles in $-\frac{1}{2}-\mathbb{N}_0$ (see also [22, 2.4, p. 193]). Furthermore,

$$T : \big\{\nu \in \mathbb{C};\ \operatorname{Re}\nu < -\tfrac{1}{2},\ \nu \notin -\tfrac{1}{2}-\mathbb{N}\big\} \longrightarrow \mathcal{D}'_{L^1}(\mathbb{R}^1_\rho)$$

is also well-defined and analytic. A classical formula (see [29, II, 13.9, p. 164]) furnishes

$$\int_0^\infty \rho^\nu J_\nu(\rho|\xi|)\sin(t\rho)\,\mathrm{d}\rho = \sqrt{\pi}(2|\xi|)^\nu \chi^{-1/2-\nu}(t^2-|\xi|^2) \tag{2.10}$$

if $-1 < \operatorname{Re}\nu < -\frac{1}{2}$. By analytic continuation we deduce from (2.10)

$$\begin{aligned}\int_0^\infty &\rho^{-n/2+1}J_{-n/2+1}(\rho|\xi|)\sin(t\rho)\,\mathrm{d}\rho\\ &= {}_{\mathcal{D}_{L^\infty}(\mathbb{R}^1_\rho)}\langle\sin(t\rho), \rho^{-n/2+1}J_{-n/2+1}(\rho|\xi|)\rangle_{\mathcal{D}'_{L^1}(\mathbb{R}^1_\rho)}\\ &= \sqrt{\pi}(2|\xi|)^{-n/2+1}\chi^{(n-3)/2}(t^2-|\xi|^2),\end{aligned}$$

and this yields

$$E=\frac{1}{(n-2)!\,|\mathbb{S}^{n-1}|}\,\partial_t^{n-2}\big[Y(t-|x|)|x|^{2-n}(t^2-|x|^2)^{(n-3)/2}\big] \tag{2.11}$$

exactly as in the case of odd n. Let us mention that a unified deduction of (2.11) independent of the parity of n is given in [31, Lemma 3.3.5, p. 218].

For some examples (and in particular for the application to the EPD-operator in Sect. 3), we need a different family of kernels $\tilde{K}_\lambda$. Of course, they are chosen such that $\tilde{K}_{n/2-1}$ again coincides with $J_{n/2-1}(|x|\cdot|\xi|)/(|x|\cdot|\xi|)^{n/2-1}$.

Theorem 2.3 *Let K_λ be defined as in* Theorem 2.1 *and assume $\lambda\in\mathbb{C}\backslash(-\mathbb{N})$. Then the kernel*

$$\tilde{K}_\lambda(x,\xi)=|x|^{2\lambda-n+2}\cdot K_\lambda(x,\xi)=(|x|\cdot|\xi|)^{\lambda-n+2}J_\lambda(|x|\cdot|\xi|)$$

belongs to $\mathcal{O}'_C(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$ and it depends therein holomorphically on $\lambda\in\mathbb{C}\backslash(-\mathbb{N})$.

If $S\in\mathcal{S}'_r(\mathbb{R}^n)$ and S is $\mathcal{C}^\infty$ in a neighborhood of 0, then $S(x)\cdot\tilde{K}_\lambda(x,\xi)$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$ and depends therein holomorphically on $\lambda\in\mathbb{C}\backslash(-\mathbb{N})$. Finally,

$$\mathcal{F}S=2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right)\cdot\tilde{U}_{n/2-1}$$

where the function

$$\tilde{U}:\mathbb{C}\backslash(-\mathbb{N})\to\mathcal{S}'(\mathbb{R}^n_\xi):\lambda\mapsto\tilde{U}_\lambda(\xi)={}_{\mathcal{D}_{L^\infty}}\langle 1(x),S(x)\cdot\tilde{K}_\lambda(x,\xi)\rangle_{\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)} \tag{2.12}$$

is holomorphic.

Proof From Theorem 2.1, we infer then

$$\tilde{K}_\lambda(x,\xi)=|x|^{2\lambda-n+2}\cdot K_\lambda(x,\xi)\in\mathcal{S}'(\mathbb{R}^n_x)\cdot\big(\mathcal{S}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)\big)\subset\mathcal{O}'_C(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)$$

due to $\mathcal{S}\cdot\mathcal{S}'\subset\mathcal{O}'_C$ and [34, Prop. 20 bis, p. 70]. Furthermore, $\tilde{K}_\lambda$ depends holomorphically on $\lambda\in\mathbb{C}\backslash(-\mathbb{N})$ since the same holds for $|x|^{2\lambda-n+2}$. (One can show that $\tilde{K}_\lambda$ has simple poles at $\lambda=-k$, $k\in\mathbb{N}$. E.g., it holds $\operatorname{Res}_{\lambda=-1}\tilde{K}_\lambda=\frac{1}{2}|\mathbb{S}^{n-1}|^2\cdot\delta(x,\xi)$.)

Analogously, also the distribution-valued function

$$\begin{aligned}\lambda\mapsto S(x)\cdot\tilde{K}_\lambda(x,\xi)=(S(x)|x|^{2\lambda-n+2})\cdot K_\lambda(x,\xi)&\in\mathcal{O}'_C(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)\\&\subset\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)\end{aligned}$$

is holomorphic on $\mathbb{C}\backslash(-\mathbb{N})$. Finally, $\mathcal{F}S=2^{n/2-1}\Gamma(\frac{n}{2})\cdot\tilde{U}_{n/2-1}$ since $\tilde{K}_{n/2-1}=K_{n/2-1}$. This completes the proof. □

Example 2.4 Let us illustrate the difference of the representations for $\mathcal{F}S$ in Theorem 2.1 and Theorem 2.3, respectively, by considering again the forward fundamental solution E of the wave operator $\partial_t^2-\Delta_n$. If $S=\sin(t|x|)/|x|$, $t>0$ fixed, then (2.12) in Theorem 2.3 yields

$$\tilde{U}_\lambda(\xi)={}_{\mathcal{D}_{L^\infty}(\mathbb{R}^n_x)}\langle 1(x),\sin(t|x|)|\xi|^{\lambda-n+2}|x|^{\lambda-n+1}J_\lambda(|x|\cdot|\xi|)\rangle_{\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)}.$$

For $-1<\operatorname{Re}\lambda<-\frac{1}{2}$ and $|\xi|\le N$, $N\in\mathbb{N}$, the moduli of the functions $f_\xi(x)=\sin(t|x|)|\xi|^{-\lambda}|x|^{\lambda-n+1}J_\lambda(|x|\cdot|\xi|)$ are bounded, independently of ξ, by the integrable function

$$g_{\lambda,N}(x) = C_\lambda |\sin(t|x|)| \cdot |x|^{2\mathrm{Re}\,\lambda-n+1} \cdot (1+N|x|)^{-1/2-\mathrm{Re}\,\lambda} \in L^1(\mathbb{R}^n_x),$$
$$C_\lambda = \|u^{-\lambda}(1+u)^{1/2+\lambda} J_\lambda(u)\|_{L^\infty((0,\infty))}.$$

Therefore, $\tilde{U}_\lambda(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n_\xi)$ for $-1 < \mathrm{Re}\,\lambda < -\frac{1}{2}$, and $\tilde{U}_\lambda(\xi)$ is given, for $\xi \neq 0$, by the absolutely convergent integral

$$\tilde{U}_\lambda(\xi) = |\mathbb{S}^{n-1}| \cdot |\xi|^{\lambda-n+2} \int_0^\infty \rho^\lambda J_\lambda(\rho|\xi|) \sin(t\rho)\,\mathrm{d}\rho$$

that we have encountered already in (2.10). By analytic continuation, we thus obtain

$$\tilde{U}_\lambda(\xi) = \frac{2^{\lambda+1}\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2}\right)} |\xi|^{2\lambda-n+2} \cdot \chi^{-1/2-\lambda}(t^2 - |\xi|^2), \qquad \lambda \in \mathbb{C}\backslash(-\mathbb{N}).$$

Hence we deduce from Theorem 2.3 the following expression for the forward fundamental solution E of $\partial_t^2 - \Delta_n$:

$$E = (2\pi)^{-n} Y(t) 2^{n/2-1}\Gamma\left(\tfrac{n}{2}\right) \tilde{U}_{n/2-1}(x) = \tfrac{1}{2}\pi^{(1-n)/2} Y(t) \chi^{(1-n)/2}(t^2 - |x|^2). \quad (2.13)$$

(As said above, we interpret E as a continuous function of t with values in $\mathcal{S}'(\mathbb{R}^n_x)$ and vanishing for $t \le 0$. Furthermore, for $t > 0$, the composition h^*T of $T = \chi^{(1-n)/2} \in \mathcal{D}'(\mathbb{R}^1)$ with the submersive $\mathcal{C}^\infty$ function $h(x) = t^2 - |x|^2$ is well-defined, see (1.4).) The representation of E in (2.13) was given already in [10, Lemme 4.2, p. 317], see also [18, Thm. 6.2.1, (6.2.1)$'$, p. 138] or [31, Ex. 1.6.17, p. 106].

Remark 2.5 Let us eventually observe that we could also employ the kernel

$$K^0_\lambda(x,\xi) = \frac{J_\lambda(|x|\cdot|\xi|)}{(|x|\cdot|\xi|)^\lambda}$$

for the analytic continuation of the Poisson–Bochner formula, yet only for a restricted class of distributions S. In fact, by partial Fourier transformation, it follows that

$$K^0_\lambda(x,\xi) \in \mathcal{D}_{L^\infty,n}(\mathbb{R}^n_x) \hat{\otimes} \mathcal{S}'(\mathbb{R}^n_\xi) \text{ where } \mathcal{D}_{L^\infty,n}(\mathbb{R}^n_x) = (1+|x|^2)^{-n/2} \cdot \mathcal{D}_{L^\infty}(\mathbb{R}^n_x).$$

Hence, for $S \in \mathcal{D}'_{L^1,-n}(\mathbb{R}^n_x) = (1+|x|^2)^{n/2} \cdot \mathcal{D}'_{L^1}(\mathbb{R}^n_x)$, the function

$$U^0 : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \longmapsto U^0_\lambda(\xi) = {}_{\mathcal{D}'_{L^1,-n}(\mathbb{R}^n_x)}\langle S(x), K^0_\lambda(x,\xi)\rangle_{\mathcal{D}_{L^\infty,n}(\mathbb{R}^n_x)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_\xi)} \quad (2.14)$$

is entire and $\mathcal{F}S = 2^{n/2-1}\Gamma(\frac{n}{2}) \cdot U^0_{n/2-1}$ if S is rotationally invariant.

If $S = \sin(t|x|)/|x|$, $t > 0$ fixed, then the assumption $S \in \mathcal{D}'_{L^1,-n}(\mathbb{R}^n)$ is satisfied and U^0_λ in (2.14) would yield the same representation of $\mathcal{F}S$ as in Example 2.2. If, in contrast, $S = 1$, then $S \notin \mathcal{D}'_{L^1,-n}(\mathbb{R}^n)$ and the entire distribution-valued function U^0 in (2.14) does not exist. Note, however, that U and $\tilde{U}$ in (2.5) and in (2.12), respectively, remain meaningful and yield

$$U_\lambda = \frac{2^{n-\lambda}\pi^{n/2}|\xi|^{2\lambda-2n+2}}{\Gamma\left(\lambda - \frac{n}{2} + 1\right)}, \qquad \mathrm{Re}\,\lambda > \tfrac{n}{2} - 1,$$

and

$$\tilde{U}_\lambda = \frac{2^{\lambda+2}\pi^n\Gamma(\lambda+1)}{\Gamma\left(\frac{n}{2}\right)^2}\,\delta, \qquad \lambda \in \mathbb{C}\backslash(-\mathbb{N}),$$

and $\mathcal{F}1 = 2^{n/2-1}\Gamma(\frac{n}{2})U_{n/2-1} = 2^{n/2-1}\Gamma(\frac{n}{2})\tilde{U}_{n/2-1} = (2\pi)^n\delta$ as expected.

3 The fundamental solution of the Euler–Poisson–Darboux operator

Let us turn now to the Euler–Poisson–Darboux operator

$$P_\alpha(t,\partial_t,\partial_x) = \partial_t^2 + \frac{2\alpha+1}{t}\,\partial_t - \Delta_n \tag{3.1}$$

acting on the space of distributions defined in the right half-space $\{(t,x)\in\mathbb{R}^{n+1};\ t>0\}$. Since $P_\alpha(t,\partial_t,\partial_x)$ is strictly hyperbolic with respect to t, it has a unique fundamental solution $E_{\alpha,\tau}(t,x)\in\mathcal{D}'(\mathbb{R}^{n+1})$ for each $\tau>0$ fulfilling

$$P_\alpha(t,\partial_t,\partial_x)E_{\alpha,\tau}(t,x) = \delta_\tau(t)\otimes\delta(x) \text{ and } \operatorname{supp} E_{\alpha,\tau}\subset\{(t,x)\in\mathbb{R}^{n+1};\ t\geq\tau\}, \tag{3.2}$$

see [19, Thm. 23.2.2, p. 392], [7, Ch. 6, Thm. and Def. 4.9, p. 379].

Moreover, the strict hyperbolicity of $P_\alpha(t,\partial_t,\partial_x)$ implies that $E_{\alpha,\tau}$ depends C^∞ on t for $t\geq\tau$ and that the support of $E_{\alpha,\tau}$ is contained in the propagation cone $\{(t,x)\in\mathbb{R}^{n+1};\ t\geq\tau+|x|\}$. In particular, $E_{\alpha,\tau}\in\mathcal{C}^\infty([\tau,\infty))\,\hat{\otimes}\,\mathcal{E}'(\mathbb{R}^n)$ and the partial Fourier transform $S_{\alpha,\tau}$ of $E_{\alpha,\tau}$ with respect to x fulfills

$$S_{\alpha,\tau} = \mathcal{F}_x(E_{\alpha,\tau}) \in \mathcal{C}^\infty([\tau,\infty))\,\hat{\otimes}\,\mathcal{O}_M(\mathbb{R}^n_x),$$

i.e., $S_{\alpha,\tau}$ is an infinitely differentiable mapping from $[\tau,\infty)$ into $\mathcal{O}_M(\mathbb{R}^n)$. By constructing the Green function of the ordinary differential operator $\partial_t^2+(2\alpha+1)t^{-1}\partial_t+|x|^2$, we next derive an explicit representation of $S_{\alpha,\tau}$.

Proposition 3.1 *For $\tau>0$ and $\alpha\in\mathbb{C}$, we have*

$$S_{\alpha,\tau} = \frac{\pi}{2}\,Y(t-\tau)\tau^{\alpha+1}t^{-\alpha}\big[-N_\alpha(\tau|x|)J_\alpha(t|x|) + J_\alpha(\tau|x|)N_\alpha(t|x|)\big]. \tag{3.3}$$

[*Here N_α, $\alpha\in\mathbb{C}$, denote the Bessel functions of the second kind.*]

Proof Upon Fourier transform with respect to x, (3.2) yields

$$\Big(\partial_t^2 + \frac{2\alpha+1}{t}\,\partial_t + |x|^2\Big)S_{\alpha,\tau}(t,x) = \delta_\tau(t). \tag{3.4}$$

This ordinary differential equation arises by substitution from Bessel's equation, and the vector space of its homogeneous solutions is generated by $t^{-\alpha}J_\alpha(t|x|)$ and $t^{-\alpha}N_\alpha(t|x|)$, see [21, C, 2.162, (9), p. 440] or [15, 8.491.6, p. 971].

Equation (3.4) implies that $S_{\alpha,\tau}$ has the two initial values $S_{\alpha,\tau}(\tau,x) = 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t}S_{\alpha,\tau})(\tau,x) = 1$. If therefore

$$S_{\alpha,\tau} = Y(t-\tau)\big[C_1t^{-\alpha}J_\alpha(t|x|) + C_2t^{-\alpha}N_\alpha(t|x|)\big],$$

then the constants C_1, C_2 are determined by the following system of linear equations:

$$0 = C_1\tau^{-\alpha}J_\alpha(\tau|x|) + C_2\tau^{-\alpha}N_\alpha(\tau|x|),$$

$$1 = C_1 \partial_\tau(\tau^{-\alpha} J_\alpha(\tau|x|)) + C_2 \partial_\tau(\tau^{-\alpha} N_\alpha(\tau|x|)).$$

The Wronskian determinant

$$W(\tau) = \det \begin{pmatrix} \tau^{-\alpha} J_\alpha(\tau|x|) & \tau^{-\alpha} N_\alpha(\tau|x|) \\ \partial_\tau(\tau^{-\alpha} J_\alpha(\tau|x|)) & \partial_\tau(\tau^{-\alpha} N_\alpha(\tau|x|)) \end{pmatrix}$$

fulfills $W(\tau) = C\tau^{-2\alpha-1}$, see [21, A, 17.1, p. 72], and the power series of J_α and N_α yield $C = \frac{2}{\pi}$. Hence

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{\pi}{2}\, \tau^{2\alpha+1} \begin{pmatrix} -\tau^{-\alpha} N_\alpha(\tau|x|) \\ \tau^{-\alpha} J_\alpha(\tau|x|) \end{pmatrix}$$

and this furnishes formula (3.3). □

Formula (3.3) is essentially equivalent with Eq. (5.5) in [8, p. 332], see also [6, pp. 30–32].

In order to evaluate the Fourier transform $E_{\alpha,\tau} = (2\pi)^{-n}\mathcal{F}_x(S_{\alpha,\tau})$, we use analytic continuation of the Poisson–Bochner formula according to Theorem 2.3. The representation of the fundamental solution $E_{\alpha,\tau}$ in Proposition 3.2 below coincides with formula (7.4) in [10, Lemme 7.1, p. 327] taking into account [10, (5.9), p. 319]. This result of S. Delache and J. Leray was also obtained (with the same method of proof) in [4, Thm. 2.1, (2.16), p. 501]. Our deduction of $E_{\alpha,\tau}$ is different and seems to be new.

Let us mention that an earlier appearance of this fundamental solution in the form of a "Riemann function" can be found in [40, p. 361, last line]. In fact, $E_{\alpha,\tau} = (\tau/t)^{1/2+\alpha} \cdot U^2(0,\tau;x,t)$ where, in Young's formula, $m = n+1$, $\lambda = \alpha$, $\Gamma = (t-\tau)^2 - |x-\xi|^2$, $H_m(2) = 2\pi^{(n-1)/2}\Gamma(\frac{3-n}{2})$ and hence

$$U^2(0,\tau;x,t) = \tfrac{1}{2}\pi^{(1-n)/2}\chi^{(1-n)/2}\big((t-\tau)^2 - |x|^2\big)\, {}_2F_1\big(\tfrac{1}{2}-\alpha, \tfrac{1}{2}+\alpha; \tfrac{3-n}{2}; -\tfrac{(t-\tau)^2-|x|^2}{4\tau t}\big).$$

(Note that $H_n(\alpha+2)$ is defined erroneously in [40, p. 357] and that the equation $\lambda = (1-k)/2$ in [40, p. 361] should read $\lambda = (k-1)/2$.)

Proposition 3.2 *Let $\tau > 0$ and assume that $n = 1$ or $n \in \mathbb{N}$ is even. Then the fundamental solution $E_{\alpha,\tau} \in \mathcal{S}'(\mathbb{R}^{n+1}_{t,x})$ of the Euler–Poisson–Darboux operator $P_\alpha(t,\partial_t,\partial_x) = \partial_t^2 + \frac{2\alpha+1}{t}\partial_t - \Delta_n$, i.e., the unique solution of (3.2), is given by*

$$E_{\alpha,\tau}(t,x) = \frac{1}{2}\pi^{(1-n)/2}\Big(\frac{\tau}{t}\Big)^{1/2+\alpha} Y(t-\tau)\chi^{(1-n)/2}((t-\tau)^2 - |x|^2)$$
$$\times {}_2F_1\Big(\frac{1}{2}-\alpha, \frac{1}{2}+\alpha; \frac{3-n}{2}; -\frac{(t-\tau)^2-|x|^2}{4\tau t}\Big). \tag{3.5}$$

[As said above, we interpret $E_{\alpha,\tau}$ as a continuous function of t with values in $\mathcal{E}'(\mathbb{R}^n_x)$ and vanishing for $t \le \tau$. Furthermore, for $t > \tau$, the composition h^*T of $T = \chi^{(1-n)/2} \in \mathcal{D}'(\mathbb{R}^1)$ with the submersive $\mathcal{C}^\infty$ function $h(x) = (t-\tau)^2 - |x|^2$ is well-defined, see (1.4), and so is the multiplication with the $\mathcal{C}^\infty$ function given by ${}_2F_1$.]

Proof (a) In order to apply Theorem 2.3, let us first check that $S_{\alpha,\tau}(t,x)$ in (3.5) is a $\mathcal{C}^\infty$ function of x in a neighborhood of 0. In fact, for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, we have

$$S_{\alpha,\tau}(t,x) = \frac{\pi}{2}\, Y(t-\tau)\tau^{\alpha+1}t^{-\alpha}\big[-N_\alpha(\tau|x|)J_\alpha(t|x|) + J_\alpha(\tau|x|)N_\alpha(t|x|)\big]$$
$$= \frac{\pi\tau^{\alpha+1}Y(t-\tau)}{2\sin(\alpha\pi)t^\alpha}\big[J_{-\alpha}(\tau|x|)J_\alpha(t|x|) - J_\alpha(\tau|x|)J_{-\alpha}(t|x|)\big]$$

which is an analytic function of $|x|^2$, see [15, Equs. 8.402 and 8.403.1, p. 951]. The same holds for $\alpha \in \mathbb{Z}$ by taking limits. Therefore, Theorem 2.3 applies and yields (for $t > \tau$ fixed) $\mathcal{F}_x S_{\alpha,\tau} = 2^{n/2-1}\Gamma(\frac{n}{2})\tilde{U}_{n/2-1}$, where

$$\tilde{U}_\lambda(\xi) = {}_{\mathcal{D}_{L^\infty}(\mathbb{R}^n_x)}\langle 1(x), S_{\alpha,\tau}(t,x)\cdot(|x|\cdot|\xi|)^{\lambda-n+2}J_\lambda(|x|\cdot|\xi|)\rangle_{\mathcal{D}'_{L^1}(\mathbb{R}^n_x)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_\xi)} \tag{3.6}$$

for $\lambda \in \mathbb{C}\setminus(-\mathbb{N})$.

(b) If $-1 < \operatorname{Re}\lambda < -\frac{1}{2}$, then we see, analogously as in Example 2.4, which is the special case of $\alpha = -\frac{1}{2}$, that $\tilde{U}_\lambda(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that the evaluation in (3.6) furnishes an absolutely convergent integral for $\xi \neq 0$. Hence we obtain the following for $-1 < \operatorname{Re}\lambda < -\frac{1}{2}$, $0 < \tau < t$ and $\xi \neq 0$ fixed:

$$\begin{aligned}(\tilde{U}_\lambda(\xi) = {} & \frac{\pi^{n/2+1}\tau^{\alpha+1}}{\Gamma\left(\frac{n}{2}\right)t^\alpha}|\xi|^{\lambda-n+2} \\ & \times \int_0^\infty \rho^{\lambda+1}\big[-N_\alpha(\tau\rho)J_\alpha(t\rho) + J_\alpha(\tau\rho)N_\alpha(t\rho)\big]\cdot J_\lambda(\rho|\xi|)\,\mathrm{d}\rho.\end{aligned}$$

According to [28, 10.51, 10.52, p. 93] (see also [1, (7.1), p. 45]), this integral yields

$$\tilde{U}_\lambda(\xi) = \frac{\sqrt{2}\,\pi^{(n+1)/2}\tau^{\alpha-\lambda}}{\Gamma\left(\frac{n}{2}\right)t^{\alpha+\lambda+1}}\,Y(t-\tau-|\xi|)|\xi|^{2\lambda-n+2}(u^2-1)^{-\lambda/2-1/4}p^{1/2+\lambda}_{-1/2+\alpha}(u)$$

where

$$u = \frac{t^2+\tau^2-|\xi|^2}{2t\tau} \geq 1 \text{ for } t-\tau \geq |\xi|.$$

The Legendre function $p^{1/2+\lambda}_{-1/2+\alpha}$ can be expressed by Gauß' hypergeometric function, see [28, p. 279], and this leads to

$$\begin{aligned}\tilde{U}_\lambda(x) = {} & \frac{2^{\lambda+1}\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\tau}{t}\right)^{1/2+\alpha}|x|^{2\lambda-n+2}\chi^{-1/2-\lambda}((t-\tau)^2-|x|^2) \\ & \times {}_2F_1\Big(\frac{1}{2}-\alpha, \frac{1}{2}+\alpha; \frac{1}{2}-\lambda; -\frac{(t-\tau)^2-|x|^2}{4\tau t}\Big).\end{aligned} \tag{3.7}$$

(c) Since $\tilde{U}_\lambda$ is holomorphic in $\mathbb{C}\setminus(-\mathbb{N})$, formula (3.7) persists by analytic continuation for those $\lambda \in \mathbb{C}\setminus(-\mathbb{N})$ where the right-hand side is, in this form, well-defined, i.e., for $\lambda \notin \frac{1}{2}+\mathbb{N}_0$. Note that ${}_2F_1(a,b;c;u)$ has poles for $c \in -\mathbb{N}_0$. In particular, if $n = 1$ or n is even and $\lambda = \frac{n}{2}-1$, then the hypergeometric series is regular at $c = \frac{1}{2}-\lambda = \frac{3-n}{2}$ and we obtain straightforwardly formula (3.5) in the proposition by using $E_{\alpha,\tau} = (2\pi)^{-n}2^{n/2-1}\Gamma(\frac{n}{2})\tilde{U}_{n/2-1}(x)$. This completes the proof. □

For odd space dimensions $n \geq 3$, Eq. (3.7) in the proof of Proposition 3.2 is still valid, but the analytic continuation into the point $\lambda = \frac{n}{2}-1$ needs more care due to the simple poles of the function $c \mapsto {}_2F_1(a,b;c;u)$ at $c = -k \in \mathbb{N}_0$. The representation of the fundamental solution $E_{\alpha,\tau}$ in Proposition 3.3 below coincides with formula (7.4) in [10, Lemme 7.1, p. 327] taking into account [10, (5.10), p. 320]. This result of S. Delache and J. Leray was also obtained (with the same method of proof) in [4, Thm. 2.2, (2.18), p. 502].

Proposition 3.3 *Let $\tau > 0$ and assume that $n \geq 3$ and $n \in \mathbb{N}$ is odd. Then the fundamental solution $E_{\alpha,\tau} \in \mathcal{S}'(\mathbb{R}^{n+1}_{t,x})$ of the Euler–Poisson–Darboux operator $P_\alpha(t,\partial_t,\partial_x) = \partial_t^2 +$*

$\frac{2\alpha+1}{t}\,\partial_t-\Delta_n$, *i.e., the unique solution of* (3.2), *is given by*

$$E_{\alpha,\tau}=\frac{1}{2\pi^{k+1}}\Big(\frac{\tau}{t}\Big)^{1/2+\alpha}Y(t-\tau)\Big[\sum_{j=0}^{k}c_j\,\delta^{(k-j)}\big((t-\tau)^2-|x|^2\big)$$
$$+c_{k+1}\,Y\big((t-\tau)^2-|x|^2\big)\,{}_2F_1\Big(\frac{n}{2}-\alpha,\frac{n}{2}+\alpha;\frac{n+1}{2};u\Big)\Big] \tag{3.8}$$

where

$$k=\frac{n-3}{2},\quad c_j=\frac{\big(\frac12-\alpha\big)_j\big(\frac12+\alpha\big)_j}{j!(-4\tau t)^j},\quad u=-\frac{(t-\tau)^2-|x|^2}{4\tau t}$$

and the Pochhammer symbol $(w)_j$ *is as in the introduction.*

[*As in* Proposition 3.2, $E_{\alpha,\tau}$ *is interpreted as a continuous function of* t *with values in* $\mathcal{E}'(\mathbb{R}^n_x)$ *and vanishing for* $t\le\tau$.]

Proof (a) From the series expansion in [15, 9.100, p. 1039] and the transformation formula [15, 9.131.1, p. 1043], we see that

$$f:\mathbb{C}\backslash(-\mathbb{N}_0)\longrightarrow\mathcal{E}((-\infty,1)):c\longmapsto f_c(u)={}_2F_1(a,b;c;u)$$

is a holomorphic function for fixed $a,b\in\mathbb{C}$ having at most simple poles in $c=-k$, $k\in\mathbb{N}_0$. The formula [15, 9.101.1, p. 1039] furnishes the residues:

$$\begin{aligned}\operatorname{Res}_{c=-k}f_c(u)&=\lim_{c\to-k}(c+k)\,{}_2F_1(a,b;c;u)\\&=\Big(\operatorname{Res}_{c=-k}\Gamma(c)\Big)\cdot\lim_{c\to-k}\frac{{}_2F_1(a,b;c;u)}{\Gamma(c)}\\&=\frac{(-1)^k(a)_{k+1}(b)_{k+1}u^{k+1}}{k!(k+1)!}\,{}_2F_1(a+k+1,b+k+1;k+2;u).\end{aligned}\tag{3.9}$$

Here we have used Pochhammer's symbol $(w)_j$ as defined in Sect. 1.

Furthermore, the Taylor series of $\operatorname{Pf}_{c=-k}f_c(z)$ up to order k is given by

$$\operatorname{Pf}_{c=-k}f_c(u)=\sum_{j=0}^{k}\frac{(a)_j(b)_j}{(-k)_j\,j!}\,u^j+O(u^{k+1}). \tag{3.10}$$

(b) Let us next investigate the holomorphic distribution-valued function

$$T:\{c\in\mathbb{C};\ 2c-3\notin\mathbb{N}_0\}\longrightarrow\mathcal{S}'(\mathbb{R}^n_x):c\longmapsto T_c=2^{-c}|x|^{-2c-n+3}\chi^{c-1}\big((t-\tau)^2-|x|^2\big)$$

for fixed $0<\tau<t$ and near a point $c=-k$, $k\in\mathbb{N}_0$. On the one hand, $\chi^{-k-1}=\delta^{(k)}$ (see [18, (3.2.17)′, p. 74]) implies

$$T_{-k}=2^k|x|^{2k-n+3}\delta^{(k)}\big((t-\tau)^2-|x|^2\big). \tag{3.11}$$

On the other hand, for $\operatorname{Re}c>0$, we have

$$\frac{\mathrm{d}\chi^{c-1}(s)}{\mathrm{d}c}=\frac{\mathrm{d}}{\mathrm{d}c}\Big(\frac{Y(s)s^{c-1}}{\Gamma(c)}\Big)=\frac{Y(s)s^{c-1}}{\Gamma(c)}\cdot[\log s-\psi(c)]$$

and in particular

$$\frac{\mathrm{d}\chi^{c-1}(s)}{\mathrm{d}c}\bigg|_{c=1} = Y(s)\cdot[\log s - \psi(1)] \in L^1_{\mathrm{loc}}(\mathbb{R}^1_s).$$

For $k \in \mathbb{N}_0$, this implies

$$\begin{aligned}\frac{\mathrm{d}\chi^{c-1}(s)}{\mathrm{d}c}\bigg|_{c=-k} &= \partial_s^{k+1}\Big(\frac{\mathrm{d}\chi^{c-1}(s)}{\mathrm{d}c}\bigg|_{c=1}\Big) = \partial_s^{k+1}\big(Y(s)\cdot[\log s - \psi(1)]\big)\\ &= \partial_s^k\big(s_+^{-1} - \psi(1)\delta\big) = (-1)^k k!\, s_+^{-k-1} - \psi(k)\delta^{(k)},\end{aligned}$$

see [31, p. 50] for the last equation. Here $s_+^{-k-1} = \mathrm{Pf}_{c=-k-1}\, Y(s)s^c \in \mathcal{S}'(\mathbb{R}^1_s)$ and $\psi(k) = \Gamma'(k)/\Gamma(k)$. So eventually, we obtain

$$\begin{aligned}\frac{\mathrm{d}T_c}{\mathrm{d}c}\bigg|_{c=-k} &= 2^k|x|^{2k-n+3}\Big[(-1)^k k!\, s_+^{-k-1}\big((t-\tau)^2 - |x|^2\big)\\ &\quad -\big(\psi(k) + \log(2|x|^2)\big)\cdot\delta^{(k)}\big((t-\tau)^2 - |x|^2\big)\Big].\end{aligned} \tag{3.12}$$

(c) As we have observed above, the distribution-valued function $\tilde{U}_\lambda$ is holomorphic in $\mathbb{C}\backslash(-\mathbb{N})$ and thus is regular at $\lambda = \frac{n}{2} - 1$. In fact, setting $n = 2k+3$, $k \in \mathbb{N}_0$, $a = \frac{1}{2} - \alpha$, $b = \frac{1}{2} + \alpha$, $c = \frac{1}{2} - \lambda$ and $u = -[(t-\tau)^2 - |x|^2]/(4\tau t)$ and assuming $0 < \tau < t$ fixed, we obtain

$$\mathrm{Res}_{\lambda=n/2-1}\,\tilde{U}_\lambda = -\frac{2^{3/2}\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2}\right)}\Big(\frac{\tau}{t}\Big)^{1/2+\alpha} T_{-k}\cdot \mathrm{Res}_{c=-k}\, f_c(u) = 0$$

since $T_{-k} = 2^k\delta^{(k)}(-4\tau t u)$ by (3.11) and $\mathrm{Res}_{c=-k}\, f_c(u)$ vanishes of order $k+1$ at $u = 0$ by (3.9).

Similarly, using [30, Prop. 1.6.3, p. 28] we conclude that

$$\begin{aligned}\tilde{U}_{n/2-1} &= \mathrm{Pf}_{\lambda=n/2-1}\,\tilde{U}_\lambda = \frac{2^{3/2}\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2}\right)}\Big(\frac{\tau}{t}\Big)^{1/2+\alpha}\mathrm{Pf}_{c=-k}[T_c(x)\cdot f_c(u)]\\ &= \frac{2^{3/2}\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2}\right)}\Big(\frac{\tau}{t}\Big)^{1/2+\alpha}\Big[\frac{\mathrm{d}T_c}{\mathrm{d}c}\bigg|_{c=-k}\\ &\quad\cdot \mathrm{Res}_{c=-k}\, f_c(u) + T_{-k}(x)\cdot\mathrm{Pf}_{c=-k}\, f_c(u)\Big].\end{aligned}$$

Equations (3.9/3.12) yield

$$\begin{aligned}\frac{\mathrm{d}T_c}{\mathrm{d}c}\bigg|_{c=-k}\cdot\mathrm{Res}_{c=-k}\, f_c(u) &= \frac{(-1)^{k+1}\left(\frac{1}{2}-\alpha\right)_{k+1}\left(\frac{1}{2}+\alpha\right)_{k+1}}{2^{k+2}(k+1)!(\tau t)^{k+1}}\\ &\quad\times Y\big((t-\tau)^2 - |x|^2\big)\,{}_2F_1\Big(\frac{n}{2}-\alpha, \frac{n}{2}+\alpha; \frac{n+1}{2}; u\Big)\end{aligned}$$

and Eqs. (3.10/3.11) yield

$$T_{-k}(x)\cdot\mathrm{Pf}_{c=-k}\, f_c(u) = 2^k\delta^{(k)}\big((t-\tau)^2 - |x|^2\big)\cdot\sum_{j=0}^{k}\frac{(a)_j(b)_j}{(-k)_j\, j!}u^j.$$

Upon using the identity

$$s^j\delta^{(k)}(s) = (-k)_j\delta^{(k-j)}(s) \in \mathcal{D}'(\mathbb{R}^1_s), \qquad j, k \in \mathbb{N}_0,\ j \le k,$$

this results in

$$T_{-k}(x)\cdot \mathrm{Pf}_{c=-k}\, f_c(u) = 2^k \cdot \sum_{j=0}^{k} \frac{(a)_j (b)_j}{j!(-4\tau t)^j}\, \delta^{(k-j)}\big((t-\tau)^2 - |x|^2\big).$$

Finally, we make use of $E_{\alpha,\tau} = (2\pi)^{-n} 2^{n/2-1} \Gamma(\frac{n}{2}) \tilde{U}_{n/2-1}$ in order to conclude the representation of $E_{\alpha,\tau}$ in (3.8). This completes the proof. □

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